Free Boundary Problems

for an Incompressible Ideal Fluid

2002

Masao OGAWA

Contents

Chapter 1. Introduction **1**

Chapter 2. Problem Close to Equilibrium **4**

2.1. Main result 4

2.2. Preliminaries 5

2.3. Representation of K and H 7

2.4. Estimates for K and H 10

2.5. Problem on the surface 12

2.6. Problem in the interior 18

2.7. Proof of Theorem 2.1 24

Chapter 3. Problem with Surface Tension **28**

3.1. Main results 28

3.2. Problem on the surface 29

3.3. Proof of Theorem 3.1 36

3.4. Proof of Theorem 3.2 38

Chapter 4. Problem Far from Equilibrium **41**

4.1. Main Result 41

4.2. Notations 42

4.3. Representation of K and H 43

4.4. Estimates for K and H 46

4.5. Problem on the surface 53

4.6. Problem in the interior 59

4.7. Proof of Theorem 4.1 65

References 68

Chapter 1. Introduction

To study the motion of water waves is one of classical problems in fluid mechanics. However, it is rather hard to solve the full problem of water waves, with no approximation based on the assumption that water waves have small amplitude. This fact comes from not only its nonlinearity, but also an unknown free boundary to be determined as a part of the solution.

Several papers addressed the well-posedness for the exact problem of water waves, in the sense of existence and uniqueness of solution. Nekrasov [30], Levi-Civita [24] and Struik [38] considered progressing waves. The papers of Lavrentiev [23], Ter-Krikorov [41], [42], Friedrichs and Hyers [10], Beale [3], Amick and Toland [2] concerned solitary wave solutions. Later, Gerber [11] examined steady waves over periodic and over monotone bottoms.

Using the abstract Cauchy-Kovalevskaya theorem, Nalimov [27], Ovsjannikov [34] and Shinbrot [36] showed the well-posedness for the general initial value problem of surface waves with analytic data. Moreover we see the similar assertions in [17], [18], [19], [35], [39], [40].

As for the initial data in a class of functions with finite smoothness, unique solvability of the plane problem of vortex-free water waves of infinite depth was proved by Nalimov [28]. Here the direction of the pressure gradient on the free surface plays a crucial role for well-posedness of the problem. That is to say, if it points inside the fluid at the initial time then there is a unique smooth solution at small time. Yosihara solved the problem when the domain is of finite depth, without and with the surface tension in [46] and [47], respectively. On the basis of their papers, two-phase problem in a Sobolev class was considered in [13] and [15]. In these articles, we required that the initial surface and the bottom were almost flat. In [44] Wu removed this restriction for the problem of gravity waves in case of infinite depth. She established the unique solvability even when the initial surface is not a single-valued graph, by showing the fact that the sign condition relating to Rayleigh-Taylor instability always holds for nonself-intersecting interface. This condition implies that for any solutions of the water wave problem, it is necessary that the pressure gradient in the inner normal direction on the free surface is positive. In [8], we see that the problem is actually ill-posed without the sign condition. Recently, Wu [45] extended her result to the problem for three-dimensional space. The problem of capillary-gravity waves in the two-dimensional space with a bottom and the large initial data was treated by Iguchi [14].

On the other hand, Nalimov [29] and Iguchi, Tanaka and Tani [16] investigated the problem describing the dynamics of planar vortical surface waves of infinite depth. When the flow is irrotational, we can reduce the free boundary problem to an initial value problem on the free surface. Then the solvability for the reduced problem leads to that for the original problem. However, for the rotational flow, we cannot deduce the problem only on the surface. We must investigate both the problems in the interior and on the boundary of the domain.

In this thesis, we address water waves for rotational flow in the plane domain with a fixed bottom. We will prove the temporary local existence and uniqueness of the solution in classes of finite smoothness.

Let the fluid occupy the domain $\Omega(t)$ bounded by the free surface $\Gamma_s(t)$ and the bottom Γ_b :

$$\begin{aligned} \Omega(t) &= \{z = (z_1, z_2);\ -h + b(z_1) < z_2 < \eta(t, z_1),\ z_1 \in \mathbf{R}^1\}, \\ \Gamma_b &= \{z = (z_1, z_2);\ z_2 = -h + b(z_1),\ z_1 \in \mathbf{R}^1\}, \\ \Gamma_s(t) &= \{z = (z_1, z_2);\ z_2 = \eta(t, z_1),\ z_1 \in \mathbf{R}^1\}, \end{aligned}$$

where h is a positive constant. Then the motion of the fluid is described by

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla_z)\mathbf{v}\right) + \nabla_z p = -\rho(0, g) \quad \text{in } \Omega(t),\ t > 0, \tag{1}$$

$$\nabla_z \cdot \mathbf{v} = 0 \quad \text{in } \Omega(t),\ t > 0, \tag{2}$$

$$p - p_e = -\sigma\mathcal{H} \quad \text{on } \Gamma_s(t),\ t > 0, \tag{3}$$

$$\frac{\partial \eta}{\partial t} + v_1\frac{\partial \eta}{\partial z_1} - v_2 = 0 \quad \text{on } \Gamma_s(t),\ t > 0, \tag{4}$$

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_b,\ t > 0, \tag{5}$$

$$\eta(0, z_1) = \eta_0(z_1),\ \mathbf{v}(0, z) = \mathbf{v}_0(z) \quad \text{on } \Omega \equiv \Omega(0). \tag{6}$$

Here ρ is density (constant), $\mathbf{v} = (v_1, v_2)$ is the velocity, p is the pressure, g is a gravitational constant, p_e is an atmospheric pressure (constant), σ is the coefficient of surface tension, $\mathcal{H} = (\partial/\partial z_1)\{(\partial\eta/\partial z_1)(1 + (\partial\eta/\partial z_1)^2)^{-1/2}\}$ is a curvature of $\Gamma_s(t)$ and $\mathbf{n}$ is the unit outer normal to Γ_b.

We introduce a function P defined by

$$P = \frac{p - p_e}{\rho} + g z_2.$$

Then by the Lagrangian coordinates (t, x)

$$z = x + \int_0^t \mathbf{u}(\tau, x)d\tau \equiv \Phi_{\mathbf{u}}(x; t), \quad \mathbf{u}(t, x) = \mathbf{v}(t, \Phi_{\mathbf{u}}(x; t)), \tag{7}$$

problem (1) – (6) becomes

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}} q = 0 && \text{in } \Omega,\ t > 0, && (8)\\
&\nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 && \text{in } \Omega,\ t > 0, && (9)\\
&q = g\left(x_2 + \int_0^t u_2(\tau, x)\mathrm{d}\tau\right) - \sigma\mathcal{H}(\Phi_{\mathbf{u}}(x;t)) && \text{on } \Gamma_s \equiv \Gamma_s(0),\ t > 0, && (10)\\
&\mathbf{u} \cdot \mathbf{n}(\Phi_{\mathbf{u}}(x;t)) = 0 && \text{on } \Gamma_b,\ t > 0, && (11)\\
&\mathbf{u}|_{t=0} = \mathbf{v}_0 && \text{on } \Omega. && (12)
\end{aligned}
$$

Here $q(t,x) = P(t, \Phi_{\mathbf{u}}(x;t)), \nabla_{\mathbf{u}} = A_{\mathbf{u}} \nabla_x$ and

$$
A_{\mathbf{u}} = {}^t\left(\frac{\partial \Phi_{\mathbf{u}}}{\partial x}\right)^{-1} = \begin{pmatrix} 1 + \int_0^t \frac{\partial u_2}{\partial x_2}\mathrm{d}\tau & -\int_0^t \frac{\partial u_2}{\partial x_1}\mathrm{d}\tau \\ -\int_0^t \frac{\partial u_1}{\partial x_2}\mathrm{d}\tau & 1 + \int_0^t \frac{\partial u_1}{\partial x_1}\mathrm{d}\tau \end{pmatrix}.
$$

Throughout this thesis we use the notation in vector analysis.

Once the solution $(\mathbf{u}, q)$ of problem (8) – (12) is determined, the solution of problem (1) – (6) is given by

$$
\mathbf{v}(t,z) = \mathbf{u}(t, \Phi_{\mathbf{u}}^{-1}(z;t)), \quad P(t,z) = q(t, \Phi_{\mathbf{u}}^{-1}(z;t)), \quad \Omega(t) = \Phi_{\mathbf{u}}(\Omega;t).
$$

Therefore we will construct the solution of the problem (8) – (12).

In Chapter 2, we study the free boundary problem in case that surface tension is not effective. It is shown that if the initial surface and the bottom are almost flat, the unique solution exists, locally in time, in a class of functions of finite smoothness.

In Chapter 3, the problem with surface tension is studied. If the assumptions mentioned above are satisfied, the problem is well-posed. Furthermore it will be shown that this solution converges to the solution of the problem without surface tension as the coefficient of surface tension tends to zero.

In Chapter 4, we study the problem without surface tension again. Here we find that for the well-posedness of the problem it is not necessary to assume the almost flatness of the boundaries. Therefore, the result in Chapter 4 is a generalization of that in Chapter 2.

