The metrical theory of non-archimedean diophantine approximations

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Chapter 1

Introduction

In this thesis, we study the metrical theory of the non-archimedean diophantine approximation. In Chapter 2 and Chapter 3, we discuss about the conditions for having infinitely many solutions. In Chapter 4 and Chapter 5, we study the convergent rate of some multi-dimensional continued fraction expansions which give some simultaneous approximation sequences. We give a short sketch of the metrical theory of diophantine approximations for real numbers as a historical ground in this chapter and then state main results with some basic definitions and notations.

1.1 Background

In the studies of the metric diophantine approximation, there are the following two important questions.

(i) Whether $|x - \frac{p}{q}| < \frac{\psi(q)}{q}$ has infinitely many solutions for a.e. $x \in [0, 1)$ or not.

(ii) Whether some solutions which give good convergences exist or not.

A.Khintchine is the first author who proved a theorem concerned (i).

The Khintchine Theorem (1925)

Let $\psi(q)$ be a positive continuous function of a positive integer q, and suppose

$q\,\psi(q)$ is non-increasing. Then

$$\left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{q} \tag{1.1}$$

with $p,\ q \in \mathbb{Z}_+$ has infinitely many solutions for a.e. $\alpha \in [0,1)$, provided that the sum

$$\sum_{q=1}^{\infty} \psi(q) \tag{1.2}$$

diverges. On the other hand, if (1.2) converges, (1.1) has only finitely many solutions for almost every α.

After this theorem, some attempts were made by many people to weaken the condition (1.2). In 1941, R.J.Duffin and A.C.Schaeffer showed (1.1) has infinitely many solutions under a weaker condition [6].

The Duffin-Schaeffer Theorem (1941)

Let $\psi(q)$, $q \in \mathbb{N}$, be an arbitrary sequence of non-negative real numbers less than $\frac{1}{2}$ such that

$$\sum_{q=1}^{\infty} \psi(q) = \infty$$

and suppose there exists an infinite set of positive integers Q such that

$$\sum_{q \le Q} \psi(q) < c_1 \sum_{q \le Q} \psi(q) \frac{\phi(q)}{q},$$

where $\phi(q)$ is the Euler function and c_1 is a positive constant. Then for a.e. $\alpha \in [0,1)$

$$\left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{q}$$

with $(p,q) = 1$, $p,\ q \in \mathbb{Z}_+$ has infinitely many solutions.

Note that $\phi(q)$ is the number of q' such that $(q,q') = 1$ and $q' < q$.

At the same time, they also gave the following conjecture.

The Duffin-Schaeffer Conjecture (1941)

Let $\psi(q)$, $q \in \mathbb{N}$, be an arbitrary sequence of non-negative real numbers less than $\frac{1}{2}$. Then

$$\left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{q}, \quad (p,q) = 1,$$

has infinitely many solutions for a.e. $\alpha \in [0,1)$ if and only if

$$\sum_{q=1}^{\infty} \psi(q) \frac{\phi(q)}{q} = \infty. \tag{1.3}$$

This conjecture has not been proved yet, but in 1978, J.D.Vaaler showed that (1.3) could be replaced with $\psi(q) = o(\frac{1}{q})$.

In the meantime, the multi-dimensional version of this problem has been done, which we call the simultaneous approximation problem. A.Khintchine showed a theorem concerned the simultaneous approximation in 1926.

The Khintchine Theorem (1926)

Let $r \in \mathbb{N}$ and $\psi(q)$ be a positive continuous function of a positive q such that $q\,\psi^r(q)$ converges monotonically to 0 as $q \to \infty$. Then for a.e. $(\alpha_1, \ldots, \alpha_r) \in [0,1)^r$,

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{\psi(q)}{q}, \quad (p_i, q) = 1,\ p_i, q \in \mathbb{Z}, \tag{1.4}$$

for $1 \le i \le r$ has infinitely many solutions provided

$$\sum_{q=1}^{\infty} \psi^r(q) \tag{1.5}$$

diverges. On the other hand, if (1.5) converges, (1.4) has only at most finitely many solutions.

Similarly, there is a multi-dimensional version of the Duffin-Schaeffer condition which was given by Sprindžuk [27].

Theorem (1979)

Let $\psi(q)$, $q \in \mathbb{Z}$, be any sequence of non-negative real numbers, which is less than

$\frac{1}{2}$, *such that*

$$\sum_{q=1}^{\infty} \psi^r(q) = \infty.$$

Suppose there are infinitely many sets of $Q \in \mathbb{Z}_+$ such that

$$\sum_{q \le Q} \psi^r(q) < c_2 \sum_{q \le Q} \psi^r(q) \left(\frac{\phi(q)}{q}\right)^r.$$

Here, c_2 is a positive constant and ϕ is the Euler function. Then, for a.e. $(\alpha_1, \ldots, \alpha_r) \in [0,1)^r$,

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{\psi(q)}{q} \qquad (p_i, q) = 1,\ p_i, q \in \mathbb{Z}$$

for $1 \le i \le r$ has infinitely many solutions.

Also Sprindžuk proposed a multi-dimensional version of the Duffin-Schaeffer conjecture.

The r-dimensional Duffin-Schaeffer Conjecture (1979)

Let $\psi(q)$, $q \in \mathbb{N}$ be an arbitrary sequence of non-negative real numbers less than $\frac{1}{2}$. Then

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{\psi(q)}{q} \qquad (p_i, q) = 1,\ p_i, q \in \mathbb{Z}$$

for $1 \le i \le r$ has infinitely many solutions for almost every $(\alpha_1, \ldots \alpha_r) \in [0,1)^r$ if and only if

$$\sum_{q=1}^{\infty} \psi^r(q) \left(\frac{\phi(q)}{q}\right)^r = \infty.$$

In 1990, A.D.Pollington and R.C.Vaughan proved that an r-dimensional version of the Duffin-Schaeffer conjecture is true for $r > 1$ (see [24]), however, the original one-dimensional Duffin-Schaeffer conjecture still remains open until now.

Now we turn to the problem concerned (ii). In the one-dimensional case, it is well-known that the continued fraction expansion gives a good convergent sequence of rational numbers. Because continued fractions are related to the

Euclidian algorithm (see [2]), it seemed to be natural to extend the notion of continued fractions to the multi-dimensional case as the higher dimensional Euclidean algorithm.

Then we get multi-dimensional maps which induce various multi-dimensional continued fractions. The Jacobi-Perron algorithm is one of the most natural one in the sense that it comes from the Euclidean algorithm. Rational vectors induced from this algorithm have a good property as the simultaneous approximation. Here we give the definition of map T associated to the Jacobi-Perron algorithm.

$$T(\alpha_1, \ldots, \alpha_r) = \left(\frac{\alpha_2}{\alpha_1} - \left[\frac{\alpha_2}{\alpha_1} \right], \ldots, \frac{\alpha_r}{\alpha_1} - \left[\frac{\alpha_r}{\alpha_1} \right], \frac{1}{\alpha_1} - \left[\frac{1}{\alpha_1} \right] \right)$$

for $(\alpha_1, \ldots, \alpha_r) \in [0,1)^r$. From this map, we can get a simultaneous approximation sequence which converges to $(\alpha_1, \ldots, \alpha_r)$. F.Schweiger proved that the existence of an absolutely continuous invariant measure and its ergodicity, and showed that the convergent rate of the approximation sequence is exponential in the two-dimensional case [26].

Theorem (2-dimensional case: 1996)

There exists a constant $\delta > 0$ *such that for a.e.* $(\alpha_1, \alpha_2) \in [0,1)^2$ *there exists* $n_0 = n_0(\alpha_1, \alpha_2)$ *such that for any* $n \geq n_0$

$$|q_n| \left| \alpha_1 - \frac{p_n}{q_n} \right| < \frac{1}{q_n^\delta}, \quad |q_n| \left| \alpha_2 - \frac{r_n}{q_n} \right| < \frac{1}{q_n^\delta},$$

where the integers p_n, q_n, r_n *are provided by the Jacobi-Perron algorithm.*

After F.Schweiger, this convergent exponent was studied by K. Nakaishi [19], A.B. Alamichel and Y. Guivarc'h [3] etc.

An algorithm similar to the Jacobi-Perron algorithm, E.V.Podsypanin considered the following map S, which is called the modified Jacobi-Perron algorithm [23]. This expansion is associated with the following map:

$$S(\alpha_1, \ldots, \alpha_r) = \left(\frac{\alpha_1}{\alpha_j} - \left[\frac{\alpha_1}{\alpha_j} \right], \ldots, \frac{1}{\alpha_j} - \left[\frac{1}{\alpha_j} \right], \ldots, \frac{\alpha_r}{\alpha_j} - \left[\frac{\alpha_r}{\alpha_j} \right] \right)$$

if $\alpha_j > \alpha_{i_1}$ for $1 \leq i_1 \leq j-1$ and $\alpha_j \geq \alpha_{i_2}$ for $j+1 \leq i_2 \leq r$. F.Schweiger also proved that the existence of an absolutely continuous invariant measure and its ergodicity and then the exponential convergent property of the modified Jacobi-Perron algorithm was shown by S.Ito, M.Keane and M.Otsuki in 1993 for the two-dimensional case [15] and T.Fujita and others in 1996 [9].

Theorem (2-dimensional case: 1993 and 1996)

There exists a constant $\delta' > 0$ such that for a.e. $(\beta_1, \beta_2) \in [0,1)^2$ there exists $n_0 = n_0(\beta_1, \beta_2)$ such that for any $n \geq n_0$

$$|q'_n| \left|\beta_1 - \frac{p'_n}{q'_n}\right| < \frac{1}{{q'}_n^{\delta'}}, \quad |q'_n| \left|\beta_2 - \frac{r'_n}{q'_n}\right| < \frac{1}{{q'}_n^{\delta'}},$$

where the integers p'_n, ${q'}_n$, r'_n are provided by the modified Jacobi-Perron algorithm.

Later, a simple proof of this theorem was given by R.Meester [17], however, it seems to be very hard to get the exponential convergent estimate for higher dimensional case.

In the sequel, we discuss the metric property of diophantine inequality (1.1) for the formal Laurent power series and get a necessary and sufficient condition or a sufficient condition for having infinitely many solutions in the one-dimensional and higher dimensional cases. Then we also discuss the exponential convergent property of the Jacobi-Perron algorithm and the modified Jacobi-Perron algorithm for the formal Laurent power series. In the formal Laurent power series' situation, the problem is simpler than that of the classical real number case. So, we have the exponential convergent property for any dimensional Jacobi-Perron algorithm and modified Jacobi-Perron algorithm in the formal Laurent power series.

1.2 Definitions

Throughout this thesis, we use the following definitions and notations. Let $\mathbb{F}_q$ be a finite fields with q elements and we consider the following:

$\mathbb{F}_q[X] = \{a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \, , a_i \in \mathbb{F}_q, \, 0 \le i \le n\}$

: the set of polynomials of $\mathbb{F}_q$-coefficients,

$\mathbb{F}_q(X) = \{\frac{P}{Q} : \, P, Q \in \mathbb{F}_q[X], \, Q \neq 0\}$

: the set of rational functions,

$\mathbb{F}_q((X^{-1})) = \{a_n X^n + a_{n-1} X^{n-1} + \cdots \, , \, a_i \in \mathbb{F}_q, \, i \le n, \, a_n \neq 0, \, n \in \mathbb{Z}\}$

: the set of formal Laurent power series of $\mathbb{F}_q$- coefficients.

We regard $\mathbb{F}_q[X]$, $\mathbb{F}_q(X)$ and $\mathbb{F}_q((X^{-1}))$ as the set of integers, of rational numbers and of real numbers, respectively. We denote 0 and 1 by the additive unity and the multiplicative unity of $\mathbb{F}_q$, respectively. Note that we identify $a_0X^0 \in \mathbb{F}_q[X]$ with $a_0 \in \mathbb{F}_q$. For $f = a_nX^n + a_{n-1}X^{n-1} + \cdots \in \mathbb{F}_q((X^{-1}))$, we put

$$\deg f \; = \begin{cases} n & \text{if} \quad a_n \neq 0, \\ -\infty & \text{if} \quad f \equiv 0. \end{cases}$$

We define the valuation of f by

$$|f| \; = \; q^{\deg f}.$$

Also we put

$$[f] \; = \; a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \qquad \text{for} \quad f \in \mathbb{F}_q((X^{-1})).$$

We define

$$\mathbb{L} = \{ \, f = a_{-1}X^{-1} + \cdots + a_{-i}X^{-i} + \cdots \, , \, a_i \in \mathbb{F}_q \quad \text{for } i \le -1\},$$

which is a compact abelian group with the metric $d(f,g) = |f - g|$. We denote by m the normalized Haar measure on $\mathbb{L}$. Note that

$$m\{f = c_{-1}X^{-1} + c_{-2}X^{-2} + \cdots : c_{-1} = c_1',\, c_{-2} = c_2',\, \ldots ,c_{-l} = c_l' \} = \frac{1}{q^l} \quad (1.6)$$

for any c_1', c_2', $\ldots$ $c_l' \in \mathbb{F}_q$. Then, we put m^r be the normalized Haar measure on $\mathbb{L}^r$.

When P and Q are coprime, which means P and Q have no non-trivial common factor, we write $(P,Q) = 1$. We define $\Phi(Q)$ be the number of the polynomials P such that

$$\deg P < \deg Q\,, \qquad (P,Q) = 1$$

1.3 Main results

In Chapter 2, we consider the problem whether

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(Q)}{|Q|}, \qquad (P,Q) = 1, \quad P,\, Q \in \mathbb{F}_q[X], \qquad (1.7)$$

has infinitely many solutions $\frac{P}{Q}$ or not for m-a.e. $f \in \mathbb{L}$. First, we assume ψ be a function which depends only on the degree of $Q \in \mathbb{F}_q[X]$. In this case, we get a necessary and sufficient condition for having infinitely many solutions by using a continued fraction algorithm [13].

Theorem 2.2.1 *Let ψ be a non-negative function defined on $\mathbb{F}_q[X]$ such that $\psi(Q)$ depends only on the degree of $Q \in \mathbb{F}_q[X]$. For any set S of positive integers,* (1.7) *with* $\deg Q \in S$ *has infinitely many solutions for m-a.e. $f \in \mathbb{L}$ if and only if*

$$\sum_{n \in S} q^n \psi(X^n) = \infty.$$

By this theorem, we would be able to say that we get the complete answer to (i) in §2 when $\psi(Q)$ depends only on the degree of Q for the non-archimedean case.

Next, we generalize ψ, that is, we assume that ψ is a function which depends not only on the degree of Q but also Q itself. Then we have the following theorem [13].

Theorem 2.3.1 (Gallagher type theorem)

For any ψ, (1.7) has infinitely many solutions $\frac{P}{Q}$ for a.e. $f \in \mathbb{L}$ or (1.7) has at most finitely many solutions $\frac{P}{Q}$ for a.e. $f \in \mathbb{L}$.

From this theorem, if we show the set of f such that (1.7) has infinitely many solutions has a positive measure, then we see that it is a set of full measure. In this way, we have the following theorem which is a non-archimedean version of the Duffin-Schaeffer theorem [13].

Theorem 2.3.2 (Duffin-Schaeffer type theorem)

Let ψ be a $\{q^{-n} : n \geq 0\} \cup \{0\}$-valued function which satisfies

$$\sum_{n=1}^{\infty} \sum_{\substack{\deg Q = n \\ Q:\text{monic}}} \psi(Q) = \infty.$$

Suppose there are infinitely many positive integers n such that

$$\sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi(Q) < C \sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi(Q) \frac{\Phi(Q)}{|Q|}$$

holds for a constant C. Then

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(Q)}{|Q|}, \qquad (P, Q) = 1,$$

has infinitely many solutions $\frac{P}{Q}$ for m-a.e. $f \in \mathbb{L}$.

In Chapter 3, we extend the Duffin-Schaeffer type theorem to the multi-dimensional case, that is, the simultaneous approximation problems. As in the one-dimensional case, we first show the Gallagher type theorem [11].

Theorem 3.1.1 (Gallagher type theorem)

For any ψ,

$$\left|f_1 - \frac{P_1}{Q}\right| < \frac{\psi(Q)}{|Q|}, \dots, \left|f_r - \frac{P_r}{Q}\right| < \frac{\psi(Q)}{|Q|}$$

$$(P_1, Q) = (P_2, Q) = \cdots = (P_r, Q) = 1.$$

has infinitely many solutions of $(Q, P_1, \dots, P_r)$ *for* m^r*-a.e.* $(f_1, \dots, f_r) \in \mathbb{L}^r$ *or has only finitely many solutions for* m^r*-a.e.* $(f_1, \dots, f_r) \in \mathbb{L}^r$.

By using Theorem 3.1.1, we also have the Duffin-Schaeffer type theorem in the multi-dimensional case [11].

Theorem 3.1.2 (Duffin-Schaeffer type theorem)

Let ψ *be a* $\{q^{-n} \mid n \geq 0\} \cup \{0\}$*-valued function which satisfies*

$$\sum_{n=1}^{\infty} \sum_{\substack{\deg Q = n \\ Q: monic}} \psi^r(Q) = \infty.$$

Suppose for a positive constant C, *there are infinitely many positive integers* n *such that*

$$\sum_{\substack{\deg Q \leq n \\ Q: monic}} \psi^r(Q) < C \sum_{\substack{\deg Q \leq n \\ Q: monic}} \psi^r(Q) \frac{\Phi^r(Q)}{|Q|^r}$$

holds. Then

$$\left|f_1 - \frac{P_1}{Q}\right| < \frac{\psi(Q)}{|Q|}, \dots, \left|f_r - \frac{P_r}{Q}\right| < \frac{\psi(Q)}{|Q|}$$

$$(P_1, Q) = (P_2, Q) = \cdots = (P_r, Q) = 1.$$

has infinitely many solutions $\left(\frac{P_1}{Q}, \dots, \frac{P_r}{Q}\right)$ *for* m^r*-a.e.* $(f_1, \dots, f_r) \in \mathbb{L}^r$.

In Chapter 4 and Chapter 5, we consider a problem concerned (ii) in §1 for the non-archimedean case. First, we consider the Jacobi-Perron algorithm for the formal Laurent power series in Chapter 4. In this case, we can associate the following map with the Jacobi-Perron algorithm:

$$T(f_1, \dots, f_r) = \left(\frac{f_2}{f_1} - \left[\frac{f_2}{f_1}\right], \dots, \frac{f_r}{f_1} - \left[\frac{f_r}{f_1}\right], \frac{1}{f_1} - \left[\frac{1}{f_1}\right]\right)$$

for $(f_1, \dots, f_r) \in \mathbb{L}^r$. The study of this algorithm for formal Laurent power series have been already done by R.Paysant-Leroux, E.Dubois [20], K.Feng and F.Wang

[7]. They showed the existence of its convergence and the ergodicity. Here we consider the rate of its convergence. The following is an a priori estimate [12].

Theorem 4.2.1 *For any $\nu \geq 1$, there exists a positive constant C such that*

$$|A_0^{(\nu)}| \left| f_i - \frac{A_i^{(\nu)}}{A_0^{(\nu)}} \right| < \frac{C}{q^{\frac{\nu}{r}}} \qquad 1 \leq i \leq r$$

for $(f_1, \ldots, f_r) \in \mathbb{L}^r$ where $T^\nu(f_1, \ldots, f_r) \in \mathbb{L}^r$ for $\nu \geq 1$.

Note that $\frac{A_i^{(\nu)}}{A_0^{(\nu)}}$ is ν-th convergence by the Jacobi-Perron algorithm (refer to Chapter 4, §1).

We discuss the stochastic property of the Jacobi-Perron algorithm digits and then get on better estimate. In particular, the degree of the denominator of convergent fractions [12].

Theorem 4.2.2 *For any $\nu \geq 1$, there exists a positive constant C' such that*

$$|A_0^{(\nu)}| \left| f_i - \frac{A_i^{(\nu)}}{A_0^{(\nu)}} \right| < \frac{C'}{|A_0^{(\nu)}|^{\frac{1}{r}(\frac{\gamma}{\rho} - \varepsilon)}} \qquad \forall \varepsilon > 0$$

for m^r-a.e. $(f_1, \ldots, f_r) \in \mathbb{L}^r$, where

$$\gamma = \frac{q^{r^2}}{q^{r^2} - 1}, \qquad \rho = \frac{q}{q-1}.$$

Next, we consider the modified Jacobi-Peron algorithm. We can associate the following map with the modified Jacobi-Perron algorithm:

$$S(f_1, \ldots, f_r) = \left(\frac{f_1}{f_j} - \left[\frac{f_1}{f_j} \right], \ldots, \frac{1}{f_j} - \left[\frac{1}{f_j} \right], \ldots, \frac{f_r}{f_j} - \left[\frac{f_r}{\alpha_j} \right] \right)$$

if $\deg f_j > \deg f_{i_1}$ for $1 \leq i_1 \leq j-1$ and $\deg f_j \geq \deg f_{i_2}$ for $j+1 \leq i_2 \leq r$. In this case, some converges but not exponential rate because the associated map depends on the degrees of $f_1, \ldots, f_{r-1}$ and f_r. For example, if the degree of the first component of $S^\nu(f_1, \ldots, f_r)$ is always greater than the others for $\nu \geq 1$, then the speed of convergence of the i-th component, $2 \leq i \leq r$, gets to be very slow.

