THE SUMMARY OF Ph. D. DISSERTATION

Major		SURNAME, Firstname
Mathematics		NAKASUJI, Maki

Title

Spectral Theory and Prime Geodesic Theorem of Hyperbolic Spaces

Abstract

The study of the asymptotic distribution of prime closed geodesics, which is called the *prime geodesic theorem*, is an interesting subject, for it is analogous to the distribution of prime numbers known as a long-standing subject in Mathematics.

Let Γ be the fundamental group of the Riemannian manifold and $\pi_\Gamma(x)$ be the number of prime geodesics P whose length $l(P)$ satisfies $e^{l(P)} \leq x$. The problem is to give precise terms of the function $\pi_\Gamma(x)$. The prime geodesic theorem states that

$$\lim_{x\to\infty} \frac{\pi_\Gamma(x)}{\mathrm{li}(x^d)} = 1,$$

where $\mathrm{li}(x) := \int_2^x \frac{dt}{\log t}$, when the manifold is of finite volume (but not necessarily compact) with constant curvature (say equal to -1) and of dimension $(d+1)$.

Our present aim is to obtain the best estimate of the remainder term $\pi_\Gamma(x) - \mathrm{li}(x^d)$. Most of the known estimates so far of the remainder term are the upper bounds. The only known result for the lower bound, due to Hejhal, is for a 2-dimensional hyperbolic manifold, with an extra arithmetic assumption concerning the zeros of the Selberg zeta function for $\Gamma \subset PSL(2, \mathbf{R})$ being a discrete subgoup.

In this thesis, we give the lower bounds for general hyperbolic manifolds. In contrast with the arithmetic assumption by Hejhal, we consider the problem from the viewpoint of the contribution of the spectra of the Laplace-Beltrami operator on the hyperbolic manifold.

We obtain results both (1) without any extra assumption about the spectrum and (2) with an added assumption, but one which is weaker than Hejhal's. This includes

a generalization to the dimension 3 case in (1), and to general $(d+1)$ in (2). More precisely, we obtain

(1)
Theorem 1.[Theorems 1.4.1 and 1.4.3]
For $d = 1$ (resp. $d = 2$), when $\Gamma \subset PSL(2, \mathbf{R})$ (resp. $PSL(2, \mathbf{C})$) is a discrete subgroup with finite covolume, we have

$$\pi_\Gamma(x) = \mathrm{li}(x^d) + \Omega(x^{\frac{d}{2}-\varepsilon}), \qquad \text{as} \quad x \to \infty,$$

where ε is any positive constant.

(2)
Theorem 2.[Theorem 1.4.5]
For any positive integer d, let $\Gamma \subset SO_e(d+1, 1)$ be a discrete subgroup with finite covolume. If the contribution of the discrete spectrum of Γ is larger than that of the continuous one in the Weyl's law, we have

$$\pi_\Gamma(x) = \mathrm{li}(x^d) + \Omega_\pm\left(\frac{x^{\frac{d}{2}}(\log\log x)^{\frac{1}{d+1}}}{\log x}\right) \qquad \text{as} \quad x \to \infty.$$

Since the conjectural exponent of x of the remainder term is $d/2$, the both theorems give sharp estimates in this sense.

This thesis is organized as follows: In Chapter 1 we describe the prime geodesic theorem and the spectral theory for our Γ, and state the main results. Chapter 2 contains the 2-dimensional cases of the prime geodesic theorem. In Chapter 3, we discuss the 3-dimensional cases without any arithmetic assumption (1) and in Chapter 4, with the spectral assumption (2). Chapter 5 is devoted to a study of higher-dimensional cases.