Chapter 2. Problem Close to Equilibrium

In this chapter, we consider the free boundary problem when the effect of surface tension is negligible. Then the problem is solved under the condition that the initial surface and the bottom are almost flat and that the initial velocity is suitably small. Furthermore, we find that the existence time of the solution increases unboundedly, as the initial data tend to zero.

2.1. Main result

Theorem 2.1. *Let $\sigma = 0$, $g > 0$ and $s \geq 3 + 1/2$. There exist positive constants $\delta_1 = \delta_1(g, s)$ and $\delta_2 = \delta_2(s)$ such that if*

$$\begin{cases} \eta_0 \in H^{s+3/2}(\mathbf{R}^1), \quad b \in H^{s+3}(\mathbf{R}^1), \quad \mathbf{v}_0 \in H^{s+5/2}(\Omega), \\ \|\eta_0\|_{H^4(\mathbf{R}^1)} + \|b\|_{H^3(\mathbf{R}^1)} + \|\mathbf{v}_0\|_{H^{3+1/2}(\Omega)} + \|\omega_0\|_{H^{3+1/2}(\Omega)} \leq \delta_1, \\ \|b\|_{H^{s+2}(\mathbf{R}^1)} \leq \delta_2, \end{cases} \tag{2.1.1}$$

where $\omega_0 = \nabla_x^\perp \cdot \mathbf{v}_0, \nabla_x^\perp = (-\partial/\partial x_2, \partial/\partial x_1)$, and $\mathbf{v}_0$ satisfies the compatibility conditions, then problem (8) – (12) has a unique solution $(\mathbf{u}, q)$ on some time interval $[0, T]$ satisfying

$$\begin{cases} \mathbf{u} \in C^j([0,T]; H^{s+3/2-j/2}(\Omega)), \quad j = 0, 1, 2, \\ q \in C^j([0,T]; H^{s+3/2-j/2}(\Omega)), \quad j = 0, 1. \end{cases} \tag{2.1.2}$$

Remark. *The magnitude of T in the above theorem can be taken such that*

$$T \to \infty \qquad as \quad \|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)} + \|\mathbf{v}_0\|_{H^{s+3/2}(\Omega)} + \|\omega_0\|_{H^{s+3/2}(\Omega)} \to 0.$$

We give a brief sketch of the proof.

In the Lagrangian coordinates, vorticity

$$\nabla^\perp \cdot \mathbf{v} = \omega$$

can be written as

$$\nabla_{\mathbf{u}}^\perp \cdot \mathbf{u} = \omega_0 \qquad \text{in } \Omega, \quad t \geq 0. \tag{2.1.3}$$

In order to investigate this together with (9) it is convenient to use the coordinate transformation mapping

$$x = y + (0, \widetilde{\eta_0}(y)) \equiv \Psi(y)$$

from Ω onto the horizontal slab

$$\Sigma = \{y = (y_1, y_2);\ -h < y_2 < 0,\ y_1 \in \mathbf{R}^1\},$$

where $\widetilde{\eta_0}$ is a function such that $\widetilde{\eta_0}(\cdot, 0) = \eta_0(\cdot)$ and $\widetilde{\eta_0}(\cdot, -h) = b(\cdot)$. Therefore from (7)

$$z = \Phi_{\mathbf{u}}(\Psi(y); t) \equiv y + X(t, y), \qquad X(t, y) = (0, \widetilde{\eta_0}(y)) + \int_0^t \mathbf{u}(\tau, \Psi(y))\mathrm{d}\tau. \quad (2.1.4)$$

Putting

$$\bar{X}(t, y_1) = X(t, y_1, 0), \qquad (2.1.5)$$

we derive from (8), (10)

$$\left(1 + \frac{\partial \bar{X}_1}{\partial y_1}\right) \frac{\partial^2 \bar{X}_1}{\partial t^2} + \frac{\partial \bar{X}_2}{\partial y_1} \left(g + \frac{\partial^2 \bar{X}_2}{\partial t^2}\right) = 0 \qquad \text{for } t \geq 0$$

(see [16]) and from (9), (2.1.3)

$$\bar{X}_{2t} = K\bar{X}_{1t} + H \qquad \text{for } t \geq 0$$

with an operator $K = K(\bar{X}, b)$ and a function $H = H(X, \omega_1)$, $\omega_1(y) = \omega_0(\Psi(y))$, being given explicitly in Section 2.3. In Section 2.4 the properties of K and H will be investigated. In Section 2.5, assuming that an H is given, we consider the Cauchy problem for $\bar{X}$ with the initial conditions determined by (2.1.4), (2.1.5). In order to solve it, we will quasi-linearize the equations on the surface. Then we obtain the system which contains a weakly hyperbolic equation. For the well-posedness of the initial value problem for this weakly hyperbolic equation, we need a kind of sign condition, which requires the condition for gravity in Theorem 2.1. Further, we will show that the solution of quasi-linear system satisfies the nonlinear Cauchy problem on the free surface. In Section 2.6, for a given $\bar{X}$, we find $\mathbf{u}$ (in Ω) by solving the boundary value problem for (9), (2.1.3). Here we apply the partial Fourier transform to reduce the problem to the boundary value problem for the system of ordinary differential equations. Then X is determined through (2.1.4). In Section 2.7 by repeating this procedure, the solution $(\mathbf{u}, X, \bar{X})$ is obtained. Moreover q can be obtained from (8).

2.2. Preliminaries

Let j be a nonnegative integer, $0 < T < \infty$ and B a Banach space. We say that $u \in C^j([0, T]; B)$ if u is a j-times continuously differentiable function on $[0, T]$ with values in B. Let D be a domain in $\mathbf{R}^n$, m a nonnegative integer and $0 < r < 1$. By $H^m(D)$ we denote the usual Sobolev space on D of order m. By $H^{m+r}(D)$ we denote the Sobolev-Slobodetskiĭ space.

From [1, Lemmas 7.44 – 7.45] it follows that the semi-norm

$$\left(\iint_{\Sigma \times \Sigma} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2r}} \mathrm{d}x\mathrm{d}y\right)^{1/2}$$

is equivalent to

$$\left(\int_\Sigma\int_{-\infty}^{\infty}\frac{|u(x_1,x_2)-u(y_1,x_2)|^2}{|x_1-y_1|^{1+2r}}\mathrm{d}y_1\mathrm{d}x\right)^{1/2}$$
$$+\left(\int_\Sigma\int_{-h}^{0}\frac{|u(x_1,x_2)-u(x_1,y_2)|^2}{|x_2-y_2|^{1+2r}}\mathrm{d}y_2\mathrm{d}x\right)^{1/2}\equiv\|u\|_{\dot{H}^r(\Sigma)}+\|u\|_{\ddot{H}^r(\Sigma)}.$$

Moreover, we introduce the norm $\|\cdot\|_{s,\lambda_1,\lambda_2}$ $(\lambda_1,\lambda_2\geq 1)$:

$$\|u\|_{s,\lambda_1,\lambda_2}=\begin{cases}\sum_{|\alpha|\leq m}\lambda_2^{m-|\alpha|}\|\partial_1^{\alpha_1}(\lambda_1^{-1}\partial_2)^{\alpha_2}u\|_{L^2(\Sigma)} & \text{for } s=m,\\ \lambda_2^r\|u\|_{m,\lambda_1,\lambda_2}+\sum_{|\alpha|=m}(\lambda_1^{-\alpha_2}\|\partial^\alpha u\|_{\dot{H}^r(\Sigma)}+\lambda_1^{-(\alpha_2+1)}\|\partial^\alpha u\|_{\ddot{H}^r(\Sigma)}) & \\ & \text{for } s=m+r,\end{cases}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ is a multi-index, $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ and $\partial_j=\partial/\partial x_j$. Then it holds that

Lemma 2.2.1. *For any $s\geq 0$, $\lambda_1,\lambda_2\geq 1$ and $u,v\in C_0^\infty(\overline{\Sigma})$ we have*

$$\|uv\|_{s,\lambda_1,\lambda_2}\leq\left(\|u\|_{L^\infty(\Sigma)}+\lambda_2^{-\gamma}C\|u\|_{H^{s_0}(\Sigma)}\right)\|v\|_{s,\lambda_1,\lambda_2},$$

where

$$\gamma=\begin{cases}1 & \text{if } s\in\mathbf{Z},\\ s-[s] & \text{if } s\notin\mathbf{Z},\end{cases}\qquad s_0=\begin{cases}2+\varepsilon\ (^\forall\varepsilon>0) & \text{if } 0\leq s\leq 2,\\ s & \text{if } s>2,\end{cases}$$

and $C=C(s,s_0,\lambda_1)>0$.

Under the appropriate assumptions on $\widetilde{\eta_0}$, Ψ is a diffeomorphism from Σ onto Ω. Hence we define

$$\dot{H}^r(\Omega)=\{u;u\circ\Psi\in\dot{H}^r(\Sigma)\}\quad\text{with}\quad\|u\|_{\dot{H}^r(\Omega)}=\|u\circ\Psi\|_{\dot{H}^r(\Sigma)},$$
$$H^s(\Gamma_s)=\{u;u\circ\Psi(y_1,0)\in H^s(\mathbf{R}^1)\}\quad\text{with}\quad\|u\|_{H^s(\Gamma_s)}=\|u\circ\Psi(\cdot,0)\|_{H^s(\mathbf{R}^1)}$$

and so on.