However, for a.e. $(f_1, \dots, f_r)$, we see exponential rate of convergence [14].

Theorem 5.1.2

(i) *If* $S^\nu(f_1, \dots, f_r) \not\equiv 0$ *for any* $\nu \geq 1$,

$$\lim_{\nu \to \infty} \frac{B_i^{(\nu)}}{B_0^{(\nu)}} = f_i \qquad \textit{for} \quad 1 \leq i \leq r,$$

on the other hand, if $S^{\nu-1}(f_1, \dots, f_r) \not\equiv 0$ *and* $S^\nu(f_1, \dots, f_r) \equiv 0$, *then*

$$\frac{B_i^{(\nu)}}{B_0^{(\nu)}} = f_i \qquad \textit{for} \quad 1 \leq i \leq r.$$

(ii) *For a given sequence of arrays* $\{b_i^{(\nu)} : 1 \leq i \leq r+1, \nu \geq 1\}$;

$$b_{r+1}^{(\nu)} \in \mathbb{F}_q[X], \qquad \deg b_{r+1}^{(\nu)} \geq 1,$$

$$b_i^{(\nu)} = 0 \qquad \textit{for} \quad 1 \leq i < j(\nu), \qquad b_i^{(\nu)} \in \mathbb{F}_q \qquad \textit{for} \quad j(\nu) \leq i \leq r$$

with a sequence $j(1), j(2), \dots$ $(1 \leq j(\nu) \leq r,\ \nu \geq 1)$, *there exists* $(f_1, \dots, f_r) \in \mathbb{L}^r$ *such that* $\kappa(\nu) = j(\nu)$.

Note that $\frac{B_i^{(\nu)}}{B_0^{(\nu)}}$ is the map associated with the modified Jacobi-Perron algorithm. For the map S, we can prove the ergodicity [14].

Theorem 5.2.2 (i) *For any Borel set* $B \subset \mathbb{L}^r$,

$$m^r(S^{-1}B) = m^r(B),$$

that is, m^r *is an invariant probability measure for* S.

(ii) $\left\{ \begin{pmatrix} b_1^{(\nu)} \\ \vdots \\ b_{r+1}^{(\nu)} \end{pmatrix} : \nu \geq 1 \right\}$ *is an independent and identically distributed sequence as a sequence of random variables.*

Here, $b_1^{(\nu)}, \dots, b_{r+1}^{(\nu)}$ are the coefficients of ν-th modified Jacobi-Perron expansions induced by the map S (refer to Chapter 5, §1). From these theorems, we can show the exponential convergent rate [14].

Proposition 5.2.3 *For a.e.* $(f_1, \ldots, f_r) \in \mathbb{L}^r$, *there exists a positive constant* $C_1 = C_1(\varepsilon)$ *such that*

$$|B_0^{(\nu)}| \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| < \frac{C_1}{q^{\nu\alpha(1-\varepsilon)}} \qquad \text{for any } \varepsilon > 0, \quad 1 \leq i \leq r.$$

In the Jacobi-Perron algorithm, we have a priori estimate of the convergent rate for all $(f_1, \ldots, f_r) \in \mathbb{L}^r$. But in the modified Jacobi-Perron algorithm, we have the estimate only for almost all $(f_1, \ldots, f_r) \in \mathbb{L}^r$.

Finally, we have the estimate associated to the degree of the denominator of convergent fractions [14].

Theorem 5.2.3 *For a.e.* $(f_1, \ldots, f_r) \in \mathbb{L}^r$, *there exists a positive constant* $C_2 = C_2(\varepsilon)$ *such that*

$$\left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| < \frac{C_2}{|B_0^{(\nu)}|^{1+\frac{\alpha}{\gamma}(1-\varepsilon)}} \qquad \text{for any } \varepsilon > 0, \quad 1 \leq i \leq r,$$

where

$$\gamma = \frac{q^r}{q^r - 1}$$

and α *is a positive constant which is given in Chapter 5 §2.*

Chapter 2

Diophantine approximation for one-dimensional case

2.1 Continued fraction expansion

In this section, we see the continued fraction expansion for the formal Laurent power series. We refer Berthé and Nakada [1].

Let T be the map of $\mathbb{L}$ onto itself defined by

$$Tf = f^{-1} - [f^{-1}], \qquad f \in \mathbb{L}.$$

Then we have

$$f = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \ddots}} =: [0; p_1, p_2, \ldots]$$

with

$$p_n = [(T^{n-1}f)^{-1}].$$

As in the classical case, we define

$$\begin{cases} P_n = p_n P_{n-1} + P_{n-2}, & P_0 = 0, \quad P_1 = 1, \\ Q_n = p_n Q_{n-1} + Q_{n-2}, & Q_0 = 1, \quad Q_1 = p_1, \end{cases} \tag{2.1}$$

and have the following:

$$P_nQ_{n-1} - Q_nP_{n-1} = \pm 1,$$

$$\frac{P_n}{Q_n} = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \ddots + \cfrac{1}{p_n}}} =: [0\,;\, p_1,\, \dots\, ,\, p_n]$$

for $n \geq 1$. We call $\frac{P_n}{Q_n}$ the n-th convergent fraction of f. Since

$$f = \frac{P_n + T^n f \cdot P_{n-1}}{Q_n + T^n f \cdot Q_{n-1}},$$

it is easy to see that

$$\left| f - \frac{P_n}{Q_n} \right| < \frac{1}{|Q_n|^2} \qquad \text{for } n \geq 1.$$

Moreover, we have the following:

Lemma 2.1.1 *If coprime two non-zero polynomials P and Q satisfy*

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2},$$

then

$$\frac{P}{Q} = \frac{P_n}{Q_n}$$

for some $n \geq 1$.

We put

$$W_n = \left\{ \frac{P}{Q} \in \mathbb{L} \,:\, \deg Q = n,\, (P\,,\, Q) = 1,\, P, Q \in \mathbb{F}_q[X] \right\}$$

for $n \geq 1$. The following is essential in the next chapter. This lemma was shown in [4] and we prove it here by using continued fractions.

Lemma 2.1.2

$$\# W_n = q^{2n} - q^{2n-1} \quad \textit{for} \quad n \geq 1.$$

Proof. If $n = 1$, all elements in W_1 are of the form

$$\frac{P}{Q} = \frac{a}{X+b}, \quad \text{with } a,\, b \in \mathbb{F}_q,\ a \neq 0.$$

This implies the assertion. Now we suppose

$$\#W_i = q^{2i} - q^{2i-1} \quad \text{for } 1 \leq i \leq n.$$

Fix $\frac{P}{Q} \in W_{n+1}$. Then we have its continued fraction expansion uniquely:

$$\frac{P}{Q} = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \ddots + \cfrac{1}{p_m}}} = [0\,;\, p_1,\, p_2,\, \ldots,\, p_m].$$

So we get a unique element $\frac{P'}{Q'} \in W_j$ for some j, $1 \leq j \leq n$ by

$$\frac{P'}{Q'} = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \ddots + \cfrac{1}{p_{m-1}}}} = [0\,;\, p_1,\, p_2,\, \ldots,\, p_{m-1}]$$

unless $m = 1$. On the other hand, for any $\frac{P'}{Q'} \in W_j$, $1 \leq j \leq n$, we have $q^{n+1-j}(q-1)$ numbers of $\frac{P}{Q} \in W_{n+1}$ by (2.1). The number of $\frac{P}{Q}$ with $\deg Q = n+1$ and $\deg P = 0$ is $q^{n+1}(q-1)$. Thus we see

$$\#W_{n+1} = \sum_{k=1}^{n} q^k(q-1)(q^{2n-2k+2} - q^{2n-2k+1}) + q^{n+1}(q-1).$$

Then we have

$$\#W_{n+1} = q^{2n+2} - q^{2n+1},$$

which is the assertion of this lemma.

□

2.2 Khintchine type theorem

Now we prove Khintchine type theorem. Here, we put $\psi(Q)$ is non-negative function which depends only on the degree of Q. In this case, it is easy to give a necessary and sufficient condition on ψ for having infinitely many solutions for a.e. $f \in \mathbb{L}$. We refer to [5] and [8].

Theorem 2.2.1 *Let ψ be a non-negative function defined on $\mathbb{F}_q[X]$ such that $\psi(Q)$ depends only on the degree of $Q \in \mathbb{F}_q[X]$. For any set S of positive integers, the inequality*

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(Q)}{|Q|}$$

with P, Q coprime and $\deg Q \in S$ has infinitely many solutions for a.e. $f \in \mathbb{L}$ if and only if

$$\sum_{n \in S} q^n \psi(X^n) = \infty.$$

Proof. In the sequel, we always assume that P and Q are non-zero coprime polynomials whenever we denote by $\frac{P}{Q}$ a rational function and that Q is monic. For $\frac{P}{Q}$ with deg $Q = n$, we put

$$E_n\left(\frac{P}{Q}\right) = \left\{ f \in \mathbb{L} : \left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n}} \right\}$$

and also put

$$E_n = \left\{ f \in \mathbb{L} : \exists \frac{P}{Q},\ \deg Q = n,\ \left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n}} \right\}.$$

Lemma 2.2.1 *For a fixed integer $n \geq 1$, if $\frac{P}{Q} \neq \frac{P'}{Q'}$ with* deg Q = deg $Q' = n$, *then*

$$E_n\left(\frac{P}{Q}\right) \cap E_n\left(\frac{P'}{Q'}\right) = \emptyset.$$

Proof. Since $|\cdot|$ is ultrametric, we see that if the intersection were non-empty, then

$$\left|\frac{P}{Q}-\frac{P'}{Q'}\right|<\frac{1}{q^{2n}}.$$

However,

$$\left|\frac{P}{Q}-\frac{P'}{Q'}\right|\geq\frac{1}{|Q||Q'|}=\frac{1}{q^{2n}},$$

which gives a contradiction.

□

Lemma 2.2.2 *For any $n\geq 1$*

$$m(E_n)=1-\frac{1}{q}.$$

Proof Since $m\left\{f\in\mathbb{L}:\left|f-\frac{P}{Q}\right|<\frac{1}{q^{2n}}\right\}=\frac{1}{q^{2n}}$ for a fixed $\frac{P}{Q}$ with $\deg Q=n$ and the number of $\frac{P}{Q}$ is $q^{2n}-q^{2n-1}$ from Lemma 2.1.2, we have the assertion.

□

Lemma 2.2.3 *For any $n\geq 1$ and $k\geq 1$, we have*

$$m(E_n\cap E_{n+k})=m(E_n)m(E_{n+k})=\left(1-\frac{1}{q}\right)^2.$$

Proof. If $f\in E_n\cap E_{n+k}$, say

$$\left|f-\frac{P}{Q}\right|<\frac{1}{q^{2n}},\qquad\left|f-\frac{P'}{Q'}\right|<\frac{1}{q^{2n+2k}}$$

with $\deg Q=n$, $\deg Q'=n+k$, then $|\frac{P'}{Q'}-\frac{P}{Q}|<\frac{1}{q^{2n}}$, so that by Lemma 2.1.1, $\frac{P}{Q}$ is a convergence of the continued fraction of $\frac{P'}{Q'}$. Conversely, when $|\frac{P'}{Q'}-\frac{P}{Q}|<\frac{1}{q^{2n}}$ and $|f-\frac{P'}{Q'}|<\frac{1}{q^{2n+2k}}$, then $f\in E_n\cap E_{n+k}$. Therefore

$$m\left(E_n\cap E_{n+k}\right)=Z\left(n,n+k\right)\frac{1}{q^{2n+2k}},\tag{2.2}$$

where $Z(n, n+k)$ is the number of pairs $\frac{P}{Q}$, $\frac{P'}{Q'}$ with $\frac{P}{Q}$ a convergent to $\frac{P'}{Q'}$, and $\deg Q = n$, $\deg Q' = n+k$. $\#W_n$, the number of choices for $\frac{P}{Q}$, is $q^{2n}(1-\frac{1}{q})$. For a given $\frac{P}{Q}$, we will show the number of choices for $\frac{P'}{Q'}$. Suppose that $\frac{P'}{Q'}$ satisfies

$$\left| f - \frac{P'}{Q'} \right| < \frac{1}{q^{2n+2k}}, \ \deg Q' = n+k \text{ for } f \in E_n\left(\frac{P}{Q}\right).$$

We see that there exist $n = j_0 < j_1 < j_2 < \cdots < j_{l-1} < j_l = n+k$ (uniquely) such that

$$\frac{P'}{Q'} = \frac{P_{m+l}}{Q_{m+l}} = [0\,;\, p_1,\, p_2,\, \ldots\,,\, p_m,\, \ldots, p_{m+l}]$$

with

$$\deg p_{m+i} = j_i - j_{i-1}, \qquad 1 \le i \le l.$$

Since

$$\#\{p \in \mathbb{F}_q[X] \,:\, \deg p = u\} = q^u(q-1),$$

we have

$$\begin{aligned} \#\left\{\frac{P'}{Q'} \,:\, \deg p_{m+i} = j_i - j_{i-1},\ 1 \le i \le l\right\} & \\ &= q^{j_1-j_0}(q-1)q^{j_2-j_1}(q-1)\cdots q^{j_l-j_{l-1}}(q-1) \\ &= q^k(q-1)^l \end{aligned}$$

for each fixed $(j_1, \ldots, j_l)$. All choices for $n < j_1 < \cdots < j_{l-1} < n+k$ are $\binom{k-1}{l-1}$ and l runs 1 to k. Hence we have

$$\begin{aligned} \#\left\{\frac{P'}{Q'} \,:\, \left| f - \frac{P'}{Q'} \right| < \frac{1}{q^{2n+2k}} \text{ for some } f \in E_n\left(\frac{P}{Q}\right)\right\} & \\ &= \sum_{l=1}^{k} \binom{k-1}{l-1} q^k(q-1)^l \\ &= q^{2k}\left(1 - \frac{1}{q}\right). \end{aligned}$$

Consequently, we see

$$Z(n, n+k) = q^{2n+2k}(1 - \frac{1}{q})^2$$

and by (2.2), we get

$$m(E_n \cap E_{n+k}) = \left(1 - \frac{1}{q}\right)^2 = m(E_n)\, m(E_{n+k}).$$

□

By the Borel-Cantelli lemma, this implies the following:

Proposition 2.2.1 *For any subsequence of positive integers*

$$n_1 < n_2 < \cdots < n_k < \cdots ,$$

we have

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2}, \ \deg Q = n_i,$$

has infinitely many solutions for m-a.e. $f \in \mathbb{L}$.

According to this proposition, we can assume that $\psi(Q) < \frac{1}{q^n}$ for any $n \geq 1$. Then we rewrite Theorem 2.2.1 to the following.

Theorem 2.2.2 *For any subsequence of positive integers*

$$n_1 < n_2 < \cdots < n_k < \cdots ,$$

and a sequence of positive integers

$$l_1, l_2, \ldots, l_k, \ldots,$$

we have

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n_i + l_i}}, \ \deg Q = n_i,$$

has infinitely many solutions for m-a.e. $f \in \mathbb{L}$ *if and only if*

$$\sum_{i=1}^{\infty} q^{-l_i} = \infty.$$

Proof. Put

$$F_i = \left\{ f \in \mathbb{L} : \exists \frac{P}{Q},\ \left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n_i+l_i}},\ \deg Q = n_i \right\}.$$

Given $\frac{P}{Q}$, the measure of $f \in \mathbb{L}$ with $\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n_i+l_i}}$ is $\frac{1}{q^{2n_i+l_i}}$. The number of $\frac{P}{Q}$ in W_{n_i} is $(q^{2n_i} - q^{2n_i-1})$, therefore

$$m(F_i) = \frac{q-1}{q}\frac{1}{q^{l_i}}. \tag{2.3}$$

Now the assertion of Theorem 2.2.2 follows from the next lemma together with (2.3) by Theorem 3 in [22].

Lemma 2.2.4

(a) $F_i \cap F_{i+j} = \emptyset$ *if* $n_i + l_i \geq n_{i+j}$.

(b) $m(F_i \cap F_{i+j}) = m(F_i)\, m(F_{i+j})$ *if* $n_i + l_i < n_{i+j}$.

Proof If $f \in F_i \cap F_{i+j}$, say

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n_i+l_i}}, \qquad \left| f - \frac{P'}{Q'} \right| < \frac{1}{q^{2n_{i+j}+l_{i+j}}}$$

with $\deg Q = n_i$, $\deg Q' = n_{i+j}$, then

$$\left| \frac{P}{Q} - \frac{P'}{Q'} \right| < \frac{1}{q^{2n_i+l_i}}, \tag{2.4}$$

and on the other hand

$$\left| \frac{P}{Q} - \frac{P'}{Q'} \right| \geq \frac{1}{|Q||Q'|} = \frac{1}{q^{n_i+n_{i+j}}}.$$

When $n_i+l_i \geq n_{i+j}$ these inequalities contradict each other, so that $F_i \cap F_{i+j} = \emptyset$.

Suppose, then, that $n_i + l_i < n_{i+j}$. It follows from (2.4) that $\frac{P}{Q}$ is a convergent to $\frac{P'}{Q'}$. Write again

$$\frac{P}{Q} = [0; p_1, \ldots, p_m], \qquad \frac{P'}{Q'} = [0; p_1, \ldots, p_m, p_{m+1}, \ldots, p_{m+l}],$$

and then by a well-known formula,

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| = \frac{1}{|Q|^2|p_{m+1}|} = \frac{1}{q^{2n_i + \deg p_{m+1}}},$$

we see $\deg p_{m+1} > l_i$. In analogue to (2.2) we obtain

$$m(F_i \cap F_{i+j}) = Z(n_i, n_{i+j}, l_i)\frac{1}{q^{2n_{i+j}+l_{i+j}}} \tag{2.5}$$

where $Z(n_i, n_{i+j}, l_i)$ is the number of pairs $\frac{P}{Q}$, $\frac{P'}{Q'}$ as above with $\deg p_{m+1} > l_i$. Now, the number of choices for p_{m+1} $\dots$, p_{m+l} is

$$q^{\deg p_{m+1}}(q-1)q^{\deg p_{m+2}}(q-1)\cdots q^{\deg p_{m+l}}(q-1) \;=\; q^{n_{i+j}-n_i}(q-1)^l.$$

Then,

$$\begin{aligned} & Z(n_i, n_{i+j}, l_i) \\ =\;& (q^{2n_i} - q^{2n_i - 1}) \sum_{l=1}^{n_{i+j}-n_i-l_i} \binom{n_{i+j} - n_i - l_i - 1}{l-1} q^{n_{i+j}-n_i}(q-1)^l \\ =\;& (q^{2n_i} - q^{2n_i-1})q^{n_{i+j}-n_i}(q-1)q^{n_{i+j}-n_i-l_i-1} \\ =\;& q^{2n_{i+j}-l_i}\left(1-\frac{1}{q}\right)^2, \end{aligned}$$

which yields the lemma with (2.5).

□

On this type theorem in the case of real numbers, we get only a sufficient condition. But in the case of formal Laurent power series, we can get a necessary and sufficient condition where $\psi(Q)$ depends only on the degree of Q. In this sense, we can get a better results than in the case of real numbers.

Example Put

$$\psi(Q) \;=\; \begin{cases} \frac{1}{|Q|} & \text{if deg Q is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that there are infinitely many solutions of

$$\left| f \;-\; \frac{P}{Q}\right| \;<\; \frac{1}{|Q|^2}, \;\; \deg Q \text{ is prime}$$

for a.e. $f \in \mathbb{L}$.

2.3 Duffin-Schaeffer type theorem

In the previous section, we see the diophantine approximation where $\psi(Q)$ depends on the degree of Q. However, in this case, there is a gap on the hypothesis comparing with the case of real numbers. Then in this section, we put $\psi(Q)$ depends on Q itself and prove the Duffin-Schaeffer type theorem.