The following classes of operators have already been introduced and used to simplify the estimates for K and H in [16], [46].

Definition. For $0\leq r,t\leq s$,

(1) $L(r,s;t)$ is the totality of M satisfying the conditions:

(i) $M=M(P;P(J))$ is a linear operator depending on $P=P(P_1,\ldots,P_k)$, where P_j are real-valued functions, J is the subset of $\{1,\ldots,k\}$, $P(J)=(P_{j_1},\ldots,P_{j_l})$ if $J=\{j_1,\ldots,j_l\}$ and $P(J)=0$ if J is empty,

(ii) There exists $d=d(M,t)>0$ such that if $P,P^0\in H^s(\mathbf{R}^1)$ satisfy

$$\|P(J)\|_{H^t(\mathbf{R}^1)},\|P^0(J)\|_{H^t(\mathbf{R}^1)}\leq d,\quad\|P\|_{H^s(\mathbf{R}^1)},\|P^0\|_{H^s(\mathbf{R}^1)}\leq d_0$$

for some $d_0 > 0$, then for any $u \in H^r(\mathbf{R}^1)$

$$\|M(P;P(J))u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^r(\mathbf{R}^1)},$$

$$\|M(P;P(J))u - M(P^0;P^0(J))u\|_{H^s(\mathbf{R}^1)} \leq C\|P - P^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^r(\mathbf{R}^1)},$$

where $C = C(r,s,t,d,d_0) > 0$,

(2) $L_0(r,s;t)$ consists of $M \in L(r,s;t)$ such that

$$\|M(P;P(J))u\|_{H^s(\mathbf{R}^1)} \leq C\|P\|_{H^s(\mathbf{R}^1)}\|u\|_{H^r(\mathbf{R}^1)}.$$

Lemma 2.2.2 ([16, Lemma 2.9]). *Suppose that $0 \leq r,t \leq s \leq s_1$. Then*

(1) *$L(r,s;t)$ and $L_0(r,s;t)$ are algebras,*

(2) *$L_0(r,s;t)$ is a two-sided $L(r,s;t)$-module,*

(3) *If f is smooth in a neighbourhood of $0 \in \mathbf{R}^k$, then the operator M defined by $M(P;P)u = f(P)u$ belongs to $L(s,s;t)$ for $1/2 < t \leq s$ and $s \geq 1$,*

(4) *If $M = M(P;P) \in L_0(q,q;t)$ for any $q \in [s,s_1]$ and $T_yM(P;P) = M(T_yP;T_yP)T_y$ for $y \in \mathbf{R}^1$, where $(T_yu)(x) = u(x+y)$, then $(1+M)^{-1}(P;P) \in L(q,q;s)$ for any $q \in [s,s_1]$.*

2.3. Representation of K and H

Throughout this section let the time $t \geq 0$ be arbitrarily fixed. We assume that $\mathbf{v}$ and X are smooth and tend to zero as variables tend to infinity. We identify $\mathbf{R}^2_{z_1,z_2}$ with the complex $z = z_1 + iz_2$ plane. Then $\Gamma_s(t)$ and Γ_b are given by

$$\begin{cases} \Gamma_s(t) : w_s(y_1) = y_1 + \bar{X}_1(y_1) + i\bar{X}_2(y_1), \\ \Gamma_b \quad : w_b(y_1) = y_1 + i(-h + b(y_1)), \qquad -\infty < y_1 < \infty. \end{cases}$$

Further let $\mathbf{v}$ satisfy the equations

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla^\perp \cdot \mathbf{v} = \omega \quad \text{in } \Omega(t), \qquad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_b$$

and put

$$\begin{cases} F = v_1 - iv_2, \\ f(y_1) = f_1(y_1) + if_2(y_1) = F(w_s(y_1)), \\ g(y_1) = g_1(y_1) + ig_2(y_1) = F(w_b(y_1)). \end{cases}$$

Now let us take $w_s^0 \in \Gamma_s(t)$ and the closed path γ in $\Omega(t)$. As γ tends to $\Gamma_s(t) \cup \Gamma_b$, the Cauchy integral formula yields

$$\frac{1}{2\pi i}\int_\gamma \frac{F(z)}{z - w_s^0}\mathrm{d}z \to -\frac{\pi i}{2\pi i}F(w_s^0) - \frac{1}{2\pi i}\text{v.p.}\int_{\Gamma_s(t)} \frac{F(z)}{z - w_s^0}\mathrm{d}z + \frac{1}{2\pi i}\int_{\Gamma_b} \frac{F(z)}{z - w_s^0}\mathrm{d}z$$

and the Green formula yields

$$\frac{1}{2\pi i}\int_\gamma \frac{F(z)}{z-w_s^0}\mathrm{d}z \to -i\iint_{\Omega(t)}\omega\frac{\partial E(z-w_s^0)}{\partial z_1}\mathrm{d}z_1\mathrm{d}z_2 - \iint_{\Omega(t)}\omega\frac{\partial E(z-w_s^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2.$$

Here $E(z)$ is the fundamental solution of Laplace's equation in two-dimensional space:

$$E(z) = \frac{1}{2\pi}\log|z|.$$

Therefore we have

$$\begin{aligned}
&-2i\iint_{\Omega(t)}\omega\frac{\partial E(z-w_s^0)}{\partial z_1}\mathrm{d}z_1\mathrm{d}z_2 - 2\iint_{\Omega(t)}\omega\frac{\partial E(z-w_s^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2\\
&+f(x_1)+\frac{1}{\pi i}\mathrm{v.p.}\int_{\Gamma_s(t)}\frac{f(y_1)}{w_s(y_1)-w_s(x_1)}\frac{\mathrm{d}w_s(y_1)}{\mathrm{d}y_1}\mathrm{d}y_1\\
&=\frac{1}{\pi i}\int_{\Gamma_b}\frac{g(y_1)}{w_b(y_1)-w_s(x_1)}\frac{\mathrm{d}w_b(y_1)}{\mathrm{d}y_1}\mathrm{d}y_1
\end{aligned}\tag{2.3.1}$$

with $x_1 \in \mathbf{R}^1$ such that $w_s^0 = w_s(x_1)$. After the integration by parts, the real part of (2.3.1) leads to the equation

$$\begin{aligned}
&-2\iint_{\Omega(t)}\omega\frac{\partial E(z-w_s^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2 + f_1 + i\mathrm{sgnD}f_2 + A_1f_1 + A_2f_2\\
&\qquad = \mathrm{e}^{-h|\mathrm{D}|}g_1 + i\mathrm{sgnD}\mathrm{e}^{-h|\mathrm{D}|}g_2 + A_3g_1 + A_4g_2,
\end{aligned}\tag{2.3.2}$$

where $\mathrm{D} = -i\partial/\partial x_1$ and

$$\begin{cases}
A_ju(x_1) = \displaystyle\int_{-\infty}^{\infty} a_j(x_1,y_1)\frac{\mathrm{d}u}{\mathrm{d}y_1}(y_1)\mathrm{d}y_1, \qquad j=1,2,3,4,\\
a_1 = -\dfrac{1}{\pi}\mathrm{Im}\log\left(1+\dfrac{\bar{X}_1(y_1)-\bar{X}_1(x_1)}{y_1-x_1}+i\dfrac{\bar{X}_2(y_1)-\bar{X}_2(x_1)}{y_1-x_1}\right),\\
a_2 = -\dfrac{1}{\pi}\mathrm{Re}\log\left(1+\dfrac{\bar{X}_1(y_1)-\bar{X}_1(x_1)}{y_1-x_1}+i\dfrac{\bar{X}_2(y_1)-\bar{X}_2(x_1)}{y_1-x_1}\right),\\
a_3 = -\dfrac{1}{\pi}\mathrm{Im}\log\left(1+\dfrac{-\bar{X}_1(x_1)+ib(y_1)-i\bar{X}_2(x_1)}{y_1-x_1-ih}\right),\\
a_4 = -\dfrac{1}{\pi}\mathrm{Re}\log\left(1+\dfrac{-\bar{X}_1(x_1)+ib(y_1)-i\bar{X}_2(x_1)}{y_1-x_1-ih}\right).
\end{cases}$$

Taking $w_b^0 \in \Gamma_b$ and proceeding in the same way as above, we obtain

$$\begin{aligned}
&-2\iint_{\Omega(t)}\omega\frac{\partial E(z-w_b^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2 + g_1 - i\mathrm{sgnD}g_2 + A_5g_1 + A_6g_2\\
&\qquad = \mathrm{e}^{-h|\mathrm{D}|}f_1 - i\mathrm{sgnD}\mathrm{e}^{-h|\mathrm{D}|}f_2 + A_7f_1 + A_8f_2,
\end{aligned}\tag{2.3.3}$$

where

$$\begin{cases} A_j u(x_1) = \int_{-\infty}^{\infty} a_j(x_1, y_1) \frac{\mathrm{d}u}{\mathrm{d}y_1}(y_1)\mathrm{d}y_1, \qquad j = 5, 6, 7, 8, \\ a_5 = -\frac{1}{\pi}\mathrm{Im}\log\left(1 + i\frac{b(y_1) - b(x_1)}{y_1 - x_1}\right), \\ a_6 = -\frac{1}{\pi}\mathrm{Re}\log\left(1 + i\frac{b(y_1) - b(x_1)}{y_1 - x_1}\right), \\ a_7 = -\frac{1}{\pi}\mathrm{Im}\log\left(1 + \frac{\bar{X}_1(y_1) + i\bar{X}_2(y_1) - ib(x_1)}{y_1 - x_1 + ih}\right), \\ a_8 = -\frac{1}{\pi}\mathrm{Re}\log\left(1 + \frac{\bar{X}_1(y_1) + i\bar{X}_2(y_1) - ib(x_1)}{y_1 - x_1 + ih}\right). \end{cases}$$

Eliminating g_1 and g_2 from (2.3.2), (2.3.3) and $\mathbf{v} \cdot \mathbf{n} = 0$, we have

$$\begin{aligned} &\left\{1 - \mathrm{e}^{-2h|\mathrm{D}|} - i\mathrm{sgnD}(1 + \mathrm{e}^{-2h|\mathrm{D}|})B_2\right\} f_1 - 2\iint_{\Omega(t)} \omega \frac{\partial E(z - w_s^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2 \\ &\quad - 2(\mathrm{e}^{-h|\mathrm{D}|} + B_3)(1 + B_4)^{-1}\iint_{\Omega(t)} \omega \frac{\partial E(z - w_b^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2 \\ &= -i\mathrm{sgnD}(1 + \mathrm{e}^{-2h|\mathrm{D}|})(1 + B_1)f_2, \end{aligned}$$

where

$$\begin{cases} B_1 = i\mathrm{sgnD}(1 + \mathrm{e}^{-2h|\mathrm{D}|})^{-1}\left\{-A_2 + \mathrm{e}^{-h|\mathrm{D}|}A_8 + B_3(-i\mathrm{sgnD}\mathrm{e}^{-h|\mathrm{D}|} + A_8)\right. \\ \qquad\qquad \left. - (\mathrm{e}^{-h|\mathrm{D}|} + B_3)B_4(1 + B_4)^{-1}(-i\mathrm{sgnD}\mathrm{e}^{-h|\mathrm{D}|} + A_8)\right\}, \\ B_2 = i\mathrm{sgnD}(1 + \mathrm{e}^{-2h|\mathrm{D}|})^{-1}\left\{A_1 - \mathrm{e}^{-h|\mathrm{D}|}A_7 - B_3(\mathrm{e}^{-h|\mathrm{D}|} + A_7)\right. \\ \qquad\qquad \left. + (\mathrm{e}^{-h|\mathrm{D}|} + B_3)B_4(1 + B_4)^{-1}(\mathrm{e}^{-h|\mathrm{D}|} + A_7)\right\}, \\ B_3 = -i\mathrm{sgnD}\mathrm{e}^{-h|\mathrm{D}|}b' + A_3 - A_4 b', \\ B_4 = i\mathrm{sgnD}b' + A_5 - A_6 b'. \end{cases}$$

Since $f_1 = v_1|_{\Gamma_s(t)}$ and $f_2 = -v_2|_{\Gamma_s(t)}$, we see that $\bar{X}_{2t} = K\bar{X}_{1t} + H$ with

$$\begin{aligned} K &= -(1 + B_1)^{-1}(i\tanh(h\mathrm{D}) + B_2) \\ &= -i\tanh(h\mathrm{D}) - B_2 + B_1(1 + B_1)^{-1}(i\tanh(h\mathrm{D}) + B_2) \\ &=: -i\tanh(h\mathrm{D}) + K_1, \end{aligned} \tag{2.3.4}$$

$$H = -i\left\{\mathrm{sgnD}(1+\mathrm{e}^{-2h|\mathrm{D}|})(1+B_1)\right\}^{-1}\left\{H_1+(\mathrm{e}^{-h|\mathrm{D}|}+B_3)(1+B_4)^{-1}H_2\right\},$$

$$H_1 = 2\iint_{\Omega(t)}\omega(z)\frac{\partial E(z-w_s^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2,\quad H_2 = 2\iint_{\Omega(t)}\omega(z)\frac{\partial E(z-w_b^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2.$$

2.4. Estimates for K and H

Assuming that $\bar{X}$ depends on x_1 and t, we define $A_{j,k,l}(\bar{X},\ldots,\partial_t^k\partial_{x_1}^l\bar{X},b,\ldots,\partial_{x_1}^l b;\bar{X},b)$, $j=1,2,\ldots,8,\ k,l=0,1,2,\ldots,$ by

$$\begin{cases} A_{j,0,0}=A_j, \qquad A_{j,0,l}=\left[\dfrac{\partial}{\partial x_1},A_{j,0,l-1}\right], \quad l=1,2,3,\ldots, \\ A_{j,k,l}=\left[\dfrac{\partial}{\partial t},A_{j,k-1,l}\right], \quad k=1,2,3,\ldots, \quad l=0,1,2,\ldots \end{cases}$$

and replace $\partial_t^p\partial_{x_1}^q\bar{X}$ by $\bar{X}^{pq}$. Here $[A,B]=AB-BA$ for operators A,B and $\partial_t=\partial/\partial t$, $\partial_{x_1}=\partial/\partial x_1$. Moreover we define $K_{1,k,l}$ for $k,l=0,1,2,\ldots$ in the same way as $A_{j,k,l}$. Then the following results come from [46, Lemmas 4.14 – 4.20] and Lemma 2.2.2(4).

Lemma 2.4.1.

(1) $A_{j,k,l}(\bar{X}^{00},\ldots,\bar{X}^{kl},b,\ldots,\partial_{x_1}^l b;\bar{X}^{00},b)\in L_0(2+(s-[s]),s;2)$ *for* $s\geq 2$.

(2) $K_{1,k,l}(\bar{X}^{00},\ldots,\bar{X}^{kl},b,\ldots,\partial_{x_1}^l b;\bar{X}^{00},b)\in L_0(2+(s-[s]),s;3)$ *for* $s\geq 3$.

(3) $\{1+Z_1+Z_2K(\bar{X},Z,b;\bar{X},Z,b)\}^{-1}\in L(s,s;3)$ *for* $s\geq 3$.

For $s\geq 0$ we introduce the notation

$$|||X|||_s = \|X\|_{H^{s+1/2}(\Sigma)}+\|X(\cdot,0)\|_{H^s(\mathbf{R}^1)}+\|X(\cdot,-h)\|_{H^s(\mathbf{R}^1)}$$

and

$$\begin{cases} [H(t)]_s = \|H(t)\|_{H^{s+1}(\mathbf{R}^1)}+\|\partial_t H(t)\|_{H^{s+1/2}(\mathbf{R}^1)}+\|\partial_t^2 H(t)\|_{H^s(\mathbf{R}^1)}, \\ |H(t)|_s = \|H(t)\|_{H^{s+1}(\mathbf{R}^1)}+\|\partial_t H(t)\|_{H^{s+1}(\mathbf{R}^1)}+\|\partial_t^2 H(t)\|_{H^s(\mathbf{R}^1)}+\|\partial_t^3 H(t)\|_{H^s(\mathbf{R}^1)}, \\ \mu_s = \|\omega_1\|_{H^{s+3/2}(\Sigma)}. \end{cases} \tag{2.4.1}$$

Assumption 2.1.

(1) $\omega_1\in H^{s+3/2}(\Sigma)$.

(2) *There exist* $c_0 > 0$, $d > 0$, $l_j > 0$ $(j = 1, 2, \ldots, 5)$ *such that for* $s \geq 3$, $0 < T < \infty$, X *and* b *satisfy*

$$\begin{cases} X \in C^j([0,T]; H^{s+2-j/2}(\Sigma)), \qquad j = 1,2,3, \\ X(t,\cdot,0) \in C^j([0,T]; H^{s+3/2-j/2}(\mathbf{R}^1)), \qquad j = 1,2,3, \\ X(t,\cdot,-h) \in C^j([0,T]; H^{s+3/2-j/2}(\mathbf{R}^1)), \qquad j = 1,2,3, \\ |||X(t)|||_3 \leq c_0, \quad |||X(t)|||_{s+1} \leq d, \\ |||\partial_t^j X(t)|||_{s+3/2-j/2} \leq l_j, \qquad j = 1,2,3, \\ |||\partial_t^j X(t)|||_s \leq l_{j+3}, \qquad j = 1,2, \\ \|b\|_{H^3(\mathbf{R}^1)} \leq c_0, \quad \|b\|_{H^{s+1}(\mathbf{R}^1)} \leq d. \end{cases} \tag{2.4.2}$$

It is to be noted that c_0 is chosen sufficiently small so that

$$\|H(X,b)\|_{H^s(\mathbf{R}^1)} \leq C\|\omega_1\|_{H^{s+1/2}(\Sigma)}$$

and for X^1, X^2 satisfying (2.4.2),

$$\|H(X^1,b) - H(X^2,b)\|_{H^s(\mathbf{R}^1)} \leq C|||X^1 - X^2|||_s \|\omega_1\|_{H^{s+1/2}(\Sigma)},$$

where $C = C(s, c_0, d) > 0$.