For a given polynomial

$$h = c_l X^l + c_{l-1}X^{l-1} + \cdots + c_1 X + c_0, \; c_i \in \mathbb{F}_q, \; 0 \le i \le l, \; c_l \ne 0,$$

we denote by $\langle h \rangle$ the cylinder set defined by

$$\{f \in \mathbb{L} : [X^{l+1} \cdot f] = h\}.$$

Lemma 2.3.1 *Let h_k, $k \ge 1$, be a sequence of polynomials with*

$$\lim_{k\to\infty} \deg h_k = \infty$$

and E_k be a sequence of measurable sets of $\mathbb{L}$ for which $E_k \subset \langle h_k \rangle$. Suppose that $m(E_k) \ge \delta m(\langle h_k \rangle)$ for some $\delta > 0$. Then

$$m(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} E_k) = m(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} \langle h_k \rangle).$$

Proof. Let

$$H := \bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} \langle h_k \rangle, \quad E_l^* = \bigcup_{k=l}^{\infty} E_k, \quad H_l^* := H \setminus E_l^*.$$

We show that $m(H_l^*) = 0$ for any $l \ge 1$, which implies the assertion of this lemma. Suppose that $m(H_k^*) > 0$. For almost all $f_0 \in H_l^*$, there are infinitely many k such that $f_0 \in \langle h_k \rangle$. For $f = \sum_{i<0} a_i X^i \in \mathbb{L}$, we put $\iota(f) = \sum_{i<0} a_i q^i \in (0,1]$. The map ι is a measure isomorphism of $(\mathbb{L}, m)$ to $[0, 1]$ with the Lebesgue measure. By this isomorphism, cylinder sets $\langle h_k \rangle$ are mapped to q-adic rational intervals. So we can apply Lebesgue's density theorem and get

$$\frac{m(H_k^* \cap \langle h_k \rangle)}{m(\langle h_k \rangle)} > 1 - \frac{\delta}{2}$$

for some k. On the other hand,

$$H_k^* \cap E_k^* = \emptyset.$$

So we see

$$m(\langle h_k \rangle) \geq m(E_k) + m(H_k^* \cap \langle h_k \rangle) \geq \delta m(\langle h_k \rangle) + m(H_k^* \cap \langle h_k \rangle),$$

which says

$$m(H_k^* \cap \langle h_k \rangle) \leq (1-\delta) m(\langle h_k \rangle).$$

This is impossible.

□

Lemma 2.3.2 *For any polynomial $h \in \mathbb{F}_q[X]$ and $g \in \mathbb{L}$, the map T of $\mathbb{L}$ onto itself defined by*

$$T f = hf + g - [hf + g] \qquad \textit{for} \quad f \in \mathbb{L}$$

is ergodic.

Proof. It is easy to see that both $f \to h \cdot f$ and $f \to f + g$ for $f \in \mathbb{L}$ are m-preserving. Then it turns out that $\omega_i(f) = [h \cdot T^{i-1}]$, $1 \leq i < \infty$ is an independent and identically distributed sequence of random variables defined on $(\mathbb{L}, m)$. This implies the assertion of the lemma.

□

Let ψ be a $\{q^{-n} : n \geq 0\} \cup \{0\}$-valued function defined on the set of monic polynomials, that is, of the form

$$X^l + a_{l-1}X^{l-1} + \cdots + a_1 X + a_0, \; a_i \in \mathbb{F}_q, \; 0 \leq i \leq l-1.$$

Here $\psi(Q)$ depends on Q itself, and we put

$$E_Q = \left\{ f \in \mathbb{L} : \left| f - \frac{P}{Q} \right| < \frac{\psi(Q)}{|Q|}, \; \deg P < \deg Q, \; (P, Q) = 1 \right\}$$

for a monic polynomial Q. The following theorem is a formal power series version of [10].

Theorem 2.3.1 (Gallagher type theorem)
For any ψ, *either* $m(\cap_{n=1}^{\infty} \cup_{\deg Q \geq n} E_Q) = 0$ *or* 1 *holds.*

Proof. If

$$\limsup_{\deg Q \to \infty} \frac{\psi(Q)}{q^{\deg Q}} > 0,$$

then we can find a sequence of monic polynomials $Q_1, Q_2, Q_3, \ldots$ and a positive integer l such that $\frac{\psi(Q_k)}{q^{\deg Q}} > q^{-l}$ for any $k \geq 1$. In this case, for any $f \in \mathbb{L}$ and a sufficiently large k, we can find $P\,(\deg P < \deg Q_k)$ such that

$$\left| f - \frac{P}{Q_k} \right| < \frac{1}{q^l} \quad \left(< \frac{\psi(Q_k)}{q^{\deg Q_k}} \right)$$

and P and Q_k are coprime. Otherwise, Q_k has more than $q^{\deg Q_k - l}$ factors, which is impossible. This implies

$$m\Big(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} E_{Q_k} \Big) = 1.$$

Now we show the assertion of the theorem when

$$\limsup_{\deg Q \to \infty} \frac{\psi(Q)}{q^{\deg Q}} = 0.$$

This means we can apply Lemma 2.3.1 for the proof. We put

$$E = \bigcap_{n=1}^{\infty} \bigcup_{\deg Q \geq n} E_Q.$$

Let R be an irreducible polynomial and consider

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(Q)|R|^{n-1}}{|Q|}, \qquad (P, Q) = 1 \tag{2.6}$$

for $n \geq 1$. We put

$$E_0(n : R) = \left\{ \begin{array}{r} f \in \mathbb{L} : \ (2.6) \text{ has infinitely many solutions of} \\ P, Q \text{ with } R \nmid Q \end{array} \right\}$$

and

$$E_1(n:R) = \left\{ \begin{array}{r} f \in \mathbb{L} : \ (2.6) \text{ has infinitely many solutions of} \\ P, Q \text{ such that} R \parallel Q \end{array} \right\}.$$

Then we see

$$E_i(1:R) \subset E_i(2:R) \subset E_i(3:R) \subset \cdots$$

and

$$E_i(1:R) \subset E$$

for $i = 0,\ 1$. From Lemma 2.3.1, we find that

$$m(E_i(n:R)) \ = \ m(E_i(1:R))$$

for $n \geq 1$. Thus

$$m\Big(\bigcup_{n\geq 1} E_i(n:R)\Big) \ = \ m(E_i(1:R)).$$

Let

$$T_1(f) \ = \ R \cdot f - [R \cdot f]$$

for $f \in \mathbb{L}$. Then

$$T_1\Big(\bigcup_{n\geq 1} E_0(n:R)\Big) \ = \ \bigcup_{n\geq 2} E_0(n:R)).$$

From Lemma 2.3.2, we have

$$m\Big(\bigcup_{n\geq 1} E_0(n:R)\Big) \ = 0 \text{ or } 1.$$

Next we let

$$T_2(f) \ = \ R \cdot f \ + \ \frac{1}{R} - \left[R \cdot f \ + \ \frac{1}{R}\right]$$

for $f \in \mathbb{L}$. Suppose (2.6) holds, we have

$$\left|\left(R \cdot f \ + \ \frac{1}{R}\right) \ - \ \frac{R \cdot P + \frac{Q}{R}}{Q}\right| \ < \ \frac{\psi(Q)|R|^n}{|Q|}, \qquad \left(R \cdot P + \frac{Q}{R}, Q\right) \ = \ 1,$$

and see that

$$T_2\Big(\bigcup_{n\geq 1} E_1(n:R)\Big) \ = \ \bigcup_{n\geq 2} E_1(n:R).$$

Thus we have, again by Lemma 2.3.2,

$$m\Big(\bigcup_{n\geq 1} E_1(n:R)\Big) = 0 \text{ or } 1.$$

Thus, if either $m(E_0(1:R)) > 0$ or $m(E_1(1:R)) > 0$ for some irreducible polynomial R, then we have $m(E) = 1$. Assume that $m(E_0(1:R)) = m(E_1(1:R)) = 0$ for any irreducible polynomial R. We put $F(R)$ is the set of $f \in \mathbb{L}$ such that

$$\left|f - \frac{P}{Q}\right| < \frac{\psi(Q)}{|Q|}, \qquad (P, Q) = 1,$$

has infinitely many solutions where $R^2 \mid Q$. If $f \in F(R)$, then

$$\left|\left(f + \frac{U}{R}\right) - \frac{P + \frac{QU}{R}}{Q}\right| < \frac{\psi(Q)}{|Q|}, \qquad \left(P + \frac{QU}{R}, Q\right) = 1$$

for any polynomial U with $0 \leq \deg U < \deg R$. This means that $f \in F(R)$ implies $f + \frac{U}{R} \in F(R)$. If we put $S(U;R) = \{f \in \mathbb{L} : [Rf] = U\}$, then

$$\bigcup_{U:0\leq \deg U\leq \deg R} S(U;R) \bigcup \{f \in \mathbb{L} \mid \deg f < -\deg R\} = \mathbb{L}$$

and each measure is equal to $\frac{1}{q^{\deg R}}$. Since $F(R)$ is $(\cdot + \frac{U}{R})$-invariant,

$$m(F(R) \cap S(U;R)) = \frac{m(F(R))}{q^{\deg R}}.$$

This implies

$$\frac{m(F(R) \cap S(U;R))}{m(S(U;R))} = m(F(R)).$$

By the density theorem, we have $m(E) = m(F(R)) = 1$ whenever $m(F(R)) > 0$ for some irreducible polynomial R, otherwise, $m(E) = 0$, since $E = F(R) \cup E_0(1,R) \cup E_1(1,R)$. This concludes the assertion of the theorem.

□

Theorem 2.3.2 (Duffin-Schaeffer type theorem)

Let ψ be a $\{q^{-n} : n \geq 0\} \cup \{0\}$-valued function which satisfies

$$\sum_{n=1}^{\infty} \sum_{\substack{\deg Q = n \\ Q:\text{monic}}} \psi(Q) = \infty.$$

Suppose there are infinitely many positive integers n such that

$$\sum_{\substack{\deg Q \le n \\ Q:monic}} \psi(Q) < C \sum_{\substack{\deg Q \le n \\ Q:monic}} \psi(Q)\frac{\Phi(Q)}{|Q|} \tag{2.7}$$

holds for a constant C. Then

$$\left|f - \frac{P}{Q}\right| < \frac{\psi(Q)}{|Q|}, \qquad (P,Q) = 1,$$

has infinitely many solutions $\frac{P}{Q}$ for m-a.e. $f \in \mathbb{L}$.

Proof. In the sequel, we always assume that Q, Q_1, Q' and Q'_1 are monic. By the definition of E_Q, we see

$$m(E_Q) = \psi(Q)\frac{\Phi(Q)}{|Q|}. \tag{2.8}$$

Now consider the measure of the intersection of E_{Q_1} and E_Q ($\deg Q_1 \le \deg Q$). We put $N(Q_1, Q)$ is the number of pairs of polynomials P and P_1. For these polynomials, the conditions

$$\left|\frac{P}{Q} - \frac{P_1}{Q_1}\right| < \frac{\psi(Q)}{|Q|} + \frac{\psi(Q_1)}{|Q_1|}, \tag{2.9}$$

$$(P,Q) = (P_1, Q_1) = 1, \quad \deg P < \deg Q, \quad \deg P_1 < \deg Q_1,$$

hold for given Q and Q_1. Then we can show the measure as follows

$$m(E_{Q_1} \cap E_Q) \le \min\left(\frac{\psi(Q_1)}{|Q_1|}, \frac{\psi(Q)}{|Q|}\right) N(Q_1, Q).$$

If the equality

$$PQ_1 - P_1Q = R \tag{2.10}$$

holds for some polynomial R, then $D = (Q_1, Q)$ divides R. Setting $Q_1 = DQ'_1, Q = DQ', R = DR'$, we have

$$PQ'_1 - P_1Q' = R', \qquad (Q'_1, Q') = 1. \tag{2.11}$$

If P' and P_1' also satisfy (2.10),

$$P'Q_1' - P_1'Q' = R'. \tag{2.12}$$

From (2.11) and (2.12),

$$P = P' + K\,Q' \qquad K : \text{a polynomial.} \tag{2.13}$$

From (2.13), we see

$$|P - P'| = |K||Q'| < |Q| = |D||Q|,$$

which implies $|K| < |D|$ must hold. The number of possible polynomials P satisfying (2.10) for a given R is no greater than $q^{\deg D}$. (2.9) implies

$$0 \neq |R| < |Q_1|\psi(Q) + |Q|\psi(Q_1)$$

and we must only take polynomials R divisible by D, we find that

$$\begin{aligned} N(Q_1, Q) &\leq \frac{|Q_1|\psi(Q) + |Q|\psi(Q_1)}{|D|}|D| \\ &= |Q_1|\psi(Q) + |Q|\psi(Q_1). \end{aligned}$$

Then

$$m\,(E_{Q_1} \cap E_Q) \leq 2\psi(Q_1)\psi(Q).$$

Since $\sum_{\deg Q \leq n} \psi(Q)$ diverges,

$$\sum_{\deg Q \leq n} \psi(Q) \leq \left(\sum_{\deg Q \leq n} \psi(Q) \right)^2$$

holds for sufficiently large n. Therefore we have

$$\begin{aligned} \sum_{\deg Q_1, \deg Q \leq n} m\,(E_{Q_1} \cap E_Q) &\leq 2 \sum_{\substack{\deg Q_1,\, \deg Q \,\leq\, n \\ Q \neq Q_1}} \psi(Q_1)\psi(Q) + \sum_{\deg Q \leq n} \psi(Q) \\ &< 3 \left(\sum_{\deg Q \leq n} \psi(Q) \right)^2 \end{aligned}$$

for all sufficiently large $\deg Q$. From (2.7) and (2.8), we have

$$\sum_{\deg Q_1, \deg Q \leq n} m\left(E_{Q_1} \cap E_Q\right) < 3C^2 \left(\sum_{\deg Q \leq n} m\left(E_Q\right) \right)^2$$

for infinitely many Q. Then we get $m\left(E\right) > (3C^2)^{-1}$ by Lemma 5 of [27](p17-18). Finally, applying Theorem 2.3.1, we have the assertion of the theorem.

□

By putting $\psi(Q)$ depends on Q itself, we generalize the theorem and get the similar results as in the case of real numbers.

Example Put

$$\psi(Q) = \begin{cases} \frac{1}{|Q|} & \text{if } Q \text{ is irreducible} \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$\sum_{n=1}^{\infty} \sum_{Q:\deg Q = n} \psi(Q) > \sum_{k=1}^{\infty} \frac{1}{q^k} \cdot \frac{1}{k} \cdot q^k = \infty$$

and it is easy to see that

$$\sum_{\deg Q \leq n} \psi(Q) \leq C \sum_{\deg Q \leq n} \psi(Q) \frac{\Phi(Q)}{|Q|}$$

holds. Thus we see there are infinitely many solutions $\frac{P}{Q}$ of

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2}, \quad Q \text{ is irreducible}$$

for a.e. $f \in \mathbb{L}$.

Chapter 3

Multi-dimensional diophantine approximation

In Chapter 2, we see the diophantine approximation for the one-dimensional version. In this chapter, we extend to the multi-dimensional version.

3.1 Duffin-Schaeffer type theorem

For given $h_i \in \mathbb{F}_q[X]$ such that

$$h_i = a_{i\,l_i} X^{l_i} + a_{i\,l_i-1} X^{l_i-1} + \cdots + a_{i\,1} X + a_{i\,0},$$

$$a_{i\,j} \in \mathbb{F}_q,\ 1 \le i \le r,\ 0 \le j \le l_i,\ a_{i\,l_i} \ne 0,$$

we define the cylinder set $\langle h_1, \ldots, h_r \rangle$ as follows:

$$\langle h_1, \ldots, h_r \rangle := \{(f_1, \ldots, f_r) \in \mathbb{L}^r : [X^{l_1+1} \cdot f_1] = h_1, \ldots, [X^{l_r+1} \cdot f_r] = h_r\}.$$

Then we see the following,

$$m^r(\langle h_1, \ldots, h_r \rangle) = \frac{1}{q^{l_1+1}} \cdots \frac{1}{q^{l_r+1}}.$$

Lemma 3.1.1 *Let* $\{\langle h_{1\,k}, h_{2\,k}, \ldots, h_{r\,k} \rangle : k \ge 1\}$ *be a sequence of cylinder sets defined as above with*

$$\lim_{k \to \infty} \deg h_{i\,k} = \infty$$

and $\{ E_k \mid k \geq 1\}$ *be a sequence of measurable sets of* $\mathbb{L}^r$ *for which* $E_k \subset \langle h_{1\,k}, \ldots, h_{r\,k} \rangle$. *Suppose there exists* $\delta > 0$ *such that* $m^r(E_k) \geq \delta m^r(\langle h_{1\,k}, \ldots, h_{r\,k}\rangle)$ *for any* $k \geq 1$. *Then*

$$m^r\left(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty} E_k\right) = m^r\left(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty} \langle h_{1\,k}, \ldots, h_{r\,k}\rangle\right).$$

Proof. Let

$$H := \bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}\langle h_{1\,k}, \ldots, h_{r\,k}\rangle, \quad E_l^* = \bigcup_{k=l}^{\infty} E_k, \quad H_l^* := H \setminus E_l^*.$$

We show that $m^r(H_l^*) = 0$ for any $l \geq 1$, which implies the assertion of this lemma. Suppose there exists a $k_0 \in \mathbb{Z}_+$ such that $m^r(H_k^*) > 0$ for $k \geq k_0$. There is a natural correspondence between cylinder sets defined for $\mathbb{L}$ as in (1.6) and q-adic rational intervals, and so $(\mathbb{L}, m)$ is isomorphic to $[0,1]$ with the Lebesgue measure. Similarly, $(\mathbb{L}^r, m^r)$ is isomorphic to $[0,1]^r$ with the Lebesgue measure. So by using cylinder sets $\langle h_1, \ldots, h_r\rangle \subset \mathbb{L}^r$ instead of $I_1 \times \cdots \times I_s \subset [0,1]^r$, we can apply Lebesgue density theorem. Then we get, since $\{H_l^* \mid l \geq 1\}$ is an increasing sequence of sets,

$$\frac{m^r(H_k^* \cap \langle h_{1\,k}, \ldots, h_{r\,k}\rangle)}{m^r(\langle h_{1\,k}, \ldots, h_{r\,k}\rangle)} > 1 - \frac{\delta}{2} \tag{3.1}$$

for some k. On the other hand,

$$H_k^* \cap E_k^* = \emptyset.$$

From the assumption of this lemma,

$$\begin{aligned} & m^r(\langle h_{1\,k}, \ldots, h_{r\,k}\rangle) \\ & \quad \geq m^r(E_k) + m^r(H_k^* \cap \langle h_{1\,k}, \ldots, h_{r\,k}\rangle) \\ & \quad \geq \delta m^r(\langle h_{1\,k}, \ldots, h_{r\,k}\rangle) + m^r(H_k^* \cap \langle h_{1\,k}, \ldots, h_{r\,k}\rangle). \end{aligned}$$

That is

$$(1-\delta)\, m^r(\langle h_{1\,k}, \ldots, h_{r\,k}\rangle) \geq m^r(H_k^* \cap \langle h_{1\,k}, \ldots, h_{r\,k}\rangle),$$

which contradicts (3.1).

□

Lemma 3.1.2 *For any polynomial* $h_i \in \mathbb{F}_q[X]\,(h_i \neq 0)$ *and* $g_i \in \mathbb{L}$, $1 \leq i \leq r$, *the map* T *of* $\mathbb{L}^r$ *onto itself defined by*

$$T(f_1, \ldots, f_r) = (h_1 f_1 + g_1 - [h_1 f_1 + g_1], \ldots, h_r f_r + g_r - [h_r f_r + g_r])$$

for $(f_1, \ldots, f_r) \in \mathbb{L}^r$ *is ergodic.*

Proof. It is easy to see that each map

$$T_i(f_i) = h_i f_i + g_i - [h_i f_i + g_i], \qquad 1 \leq i \leq r$$

is a Bernoulli transformation of $\mathbb{L}$. In other words, if we put

$$W_k(f_i) = [h_i \cdot T_i^{k-1} f_i + g_i], \qquad \text{for } f_i \in \mathbb{L},$$

then $\{W_k \mid k \geq 1\}$ gives a sequence of independent and identically distributed random variables. In particular, T_i is weak mixing. Since the $r$fold product of weak mixing transformations is ergodic (see [18] Prop. 4.2.), this yields the assertion of the lemma.

□

Theorem 3.1.1 (Gallagher type theorem)
For any ψ,

$$\left|f_1 - \frac{P_1}{Q}\right| < \frac{\psi(Q)}{|Q|}, \ldots, \left|f_r - \frac{P_r}{Q}\right| < \frac{\psi(Q)}{|Q|}$$

$$(P_1, Q) = (P_2, Q) = \cdots = (P_r, Q) = 1,$$

has infinitely many solutions of $(Q, P_1, \ldots, P_r)$ *for* m^r*-a.e.* $(f_1, \ldots, f_r) \in \mathbb{L}^r$ *or has only finitely many solutions for* m^r*-a.e.* $(f_1, \ldots, f_r) \in \mathbb{L}^r$.