Proposition 2.4.1. *Under* Assumption 2.1 *we have*

$$H = H(X,b) \in C^j([0,T]; H^{s+3/2-j/2}(\mathbf{R}^1)), \qquad j = 1,3,$$

$$[H]_s \leq C_1\mu_s, \quad |H|_s \leq C_2\mu_s, \qquad 0 \leq t \leq T. \tag{2.4.3}$$

Moreover, for X^1 *and* X^2 *satisfying* (2.4.2), *we have*

$$[H(X^1,b) - H(X^2,b)]_s \leq C_1\mu_s \sum_{j=0}^{2} |||\partial_t^j X^1(t) - \partial_t^j X^2(t)|||_{s+1-j/2},$$

$$|H(X^1,b) - H(X^2,b)|_s \leq C_2\mu_s\Big(|||X^1(t) - X^2(t)|||_{s+1} + \sum_{j=1}^{3} |||\partial_t^j X^1(t) - \partial_t^j X^2(t)|||_{s+3/2-j/2}\Big),$$
$$0 \leq t \leq T,$$

where $C_1 = C_1(s, c_0, d, l_4, l_5) > 0$ *and* $C_2 = C_2(s, c_0, d, l_1, l_2, l_3) > 0$.

2.5. Problem on the surface

In this section we consider Cauchy problem

$$\left(1+\frac{\partial \bar{X}_1}{\partial y_1}\right)\frac{\partial^2 \bar{X}_1}{\partial t^2}+\frac{\partial \bar{X}_2}{\partial y_1}\left(g+\frac{\partial^2 \bar{X}_2}{\partial t^2}\right)=0, \qquad t\geq 0, \tag{2.5.1}$$

$$\bar{X}_{2t}=K\bar{X}_{1t}+H, \qquad t\geq 0, \tag{2.5.2}$$

$$\bar{X}|_{t=0}=(0,\eta_0), \quad \bar{X}_{1t}|_{t=0}=u_{01}|_{y_2=0} \tag{2.5.3}$$

for a given function H. First we reduce problem (2.5.1) – (2.5.3) to the initial value problem for a quasi-linear system. Then by solving this reduced initial value problem, we show that problem (2.5.1) – (2.5.3) is solvable. For simplicity we will use X and y instead of $\bar{X}$ and y_1 in the following.

From (2.5.2) and (2.3.4) it follows that

$$\partial_t^k X_{2t}=K(X)\partial_t^k X_{1t}+F_{k0}(X,\ldots,\partial_t^k X)+\partial_t^k H, \tag{2.5.4}$$

$$\partial_t^k\partial_y^l X_{2t}=K(X)\partial_t^k\partial_y^l X_{1t}+F_{kl}(X,\ldots,\partial_t^k\partial_y^l X,\partial_t^{k+1}X_1)+\partial_t^k\partial_y^l H, \tag{2.5.5}$$

where $k=0,1,2,\ldots,\ l=1,2,3,\ldots$ and $F_{kl}=[\partial_t^k\partial_y^l,K_1]X_{1t}$. Put

$$Y=X_{tt}, \quad Z=X_y, \quad W=(X,Y,Z), \quad W'=(X,Y_1).$$

In virtue of (2.5.4) with $k=2$ we have

$$Y_{2t}=K(X)Y_{1t}+F_{20}(X,X_t,Y)+H_{tt}=:f_2(W,W_t',H). \tag{2.5.6}$$

From (2.5.5) with $k=0,\ l=1$ and (2.3.4) it follows that

$$\begin{aligned}
X_{2ty}=&KX_{1ty}+F_{01}(X,X_y,X_{1t})+H_y\\
=&-i\mathrm{sgnD}X_{1ty}+i(\mathrm{sgnD}-\tanh(h\mathrm{D}))\partial_y X_{1t}+K_1\partial_y X_{1t}+F_{01}+H_y\\
\equiv&-i\mathrm{sgnD}X_{1ty}+F_{010}+H_y,
\end{aligned}$$

hence we obtain

$$Z_{2t}=-i\mathrm{sgnD}Z_{1t}+F_{010}+H_y. \tag{2.5.7}$$

Differentiating (2.5.1) with respect to t and using (2.5.7), we have

$$\begin{aligned}
Z_{1t}=&-\{(g+Y_2)(-i\mathrm{sgnD})+Y_1\}^{-1}\\
&\times\{(g+Y_2)(F_{010}+H_y)+(1+Z_1)Y_{1t}+Z_2f_2(W,W_t',H)\}\\
=:&\,f_3(W,W_t',H).
\end{aligned} \tag{2.5.8}$$

Putting (2.5.8) into (2.5.7) leads to

$$Z_{2t}=-i\mathrm{sgnD}f_3(W,W_t',H)+F_{010}+H_y=:f_4(W,W_t',H). \tag{2.5.9}$$

Next, differentiating (2.5.1) twice with respect to t implies

$$(1+Z_1)Y_{1tt}+Z_2Y_{2tt}+Y_1Y_{1y}+(g+Y_2)Y_{2y}+2Z_t\cdot Y_t=0. \tag{2.5.10}$$

Since (2.5.4) with $k=3$ and (2.5.5) with $k=l=1$ yield

$$\begin{cases} Y_{2tt}=K(X)Y_{1tt}+F_{30}(X,X_t,Y,Y_t)+H_{ttt}, \\ Y_{2y}=K(X)Y_{1y}+F_{11}(X,X_t,Z,Z_t,Y_1)+H_{ty}, \end{cases}$$

one can rewrite (2.5.10) in the form

$$\begin{aligned} &Y_{1tt}+(1+Z_1+Z_2K)^{-1}\{Y_1+(g+Y_2)K\}\partial_yY_1 \\ &=-(1+Z_1+Z_2K)^{-1}\{2Z_t\cdot Y_t+Z_2(F_{30}+H_{ttt})+(g+Y_2)(F_{11}+H_{ty})\}. \end{aligned} \tag{2.5.11}$$

The identity

$$\begin{aligned} &(1+Z_1+Z_2K)^{-1}\{Y_1+(g+Y_2)K\} \\ &=\{(1+Z_1)^2+Z_2^2\}^{-1}\{(1+Z_1)Y_1+Z_2(g+Y_2)\} \\ &\quad+\{(1+Z_1)^2+Z_2^2\}^{-1}\{(1+Z_1)(g+Y_2)-Z_2Y_1\}\{-i\mathrm{sgnD}+i(\mathrm{sgnD}-\tanh(h\mathrm{D}))\}+P_1, \end{aligned}$$

$$\begin{aligned} P_1 =&P_1(W;X,Z) \\ =&\{(1+Z_1)^2+Z_2^2\}^{-1}\{(1+Z_1)(g+Y_2)-Z_2Y_1\}K_1 \\ &-\{(1+Z_1)^2+Z_2^2\}^{-1}Z_2\{[K,Y_1]+[K,Y_2]K+(g+Y_2)(1+K^2)\} \\ &+\{(1+Z_1)^2+Z_2^2\}^{-1}Z_2\{[K,Z_1]+[K,Z_2]K+Z_2(1+K^2)\}(1+Z_1+Z_2K)^{-1} \\ &\times\{Y_1+(g+Y_2)K\}, \end{aligned}$$

and using (2.5.6), (2.5.8), (2.5.9) lead the equivalent equation to (2.5.11)

$$Y_{1tt}+a(W)|\mathrm{D}|Y_1=f_1(W,W_t',H)$$

with

$$\begin{cases} a(W)=\{(1+Z_1)^2+Z_2^2\}^{-1}\{(1+Z_1)(g+Y_2)-Z_2Y_1\}, \\ f_1=-P_1\partial_yY_1-(1+Z_1+Z_2K)^{-1}\{2Z_t\cdot Y_t+Z_2(F_{30}(X,X_t,Y,Y_t)+H_{ttt}) \\ \qquad +(g+Y_2)(F_{11}(X,X_t,Z,Z_t,Y_1)+H_{ty})\}-a(W)(i\mathrm{sgnD}-i\tanh(h\mathrm{D}))\partial_yY_1. \end{cases}$$

Thus the required quasi-linear system is of the form

$$\begin{cases} X_{tt}=Y, \quad Y_{1tt}+a(W)|\mathrm{D}|Y_1=f_1(W,W_t',H), \\ Y_{2t}=f_2(W,W_t',H), \quad Z_{1t}=f_3(W,W_t',H), \quad Z_{2t}=f_4(W,W_t',H). \end{cases} \tag{2.5.12}$$

Lemma 2.5.1. *Let $s \geq 3$ and $0 < T < \infty$. There exists a positive constant $c_1 = c_1(g)$ such that if W, W'_t, H, b satisfy*

$$\begin{cases} W, W'_t \in C^0([0,T]; H^s(\mathbf{R}^1)), \\ H \in C^j([0,T]; H^{s+3/2-j/2}(\mathbf{R}^1)), \quad j = 1, 3, \\ b \in H^{s+1}(\mathbf{R}^1), \\ \|W(t)\|_{H^3(\mathbf{R}^1)} \leq c_1, \quad \|W(t)\|_{H^s(\mathbf{R}^1)} + \|W'_t(t)\|_{H^s(\mathbf{R}^1)} \leq d_0, \\ |H(t)|_s \leq d_2, \quad \|b\|_{H^3(\mathbf{R}^1)} \leq c_0, \quad \|b\|_{H^{s+1}(\mathbf{R}^1)} \leq d' \end{cases} \tag{2.5.13}$$

for $0 \leq t \leq T$ and some constants $d_0, d_2, d' > 0$, then

$$a(W) - g \in C^1([0,T]; H^s(\mathbf{R}^1)), \quad f = f(W, W'_t, H, b) \in C^0([0,T]; H^s(\mathbf{R}^1)),$$

$$\begin{cases} \|f(W, W'_t, H)\|_{H^s(\mathbf{R}^1)} \leq C_3(\|W\|_{H^s(\mathbf{R}^1)} + \|W'_t\|_{H^s(\mathbf{R}^1)} + |H|_s), \\ \|(f_2, f_3, f_4)(W, W'_t, H)\|_{H^s(\mathbf{R}^1)} \leq C_4(\|W\|_{H^s(\mathbf{R}^1)} + \|W'_t\|_{H^s(\mathbf{R}^1)} + [H]_s). \end{cases}$$

Moreover, for $W^0, {W^0}'_t$ and H^0 satisfying (2.5.13)

$$\begin{cases} \|a(W) - a(W^0)\|_{H^s(\mathbf{R}^1)} \leq C_4\|W - W^0\|_{H^s(\mathbf{R}^1)}, \\ \|f(W, W'_t, H) - f(W^0, {W^0}'_t, H^0)\|_{H^s(\mathbf{R}^1)} \\ \qquad \leq C_3(\|W - W^0\|_{H^s(\mathbf{R}^1)} + \|W'_t - {W^0}'_t\|_{H^s(\mathbf{R}^1)} + |H - H^0|_s), \end{cases}$$

where $C_3 = C_3(c_1, g, d_0, d_2, s, c_0, d') > 0$ and $C_4 = C_4(c_1, g, d_0, s, c_0, d') > 0$.

Proof. The properties of a were shown in [46, Lemmas 5.18 – 5.20]. Other estimates are easily derived from the lemmas in Section 2.4. ∎

The initial value problem

$$\begin{cases} u_{tt} + a(W)|\mathrm{D}|u = f & \text{for } 0 \leq t \leq T, \\ u = u_0, \quad u_t = u_1 & \text{at } t = 0 \end{cases} \tag{2.5.14}$$

was solved in [46, Theorem 6.20].

Theorem 2.5.1. *Let $s \geq 2$ and $0 < T < \infty$. There exists a positive constant $c_1 = c_1(g)$ such that if $W = (0, Y, Z) \in C^0([0,T]; H^s(\mathbf{R}^1)) \cap C^1([0,T]; H^2(\mathbf{R}^1))$ satisfies*

$$\|W(t)\|_{H^2(\mathbf{R}^1)} \leq c_1, \ \|W_t(t)\|_{H^2(\mathbf{R}^1)} \leq d_1, \ \|W(t)\|_{H^s(\mathbf{R}^1)} \leq d_0 \quad \textit{for} \ \ 0 \leq t \leq T$$

with some positive constants d_0, d_1, then for any $u_0 \in H^{s+1/2}(\mathbf{R}^1)$, $u_1 \in H^s(\mathbf{R}^1)$ and $f \in C^0([0,T]; H^s(\mathbf{R}^1))$, (2.5.14) has a unique solution $u \in C^j([0,T]; H^{s+1/2-j/2}(\mathbf{R}^1))$, $j = 0, 1, 2$, such that

$$|u(t)|_s \leq C_5 \mathrm{e}^{C_6 t}|u(0)|_s + C_5\int_0^t \mathrm{e}^{C_6(t-\tau)}\|f(\tau)\|_{H^s(\mathbf{R}^1)}\mathrm{d}\tau,$$

where

$$|u(t)|_s = \|u_t(t)\|_{H^s(\mathbf{R}^1)} + \|u(t)\|_{H^{s+1/2}(\mathbf{R}^1)},$$

$C_5 = C_5(c_1, g, s) > 0$ *and* $C_6 = C_6(c_1, g, d_0, d_1, s) > 0$.

Now we consider the initial value problem (2.5.12) with

$$W(0) = \widetilde{W} = (\widetilde{X}, \tilde{Y}, \tilde{Z}), \quad W_t'(0) = \widetilde{W_t'} = (\widetilde{X_t}, \widetilde{Y_{1t}}). \tag{2.5.15}$$

Let us introduce the new norms

$$\begin{cases} |Y_1(t)|_s = \|Y_{1t}(t)\|_{H^s(\mathbf{R}^1)} + \|Y_1(t)\|_{H^{s+1/2}(\mathbf{R}^1)}, \\ |W(t)|_s = \|X(t)\|_{H^s(\mathbf{R}^1)} + \|X_t(t)\|_{H^s(\mathbf{R}^1)} + \|Y_{1t}(t)\|_{H^s(\mathbf{R}^1)} + \|Y_1(t)\|_{H^{s+1/2}(\mathbf{R}^1)} \\ \qquad\qquad + \|Y_2(t)\|_{H^s(\mathbf{R}^1)} + \|Z(t)\|_{H^s(\mathbf{R}^1)}. \end{cases}$$

Theorem 2.5.2. *Let* $c_1 = c_1(g)$ *be the constant in* Lemma 2.5.1 *and* Theorem 2.5.1, $s \geq 3 + 1/2$ *and* $0 < T_1 < \infty$. *If* $H \in C^j([0, T_1]; H^{s+3/2-j/2}(\mathbf{R}^1))$, $j = 1, 3$, $b \in H^{s+1}(\mathbf{R}^1)$, $\|b\|_{H^3(\mathbf{R}^1)} \leq c_0$,

$$\widetilde{X}, \tilde{Z}, \widetilde{W_t'} \in H^s(\mathbf{R}^1), \quad \widetilde{Y_1} \in H^{s+1/2}(\mathbf{R}^1), \quad \|\widetilde{W}\|_{H^3(\mathbf{R}^1)} \leq c_1/2, \tag{2.5.16}$$

then there exists $T \in (0, T_1]$ *such that problem* (2.5.12), (2.5.15) *has a unique solution* $W = (X, Y, Z)$ *satisfying*

$$\begin{cases} X \in C^2([0, T]; H^s(\mathbf{R}^1)), \quad Y_2, Z \in C^1([0, T]; H^s(\mathbf{R}^1)), \\ Y_1 \in C^j([0, T]; H^{s+1/2-j/2}(\mathbf{R}^1)), \quad j = 0, 1, 2, \\ \|W(t)\|_{H^3(\mathbf{R}^1)} \leq c_1 \quad \textit{for} \ \ 0 \leq t \leq T. \end{cases} \tag{2.5.17}$$

Proof. Take the constants $J, J_0, d_0, J_2, d_2, J_1, d_1$ and d' such that

$$\begin{cases} J = (3 + C_5)|W(0)|_s, \quad J_0 > 2J, \quad d_0 = \max\{1, J_0\}, \\ J_2 \geq \sup\limits_{0 \leq t \leq T_1} |H(t)|_s, \quad d_2 = \max\{1, J_2\}, \\ J_1 = J_0 + C_4(J_0 + J_2), \quad d_1 = \max\{1, J_1\}, \\ \|b\|_{H^{s+1}(\mathbf{R}^1)} \leq d'. \end{cases} \tag{2.5.18}$$

By $\mathcal{S}_1$ we denote the totality of $W = (X, Y, Z)$ satisfying

$$\begin{cases} W \in C^1([0, T]; H^s(\mathbf{R}^1)), \quad Y_1 \in C^0([0, T]; H^{s+1/2}(\mathbf{R}^1)), \\ \|W(t)\|_{H^s(\mathbf{R}^1)} + \|W_t'(t)\|_{H^s(\mathbf{R}^1)} \leq d_0, \\ |W(t)|_s \leq J \exp(C_7 t) + J_2 C_7 t \exp(C_7 t), \\ \|W(t)\|_{H^3(\mathbf{R}^1)} \leq c_1, \quad \|Y_t(t)\|_{H^2(\mathbf{R}^1)} + \|Z_t(t)\|_{H^2(\mathbf{R}^1)} \leq J_1, \quad 0 \leq {}^\forall t \leq T, \end{cases}$$

where $C_7 = C_6 + 2 + C_4 + C_3 C_5$. For $W^0 = (X^0, Y^0, Z^0) \in \mathcal{S}_1$, by $M_1(W^0)$ we denote the solution W of the initial value problem for

$$\begin{cases} X_{tt} + X = X^0 + Y^0, \quad Y_{1tt} + a(W^0)|\mathrm{D}|Y_1 = f_1(W^0, {W^0}'_t, H), \\ Y_{2t} = f_2(W^0, {W^0}'_t, H), \quad Z_{1t} = f_3(W^0, {W^0}'_t, H), \quad Z_{2t} = f_4(W^0, {W^0}'_t, H) \end{cases}$$

with (2.5.15). Lemma 2.5.1 and Theorem 2.5.1 imply that

$$|W(t)|_s \le (3+C_5)\mathrm{e}^{C_6 t}|W(0)|_s + (2+C_4+C_3C_5)\int_0^t \mathrm{e}^{C_6(t-\tau)}(|W^0(\tau)|_s + |H(\tau)|_s)\mathrm{d}\tau.$$