Proof. Here, we put

$$E_Q = \left\{ (f_1, \ldots, f_r) \in \mathbb{L}^r : \begin{array}{l} \left|f_i - \frac{P_i}{Q}\right| < \frac{\psi(Q)}{|Q|}, \text{ for some } P_i \text{ s.t.} \\ \deg P_i < \deg Q, (P_i, Q) = 1, 1 \leq i \leq r \end{array} \right\}.$$

and

$$E = \bigcap_{n=1}^{\infty} \bigcup_{\deg Q \geq n} E_Q.$$

If

$$\lim_{\deg Q \to \infty} \frac{\psi(Q)}{q^{\deg Q}} > 0,$$

then we can find a sequence of polynomials Q_1, Q_2, Q_3, ... and a positive integer l such that $\frac{\psi(Q_k)}{q^{\deg Q_k}} > q^{-l}$ for any $k \geq 1$. In this case, for any $f_i \in \mathbb{L}$ and a sufficiently large k, we can find $P_i\,(\deg P_i < \deg Q_k)$ such that

$$\left| f_i - \frac{P_i}{Q_k} \right| < \frac{1}{q^l} \quad \left(< \frac{\psi(Q_k)}{q^{\deg Q_k}} \right) \qquad 1 \leq i \leq r,$$

and P_i and Q_k have no non-trivial common factor. Indeed, the number of polynomials $\hat{P}$ such that

$$\left| f_i - \frac{\hat{P}}{Q_k} \right| < \frac{1}{q^l}$$

is $q^{\deg Q_k - l}$. If all such polynomials $\hat{P}$ are not relatively prime to Q_k, then Q_k has more than $q^{\deg Q_k - l}$ factors, which is impossible if $\deg Q_k$ is sufficiently large. This implies

$$E = \mathbb{L}^r.$$

Now we show the assertion of the theorem when

$$\lim_{\deg Q \to \infty} \frac{\psi(Q)}{q^{\deg Q}} = 0. \tag{3.2}$$

For fixed Q, $P_1, \ldots, P_{r-1}$ and P_r, there exist polynomials $h_1, \ldots, h_r$ such that $\left\{ (f_1, \ldots, f_r) \mid \left| f_i - \frac{P_i}{Q} \right| < \frac{\psi(Q)}{|Q|} \right\} = \langle h_1, \ldots, h_r \rangle$. Then (3.2) implies $\deg h_i \to \infty$, $1 \leq i \leq r$ as $\deg Q$ tends to ∞. Thus we can apply Lemma 3.1.1 when (3.2) holds. Then we evaluate the measure of $\cap_{n=1}^{\infty} \cup_{\deg Q \geq n} E_Q$. Let R be an irreducible polynomial and consider

$$\left| f_i - \tfrac{P_i}{Q} \right| < \frac{\psi(Q)|R|^{n-1}}{|Q|}, \qquad (P_i, Q) = 1 \tag{3.3}$$

for $n \geq 1$ and $1 \leq i \leq r$.

We put

$$E_0(n:R) = \left\{ (f_1, \ldots, f_r) \in \mathbb{L}^r : \begin{array}{r} \text{(3.3) has infinitely many solutions} \\ P_i, Q \text{ with } R \nmid Q \quad \text{for } 1 \leq i \leq r \end{array} \right\}$$

and

$$E_1(n:R) = \left\{ (f_1, \ldots, f_r) \in \mathbb{L}^r : \begin{array}{r} \text{(3.3) has infinitely many solutions} \\ P_i, Q \text{ with } R \parallel Q \quad \text{for } 1 \leq i \leq r \end{array} \right\}$$

Then we see

$$E_j(1:R) \subset E_j(2:R) \subset E_j(3:R) \subset \cdots$$

and

$$E_j(1:R) \subset E$$

for $j = 0$, 1. From Lemma 3.1.1, we find that

$$m^r(E_j(n:R)) \;=\; m^r(E_j(1:R)) \;=\; m^r\Big(\bigcup_{n\geq 1} E_j(n:R)\Big).$$

Let

$$\begin{aligned} &T_j(f_1, \ldots, f_r) \\ &= \begin{cases} (R\cdot f_1 - [R\cdot f_1], \ldots, R\cdot f_r - [R\cdot f_r]) & j = 0, \\ \big(R\cdot f_1 + \frac{1}{R} - [R\cdot f_1 + \frac{1}{R}], \ldots, R\cdot f_r + \frac{1}{R} - [R\cdot f_r + \frac{1}{R}]\big) & j = 1 \end{cases} \end{aligned}$$

for $(f_1, \ldots, f_r) \in \mathbb{L}^r$. Suppose (3.3), we have

$$\left| R\cdot f_i - \frac{R\cdot P_i}{Q} \right| \;<\; \frac{\psi(Q)|R|^n}{|Q|}$$

and see

$$\Big(R\cdot P_i\,,\, Q\Big) \;=\; 1.$$

Also, we have

$$\left| \Big(R\cdot f_i + \frac{1}{R}\Big) \;-\; \frac{R\cdot P_i + \frac{Q}{R}}{Q} \right| \;<\; \frac{\psi(Q)|R|^n}{|Q|}.$$

Here,

$$\Big(R \cdot P_i + \frac{Q}{R}, Q\Big) = 1.$$

These imply

$$T_j\Big(\bigcup_{n\geq 1} E_j(n:R)\Big) = \bigcup_{n\geq 2} E_j(n:R))$$

for $j = 0, 1$. Hence from Lemma 3.1.2, we have

$$m^r\Big(\bigcup_{n\geq 1} E_j(n:R)\Big) = 0 \text{ or } 1.$$

for $j = 0, 1$. Thus, if either $m^r(E_0(1:R))$ or $m^r(E_1(1:R)) > 0$ for some irreducible polynomial R, then we have $m^r(E) = 1$.

Now we assume that $m^r(E_0(1:R)) = m^r(E_1(1:R)) = 0$ for any irreducible polynomial R. We put

$$F(R) = \left\{(f_1, \ldots, f_r) \in \mathbb{L}^r \;\middle|\; \begin{array}{r} (3.4) \text{ has infinitely many solutions} \\ P_i,\, Q \text{ such that } R^2 \mid Q \end{array}\right\},$$

where (3.4) refers to:

$$\left|f_i - \frac{P_i}{Q}\right| < \frac{\psi(Q)}{|Q|}, \qquad (P_i, Q) = 1, \qquad 1 \leq i \leq r. \tag{3.4}$$

Suppose (3.4), we have

$$\left|\Big(f_i + \frac{U}{R}\Big) - \frac{P_i + \frac{QU}{R}}{Q}\right| < \frac{\psi(Q)}{|Q|},$$

for any polynomial U with $0 \leq \deg U < \deg R$. Here, we see

$$\Big(P_i + \frac{QU}{R}, Q\Big) = 1,$$

which implies that $(f_1 + \frac{U}{R}, \ldots, f_r + \frac{U}{R}) \in F(R)$ if $(f_1, \ldots, f_r) \in F(R)$. Also we put

$$S(U;R) = \{(f_1, \ldots, f_r) \in \mathbb{L}^r : [Rf_1] = U, \ldots, [Rf_r] = U\},$$

then its measure is $\frac{1}{q^{r \deg R}}$ and

$$\bigcup_{U: 0 \leq \deg U < \deg R} S(U;R) \bigcup \{(f_1, \ldots, f_r) \in \mathbb{L}^r : \deg f_i < -\deg R\} = \mathbb{L}^r.$$

Since $F(R)$ is $(\cdot + \frac{U}{R})$-invariant, $S(U;R)$, $0 \leq \deg U < \deg R$, and $\{(f_1, \ldots, f_r) \in \mathbb{L}^r : \deg f_i < -\deg R\}$ have the same measure. Hence we have

$$m^r(F(R) \cap S(U;R)) = \frac{m^r(F(R))}{q^{r \deg R}},$$

which implies

$$\frac{m^r(F(R) \cap S(U;R))}{m^r(S(U;R))} = m^r(F(R)).$$

Suppose $m^r(E) > 0$, since $E = F(R) \cup E_0(1,R) \cup E_1(1,R)$, we see that $m^r(F(R)) > 0$ for any irreducible R. By the density theorem, we have $m^r(E) = m^r(F(R)) = 1$ where R is chosen so that $\deg R$ is sufficiently large. Otherwise, $m^r(E) = 0$.

□

From now, we generalize the theorem. That is, we prove the Duffin-Schaeffer theorem for the multi-dimensional version.

Theorem 3.1.2 (Duffin-Schaeffer type theorem)
Let ψ be a $\{q^{-n} \mid n \geq 0\} \cup \{0\}$-valued function which satisfies

$$\sum_{n=1}^{\infty} \sum_{\substack{\deg Q = n \\ Q:monic}} \psi^r(Q) = \infty.$$

Suppose for a positive constant C, there are infinitely many positive integers n such that

$$\sum_{\substack{\deg Q \leq n \\ Q:monic}} \psi^r(Q) < C \sum_{\substack{\deg Q \leq n \\ Q:monic}} \psi^r(Q) \frac{\Phi^r(Q)}{|Q|^r} \tag{3.5}$$

holds. Then

$$\left|f_1 - \tfrac{P_1}{Q}\right| < \tfrac{\psi(Q)}{|Q|}, \ldots, \left|f_r - \tfrac{P_r}{Q}\right| < \tfrac{\psi(Q)}{|Q|}$$

$$(P_1, Q) = (P_2, Q) = \cdots = (P_r, Q) = 1.$$

has infinitely many solutions $\left(\frac{P_1}{Q}, \ldots, \frac{P_r}{Q}\right)$ for m^r-a.e. $(f_1, \ldots, f_r) \in \mathbb{L}^r$.

Proof. In the sequel, we always assume that Q , Q_1 , Q' and Q'_1 are monic. From (3.2), if $\frac{\psi(Q)}{|Q|}$ is sufficiently small, then we have

$$m^r(E_Q) = \psi^r(Q)\,\frac{\Phi^r(Q)}{|Q|^r}. \tag{3.6}$$

Now consider the measure of $E_Q \cap E_{Q'}$ with $\deg Q' \le \deg Q$.We let $N(Q,Q')$ be the number of pairs of polynomials P and P' which satisfy

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| < \frac{\psi(Q)}{|Q|} + \frac{\psi(Q')}{|Q'|}\ , \tag{3.7}$$

$$(P,Q) = (P',Q') = 1, \quad \deg P < \deg Q,\ \deg P' < \deg Q'$$

for given Q and Q'. Then we show that the measure is bounded as follows,

$$m^r(E_Q \cap E_{Q'}) \le \left\{\min\left(\frac{\psi(Q)}{|Q|}, \frac{\psi(Q')}{|Q'|}\right)\cdot N(Q,Q')\right\}^r.$$

Suppose

$$PQ' - P'Q = R \tag{3.8}$$

holds for some polynomial R and $D = (Q,Q')$. If D divides R, we may write

$$Q = DQ^*, \quad Q' = DQ'^*, \quad R = DR^*,$$

and have

$$PQ'^* - P'Q^* = R^*, \qquad (Q^*, Q'^*) = 1. \tag{3.9}$$

If P^* and P'^* also satisfy (3.8), then

$$P^*Q'^* - P'^*Q^* = R^*. \tag{3.10}$$

From (3.9), (3.10), we get

$$P = P^* + KQ^*, \qquad K\ :\ \text{a polynomial.} \tag{3.11}$$

From (3.11), we see

$$|P - P^*| = |K||Q^*| < |Q| = |D||Q^*|,$$

which implies that

$$|K| < |D|$$

must hold. Thus the possible number of polynomials P satisfying (3.8) for a given R is no more than $q^{\deg D}$. Since (3.7) implies

$$0 \neq |R| < |Q'|\psi(Q) + |Q|\psi(Q')$$

and R is divisible by D, we find that

$$N(Q,Q') \leq \frac{|Q'|\psi(Q) + |Q|\psi(Q')}{|D|} \cdot |D| = |Q'|\psi(Q) + |Q|\psi(Q').$$

Then

$$\begin{aligned} m^r(E_Q \cap E_{Q'}) &\leq \left[\min\left(\frac{\psi(Q)}{|Q|}, \frac{\psi(Q')}{|Q'|}\right) \cdot \{|Q'|\psi(Q) + |Q|\psi(Q')\}\right]^r \\ &= 2^r \psi^r(Q)\psi^r(Q'). \end{aligned}$$

Because we assume $\sum_{\deg Q \leq n} \psi^r(Q) = \infty$, $\sum_{\deg Q \leq n} \psi^r(Q) \leq \left(\sum_{\deg Q \leq n} \psi^r(Q)\right)^2$ holds for sufficiently large n. Therefore we have

$$\begin{aligned} \sum_{\deg Q, \deg Q' \leq n} m^r(E_Q \cap E_{Q'}) &\leq 2^r \sum_{\substack{\deg Q' \leq \deg Q \leq n \\ Q \neq Q'}} \psi^r(Q)\psi^r(Q') + \sum_{\deg Q \leq n} \psi^r(Q) \\ &< 2^r \left(\sum_{\deg Q \leq n} \psi^r(Q)\right)^2 \end{aligned}$$

for sufficiently large $\deg Q$. From (3.5) and (3.6), we have

$$\sum_{\deg Q, \deg Q' \leq n} m^r(E_Q \cap E_{Q'}) < 2^r C^2 \left(\sum_{\deg Q \leq n} m^r(E_Q)\right)^2 \tag{3.12}$$

for infinitely many n. Then $m^r(E) > (2^r C^2)^{-1}$ by (3.12) and Lemma 5 of [27](p17-18). Finally using Theorem 3.1.1, we complete the proof of this theorem.

□

Example Put

$$\psi(Q) = \begin{cases} \frac{1}{|Q|^{\frac{1}{r}}} & \text{if Q is irreducible,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, from Theorem 2.2 of [25], we have

$$\sum_{\deg Q \leq n}^{\infty} \psi^r(Q) > C \sum_{k=1}^{\infty} \left(\frac{1}{q^{\frac{k}{r}}} \right)^r \cdot \frac{1}{k} \cdot q^k = \infty, C: \text{ constant}$$

and it is easy to see that

$$\sum_{\deg Q \leq n} \psi^r(Q) \leq C \sum_{\deg Q \leq n} \psi^r(Q) \frac{\Phi^r(Q)}{|Q|^r}$$

holds. Thus we see that there are infinitely many solutions $\left(\frac{P_1}{Q}, \ldots, \frac{P_r}{Q}\right)$ with irreducible Q's of

$$\left| f_i - \frac{P_i}{Q} \right| < \frac{1}{|Q|^{\frac{r+1}{r}}}, \quad \text{for} \quad 1 \leq i \leq r$$

for a.e. $(f_1, \ldots, f_r) \in \mathbb{L}^r$.

Remark It is natural to ask whether we can get a necessary and sufficient condition instead of (3.5) in Theorem 3.1.2. In this sense, we give the r-dimensional Duffin-Schaeffer type conjecture in the following.

Conjecture : (3.4) has infinitely many solutions $(\frac{P_1}{Q}, \ldots \frac{P_r}{Q})$ for m^r-a.e. $(f_1, \ldots, f_r) \in \mathbb{L}^r$ if and only if

$$\sum_{\substack{\deg Q = 1 \\ Q \,:\, \text{monic}}}^{\infty} \psi^r(Q) \frac{\Phi^r(Q)}{|Q|^r}$$

diverges.

In the classical case, the r-dimensional Duffin-Schaeffer conjecture was proved A.D.Pollington and R.C.Vaughan [24] for $r \geq 2$. We may also prove this conjecture for the r-dimensional formal power series, $r \geq 2$, if we estimate the lower bound of $\Phi(Q)$.

Chapter 4

On the exponential convergence of the Jacobi-Perron algorithm

In this chapter, we discuss about the (ii) in page 1. In particular, we study the convergent rate of Jacobi-Perron algorithm which gives a simultaneous approximation sequence.

4.1 Definitions and basic properties

Fast, we define a map T which is arisen from the Jacobi-Perron algorithm (JPA). For $(f_1, \dots, f_r) \in \mathbb{L}^r$, we define

$$a_i = a_i(f_1, \dots, f_r) = \begin{cases} \left[\dfrac{f_{i+1}}{f_1}\right] & 1 \le i \le r-1 \\ \\ \left[\dfrac{1}{f_1}\right] & i = r. \end{cases}$$

By the definition, it is easy to see that

$$a_i \in \mathbb{F}_q[X] \qquad \text{for } 1 \le i \le r$$

and

$$\deg a_r > \deg a_i \qquad \text{for} \quad 1 \le i \le r-1. \tag{4.1}$$

Now we define the map $T : \mathbb{L}^r \to \mathbb{L}^r$ by

$$T(f_1, \ldots, f_r) = \left(\frac{f_2}{f_1} - \left[\frac{f_2}{f_1}\right], \ldots, \frac{f_r}{f_1} - \left[\frac{f_r}{f_1}\right], \frac{1}{f_1} - \left[\frac{1}{f_1}\right]\right)$$

for $(f_1, \ldots, f_r) \in \mathbb{L}^r$ with $f_1 \neq 0$ and

$$T(0, f_2, \ldots, f_r) = (0, 0, \ldots, 0).$$

For we are going to discuss metrical theory of the JPA, we always assume that $f_1^{(\nu)} \neq 0$ for $\nu \geq 0$, that is, $T^\nu(f_1, \ldots, f_r) \in \mathbb{L}^r$. We put

$$(f_1^{(\nu)}, \ldots, f_r^{(\nu)}) = T^\nu(f_1, \ldots, f_r) \qquad \text{for} \quad \nu \geq 1$$

and

$$a_i^{(\nu)} = a_i(f_1^{(\nu-1)}, \ldots, f_r^{(\nu-1)}) \qquad \text{for} \quad 1 \leq i \leq r.$$

We define a $(r+1) \times (r+1)$ matrix $J = (m_{i_1 i_2})$ associated with $(f_1, \ldots, f_r) \in \mathbb{L}^r$ by the following way;

(i) $1 \leq i_1 \leq r+1, \ 1 \leq i_2 \leq r$

$$m_{i_1 i_2} = \begin{cases} 1 & i_1 = i_2 + 1, \\ 0 & \text{otherwise}, \end{cases}$$

(ii) $i_2 = r+1$

$$m_{i_1 i_2} = \begin{cases} 1 & i_1 = 1, \\ a_{i_1 - 1} & 2 \leq i_1 \leq r+1, \end{cases}$$

that is,

$$J = J(f_1, \ldots, f_r) = \begin{pmatrix} 0 & & 0 & & 1 \\ 1 & & & & a_1 \\ & \ddots & & & \vdots \\ & & \ddots & & a_{r-1} \\ 0 & & & 1 & a_r \end{pmatrix}.$$

We put

$$J^{(0)} = I_{r+1}$$

and

$$J^{(\nu)} = J\,(f_1^{(\nu-1)}, \ldots, f_r^{(\nu-1)}) \quad \text{for } \nu \geq 1,$$

where I_{r+1} denotes the $(r+1)\times(r+1)$ unit matrix. Since we consider the columns of the matrix $J^{(1)}\cdots J^{(\nu)}$, we denote

$$J^{(1)}\ldots J^{(\nu)} = \begin{pmatrix} A_1^{(\nu-r)} & \cdots & A_1^{(\nu)} \\ \vdots & & \vdots \\ A_r^{(\nu-r)} & \cdots & A_r^{(\nu)} \\ A_0^{(\nu-r)} & \cdots & A_0^{(\nu)} \end{pmatrix}$$

and

$$J^{(0)} = \begin{pmatrix} A_1^{(-r)} & \cdots & A_1^{(-1)} & A_1^{(0)} \\ \vdots & & \vdots & \vdots \\ A_r^{(-r)} & \cdots & A_r^{(-1)} & A_r^{(0)} \\ A_0^{(-r)} & \cdots & A_0^{(-1)} & A_0^{(0)} \end{pmatrix}.$$

Evidently,

$$\begin{aligned} &J^{(1)}\ldots J^{(\nu)} \\ &= \begin{pmatrix} A_1^{(\nu-1-r)} & \cdots & A_1^{(\nu-1)} \\ \vdots & & \vdots \\ A_r^{(\nu-1-r)} & \cdots & A_r^{(\nu-1)} \\ A_0^{(\nu-1-r)} & \cdots & A_0^{(\nu-1)} \end{pmatrix} \begin{pmatrix} 0 & & \mathbf{0} & 1 \\ 1 & & & a_1^{(\nu)} \\ & \ddots & & \vdots \\ & & \ddots & a_{r-1}^{(\nu)} \\ \mathbf{0} & & 1 & a_r^{(\nu)} \end{pmatrix} \\ &= \begin{pmatrix} A_1^{(\nu-r)} & \cdots & A_1^{(\nu-1)} & A_1^{(\nu-1-r)} + \sum_{k=1}^{r} a_k^{(\nu)} A_1^{(\nu-1-r+k)} \\ \vdots & & \vdots & \vdots \\ A_r^{(\nu-r)} & \cdots & A_r^{(\nu-1)} & A_r^{(\nu-1-r)} + \sum_{k=1}^{r} a_k^{(\nu)} A_r^{(\nu-1-r+k)} \\ A_0^{(\nu-r)} & \cdots & A_0^{(\nu-1)} & A_0^{(\nu-1-r)} + \sum_{k=1}^{r} a_k^{(\nu)} A_0^{(\nu-1-r+k)} \end{pmatrix}. \end{aligned} \tag{4.2}$$

Since $\det(J^{(1)}\cdots J^{(\nu)}) = (-1)^{r\nu}$, $A_0^{(\nu)}, \ldots, A_{r-1}^{(\nu)}$ and $A_r^{(\nu)}$ are coprime denoting by $(A_0, A_1, \ldots, A_r) = 1$. By a simple calculation, for $(f_1^{(\nu-1)}, \ldots, f_r^{(\nu-1)}) \in \mathbb{L}^r$, we see that

$$\deg A_i^{(\nu)} = \deg a_r^{(\nu)} + \deg A_i^{(\nu-1)} \qquad \text{for} \quad 0 \le i \le r.$$

Now we put

$$J^{(1)} \cdots J^{(\nu)} \begin{pmatrix} f_1^{(\nu)} \\ \vdots \\ f_r^{(\nu)} \\ 1 \end{pmatrix} = \begin{pmatrix} A_1^{(\nu-r)} f_1^{(\nu)} + \cdots + A_1^{(\nu-1)} f_r^{(\nu)} + A_1^{(\nu)} \\ \vdots \\ A_r^{(\nu-r)} f_1^{(\nu)} + \cdots + A_r^{(\nu-1)} f_r^{(\nu)} + A_r^{(\nu)} \\ A_0^{(\nu-r)} f_1^{(\nu)} + \cdots + A_0^{(\nu-1)} f_r^{(\nu)} + A_0^{(\nu)}, \end{pmatrix}$$

and have

$$f_i = \frac{A_i^{(\nu-r)} f_1^{(\nu)} + \cdots + A_i^{(\nu-1)} f_r^{(\nu)} + A_i^{(\nu)}}{A_0^{(\nu-r)} f_1^{(\nu)} + \cdots + A_0^{(\nu-1)} f_r^{(\nu)} + A_0^{(\nu)}}, \qquad 1 \le i \le r \tag{4.3}$$

for any $(f_1, \ldots, f_r) \in \mathbb{L}^r$. Here we call $\frac{A_i^{(\nu)}}{A_0^{(\nu)}}$ the ν-th convergence of the JPA and $J^{(1)}, \ldots, J^{(\nu)}$ the expansions by this algorithm. We see the following in [20]

(i) For any $\nu \ge 1$,

$$\lim_{\nu \to \infty} \frac{A_i^{(\nu)}}{A_0^{(\nu)}} = f_i \qquad \text{for} \quad 1 \le i \le r,$$

on the other hand, if $T^{\nu-1}(f_1, \ldots, f_r) \not\equiv 0$ and $T^{\nu}(f_1, \ldots, f_r) \equiv 0$, then

$$\frac{A_i^{(\nu)}}{A_0^{(\nu)}} = f_i \qquad \text{for} \quad 1 \le i \le r.$$

(ii) For a given sequence of arrays $\{a_i^{(\nu)} : 1 \le i \le r, \nu \ge 1\}$;

$$a_i^{(\nu)} \in \mathbb{F}_q[X], \qquad \text{for} \quad 1 \le i \le r$$

$$\deg a_r^{(\nu)} > \deg a_i^{(\nu)} \qquad \text{for} \quad 1 \le i < r - 1,$$

there exists $(f_1, \ldots, f_r) \in \mathbb{L}^r$ such that $a_i = a_i\left(f_1^{(\nu-1)}, \ldots, f_r^{(\nu-1)}\right)$ for $1 \le i \le r$ and $\nu \ge 1$.

4.2 The rate of convergence

At first, we show the following an a priori estimate.

Theorem 4.2.1 *For any $\nu \geq 1$, there exists a positive constant C such that*

$$|A_0^{(\nu)}|\left|f_i - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| < \frac{C}{q^{\frac{\nu}{r}}}, \qquad 1 \leq i \leq r,$$

for $(f_1, \ldots, f_r) \in \mathbb{L}^r$ where $T^\nu(f_1, \ldots, f_r) \in \mathbb{L}^r$ for $\nu \geq 1$.

Proof. From (4.3), we see

$$\begin{aligned}
|A_0^{(\nu)}|\left|f_i - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| &= |A_0^{(\nu)}|\left|\frac{A_i^{(\nu-r)}f_1^{(\nu)} + \cdots + A_i^{(\nu-1)}f_r^{(\nu)} + A_i^{(\nu)}}{A_0^{(\nu-r)}f_1^{(\nu)} + \cdots + A_0^{(\nu-1)}f_r^{(\nu)} + A_0^{(\nu)}} - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| \\
&= |A_0^{(\nu)}|\left|\frac{\sum_{k=1}^{r}(A_i^{(\nu-r-1+k)}A_0^{(\nu)} - A_i^{(\nu)}A_0^{(\nu-r-1+k)})f_k^{(\nu)}}{(A_0^{(\nu-r)}f_1^{(\nu)} + \cdots + A_0^{(\nu-1)}f_r^{(\nu)} + A_0^{(\nu)})A_0^{(\nu)}}\right| \\
&< \left|\sum_{k=1}^{r} A_0^{(\nu-r-1+k)}\left(\frac{A_i^{(\nu-r-1+k)}}{A_0^{(\nu-r-1+k)}} - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right)\right| \\
&= \left|\sum_{k=1}^{r} A_0^{(\nu-r-1+k)} \sum_{l=\nu-r-1+k}^{\nu-1}\left(\frac{A_i^{(l)}}{A_0^{(l)}} - \frac{A_i^{(l+1)}}{A_0^{(l+1)}}\right)\right| \\
&\leq \max_{\nu-r\leq l\leq \nu-1} |A_0^{(l)}|\left|\frac{A_i^{(l)}}{A_0^{(l)}} - \frac{A_i^{(l+1)}}{A_0^{(l+1)}}\right|. \qquad (4.4)
\end{aligned}$$

Now, we prove the following lemma.

Lemma 4.2.1 *For any $\nu \geq 1$, there exists a positive constant C such that*

$$|A_0^{(\nu)}|\left|\frac{A_i^{(\nu+1)}}{A_0^{(\nu+1)}} - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| < \frac{C}{q^{\frac{\nu}{r}}} \qquad 1 \leq i \leq r.$$

Proof. From (4.2), we see

$$
\begin{aligned}
&|A_0^{(\nu)}|\left|\frac{A_i^{(\nu+1)}}{A_0^{(\nu+1)}}-\frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| \\
&= |A_0^{(\nu)}|\left|\frac{A_i^{(\nu-r)}+\sum_{k=1}^{r}a_k^{(\nu+1)}A_i^{(\nu-r+k)}}{A_0^{(\nu-r)}+\sum_{k=1}^{r}a_k^{(\nu+1)}A_0^{(\nu-r+k)}}-\frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| \\
&= \frac{|A_0^{(\nu)}|}{|A_0^{(\nu+1)}|}\left|A_0^{(\nu-r)}\left(\frac{A_i^{(\nu-r)}}{A_0^{(\nu-r)}}-\frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right)+\sum_{k=1}^{r-1}a_k^{(\nu+1)}A_0^{(\nu-r+k)}\left(\frac{A_i^{(\nu-r+k)}}{A_0^{(\nu-r+k)}}-\frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right)\right| \\
&\le \frac{1}{|a_r^{(\nu+1)}|}\max\left\{|A_0^{(\nu-r)}|\left|\frac{A_i^{(\nu-r)}}{A_0^{(\nu-r)}}-\frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right|,|a_1^{(\nu+1)}||A_0^{(\nu-r+1)}|\left|\frac{A_i^{(\nu-r+1)}}{A_0^{(\nu-r+1)}}-\frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right|,\right. \\
&\qquad\left.\dots,|a_{r-1}^{(\nu+1)}||A_0^{(\nu-1)}|\left|\frac{A_i^{(\nu-1)}}{A_0^{(\nu-1)}}-\frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right|\right\} \\
&\le \frac{1}{|a_r^{(\nu+1)}|}\max\left\{\max_{0\le k\le r-1}|A_0^{(\nu-r)}|\left|\frac{A_i^{(\nu-r+k)}}{A_0^{(\nu-r+k)}}-\frac{A_i^{(\nu-r+k+1)}}{A_0^{(\nu-r+k+1)}}\right|,\right. \\
&\qquad\left.\max_{\substack{1\le l\le r-1\\0\le l'\le r-l-1}}|a_l^{(\nu+1)}||A_0^{(\nu-r+l)}|\left|\frac{A_i^{(\nu-r+l+l')}}{A_0^{(\nu-r+l+l')}}-\frac{A_i^{(\nu-r+l+l'+1)}}{A_0^{(\nu-r+l+l'+1)}}\right|\right\}
\end{aligned}
\tag{4.5}
$$

From (4.1), it is easy to see that

$$|A_0^{(0)}|\left|\frac{A_i^{(1)}}{A_0^{(1)}}-\frac{A_i^{(0)}}{A_0^{(0)}}\right|\le\frac{1}{q}$$

and

$$|A_0^{(\nu)}|\left|\frac{A_i^{(\nu+1)}}{A_0^{(\nu+1)}}-\frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right|\le\frac{1}{q^2}\qquad\text{for}\quad 1\le\nu\le r.$$

Then by induction together with (4.1) and (4.5), we have

$$|A_0^{(\nu)}|\left|\frac{A_i^{(\nu+1)}}{A_0^{(\nu+1)}}-\frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right|\le\frac{1}{q^{u+1}}\qquad\text{for}\quad (u-1)r+1\le\nu\le ur$$

for $u\in\mathbb{N}$. This shows the assertion of the lemma.

□

From (4.4) and this lemma, it is easy to see that

$$|A_0^{(\nu)}|\left|f_i - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| < \frac{C}{q^{\frac{\nu}{r}}}.$$

□

Next, we shall give an estimate of the error of the convergence by using $|A_0^{(\nu)}|$ for a.e. $(f_1, \ldots, f_r) \in \mathbb{L}^r$.

Theorem 4.2.2 *For any $\nu \geq 1$, there exists a positive constant $C' = C'(\varepsilon)$ such that*

$$|A_0^{(\nu)}|\left|f_i - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| < \frac{C'}{|A_0^{(\nu)}|^{\frac{1}{r}(\frac{\gamma}{\rho}-\varepsilon)}} \qquad \forall\varepsilon > 0$$

for m^r-a.e. $(f_1, \ldots, f_r) \in \mathbb{L}^r$, where

$$\gamma = \frac{q^{r^2}}{q^{r^2}-1}, \qquad \rho = \frac{q}{q-1}.$$

Proof. From (4.4) it is enough to estimate $|A_0^{(\nu)}|\left|\frac{A_i^{(\nu+1)}}{A_0^{(\nu+1)}} - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right|$. Then by (4.5), we get

$$\begin{aligned}
&|A_0^{(\nu)}|\left|\frac{A_i^{(\nu+1)}}{A_0^{(\nu+1)}} - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| \\
&< \max_{\nu-r\leq l\leq \nu-1} \frac{1}{|a_r^{(l+1)}|}\left\{\max_{0\leq k_1\leq r-1} |A_0^{(l-r)}|\left|\frac{A_i^{(l-r+k_1)}}{A_0^{(l-r+k_1)}} - \frac{A_i^{(l-r+k_1+1)}}{A_0^{(l-r+k_1+1)}}\right|,\right. \\
&\qquad \left.\max_{\substack{1\leq k_2\leq r-1\\ 0\leq k_2'\leq r-k_2-1}} |a_{k_2}^{(l+1)}||A_0^{(l-r+k_2)}|\left|\frac{A_i^{(l-r+k_2+k_2')}}{A_0^{(l-r+k_2+k_2')}} - \frac{A_i^{(l-r+k_2+k_2'+1)}}{A_0^{(l-r+k_2+k_2'+1)}}\right|\right\}.
\end{aligned}$$

Now we suppose $ur + 1 \leq \nu \leq (u+1)r$ for some $u \in \mathbb{N}$. Then we have

$$\begin{aligned}
&|A_0^{(\nu)}|\left|\frac{A_i^{(\nu+1)}}{A_0^{(\nu+1)}} - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| \\
&< \max_{\substack{(u-1)r+1\leq \nu'\leq ur\\ 1\leq i\leq r-1}} \frac{\max_{1\leq i\leq r-1}(|a_i^{(\nu')}|, 1)}{|a_r^{(\nu')}|} \max_{(u-1)r+1\leq\nu'\leq ur} |A_0^{(\nu')}|\left|\frac{A_i^{(\nu')}}{A_0^{(\nu')}} - \frac{A_i^{(\nu'+1)}}{A_0^{(\nu'+1)}}\right|.
\end{aligned}$$

So we can get an estimate of the error term for the index $\nu \in (u\,r+1,\,(u+1)\,r]$ by those in $((u-1)\,r+1,\,u\,r]$ and $\frac{|a_i^{(\nu)}|}{|a_r^{(\nu)}|}$, $1 \le i \le r-1$. Then we consider the stochastic behavior of $\frac{|a_i^{(\nu)}|}{|a_r^{(\nu)}|}$, $1 \le i \le r-1$. We first see the distribution of the maximum degree of $\frac{|a_i^{(\nu)}|}{|a_r^{(\nu)}|}$, $1 \le i \le r-1$, for a fixed ν. We define

$$\deg^* a_i^{(\nu)} = \max(\deg a_i^{(\nu)}\,,\,1)$$

for $a_i^{(\nu)} \in \mathbb{F}_q[X]$, $1 \le i \le r-1$. We put

$$k_j = \deg a_r^{(j)} - \max_{1\le i\le r-1} \deg^* a_i^{(j)}.$$

For a fixed i, if $k_j \ne \deg a_r^{(j)} = n$,

$$\begin{aligned}
& m^r(\{(f_1,\dots,f_r) \in \mathbb{L}^r \,:\, \deg a_r^{(j)} - \max_{1\le i\le r-1} \deg^* a_i^{(j)} = k_j,\ \deg a_r^{(j)} = n\}) \\
= & \left[\sum_{t=1}^{r-1} \binom{r-1}{t} \left(\frac{1}{q^{k_j}}\right)^{r-1-t} \left(\frac{q-1}{q^{k_j}}\right)^{t}\right] \frac{q-1}{q^n} \\
= & \left[\left(\frac{1}{q^{k_j}-1}\right)^{r-1} - \left(\frac{1}{q^{k_j}}\right)^{r-1}\right] \frac{q-1}{q^n} \\
= & \frac{(q-1)(q^{r-1}-1)}{q^{n+(r-1)k_j}} \qquad (4.6)
\end{aligned}$$

and if $k_j = \deg a_r^{(j)}$,

$$\begin{aligned}
m^r(\{(f_1,\dots,f_r) \in \mathbb{L}^r \,:\, k_j = \deg a_r^{(j)}\}) &= \left(\frac{1}{q^{k_j-1}}\right)^{r-1} \frac{q-1}{q^{k_j}} \\
&= \frac{q-1}{q^{(k_j-1)r+1}}. \qquad (4.7)
\end{aligned}$$

Then, from (4.6) and (4.7),

$$\begin{aligned}
& m^r(\{(f_1,\dots,f_r) \in \mathbb{L}^r \,:\, \deg a_r^{(j)} - \max_{1\le i\le r-1} \deg^* a_i^{(j)} = k_j\}) \\
= & \frac{q-1}{q^{(k_j-1)r+1}} + \sum_{n=k_j+1}^{\infty} \frac{(q-1)(q^{r-1}-1)}{q^{n+(r-1)k_j}} \\
= & \frac{q-1}{q^{(k_j-1)r+1}} + \frac{(q-1)(q^{r-1}-1)}{q^{(r-1)k_j}} \frac{q}{q^{k_j+1}(q-1)} \\
= & \frac{q^r-1}{q^{r\,k_j}}.
\end{aligned}$$

Next, we see the distribution for any ν. We put $k = \min_{\nu-r+1\leq j\leq \nu-1} k_j$,

$$\begin{aligned}
& m^r(\{(f_1,\ldots,f_r)\in \mathbb{L}^r : \min_{\nu-r+1\leq j\leq \nu-1} k_j = k\}) \\
&= \sum_{t=1}^{r} \binom{r}{t} \left[\sum_{s=k+1}^{\infty} \frac{q^r-1}{q^{r\,s}}\right]^{r-t} \left[\frac{q^r-1}{q^{r\,k}}\right]^{t} \\
&= \sum_{t=1}^{r} \binom{r}{t} \left[\frac{1}{q^{r\,k}}\right]^{r-t} \left[\frac{q^r-1}{q^{r\,k}}\right]^{t} \\
&= \frac{q^{r^2}-1}{q^{r^2\,k}}.
\end{aligned}$$

Let $X_\nu = \deg a_r^{(\nu)} - \max_{1\leq i\leq r-1} \deg^* a_i^{(\nu)}$ and $Y_s = \min_{1\leq s\leq r} X_{r\cdot s+\nu}$. Then the expectation of Y_s is as follows.

$$\begin{aligned}
E(Y_s) &= \sum_{k=1}^{\infty} \frac{(q^{r^2}-1)k}{q^{r^2\,k}} \\
&= \frac{q^{r^2}}{q^{r^2}-1}.
\end{aligned}$$

Because $\left\{ \begin{pmatrix} a_1^{(\nu)} \\ \vdots \\ a_r^{(\nu)} \end{pmatrix} : \nu \geq 1 \right\}$ is an independent and identically distributed sequence, see [21], we have

$$\lim_{r\to\infty} \frac{1}{r} \sum_{t=1}^{r} Y_t = \frac{q^{r^2}}{q^{r^2}-1} \qquad a.e.$$

by the strong law of large numbers. Then, for $\varepsilon_1 > 0$, there exists a positive constant C and ν_1 such that for $\nu \geq \nu_1$

$$|A_0^{(\nu)}| \left| f_i - \frac{A_i^{(\nu)}}{A_0^{(\nu)}} \right| < \frac{C}{q^{[\frac{\nu}{r}](\gamma-\varepsilon)}} \qquad a.e. \tag{4.8}$$

On the other hand, by the strong law of large numbers, we have

$$\lim_{\nu\to\infty} \frac{1}{\nu} \sum_{t=1}^{\nu} \deg a_r^{(t)} = \frac{q}{q-1} \qquad a.e.$$

(see [21]). That is, for a.e. $(f_1,\ldots,f_r)\in \mathbb{L}^r$, for $\varepsilon_2 > 0$, there exists ν_2 such that for $\nu \geq \nu_2$

$$\left| \frac{\deg A_0^{(\nu)}}{\nu} - \frac{q}{q-1} \right| < \varepsilon_2,$$

and so

$$\frac{q}{q-1}\nu - \varepsilon_2\,\nu < \deg A_0^{(\nu)} < \frac{q}{q-1}\nu + \varepsilon_2\,\nu.$$

We have

$$q^{(\rho-\varepsilon_2)\nu} < |A_0^{(\nu)}| < q^{(\rho+\varepsilon_2)\nu},$$

then

$$|A_0^{(\nu)}|^{\frac{1}{(\rho+\varepsilon_2)\nu}} < q. \tag{4.9}$$

From (4.9), (4.8) is as follows:

$$|A_0^{(\nu)}|\left|f_i - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| < \frac{C}{|A_0^{(\nu)}|^{\frac{1}{\nu(\rho+\varepsilon_2)}\left[\frac{\nu}{r}\right](\gamma-\varepsilon_1)}} \qquad a.e.$$

That is, for any $\varepsilon > 0$, there exists a positive constant C' and ν_0 such that for $\nu \geq \nu_0$

$$|A_0^{(\nu)}|\left|f_i - \frac{A_i^{(\nu)}}{A_0^{(\nu)}}\right| < \frac{C'}{|A_0^{(\nu)}|^{\frac{1}{r}\left(\frac{\gamma}{\rho}-\varepsilon\right)}} \qquad a.e.$$

This is the assertion of the theorem.

□

Remark For any $\varepsilon > 0$, it is easy to see from the Borel-Cantelli lemma, there exists a positive constant C such that

$$|Q|\left|f_i - \frac{P_i}{Q}\right| < \frac{C}{|Q|^{\frac{1}{r}+\varepsilon}}, \qquad 1 \leq i \leq r,$$

has at most finitely many $(\frac{P_1}{Q}, \ldots, \frac{P_r}{Q})$ for a.e. $(f_1, \ldots, f_r) \in \mathbb{L}^r$. In Theorem 4.2.2, it is evident that $\frac{\gamma}{\rho} < 1$. Now the question is whether there exists a positive constant C' such that

$$|Q^{(\nu)}|\left|f_i - \frac{P_i^{(\nu)}}{Q^{(\nu)}}\right| < \frac{C'}{|Q^{(\nu)}|^{\frac{1}{r}-\varepsilon}}, \qquad 1 \leq i \leq r,$$

has infinitely many solutions a.e. for any $\varepsilon > 0$ or not.