Here we choose T as

$$T = \min\left\{T_1,\ \frac{c_1}{2(J_0+J_1)},\ \frac{1}{C_7}\log\frac{J_0}{2J},\ \varphi_1^{-1}\left(\frac{J_0}{2C_7J_2}\right)\right\}, \tag{2.5.19}$$

where $\varphi_1(t) = t\exp(C_7 t)$ and φ_1^{-1} is the inverse function of φ_1. Then M_1 is a mapping from $\mathcal{S}_1$ to itself. Since $s - 1/2 \ge 3$, the successive approximation and Lemma 2.5.1 show that there is a unique solution W of (2.5.12), (2.5.15) and satisfies (2.5.17) with s replaced by $s-1/2$. Refering to [16, Theorem 6.27], we can show that W obtained above satisfies (2.5.17). ∎

By the same method used for the derivation of Lemma 2.5.1 we obtain

Lemma 2.5.2. *Let $W^0 = (X^0, Y^0, Z^0)$ be the solution of* (2.5.1), (2.5.2) *with H replaced by $H^0 \in C^j([0,T]; H^{s+3/2-j/2}(\mathbf{R}^1))$, $j = 1, 3$, whose initial data $W^0(0)$, $W^{0\prime}_t(0)$ satisfy* (2.5.16). *We have*

$$|W(t) - W^0(t)|_{s-1/2} \le C\left(|W(0) - W^0(0)|_{s-1/2} + \int_0^t |H(\tau) - H^0(\tau)|_{s-1/2}\mathrm{d}\tau\right)$$

for $0 \le t \le T$, where $C = C(c_1, g, d_0, d_1, d_2, s, T, c_0, d') > 0$.

In view of the original problem, we specify the initial data as follows:

$$\begin{cases} \widetilde{X} = (0,\eta_0),\ \widetilde{Z} = \widetilde{X}_y,\ \widetilde{X_{1t}} = u_{01}(\cdot,0),\ \widetilde{X_{2t}} = K(\widetilde{X})\widetilde{X_{1t}} + H(0), \\ \widetilde{Y_1} = -(1+\widetilde{Z_1}+\widetilde{Z_2}K(\widetilde{X}))^{-1}\widetilde{Z_2}(g + F_{10}(\widetilde{X},\widetilde{X_t}) + H_t(0)), \\ \widetilde{Y_2} = K(\widetilde{X})\widetilde{Y_1} + F_{10}(\widetilde{X},\widetilde{X_t}) + H_t(0), \\ \widetilde{Y_{1t}} = -(1+\widetilde{Z_1}+\widetilde{Z_2}K(\widetilde{X}))^{-1} \\ \qquad \times\{\widetilde{Z_2}(F_{20}(\widetilde{X},\widetilde{X_t},\widetilde{Y}) + H_{tt}(0)) + \widetilde{Y_1}\partial_y\widetilde{X_{1t}} + (g+\widetilde{Y_2})\partial_y\widetilde{X_{2t}}\}. \end{cases} \tag{2.5.20}$$

For these, one can easily prove

Lemma 2.5.3. *Let $c_1 = c_1(g)$ be the constant in* Lemma 2.5.1 *and* Theorem 2.5.1. *Then there exists a positive constant $\varepsilon_1 = \varepsilon_1(g)$ such that if*

$$\begin{cases} \eta_0 \in H^{s+3/2}(\mathbf{R}^1), \quad b \in H^{s+1}(\mathbf{R}^1), \quad u_{01}(\cdot,0) \in H^{s+1}(\mathbf{R}^1), \\ \partial_t^j H(0) \in H^{s+1-j/2}(\mathbf{R}^1),\ \ j = 0,1,2, \\ \|b\|_{H^3(\mathbf{R}^1)} \le c_0, \quad \|b\|_{H^{s+1}(\mathbf{R}^1)} \le d', \\ \|\eta_0\|_{H^4(\mathbf{R}^1)} + \|u_{01}(\cdot,0)\|_{H^3(\mathbf{R}^1)} + \|H(0)\|_{H^3(\mathbf{R}^1)} + \|H_t(0)\|_{H^3(\mathbf{R}^1)} \le \varepsilon_1, \\ \|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)} + \|u_{01}(\cdot,0)\|_{H^{s+1}(\mathbf{R}^1)} + \|H(0)\|_{H^{s+1}(\mathbf{R}^1)} + \|H_t(0)\|_{H^{s+1/2}(\mathbf{R}^1)} \le d_4 \end{cases}$$

with $s \geq 3,\ d_4 > 0$, *then the initial data* $\widetilde{W}, \widetilde{W_t'}$ *defined by* (2.5.20) *satisfy the condition* (2.5.16) *and*

$$|W(0)|_s \leq C\left(\|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)} + \|u_{01}(\cdot,0)\|_{H^{s+1}(\mathbf{R}^1)} + \sum_{j=0}^{2}\|\partial_t^j H(0)\|_{H^{s+1-j/2}(\mathbf{R}^1)}\right),$$

where $C = C(c_1, c_0, \varepsilon_1, g, d_4, s, d') > 0$.

From Lemma 2.5.3 and Theorem 2.5.2 we conclude

Theorem 2.5.3. *Let* $\varepsilon_1 = \varepsilon_1(g)$ *be the constant in* Lemma 2.5.3, $b \in H^{s+1}(\mathbf{R}^1)$, $s \geq 3 + 1/2$, $\|b\|_{H^3(\mathbf{R}^1)} \leq c_0$ *and* $0 < T_1 < \infty$. *If* $\eta_0, u_{01}(\cdot, 0)$ *and* H *satisfy the conditions*

$$\begin{cases} \eta_0 \in H^{s+3/2}(\mathbf{R}^1), \quad u_{01}(\cdot,0) \in H^{s+1}(\mathbf{R}^1), \\ \|\eta_0\|_{H^4(\mathbf{R}^1)} + \|u_{01}(\cdot,0)\|_{H^3(\mathbf{R}^1)} \leq \varepsilon_1/2, \end{cases} \tag{2.5.21}$$

$$\begin{cases} H \in C^j([0,T_1]; H^{s+3/2-j/2}(\mathbf{R}^1)), \quad j = 1,3, \\ \|H(0)\|_{H^3(\mathbf{R}^1)} + \|H_t(0)\|_{H^3(\mathbf{R}^1)} \leq \varepsilon_1/2, \end{cases} \tag{2.5.22}$$

then there exists $T \in (0, T_1]$ *such that problem* (2.5.1) – (2.5.3) *has a unique solution*

$$\bar{X} \in C^j([0,T]; H^{s+3/2-j/2}(\mathbf{R}^1)), \quad j = 1,2,3. \tag{2.5.23}$$

Now we assume that

$$[H(t)]_s \leq J_3, \quad 0 \leq t \leq T_1, \tag{2.5.24}$$

and put $d_3 = \max\{1, J_3\}$. Then we get

Lemma 2.5.4. *Let* $\bar{X}$ *be the solution of* (2.5.1) – (2.5.3) *obtained in* Theorem 2.5.3. *Then we have*

$$\begin{cases} \|\bar{X}_t(t)\|_{H^s(\mathbf{R}^1)} + \|\bar{X}_{1tt}(t)\|_{H^{s+1/2}(\mathbf{R}^1)} + \|\bar{X}_{1ttt}(t)\|_{H^s(\mathbf{R}^1)} \leq d_0, \\ \|\bar{X}_t(t)\|_{H^{s+1}(\mathbf{R}^1)} \leq (1 + C_4)\{J\exp(C_7 t) + J_2 C_7 t \exp(C_7 t)\} + C_4 J_3 \\ \qquad\qquad \leq (1 + C_4)d_0 + C_4 d_3, \quad 0 \leq t \leq T. \end{cases}$$

By Lemma 2.5.2 and the similar arguments as above we obtain

Proposition 2.5.1. *Suppose that* H^0 *satisfies the conditions in* (2.5.22) *and* $\bar{X}^0$ *is the solution of* (2.5.1), (2.5.2) *with* H *replaced by* H^0 *and* (2.5.3). *Then*

$$\sum_{j=0}^{2}\|\partial_t^{j+1}\bar{X}(t) - \partial_t^{j+1}\bar{X}^0(t)\|_{H^{s+1/2-j/2}(\mathbf{R}^1)}$$
$$\leq C_8\left([H(0) - H^0(0)]_{s-1/2} + [H(t) - H^0(t)]_{s-1/2} + \int_0^t |H(\tau) - H^0(\tau)|_{s-1/2}\mathrm{d}\tau\right)$$

for $0 \leq t \leq T$, *where* $C_8 = C_8(c_0, g, d_0, d_1, d_2, s, T, c_1, d') > 0$.