Chapter 5

On the exponential convergence of the modified Jacobi-Perron algorithm

In this chapter, we consider some problems similar to Chapter 4, we study the convergent rate of the modified Jacobi-Perron algorithm.

5.1 Definitions and basic properties

In this section, we define a map S which is arisen from the modified Jacobi-Perron algorithm (MJPA).

Now, for $1 \leq j \leq r$, we put

$$\mathbb{L}_j^r = \left\{ \begin{array}{ll} (f_1, \ldots, f_r): & \deg f_j > \deg f_i \quad \text{for } 1 \leq i < j, \\ & \deg f_j \geq \deg f_i \quad \text{for } j < i \leq r \end{array} \right\},$$

then

$$\mathbb{L}^r = \mathbb{L}_1^r \cup \cdots \cup \mathbb{L}_r^r.$$

Note that $(0, \ldots, 0) \in \mathbb{L}_1^r$. We denote by m^r the normalized Haar measure on $\mathbb{L}^r$.

For $(f_1, \dots, f_r) \in \mathbb{L}_j^r$, we define

$$b_i = b_i(f_1, \dots, f_r) = \begin{cases} \left[\dfrac{f_i}{f_j}\right] & 1 \le i \le r, \\ \\ \left[\dfrac{1}{f_j}\right] & i = r+1. \end{cases}$$

if $(f_1, \dots, f_r) \neq (0, \dots, 0)$ and

$$b_i = 0, \qquad 1 \le i \le r$$

if $(f_1, \dots, f_r) = (0, \dots, 0)$. From the above, we see

$$b_i = \begin{cases} 0 & 1 \le i \le j-1 \\ b_i \in \mathbb{F}_q & j \le i \le r \\ b_i \in \mathbb{F}_q[X], \deg b_i \ge 1 & i = r+1. \end{cases} \tag{5.1}$$

Now we define the map $S : \mathbb{L}^r \to \mathbb{L}^r$ by

$$\begin{aligned} &S(f_1, \dots, f_r) \\ &= \left(\frac{f_1}{f_j}, \dots, \frac{f_{j-1}}{f_j}, \frac{1}{f_j} - \left[\frac{1}{f_j}\right], \frac{f_{j+1}}{f_j} - \left[\frac{f_{j+1}}{f_j}\right], \dots, \frac{f_r}{f_j} - \left[\frac{f_r}{f_j}\right]\right) \\ &= \left(\frac{f_1}{f_j}, \dots, \frac{f_{j-1}}{f_j}, \frac{1}{f_j} - b_{r+1}, \frac{f_{j+1}}{f_j} - b_{j+1}, \dots, \frac{f_r}{f_j} - b_r\right) \end{aligned}$$

for $(f_1, \dots, f_r) \in \mathbb{L}_j^r$, $(f_1, \dots, f_r) \neq (0, 0, \dots, 0)$ and

$$S(0, \dots, 0) = (0, \dots, 0).$$

We put

$$(f_1^{(\nu)}, \dots, f_r^{(\nu)}) = S^\nu(f_1, \dots, f_r) \qquad \text{for} \quad \nu \ge 1$$

and

$$b_i^{(\nu)} = b_i(f_1^{(\nu-1)}, \dots, f_r^{(\nu-1)}) \qquad \text{for} \quad 1 \le i \le r+1,$$

that is,

$S^{\nu}(f_1, \ldots, f_r)$

$$= S(f_1^{(\nu-1)}, \ldots, f_r^{(\nu-1)})$$

$$= \left(\frac{f_1^{(\nu-1)}}{f_j^{(\nu-1)}}, \ldots, \frac{f_{j-1}^{(\nu-1)}}{f_j^{(\nu-1)}}, \frac{1}{f_j^{(\nu-1)}} - \left[\frac{1}{f_j^{(\nu-1)}} \right], \frac{f_{j+1}^{(\nu-1)}}{f_j^{(\nu-1)}} - \left[\frac{f_{j+1}^{(\nu-1)}}{f_j^{(\nu-1)}} \right], \ldots, \frac{f_r^{(\nu-1)}}{f_j^{(\nu-1)}} - \left[\frac{f_r^{(\nu-1)}}{f_j^{(\nu-1)}} \right] \right)$$

$$= \left(\frac{f_1^{(\nu-1)}}{f_j^{(\nu-1)}}, \ldots, \frac{f_{j-1}^{(\nu-1)}}{f_j^{(\nu-1)}}, \frac{1}{f_j^{(\nu-1)}} - b_{r+1}^{(\nu)}, \frac{f_{j+1}^{(\nu-1)}}{f_j^{(\nu-1)}} - b_{j+1}^{(\nu)}, \ldots, \frac{f_r^{(\nu-1)}}{f_j^{(\nu-1)}} - b_r^{(\nu)} \right)$$

for $(f_1^{(\nu-1)}, \ldots, f_r^{(\nu-1)}) \in \mathbb{L}_j^r$. Also we put $\kappa(\nu) := j$ such that

$$\deg f_j^{(\nu-1)} > \deg f_i^{(\nu-1)} \qquad \text{for} \quad 1 \leq i < j,$$

$$\deg f_j^{(\nu-1)} \geq \deg f_i^{(\nu-1)} \qquad \text{for} \quad j < i \leq r.$$

We define a $(d+1) \times (d+1)$ matrix $M = (m_{i_1 i_2})$, $m_{i_1 i_2} \in \mathbb{F}_q[X]$, associated to $(f_1, \ldots, f_r) \in \mathbb{L}_j^r$, $(f_1, \ldots, f_r) \neq (0, \ldots, 0)$ in the following way;

(i) $1 \leq i_2 \leq r,\ i_2 \neq j$

$$m_{i_1 i_2} = \delta_{i_1 i_2} \qquad 1 \leq i_1 \leq r+1, \tag{5.2}$$

(ii) $i_2 = j$

$$m_{i_1 j} = \begin{cases} 1 & i_1 = r+1 \\ 0 & 1 \leq i_1 \leq r, \end{cases} \tag{5.3}$$

(iii) $i_2 = d+1,\ 1 \leq i_1 \leq r+1$

$$m_{i_1 i_2} = b_{i_1},$$

that is,

$M = M(f_1, \ldots, f_r)$

$$= \left(\begin{array}{ccc|ccccc} 1 & & 0 & & & & & \\ & \ddots & & & & \mathbf{0} & & \\ 0 & & 1 & & & & & \\ \hline & & & 0 & \cdots & \cdots & 0 & 1 \\ & & & \vdots & 1 & & 0 & b_{j+1} \\ & \mathbf{0} & & \vdots & & \ddots & & \vdots \\ & & & 0 & & & 1 & b_r \\ & & & 1 & 0 & & & b_{r+1} \end{array}\right). \tag{5.4}$$

For $(f_1, \ldots, f_r) = (0, \ldots, 0)$, we define M the $(r+1) \times (r+1)$ unit matrix I_{r+1}. We put

$$M^{(0)} = I_{r+1}$$

and

$$M^{(\nu)} = M\left(f_1^{(\nu-1)}, \ldots, f_r^{(\nu-1)}\right) \quad \text{for } \nu \geq 1,$$

where $(f_1^{(0)}, \ldots, f_r^{(0)}) = (f_1, \ldots, f_r)$. Since we consider the columns of the matrix $M^{(1)} \cdots M^{(\nu)}$, we denote

$$M^{(1)} \cdots M^{(\nu)} = \begin{pmatrix} \beta_{1\,1}^{(\nu)} & \cdots & \cdots & \beta_{1\,r}^{(\nu)} & B_1^{(\nu)} \\ \vdots & & & \vdots & \vdots \\ \beta_{\kappa(\nu)\,1}^{(\nu)} & \cdots & \cdots & \beta_{\kappa(\nu)\,r}^{(\nu)} & B_j^{(\nu)} \\ \vdots & & & \vdots & \vdots \\ \beta_{r\,1}^{(\nu)} & \cdots & \cdots & \beta_{r\,r}^{(\nu)} & B_r^{(\nu)} \\ \beta_{0\,1}^{(\nu)} & \cdots & \cdots & \beta_{0\,r}^{(\nu)} & B_0^{(\nu)} \end{pmatrix}$$

and

$$M^{(0)} = \begin{pmatrix} B_1^{(-r)} & \cdots & B_1^{(-1)} & B_1^{(0)} \\ \vdots & & \vdots & \vdots \\ B_r^{(-r)} & \cdots & B_r^{(-1)} & B_r^{(0)} \\ B_0^{(-r)} & \cdots & B_0^{(-1)} & B_0^{(0)} \end{pmatrix}.$$

By the definition of $B_0^{(\nu)}$, it is easy to see that $\deg B_0^{(\nu)} = \sum_{i=1}^{\nu} \deg b_{r+1}^{(i)}$ which we use often. $B_0^{(\nu)}$ will be the denominator of the ν-th convergence and $B_i^{(\nu)}$, $1 \leq i \leq r$, will be the numerator. Evidently,

$M^{(1)}\cdots M^{(\nu)}$

$$
= \begin{pmatrix}
\beta_{1\,1}^{(\nu-1)} & \cdots & \beta_{1\,r}^{(\nu-1)} & B_1^{(\nu-1)} \\
\vdots & & \vdots & \vdots \\
\beta_{\kappa(\nu)\,1}^{(\nu-1)} & \cdots & \beta_{\kappa(\nu)\,r}^{(\nu-1)} & B_{\kappa(\nu)}^{(\nu-1)} \\
\vdots & & \vdots & \vdots \\
\beta_{r\,1}^{(\nu-1)} & \cdots & \beta_{r\,r}^{(\nu-1)} & B_r^{(\nu-1)} \\
\beta_{0\,1}^{(\nu-1)} & \cdots & \beta_{0\,r}^{(\nu-1)} & B_0^{(\nu-1)}
\end{pmatrix}
\left(\begin{array}{c|c}
\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \mathbf{0} \\
\hline
\mathbf{0} & \begin{matrix}
0 & \cdots & \cdots & 0 & 1 \\
\vdots & 1 & & 0 & b_{\kappa(\nu)+1}^{(\nu)} \\
\vdots & & \ddots & & \vdots \\
0 & & & 1 & b_r^{(\nu)} \\
1 & 0 & & & b_{r+1}^{(\nu)}
\end{matrix}
\end{array}\right) \tag{5.5}
$$

$$
= \begin{pmatrix}
\beta_{1\,1}^{(\nu-1)} & \cdots & \beta_{1\,\kappa(\nu)-1}^{(\nu-1)} & B_1^{(\nu-1)} & \beta_{1\,\kappa(\nu)+1}^{(\nu-1)} & \cdots & \beta_{1\,r}^{(\nu-1)} & \beta_{1\,\kappa(\nu)}^{(\nu-1)} + \sum_{k=\kappa(\nu)+1}^{r} b_k^{(\nu)}\beta_{1\,k}^{(\nu-1)} + b_{r+1}^{(\nu)}B_1^{(\nu-1)} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\beta_{\kappa(\nu)\,1}^{(\nu-1)} & \cdots & \beta_{\kappa(\nu)\,\kappa(\nu)-1}^{(\nu-1)} & B_{\kappa(\nu)}^{(\nu-1)} & \beta_{\kappa(\nu)\,\kappa(\nu)+1}^{(\nu-1)} & \cdots & \beta_{\kappa(\nu)\,r}^{(\nu-1)} & \beta_{\kappa(\nu)\,\kappa(\nu)}^{(\nu-1)} + \sum_{k=\kappa(\nu)+1}^{r} b_k^{(\nu)}\beta_{\kappa(\nu)\,k}^{(\nu-1)} + b_{r+1}^{(\nu)}B_{\kappa(\nu)}^{(\nu-1)} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\beta_{r\,1}^{(\nu-1)} & \cdots & \beta_{r\,\kappa(\nu)-1}^{(\nu-1)} & B_r^{(\nu-1)} & \beta_{r\,\kappa(\nu)+1}^{(\nu-1)} & \cdots & \beta_{r\,r}^{(\nu-1)} & \beta_{r\,\kappa(\nu)}^{(\nu-1)} + \sum_{k=\kappa(\nu)+1}^{r} b_k^{(\nu)}\beta_{r\,k}^{(\nu-1)} + b_{r+1}^{(\nu)}B_r^{(\nu-1)} \\
\beta_{0\,1}^{(\nu-1)} & \cdots & \beta_{0\,\kappa(\nu)-1}^{(\nu-1)} & B_0^{(\nu-1)} & \beta_{0\,\kappa(\nu)+1}^{(\nu-1)} & \cdots & \beta_{0\,r}^{(\nu-1)} & \beta_{0\,\kappa(\nu)}^{(\nu-1)} + \sum_{k=\kappa(\nu)+1}^{r} b_k^{(\nu)}\beta_{0\,k}^{(\nu-1)} + b_{r+1}^{(\nu)}B_0^{(\nu-1)}
\end{pmatrix}
$$

Since $\det(M^{(1)}\cdots M^{(\nu)}) = \pm 1$, which follows from (5.4), we see that $B_0^{(\nu)}, \ldots, B_{r-1}^{(\nu)}$ and $B_r^{(\nu)}$ have no non-trivial common factor. By a simple calculation, for $(f_1^{(\nu-1)}, \ldots, f_r^{(\nu-1)}) \in \mathbb{L}_{\kappa(\nu)}^r$, we see that

(i) $i_2 \neq \kappa(\nu)\,,\; r+1$

$$\beta_{i_1\,i_2}^{(\nu)} = \beta_{i_1\,i_2}^{(\nu-1)} \qquad 1 \le i_1 \le r, \tag{5.6}$$

(ii) $i_2 = \kappa(\nu)$

$$\beta_{i_1\,i_2}^{(\nu)} = B_{i_1}^{(\nu-1)} \qquad 0 \le i_1 \le r, \tag{5.7}$$

(iii) $i_2 = r+1$

$$\beta_{i_1\,i_2}^{(\nu)} = B_{i_1}^{(\nu)} = \beta_{i_1\,\kappa(\nu)}^{(\nu-1)} + \sum_{k=\kappa(\nu)+1}^{r} b_k^{(\nu)}\beta_{i_1\,k}^{(\nu-1)} + b_{r+1}^{(\nu)}B_{i_1}^{(\nu-1)} \qquad 0 \le i_1 \le r. \tag{5.8}$$

The above (i) and (ii) mean $\begin{pmatrix} \beta_{1i}^{(\nu')} \\ \vdots \\ \beta_{ri}^{(\nu')} \\ \beta_{0i}^{(\nu')} \end{pmatrix}$ is one of $\begin{pmatrix} B_{1i}^{(\nu')} \\ \vdots \\ B_{ri}^{(\nu')} \\ B_{0i}^{(\nu')} \end{pmatrix}$, $-r \leq \nu' \leq \nu - 1$.

From (5.5), we find that $B_i^{(\nu)}$ increases as ν increases and

$$\deg B_{i_1}^{(\nu)} > \deg \beta_{i_1\,\kappa(\nu)}^{(\nu)} > \deg \beta_{i_1\,i_2}^{(\nu)}$$

if $i_2 \neq \kappa(\nu), d+1$ for $0 \leq i_1 \leq r$. We put

$$M^{(1)} \cdots M^{(\nu)} \begin{pmatrix} f_1^{(\nu)} \\ \vdots \\ \vdots \\ f_r^{(\nu)} \\ 1 \end{pmatrix} = \begin{pmatrix} \beta_{11}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{1r}^{(\nu)} f_r^{(\nu)} + B_1^{(\nu)} \\ \vdots \\ \vdots \\ \beta_{r1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{rr}^{(\nu)} f_r^{(\nu)} + B_r^{(\nu)} \\ \beta_{01}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{0r}^{(\nu)} f_r^{(\nu)} + B_0^{(\nu)} \end{pmatrix}$$

and see the following theorem.

Theorem 5.1.1 *For any $(f_1, \ldots, f_r) \in \mathbb{L}^r$, we have*

$$f_i = \frac{\beta_{i1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{ir}^{(\nu)} f_r^{(\nu)} + B_i^{(\nu)}}{\beta_{01}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{0r}^{(\nu)} f_r^{(\nu)} + B_0^{(\nu)}} \qquad \text{for} \quad 1 \leq i \leq r,$$

whenever $S^{\nu'}(f_1, \ldots, f_r) \neq (0, \ldots 0)$ for any $0 \leq \nu' < \nu$.

Proof. From the definition, for $(f_1, \ldots, f_r) \in \mathbb{L}_j^r$,

$$\begin{aligned} S(f_1, \ldots, f_r) &= (f_1^{(1)}, \ldots, f_r^{(1)}) \\ &= \left(\frac{f_1}{f_j}, \ldots, \frac{f_{j-1}}{f_j}, \frac{1}{f_j} - b_{r+1}^{(1)}, \frac{f_{j+1}}{f_j} - b_{j+1}^{(1)}, \ldots, \frac{f_r}{f_j} - b_r^{(1)} \right). \end{aligned}$$

Then

$$f_i = \begin{cases} \dfrac{1 \cdot f_i^{(1)}}{1 \cdot f_j^{(1)} + b_{r+1}^{(1)}} & 1 \leq i < j \\ \dfrac{1}{1 \cdot f_j^{(1)} + b_{r+1}^{(1)}} & i = j \\ \dfrac{1 \cdot f_i^{(1)} + b_i^{(1)}}{1 \cdot f_j^{(1)} + b_{r+1}^{(1)}} & j < i \leq r. \end{cases} \tag{5.9}$$

On the other hand, for $(f_1, \ldots, f_r) \in \mathbb{L}_j^r$,

$$\frac{\beta_{i1}^{(1)} f_1^{(1)} + \cdots + \beta_{ir}^{(1)} f_r^{(1)} + B_i^{(1)}}{\beta_{01}^{(1)} f_1^{(1)} + \cdots + \beta_{0r}^{(1)} f_r^{(1)} + B_0^{(1)}} = \begin{cases} \dfrac{1 \cdot f_i^{(1)}}{1 \cdot f_j^{(1)} + b_{r+1}^{(1)}} & 1 \le i < j \\ \dfrac{1}{1 \cdot f_j^{(1)} + b_{r+1}^{(1)}} & i = j \\ \dfrac{1 \cdot f_i^{(1)} + b_i^{(1)}}{1 \cdot f_j^{(1)} + b_{r+1}^{(1)}} & j < i \le r. \end{cases} \tag{5.10}$$

From (5.9) and (5.10), the assertion of the theorem holds for $\nu = 1$. Now we assume that the assertion of the theorem holds by ν, and we will show that the assertion holds for $\nu+1$. Note that $\kappa(\nu+1)$ is chosen by $(f_1^{(\nu)}, \ldots, f_r^{(\nu)}) \in \mathbb{L}_{\kappa(\nu+1)}^r$.

$$\frac{\beta_{i1}^{(\nu+1)} f_1^{(\nu+1)} + \cdots + \beta_{ir}^{(\nu+1)} f_r^{(\nu+1)} + B_i^{(\nu+1)}}{\beta_{01}^{(\nu+1)} f_1^{(\nu+1)} + \cdots + \beta_{0r}^{(\nu+1)} f_r^{(\nu+1)} + B_0^{(\nu+1)}}$$

$$= \frac{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{ik}^{(\nu+1)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + \beta_{i\kappa(\nu+1)}^{(\nu+1)} \left(\frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} - b_{r+1}^{(\nu+1)} \right) + \sum_{k=\kappa(\nu+1)+1}^{r} \beta_{ik}^{(\nu+1)} \left(\frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} - b_k^{(\nu+1)} \right) + B_i^{(\nu+1)}}{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{0k}^{(\nu+1)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + \beta_{0\kappa(\nu+1)}^{(\nu+1)} \left(\frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} - b_{r+1}^{(\nu+1)} \right) + \sum_{k=\kappa(\nu+1)+1}^{r} \beta_{0k}^{(\nu+1)} \left(\frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} - b_k^{(\nu+1)} \right) + B_0^{(\nu+1)}}$$

$$= \frac{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{ik}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + B_i^{(\nu)} \left(\frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} - b_{r+1}^{(\nu+1)} \right) + \sum_{k=\kappa(\nu+1)+1}^{r} \beta_{ik}^{(\nu)} \left(\frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} - b_k^{(\nu+1)} \right) + B_i^{(\nu+1)}}{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{0k}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + B_0^{(\nu)} \left(\frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} - b_{r+1}^{(\nu+1)} \right) + \sum_{k=\kappa(\nu+1)+1}^{r} \beta_{0k}^{(\nu)} \left(\frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} - b_k^{(\nu+1)} \right) + B_0^{(\nu+1)}}.$$

From (5.8),

$$\frac{\beta_{i1}^{(\nu+1)} f_1^{(\nu+1)} + \cdots + \beta_{ir}^{(\nu+1)} f_r^{(\nu+1)} + B_i^{(\nu+1)}}{\beta_{01}^{(\nu+1)} f_1^{(\nu+1)} + \cdots + \beta_{0r}^{(\nu+1)} f_r^{(\nu+1)} + B_0^{(\nu+1)}}$$

$$= \frac{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{ik}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + B_i^{(\nu)} \frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} + \sum_{k=\kappa(\nu+1)+1}^{r} \beta_{ik}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + \beta_{i\kappa(\nu+1)}^{(\nu)}}{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{0k}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + B_0^{(\nu)} \frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} + \sum_{k=\kappa(\nu+1)+1}^{r} \beta_{0k}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + \beta_{0\kappa(\nu+1)}^{(\nu)}}$$

$$= \frac{\beta_{i1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{ir}^{(\nu)} f_r^{(\nu)} + B_i^{(\nu)}}{\beta_{01}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{0r}^{(\nu)} f_r^{(\nu)} + B_0^{(\nu)}} = f_i.$$

Thus the assertion holds for $\nu + 1$ and the proof is complete.

□

Now we call $\frac{B_i^{(\nu)}}{B_0^{(\nu)}}$ the ν-th convergent of the MJPA and $M^{(1)}, \dots, M^{(\nu)}$ the expansion by this algorithm when $S^\nu(f_1, \dots, f_r) \neq (0, \dots, 0)$ for $\nu \geq 0$. Moreover the expansion by the MJPA is said to be finite or infinite if $S^\nu(f_1, \dots, f_r) = 0$ for some $\nu \geq 0$ or $S^\nu(f_1, \dots, f_r) \neq 0$ for any $\nu \geq 0$, respectively. In the sequel, we show some lemmas about the expansion.

Lemma 5.1.1 *For* $(f_1, \dots, f_r) \in \mathbb{L}^r$ *with* $f_i \in \mathbb{F}_q(X)$, $1 \leq i \leq r$, *the expansion by the MJPA is finite.*

Proof. As $f_i \in \mathbb{F}_q(X)$, $1 \leq i \leq r$,, we can write $(f_1, \dots, f_r) = \left(\frac{P_1}{Q}, \dots, \frac{P_r}{Q}\right)$ where $P_1, \dots, P_r$ and Q are in $\mathbb{F}_q[X]$ and have no non-trivial common factor. Then it is clear that

$$S(f_1, \dots, f_r) = \left(\frac{P_1'}{P_j}, \dots, \frac{P_r'}{P_j}\right) \tag{5.11}$$

for some $P_1', \dots, P_r' \in \mathbb{F}_q[X]$ if $(f_1, \dots, f_r) \in \mathbb{L}_j^r$. If we put $S^\nu(f_1, \dots, f_r) = \left(\frac{P_1^{(\nu)}}{Q^{(\nu)}}, \dots, \frac{P_r^{(\nu)}}{Q^{(\nu)}}\right)$, $P_1^{(\nu)}, \dots, P_r^{(\nu)}$ and $Q^{(\nu)}$ have no non-trivial common factor, then (5.11) implies

$$\deg Q^{(\nu)} < \deg Q^{(\nu-1)} \qquad \text{for} \quad \nu \geq 1.$$

Consequently, for some $\nu_0 \geq 1$, $P_1^{(\nu_0)} = \cdots = P_r^{(\nu_0)} = 0$ since $\deg P_i^{(\nu)} < \deg Q^{(\nu)}$ and $Q^{(\nu)}$ is a polynomial for any $\nu \geq 1$.

□

Lemma 5.1.2 *For* $(f_1, \dots, f_r) \in \mathbb{L}^r$, *if* $f_i \notin \mathbb{F}_q(X)$ *for some* i, $1 \leq i \leq r$,, *the expansion by the MJPA is infinite.*

Proof. If the expansion of f_i is finite, which means $S^\nu(f_1, \dots, f_r) = (0, \dots, 0)$ for some $\nu \geq 0$, then from Theorem 5.1.1, we see

$$f_i = \frac{B_i^{(\nu)}}{B_0^{(\nu)}} = \frac{\beta_{i\,\kappa(\nu)}^{(\nu-1)} + \sum\limits_{k=\kappa(\nu)+1}^{r} b_k^{(\nu)} \beta_{i\,k}^{(\nu-1)} + b_{r+1}^{(\nu)} B_i^{(\nu-1)}}{\beta_{0\,\kappa(\nu)}^{(\nu-1)} + \sum\limits_{k=\kappa(\nu)+1}^{r} b_k^{(\nu)} \beta_{0\,k}^{(\nu-1)} + b_{r+1}^{(\nu)} B_0^{(\nu-1)}}.$$

But this shows that f_i is a rational function, which contradicts with the assertion $f_i \notin \mathbb{F}_q(X)$.

□

Lemma 5.1.3 *For any sequence* $M^{(1)}, \ldots, M^{(\nu+1)}, \ldots$ *of the form* (5.5),

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \leq \frac{1}{q}$$

holds for any $\nu \geq 1$.

Proof. Note that $\kappa(\nu) = \min_{1 \leq i \leq r+1} \{i : m_{ir+1}^{(\nu)} \neq 0\}$ where $m_{ir+1}^{(\nu)}$ is the $(i, r+1)$ component of $M^{(\nu)}$. Then if $1 \leq \kappa(1) < \kappa(2)$,

$$\left| \frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}} \right| = \left| \frac{b_i^{(2)}}{b_{r+1}^{(1)} b_{r+1}^{(2)}} \right|, \qquad 1 \leq i \leq r.$$

Since $\deg b_{r+1}^{(\nu)} \geq 1$ and $\deg b_{r+1}^{(\nu)} > \deg b_i^{(\nu)}$, $1 \leq i \leq r$, for $\nu \geq 1$, we have

$$|B_0^{(1)}| \left| \frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}} \right| \leq \frac{1}{q}. \tag{5.12}$$

We also see if $\kappa(1) = \kappa(2)$,

$$\left| \frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}} \right| = \begin{cases} \left| \dfrac{b_i^{(2)}}{(1 + b_{r+1}^{(1)} b_{r+1}^{(2)}) b_{r+1}^{(1)}} \right| & 1 \leq i \leq \kappa(1) \\[2ex] \left| \dfrac{b_{r+1}^{(1)} b_i^{(2)} - b_i^{(1)}}{(1 + b_{r+1}^{(1)} b_{r+1}^{(2)}) b_{r+1}^{(1)}} \right| & \kappa(1) < i \leq r \end{cases}$$

and if $\kappa(2) < \kappa(1) \leq r$,

$$\left| \frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}} \right| = \begin{cases} \left| \dfrac{b_i^{(2)}}{b_{\kappa(1)}^{(2)} + b_{r+1}^{(1)} b_{r+1}^{(2)}} \right| & 1 \leq i \leq \kappa(1) \\[2ex] \left| \dfrac{b_{r+1}^{(1)} b_i^{(2)} - b_i^{(1)} b_{\kappa(1)}^{(2)}}{(b_{\kappa(1)}^{(2)} + b_{r+1}^{(1)} b_{r+1}^{(2)}) b_{r+1}^{(1)}} \right| & \kappa(1) < i \leq r. \end{cases}$$

Then similarly, we have

$$|B_0^{(1)}| \left| \frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}} \right| \leq \frac{1}{q}.$$

Now we suppose the assertion of Lemma 5.1.3 holds by $\nu - 1$. For $\nu \geq 2$,

$$\begin{aligned}
&\left|\frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| \\
&= \left|\frac{\beta_{i\,\kappa(\nu+1)}^{(\nu)} + \sum_{k=\kappa(\nu+1)+1}^{r} b_k^{(\nu+1)}\beta_{i\,k}^{(\nu)} + b_{r+1}^{(\nu+1)}B_i^{(\nu)}}{\beta_{0\,\kappa(\nu+1)}^{(\nu)} + \sum_{k=\kappa(\nu+1)+1}^{d} b_k^{(\nu+1)}\beta_{0\,k}^{(\nu)} + b_{r+1}^{(\nu+1)}B_0^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| \\
&= \left|\frac{\beta_{i\,\kappa(\nu+1)}^{(\nu)}B_0^{(\nu)} - \beta_{0\,\kappa(\nu+1)}^{(\nu)}B_i^{(\nu)} + \sum_{k=\kappa(\nu+1)+1}^{r} b_k^{(\nu+1)}(\beta_{i\,k}^{(\nu)}B_0^{(\nu)} - \beta_{0\,k}^{(\nu)}B_i^{(\nu)})}{\left(\beta_{0\,\kappa(\nu+1)}^{(\nu)} + \sum_{k=\kappa(\nu+1)+1}^{r} b_k^{(\nu+1)}\beta_{0\,k}^{(\nu)} + b_{r+1}^{(\nu+1)}B_0^{(\nu)}\right)B_0^{(\nu)}}\right| \\
&= \frac{\left|\sum_{k=\kappa(\nu+1)}^{d} b_k^{(\nu+1)}(\beta_{i\,k}^{(\nu)}B_0^{(\nu)} - \beta_{0\,k}^{(\nu)}B_i^{(\nu)})\right|}{\left|b_{r+1}^{(\nu+1)}\,B_0^{(\nu)2}\right|} \\
&= \frac{1}{|b_{r+1}^{(\nu+1)}B_0^{(\nu)}|}\left|\sum_{k=\kappa(\nu+1)}^{r} \beta_{0\,k}^{(\nu)}\left(\frac{\beta_{i\,k}^{(\nu)}}{\beta_{0\,k}^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right)\right|,
\end{aligned}$$

here we use the facts that $\deg b_k^{(\nu+1)}\beta_{0\,k}^{(\nu)} < \deg b_{r+1}^{(\nu)}B_0^{(\nu)}$ for the third equality. By (5.6) and (5.7), we can replace $\beta_{i\,k}^{(\nu)}$ by $B_i^{(l_k)}$ for some l_k, $-r \leq l_k \leq \nu - 1$,, but $B_i^{(l_k)} = 0$ for $i \neq 0$ and $l_k < 0$,

$$\begin{aligned}
|B_0^{(\nu)}|\left|\frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| &= \frac{1}{|b_{r+1}^{(\nu+1)}|}\left|\sum_{l_k=1}^{r} B_0^{(l_k)}\left(\frac{B_i^{(l_k)}}{B_0^{(l_k)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right)\right| \\
&= \frac{1}{|b_{r+1}^{(\nu+1)}|}\left|\sum_{l_k=1}^{r} B_0^{(l_k)}\sum_{l=1}^{\nu-1}\left(\frac{B_i^{(l)}}{B_0^{(l)}} - \frac{B_i^{(l+1)}}{B_0^{(l+1)}}\right)\right| \\
&= \frac{1}{|b_{r+1}^{(\nu+1)}|}\max_{1\leq k\leq \nu-1}\max_{k\leq l\leq \nu-1}|B_0^{(k)}|\left|\frac{B_i^{(l)}}{B_0^{(l)}} - \frac{B_i^{(l+1)}}{B_0^{(l+1)}}\right|.
\end{aligned}$$

Then, from the assumption of the induction,

$$|B_0^{(\nu)}|\left|\frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| \leq \frac{1}{q^2} \leq \frac{1}{q}.$$

□

Theorem 5.1.2 (i) *If* $S^\nu(f_1,\dots,f_r)\not\equiv 0$ *for any* $\nu\ge 1$,

$$\lim_{\nu\to\infty}\frac{B_i^{(\nu)}}{B_0^{(\nu)}}=f_i \qquad \text{for} \quad 1\le i\le r,$$

on the other hand, if $S^{\nu-1}(f_1,\dots,f_r)\not\equiv 0$ *and* $S^\nu(f_1,\dots,f_r)\equiv 0$, *then*

$$\frac{B_i^{(\nu)}}{B_0^{(\nu)}}=f_i \qquad \text{for} \quad 1\le i\le r.$$

(ii) *For a given sequence of arrays* $\{b_i^{(\nu)}:1\le i\le r+1,\nu\ge 1\}$;

$$\begin{gathered} b_{r+1}^{(\nu)}\in\mathbb{F}_q[X], \qquad \deg b_{r+1}^{(\nu)}\ge 1, \\ b_i^{(\nu)}=0 \quad \text{for} \quad 1\le i<j(\nu), \qquad b_i^{(\nu)}\in\mathbb{F}_q \quad \text{for} \quad j(\nu)\le i\le r \end{gathered} \tag{5.13}$$

with a sequence $j(1),j(2),\dots$, $1\le j(\nu)\le r$ *and* $\nu\ge 1$, *there exists* $(f_1,\dots,f_r)\in\mathbb{L}^r$ *such that* $\kappa(\nu)=j(\nu)$.

Proof.

(i) We see

$$\begin{aligned}
\left|f_i-\frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| &= \left|\frac{\beta_{i1}^{(\nu)}f_1^{(\nu)}+\cdots+\beta_{ir}^{(\nu)}f_r^{(\nu)}+B_i^{(\nu)}}{\beta_{01}^{(\nu)}f_1^{(\nu)}+\cdots+\beta_{0r}^{(\nu)}f_r^{(\nu)}+B_0^{(\nu)}}-\frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| \\
&= \left|\frac{\sum_{k=1}^{r}(\beta_{ik}^{(\nu)}B_0^{(\nu)}-\beta_{0k}^{(\nu)}B_i^{(\nu)})f_k^{(\nu)}}{(\beta_{01}^{(\nu)}f_1^{(\nu)}+\cdots+\beta_{0r}^{(\nu)}f_r^{(\nu)}+B_0^{(\nu)})B_0^{(\nu)}}\right| \\
&= \left|\frac{\sum_{k=1}^{r}\left(\frac{\beta_{ik}^{(\nu)}}{\beta_{0k}^{(\nu)}}-\frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right)\beta_{0k}^{(\nu)}f_k^{(\nu)}}{(\beta_{01}^{(\nu)}f_1^{(\nu)}+\cdots+\beta_{0r}^{(\nu)}f_r^{(\nu)}+B_0^{(\nu)})}\right|.
\end{aligned}$$

For each k, $1\le k\le r$, there exists some l_k, $-r\le l_k<\nu$, such that

$$\beta_{ik}^{(\nu)}=B_i^{(l_k)}.$$

Then, we have

$$\begin{aligned}
&\left|\sum_{k=1}^{r}\beta_{0k}^{(\nu)}\left(\frac{\beta_{ik}^{(\nu)}}{\beta_{0k}^{(\nu)}}-\frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right)\right| \\
&= \left|\sum_{k=1}^{r}B_0^{(l_k)}\left(\frac{B_i^{(l_k)}}{B_0^{(l_k)}}-\frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right)\right| \\
&= \sum_{k=1}^{r}|B_0^{(l_k)}|\left|\left(\frac{B_i^{(l_k)}}{B_0^{(l_k)}}-\frac{B_i^{(l_k+1)}}{B_0^{(l_k+1)}}\right)+\cdots+\left(\frac{B_i^{(\nu-1)}}{B_0^{(\nu-1)}}-\frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right)\right| \\
&\leq \max_{1\leq l\leq \nu-1}|B_0^{(l)}|\left|\frac{B_i^{(l)}}{B_0^{(l)}}-\frac{B_i^{(l+1)}}{B_0^{(l+1)}}\right|, \quad \text{since } B_i^{(l_k)}=0 \text{ for } i\neq l_k,\ -r\leq l_k<0 \\
&\leq \frac{1}{q}.
\end{aligned}$$

Since $\deg B_0^{(\nu)} = \sum_{k=1}^{\nu} \deg b_{r+1}^{(k)} \geq \nu$ for any $\varepsilon > 0$, there exists $\nu_0 \geq 1$ such that

$$\left|f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| \leq \frac{1}{q}\frac{1}{|B_0^{(\nu)}|} < \varepsilon \qquad \text{for} \quad \forall \nu \geq \nu_0.$$

This implies

$$\lim_{\nu\to\infty}\frac{B_i^{(\nu)}}{B_0^{(\nu)}} = f_i \qquad 1 \leq i \leq r.$$

(ii) Now we suppose that such a sequence of arrays $\{b_i^{(\nu)}\}$ satisfying (5.13) is given. Since

$$\left|\frac{B_i^{(\nu)}}{B_0^{(\nu)}} - \frac{B_i^{(\nu+l)}}{B_0^{(\nu+l)}}\right| \leq \max_{\nu\leq k\leq \nu+l-1}\left|\frac{B_i^{(k)}}{B_0^{(k)}} - \frac{B_i^{(k+1)}}{B_0^{(k+1)}}\right|,$$

it is easy to see, from Lemma 5.1.3, that $\left(\frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right)$ is a Cauchy sequence for $1 \leq i \leq r$. Then we have the existence of the limit of $\left(\frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right)$, because $\mathbb{L}$ is complete.

□

5.2 The rate of convergence

In this section, we shall give a stronger estimate of the convergence than that of Lemma 5.1.3 under an assumption on $\{\kappa(\nu),\ \nu \geq 1\}$.

Theorem 5.2.1 *Suppose* $\{b_i^{(\nu)} : 1 \le \kappa(\nu) \le i \le r\,,\, \nu \ge 1\}$ *is the expansion of* $(f_1, \ldots, f_r) \in \mathbb{L}^r$. *If* $\#\{\nu : \kappa(\nu) = i\} = \infty$,

$$\lim_{\nu\to\infty} |B_i^{(\nu)} - f_i B_0^{(\nu)}| = 0 \qquad \text{for any } i, \quad 1 \le i \le r.$$

Here the condition $\#\{\nu : \kappa(\nu) = i\} = \infty$ holds for a.e. We prove it later. Before we prove Theorem 5.2.3, we give a definition and some lemmas which are necessary for the proof.

Definition 5.2.1 *For any* $(f_1^{(\nu-1)}, \ldots, f_r^{(\nu-1)}) \in \mathbb{L}^r_{\kappa(\nu)}$, *we put*

$$u(\nu) := \min_{1\le k\le r}\{l_k : \beta_{ik}^{(\nu)} = B_i^{(l_k)}, \text{ for any } 0 \le i \le r\}.$$

Also we put, for $s \ge 2$,

$$n_{s,i} := \{\nu : \min_{\tau_{s-1}\le\nu} \kappa(\nu) = i\}$$

for $s \ge 2$, $1 \le i \le r$ *and*

$$\tau_s := \max_{1\le i\le r} n_{s,i} + 1$$

with $\tau_1 = 0$.

Lemma 5.2.1 *Suppose* $\#\{\nu : \kappa(\nu) = i\} = \infty$ *for any* i, $1 \le i \le r$. *Then*

$$\tau_{s-1} \le u(\nu) < \tau_s \qquad \text{for} \quad \tau_s \le \nu < \tau_{s+1}.$$

Proof. From the definition of τ_s,

$$0 = \tau_1 < u(\nu) \qquad \text{for } \nu \ge \tau_2.$$

Note that $u(\nu)$ is non-increasing. If $\nu = \tau_3 - 1$, then $u(\nu) < \tau_2$ also by the definition of τ_s. So

$$\tau_1 \le u(\nu) < \tau_2$$

holds for $\tau_2 \le \nu < \tau_3$. In general, the assumption of the lemma implies $\tau_s < \infty$ for any $s \ge 2$ and we have $\tau_s \le u(\nu)$ for $\nu \ge \tau_{s+1}$. Also we have $u(\nu) = \tau_s$ if $\nu = \tau_{s+1} - 1$.

□

Lemma 5.2.2 *For any sequence* $M^{(1)}, \ldots, M^{(\nu+1)}, \ldots$ *of the form* (5.1) *we have*

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \le \frac{1}{q^s} \qquad \textit{for} \quad \nu \ge \tau_s.$$

Proof. From (5.12), it is clear that

$$|B_0^{(0)}| \left| \frac{B_i^{(1)}}{B_0^{(1)}} - \frac{B_i^{(0)}}{B_0^{(0)}} \right| = \left| \frac{B_i^{(1)}}{B_0^{(1)}} \right| = \frac{1}{q}. \tag{5.14}$$

For $\nu \ge 1$,

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| = \frac{1}{|b_{r+1}^{(\nu+1)}|} \left| \sum_{k=\kappa(\nu)}^{r} \beta_{0k}^{(\nu)} \left(\frac{\beta_{ik}^{(\nu)}}{\beta_{0k}^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) \right|.$$

We can replace $\beta_{ik}^{(\nu)}$ to $B_i^{(l_k)}$ for some l_k, $-r \le l_k \le \nu - 1$, but $B_i^{(l_k)} = 0$ for $i \ne l_k$, $-r \le i < 0$, then

$$\begin{aligned} |B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| &= \frac{1}{|b_{r+1}^{(\nu+1)}|} \left| \sum_{k=\kappa(\nu)}^{r} B_0^{(l_k)} \left(\frac{B_i^{(l_k)}}{B_0^{(l_k)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) \right| \\ &= \frac{1}{|b_{r+1}^{(\nu+1)}|} \left| \sum_{k=\kappa(\nu)}^{r} B_0^{(l_k)} \sum_{l=l_k}^{\nu-1} \left(\frac{B_i^{(l)}}{B_0^{(l)}} - \frac{B_i^{(l+1)}}{B_0^{(l+1)}} \right) \right| \\ &\le \frac{1}{|b_{r+1}^{(\nu+1)}|} \max_{u(\nu) \le l \le \nu-1} |B_0^{(l)}| \left| \frac{B_i^{(l)}}{B_0^{(l)}} - \frac{B_i^{(l+1)}}{B_0^{(l+1)}} \right|. \end{aligned}$$

By (5.14) and Lemma 5.1.3, for $\nu \ge \tau_2$,

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \le \frac{1}{q^2}.$$

By the induction, for $\nu \ge \tau_s$, we have

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \le \frac{1}{q^s}.$$

□

Proof of Theorem 5.2.3. For some l_k $(1 \leq k \leq r)$, we have

$$\begin{aligned} |B_i^{(\nu)} - f_i B_0^{(\nu)}| &= |B_0^{(\nu)}| \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \\ &= |B_0^{(\nu)}| \frac{1}{|B_0^{(\nu)}|} \left| \sum_{k=1}^{r} \left(\frac{\beta_{ik}^{(\nu)}}{\beta_{0k}^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) f_k^{(\nu)} \beta_{0k}^{(\nu)} \right| \\ &\leq \left| \max_{1\leq k\leq r} \left(\frac{B_i^{(l_k)}}{B_0^{(l_k)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) f_k^{(\nu)} \beta_{0k}^{(\nu)} \right| \\ &< \max_{1\leq k\leq r} \max_{l_k \leq t \leq \nu-1} |B_0^{(t)}| \left| \frac{B_i^{(t)}}{B_0^{(t)}} - \frac{B_i^{(t+1)}}{B_0^{(t+1)}} \right|. \end{aligned}$$

By Lemma 5.2.5, for $t \geq \tau_s$,

$$|B_0^{(t)}| \left| \frac{B_i^{(t)}}{B_0^{(t)}} - \frac{B_i^{(t+1)}}{B_0^{(t+1)}} \right| < \frac{1}{q^s}.$$

Then,

$$\lim_{\nu\to\infty} |B_i^{(\nu)} - f_i B_0^{(\nu)}| = 0.$$

□

We show that S is Haar measure preserving.

For a fixed $\nu \geq 1$, we denote by $\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle$ the cylinder set induced from $(\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)})$, that is, we put

$$\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle = \left\{ (f_1, \ldots, f_r) : \begin{pmatrix} b_1^{(1)} \\ \vdots \\ b_{r+1}^{(1)} \end{pmatrix} = \mathbf{b}^{(1)}, \ldots, \begin{pmatrix} b_1^{(\nu)} \\ \vdots \\ b_{r+1}^{(\nu)} \end{pmatrix} = \mathbf{b}^{(\nu)} \right\}.$$

Theorem 5.2.2 (i) *For any Borel set $B \subset \mathbb{L}^r$,*

$$m^r(S^{-1}B) = m^r(B),$$

that is, m^r, the normalized Haar measure on $\mathbb{L}^r$, is an invariant probability measure for S.

(ii) $\left\{ \begin{pmatrix} b_1^{(\nu)} \\ \vdots \\ b_{r+1}^{(\nu)} \end{pmatrix} : \nu \geq 1 \right\}$ *is an independent and identically distributed sequence of random variables with respect to m^r.*

Proof. (i) It is enough to show that

$$m^r(S^{-1}\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle) = m^r(\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle)$$

for every cylinder set $\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle$. Let

$$\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_{r+1} \end{pmatrix}$$

with

$$b_i = \begin{cases} 0 & 1 \leq i < j \\ 1 & i = j \\ b_i \in \mathbb{F}_q & j < i \leq r \\ b_i \in \mathbb{F}_q[X],\ \deg b_i \geq 1 & i = r+1, \end{cases} \tag{5.15}$$

Then we see that

$$S^{-1}\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle = \bigcup_{\mathbf{b}} \langle \mathbf{b}, \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle,$$

where $\mathbf{b}$ takes all such vectors with $1 \leq j \leq r$. If we fix $\mathbf{b}$, then $S|_{\langle \mathbf{b}, \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle}$ is 1-1 and onto $\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle$. For any $f \in \mathbb{L}$ of $\deg f = -n$, we consider

$$S_f(g) = \frac{g}{f} - \left[\frac{g}{f}\right] \qquad \text{for} \quad g \in \mathbb{L}.$$

The composition $m \circ S_f$ of the normalized Haar measure on $\mathbb{L}$ and S_f is defined by

$$(m \circ S_f)(A) = m(S_f\, A)$$

for a Borel subset A of $\mathbb{L}$. Then it is easy to see that

$$\frac{dm \circ S_f}{dm}(g) = q^n \qquad a.e.$$

holds. Also we consider

$$V(f) = \frac{1}{f} - \left[\frac{1}{f}\right]$$

and have

$$\frac{dm \circ V}{dm}(f) = q^{2n} \qquad (a.e.).$$

This means that the Radon-Nikodym derivatives of S_f and V are constants (a.e.) if $\deg f = -n$. This shows

$$\frac{dm^r \circ S}{dm^r}(f_1, \ldots, f_r) = q^{2n} \cdot q^{n(r-1)} \qquad (a.e.)$$

on $\langle \mathbf{b}, \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle$. Hence we have

$$q^{2n} q^{n(r-1)} \, m^r(\langle \mathbf{b}, \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle) = m^r(\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle)$$

when $\deg b_{r+1} = n \geq 1$. Moreover, the number of $\mathbf{b}$ with (5.7) is $q^{r-j} q^n (q-1)$. Therefore,

$$\begin{aligned}
m^r(S^{-1}\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle) &= m^r\left(\bigcup_{\mathbf{b}} \langle \mathbf{b}, \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle\right) \\
&= \sum_{\mathbf{b}} m^r(\langle \mathbf{b}, \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle) \\
&= \sum_{j=1}^{r} \sum_{n=1}^{\infty} (q-1) q^n q^{r-j} \frac{m^r(\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle)}{q^{2n} q^{n(r-1)}} \\
&= \sum_{j=1}^{r} \sum_{n=1}^{\infty} \frac{q-1}{q^{(n-1)r+j}} m^r(\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle) \\
&= m^r(\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle).
\end{aligned}$$

(ii) A similar calculation shows that

$$m^r(\langle \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\nu)} \rangle) = m^r(\langle \mathbf{b}^{(1)} \rangle) \cdots m^r(\langle \mathbf{b}^{(\nu)} \rangle).$$

This means that the coefficients of the MJPA induce an independent and identically distributed sequence of $(r+1)$-dimensional $\mathbb{F}_q[X]$-valued random variables.

$\square$

From Theorem 5.2.4, we have the following.

Proposition 5.2.1 *For a.e.* $(f_1, \ldots, f_r) \in \mathbb{L}^r$,

(i)

$$\lim_{\nu\to\infty} \frac{\#\{\eta : 1 \le \eta \le \nu,\ \kappa(\eta) = j\}}{\nu} = \frac{(q-1)q^{r-j}}{q^r - 1} \quad 1 \le i \le r,$$

(ii)

$$\lim_{\nu\to\infty} \frac{\#\{\eta : 1 \le \eta \le \nu,\ \deg b_{r+1}^{(\eta)} = n\}}{\nu} = \frac{q^r - 1}{q^{rn}},$$

(iii)

$$\lim_{\nu\to\infty} \frac{\#\{\eta : 1 \le \eta \le \nu,\ \kappa(\eta) = j,\ \deg b_{r+1}^{(\eta)} = n\}}{\nu} = \frac{q-1}{q^{(n-1)r+j}}.$$

Proof. It is easy to see that

$$m(\{f : \deg f = -n\}) = \frac{q-1}{q^n},$$

$$m(\{f : \deg f < -n\}) = \frac{1}{q^n}$$

and

$$m(\{f : \deg f \le -n\} = \frac{1}{q^{n-1}}.$$

So,

$$m^r(\{(f_1, \ldots, f_r) : (f_1, \ldots, f_r) \in \mathbb{L}_j^r,\ \deg f_j = -n\})$$

$$= \frac{q-1}{q^n}\left(\frac{1}{q^n}\right)^{j-1}\left(\frac{1}{q^{n-1}}\right)^{r-j}$$

$$= \frac{q-1}{q^{(n-1)r+j}}. \tag{5.16}$$

Then the strong law of large numbers shows (iii). Also (i) and (ii) are easily shown by

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q-1}{q^{(n-1)r+j}} &= \frac{q-1}{q^j}\sum_{n=1}^{\infty}\left(\frac{1}{q^r}\right)^{n-1} \\ &= \frac{(q-1)q^{r-j}}{q^r-1}\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{r}\frac{q-1}{q^{(n-1)r+j}} &= \frac{q-1}{q^{(n-1)r}}\sum_{j=1}^{r}\frac{1}{q^j} \\ &= \frac{q^r-1}{q^{rn}}.\end{aligned}$$

□

Proposition 5.2.2 *For a.e.* $(f_1,\ldots,f_r)\in\mathbb{L}^r$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\nu=1}^{N}\deg b_{r+1}^{(\nu)} = \frac{q^r}{q^r-1}.$$

Proof. We consider the sequence of random variables $\{X_\nu\}$ on the probability space $(\mathbb{L}^r, m^r)$ by $X_\nu(f_1,\ldots,f_r)=\deg b_{r+1}^{(\nu)}$. From (5.8), we have

$$\begin{aligned}E(X_\nu) &= \sum_{j=1}^{r}\sum_{n=1}^{\infty}\frac{1}{q^j}n\frac{q-1}{q^{(n-1)r}} \\ &= \frac{q^r}{q^r-1}\end{aligned}$$

By the strong law of large numbers, we have the conclusion.

□

Now we put

$$\gamma := \frac{q^r}{q^r-1}.$$

Lemma 5.2.3 *Let*

$$w(\nu) := \max_{\tau_s < \nu} s,$$

then there exists $\alpha > 0$ *such that*

$$\lim_{\nu \to \infty} \frac{w(\nu)}{\nu} = \alpha \qquad a.e.$$

Proof. For a fix $s \geq 1$, we put

$$A_l := \{(f_1, \ldots, f_r) \in \mathbb{L}^r : \tau_{s+1} - \tau_s = l\}, \qquad \text{for } l \geq r,$$

and $\{Y_s\}$ is the sequence of random variables on $(\mathbb{L}^r, m^r)$ defined by $Y_s(f_1, \ldots, f_r) = \tau_{s+1} - \tau_s$. Then, we have

$$\begin{aligned} E(Y_s) &= \sum_{l=r}^{\infty} l \cdot m^r(A_l) \\ &= r + \sum_{l=r}^{\infty} m^r(Y_s > l). \end{aligned}$$

Here we have

$$m^r(Y_\nu > l) < \sum_{k=1}^{r} \left(1 - \frac{(q-1)q^{r-k}}{q^r - 1}\right)^l,$$

and have

$$E(Y_s) < r + \sum_{l=r}^{\infty} \sum_{k=1}^{r} \left(1 - \frac{(q-1)q^{r-k}}{q^r - 1}\right)^l = r + \alpha_0 < \infty.$$

It is easy to see that $\{Y_s\}_{s \leq 1}$ is an independent and identically distributed sequence. The law of large numbers implies

$$\lim_{S \to \infty} \frac{1}{S} \sum_{s=1}^{S} Y_s = r + \alpha_0 \qquad a.e. \tag{5.17}$$

Since

$$\tau_{S+1} = \sum_{s=1}^{S} (\tau_{s+1} - \tau_s) = \sum_{s=1}^{S} Y_s$$

and

$$\tau_{w(\nu)} < \nu < \tau_{w(\nu)+1},$$

we have

$$\frac{S}{\sum_{s=1}^{S} Y_s} \leq \frac{w(\nu)}{\nu} < \frac{S}{\sum_{s=1}^{S-1} Y_s}$$

when $w(\nu) = S$. From (5.17), we have the assertion of the Lemma with

$$\alpha = \frac{1}{r + \alpha_0}.$$

□

Proposition 5.2.3 *For a.e.* $(f_1, \dots, f_r) \in \mathbb{L}^r$, *there exists a positive constant* $C_1 = C_1(\varepsilon)$ *such that*

$$|B_0^{(\nu)}| \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| < \frac{C_1}{q^{\nu\alpha(1-\varepsilon)}} \qquad \textit{for any } \varepsilon > 0, \quad 1 \leq i \leq r.$$

Proof. We fix $\varepsilon > 0$. For a.e. $(f_1, \dots, f_r) \in \mathbb{L}^r$, from Lemma 5.2.6,

$$\alpha - \varepsilon < \frac{w(\nu)}{\nu} < \alpha + \varepsilon$$

for sufficiently large ν, equivalently,

$$\nu\alpha - \nu\varepsilon < w(\nu) < \nu\alpha + \nu\varepsilon.$$

Then,

$$|B_0^{(\nu)}| \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \leq \frac{1}{q^{w(\nu)}} \leq \frac{1}{q^{\nu(\alpha-\varepsilon)}}$$

for sufficiently large ν.

□

Theorem 5.2.3 *For a.e.* $(f_1, \dots, f_r) \in \mathbb{L}^r$, *there exists a positive constant* $C_2 = C_2(\varepsilon)$ *such that*

$$\left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| < \frac{C_2}{|B_0^{(\nu)}|^{1+\frac{\alpha}{\gamma}(1-\varepsilon)}} \qquad \textit{for any } \varepsilon > 0, \quad 1 \leq i \leq r.$$

Proof. We fix $\varepsilon > 0$. From Proposition 3,

$$|B_0^{(\nu)}|\left|f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| < \frac{C_1}{q^{\nu\alpha(1-\frac{\varepsilon}{2})}}.$$

Since

$$\deg B_0^{(\nu)} = \sum_{i=1}^{\nu} \deg b_{r+1}^{(i)},$$

from Proposition 2, we have

$$\begin{aligned} q^{\nu\alpha(1-\frac{\varepsilon}{2})} &= q^{\nu\gamma\frac{\alpha}{\gamma}(1-\frac{\varepsilon}{2})} \\ &\geq \left(|B_0^{(\nu)}|^{(1-\frac{\varepsilon}{2})}\right)^{\frac{\alpha}{\gamma}(1-\frac{\varepsilon}{2})} \\ &= |B_0^{(\nu)}|^{\frac{\alpha}{\gamma}\left(1-\frac{\varepsilon}{2}\right)^2} \end{aligned}$$

for sufficiently large ν. Then there exists a positive constant C_2 such that

$$\begin{aligned} |B_0^{(\nu)}|\left|f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| &< \frac{C_2}{|B_0^{(\nu)}|^{\frac{\alpha}{\gamma}\left(1-\frac{\varepsilon}{2}\right)^2}} \\ &\leq \frac{C_2}{|B_0^{(\nu)}|^{\frac{\alpha}{\gamma}(1-\varepsilon)}}, \end{aligned}$$

which means

$$\left|f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| < \frac{C_2}{|B_0^{(\nu)}|^{1+\frac{\alpha}{\gamma}(1-\varepsilon)}}.$$

□

5.3 Rational functions

In this section, we study the number of $\left(\frac{B_1}{B_0}, \ldots, \frac{B_r}{B_0}\right)$ with $B_i \in \mathbb{F}_q[X]$, $\deg B_i < \deg B_0 = n \geq 1$, $1 \leq i \leq r$.

Definition 5.3.1 *For* $(B_0, B_1, \ldots, B_r) \in \mathbb{F}_q[X]^{r+1}$ *with*

$$(B_0, B_1, \ldots, B_r) = 1 \quad \textit{and} \quad \deg B_i < \deg B_0 \quad \textit{for} \quad 1 \leq i \leq r,$$

we denote by

$$L = L(B_0, B_1, \ldots, B_r)$$

the length of the expansion by the MJPA.

Definition 5.3.2 *We put*

$$E_\nu(n) = \#\left\{ \begin{array}{lcl} (B_0, B_1, \ldots, B_r) \in \mathbb{F}_q[X]^{r+1} & : & (B_0, B_1, \ldots, B_r) = 1, L = \nu, \\ & & \max_{1 \le i \le r} \deg B_i < \deg B_0 = n \end{array} \right\}$$

and

$$E(n) = \#\left\{ \begin{array}{lcl} (B_0, B_1, \ldots, B_r) \in \mathbb{F}_q[X]^{r+1} & : & (B_0, B_1, \ldots, B_r) = 1 \\ & & \max_{1 \le i \le r} \deg B_i < \deg B_0 = n \end{array} \right\}.$$

Theorem 5.3.1 *We have*

$$E_\nu(n) = \binom{n-1}{\nu-1} q^n (q^r - 1)^\nu$$

and

$$E(n) = (q^r - 1) q^{(r+1)n-r}.$$

Proof. For $(B_0, B_1, \ldots, B_r) \in \mathbb{F}_q[X]^{r+1}$, if $L = \nu$, then B_0 is determined by ν polynomials $b_{r+1}^{(1)}, \ldots, b_{r+1}^{(\nu)}$. Recall that $\deg B_0^{(\nu)} = n = \sum_{i=1}^{\nu} \deg b_{r+1}^{(i)}$. Then, the number of choices of $\deg b_{r+1}^{(i)}$, $1 \le i \le \nu$, is equal to $\binom{n-1}{\nu-1}$. Put $n_i = \deg b_{r+1}^{(i)}$ for $1 \le i \le \nu$, then the number of possible choices of $\{b_{r+1}^{(i)}\}$ is $(q-1)q^{n_i}$. So when we fix positive integers $n_1, \ldots, n_\nu$ with $\sum_{i=1}^{\nu} n_i = n$, the number of possible choices of $\{b_{r+1}^{(i)} : 1 \le i \le r\}$ is $(q-1)^\nu q^n$. Consequently the number of all choices of polynomials $b_{r+1}^{(1)}, \ldots, b_{r+1}^{(\nu)}$ is equal to

$$\binom{n-1}{\nu-1} (q-1)^\nu q^n.$$

Since the number of possible choices of $\{b_j^{(i)} : 1 \le j \le r\}$ is $q^{d-\kappa(i)}$, the one of $\{b_j^{(i)} : 1 \le j \le r, 1 \le \kappa(i) \le r\}$ is

$$\sum_{\kappa(i)=1}^{r} q^{d-\kappa(i)} = \frac{q^r - 1}{q - 1}.$$

Therefore

$$\begin{aligned} E_\nu(n) &= \binom{n-1}{\nu-1}(q-1)^\nu q^n \left(\sum_{\kappa(i)=1}^{r} q^{d-\kappa(i)}\right)^\nu \\ &= \binom{n-1}{\nu-1} q^n (q^r-1)^\nu. \end{aligned}$$

From the definition, it is clear that

$$\begin{aligned} E(n) &= \sum_{\nu=1}^{n} E_\nu(n) \\ &= \sum_{\nu=1}^{n} \binom{n-1}{\nu-1} q^n (q^r-1)^\nu \\ &= (q^r-1) q^n q^{r(n-1)} \\ &= (q^r-1) q^{(r+1)n-r}. \end{aligned}$$

□

Definition 5.3.3 *We put*

$$\hat{E}(n) = \#\left\{ (B_0, B_1, \ldots, B_r) \in \mathbb{F}_q[X]^{r+1} \ : \ \begin{array}{l} (B_0, B_1, \ldots, B_r) = 1, \\ \max_{1\le i\le r} \deg B_i \le \deg B_0 = n \end{array} \right\}.$$

Theorem 5.3.2 *We have*

$$\hat{E}(n) = (q^r-1) q^{(r+1)n}.$$

Proof. For $(B_0, B_1, \ldots, B_r) \in \mathbb{F}_q[X]^{r+1}$ satisfying

$$\deg B_i < \deg B_0 = n \qquad \text{for} \quad 1 \le i \le r$$

and

$$(B_0, B_1, \ldots, B_r) = 1,$$

there are q polynomials $\hat{B}_i$ of the form

$$\hat{B}_i = c\,B_0 + B_i \qquad c \in \mathbb{F}_q.$$

It is clear that

$$(B_0, \hat{B}_i) = 1 \qquad \text{and} \qquad \deg \hat{B}_i = n$$

unless $c = 0$. Hence for each $(B_0, B_1, \ldots, B_r) \in \mathbb{F}_q[X]^{r+1}$, we get q^r vectors $(\hat{B}_1, \ldots, \hat{B}_r)$ which satisfies

$$(B_0, \hat{B}_1, \ldots, \hat{B}_r) = 1 \qquad \text{and} \qquad \deg \hat{B}_i \leq n \quad \text{for} \quad 0 \leq i \leq r.$$

Then

$$\begin{aligned} \hat{E}(n) &= q^r\,E(n) \\ &= q^r(q^r - 1)q^{(r+1)n-r} \\ &= (q^r - 1)q^{(r+1)n}. \end{aligned}$$

□

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