2.6. Problem in the interior

In this section we will solve the boundary value problem

$$\begin{cases} \nabla_{\mathbf{u}} \cdot \mathbf{u} = 0, \quad \nabla_{\mathbf{u}}^{\perp} \cdot \mathbf{u} = \omega_0 & \text{in } \Omega,\ t \geq 0, \\ u_1 = \bar{X}_{1t} & \text{on } \Gamma_s,\ t \geq 0, \\ \mathbf{u} \cdot \mathbf{n}(\Phi_{\mathbf{u}}(x;t)) = 0 & \text{on } \Gamma_b,\ t \geq 0 \end{cases} \tag{2.6.1}$$

for a given $\bar{X}$. First let us investigate problem

$$\begin{cases} \nabla \cdot \mathbf{u} = \phi_1, \quad \nabla^{\perp} \cdot \mathbf{u} = \phi_2 & \text{in } \Sigma, \\ u_1 = \theta_1 & \text{on } \{y_2 = 0\}, \\ u_2 = \theta_2 & \text{on } \{y_2 = -h\}. \end{cases} \tag{2.6.2}$$

Applying the partial Fourier transform with respect to y_1 to (2.6.2) yields the ordinary differential equations, whose solutions are easily estimated so that

Theorem 2.6.1. *Suppose that $\phi = (\phi_1, \phi_2) \in H^s(\Sigma)$ and $\theta = (\theta_1, \theta_2) \in H^{s+1/2}(\mathbf{R}^1)$ with $s > 0$. Then the boundary value problem (2.6.2) has a unique solution $\mathbf{u} = (u_1, u_2)$ such that*

$$\mathbf{u} \in H^{s+1}(\Sigma), \quad \mathbf{u}(\cdot, 0) \in H^{s+1/2}(\mathbf{R}^1), \quad \mathbf{u}(\cdot, -h) \in H^{s+1/2}(\mathbf{R}^1),$$

$$\|\mathbf{u}\|_{s+1,\lambda_1,\lambda_2} \leq (C_9 + \lambda_1^{-1} C_{10}) \|\phi\|_{s,\lambda_1,\lambda_2} + \lambda_2^s C_{10} \|\theta\|_{H^{s+1/2}(\mathbf{R}^1)} \quad \textit{for } \lambda_1, \lambda_2 \geq 1, \tag{2.6.3}$$

$$\|\mathbf{u}(\cdot, 0)\|_{H^{s+1/2}(\mathbf{R}^1)} + \|\mathbf{u}(\cdot, -h)\|_{H^{s+1/2}(\mathbf{R}^1)} \leq C_{11}(\|\phi\|_{H^s(\Sigma)} + \|\theta\|_{H^{s+1/2}(\mathbf{R}^1)}),$$

where $C_9 > 1$ and $C_j = C_j(s) > 1,\ j = 10, 11$.

Similar estimates hold for problem

$$\begin{cases} \nabla \cdot \mathbf{u} = \phi_1, \quad \nabla^{\perp} \cdot \mathbf{u} = \phi_2 & \text{in } \Sigma, \\ u_1 = \theta_1 & \text{on } \{y_2 = 0\}, \\ u_1 = \theta_3 & \text{on } \{y_2 = -h\}. \end{cases} \tag{2.6.4}$$

Next we consider problem

$$\begin{cases} \nabla \cdot \mathbf{u} = \phi_1, \quad \nabla^{\perp} \cdot \mathbf{u} = \phi_2 & \text{in } \Omega, \\ u_1 = \theta_1 & \text{on } \Gamma_s, \\ \mathbf{u} \cdot \mathbf{n} = \mu & \text{on } \Gamma_b. \end{cases} \tag{2.6.5}$$

Assumption 2.2. *Let η_0 and b satisfy $\eta_0 \in H^{s_0+1/2}(\mathbf{R}^1)$, $b \in H^{s_0+3/2}(\mathbf{R}^1)$ with $s_0 > 2$ and*

$$\begin{cases} \|\eta_0\|_{H^2(\mathbf{R}^1)} \le \kappa_1, & \|b\|_{H^2(\mathbf{R}^1)} \le \kappa_2, \\ \|\eta_0\|_{H^{s_0+1/2}(\mathbf{R}^1)} \le d_0, & \|b\|_{H^{s_0+3/2}(\mathbf{R}^1)} \le d'. \end{cases}$$

In what follows we set $\widetilde{\eta_0}$ as

$$\widetilde{\eta_0}(y_1,y_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-\mathrm{e}^{iy_1\xi}(\mathrm{e}^{|\xi|(y_2+2h)} - \mathrm{e}^{|\xi|y_2})\hat{\eta_0}(\xi) + \mathrm{e}^{iy_1\xi}(\mathrm{e}^{|\xi|(y_2+h)} - \mathrm{e}^{|\xi|(h-y_2)})\hat{b}(\xi)}{1-\mathrm{e}^{2|\xi|h}}\mathrm{d}\xi. \tag{2.6.6}$$

Lemma 2.6.1. *If $\eta_0, b \in H^{s+1/2}(\mathbf{R}^1)$ with $s \ge 3/2$, then $\widetilde{\eta_0} \in C^1(\overline{\Sigma})$, $\widetilde{\eta_0} \in H^{s+1}(\Sigma)$,*

$$\begin{cases} \|\nabla\widetilde{\eta_0}\|_{C^0(\overline{\Sigma})} \le C_{12}(\|\eta_0\|_{H^2(\mathbf{R}^1)} + \|b\|_{H^2(\mathbf{R}^1)}), \\ |||\widetilde{\eta_0}|||_{s+1/2} \le C_{11}(\|\eta_0\|_{H^{s+1/2}(\mathbf{R}^1)} + \|b\|_{H^{s+1/2}(\mathbf{R}^1)}), \end{cases} \tag{2.6.7}$$

where $C_{12} > 1$.

Proof. First estimate of (2.6.7) is easily derived from (2.6.6). Second estimate of (2.6.7) comes from Theorem 2.6.1 since $\mathbf{u} = (u_1,u_2) = (\widetilde{\eta_0}, u_2)$ satisfies (2.6.4) with $\phi_1 = \phi_2 = 0$, $\theta_1 = \eta_0$ and $\theta_3 = b$. ∎

Put $\mathbf{w}(y) = \mathbf{u}(\Psi(y))$, $\varphi(y) = \phi(\Psi(y))$, $\vartheta_1(y_1) = \theta_1(\Psi(y_1,0))$, $\nu(y_1) = \mu(y_1, -h + b(y_1))$ in (2.6.5). Then this system is equivalent to

$$\begin{cases} \nabla\cdot\mathbf{w} = J_{\widetilde{\eta_0}}\varphi_1 + ((I - A_{\widetilde{\eta_0}})\nabla)\cdot\mathbf{w} & \text{in } \Sigma, \\ \nabla^{\perp}\cdot\mathbf{w} = J_{\widetilde{\eta_0}}\varphi_2 + ((I - A_{\widetilde{\eta_0}})\nabla)^{\perp}\cdot\mathbf{w} & \text{in } \Sigma, \\ w_1 = \vartheta_1 & \text{on } \{y_2 = 0\}, \\ w_2 = \sqrt{1+b'^2}\nu - w_1 b' & \text{on } \{y_2 = -h\}, \end{cases} \tag{2.6.8}$$

where $A_{\widetilde{\eta_0}}$ is a matrix whose (i,j)-element is the (i,j)-cofactor of the Jacobian matrix $(\partial\Psi/\partial y)$ and $J_{\widetilde{\eta_0}}$ is its Jacobian.

Theorem 2.6.2. *Under* Assumption 2.2, *if $\phi = (\phi_1,\phi_2) \in H^s(\Omega)$, $\theta_1 \in H^{s+1/2}(\Gamma_s)$ and $\mu \in H^{s+1/2}(\Gamma_b)$ with $0 < s \le s_0$, then* (2.6.5) *has a unique solution* $\mathbf{u}$ *satisfying*

$$\mathbf{u} \in H^{s+1}(\Omega), \quad \mathbf{u}|_{\Gamma_s} \in H^{s+1/2}(\Gamma_s), \quad \mathbf{u}|_{\Gamma_b} \in H^{s+1/2}(\Gamma_b),$$

$$\|\mathbf{u}\|_{H^{s+1}(\Omega)} \le C(\|\phi\|_{H^s(\Omega)} + \|\theta_1\|_{H^{s+1/2}(\Gamma_s)} + \|\mu\|_{H^{s+1/2}(\Gamma_b)}),$$

$$\begin{aligned} &\|\mathbf{u}|_{\Gamma_s}\|_{H^{s+1/2}(\Gamma_s)} + \|\mathbf{u}|_{\Gamma_b}\|_{H^{s+1/2}(\Gamma_b)} \\ &\qquad \le C(\|\phi\|_{H^s(\Omega)} + \|\theta_1\|_{H^{s+1/2}(\Gamma_s)} + \|\mu\|_{H^{s+1/2}(\Gamma_b)}), \end{aligned}$$

where $C = C(s, s_0, d_0) > 0$.

Proof. It is sufficient to solve (2.6.8). For a given $\mathbf{w}$ satisfying $\mathbf{w} \in H^{s+1}(\Sigma)$, we denote by $\tilde{\mathbf{w}} = \Phi(\mathbf{w})$ the solution $\tilde{\mathbf{w}} = (\tilde{w}_1, \tilde{w}_2)$ of the problem

$$\begin{cases} \nabla \cdot \tilde{\mathbf{w}} = J_{\tilde{\eta}_0}\varphi_1 + ((I - A_{\tilde{\eta}_0})\nabla) \cdot \mathbf{w} & \text{in } \Sigma, \\ \nabla^\perp \cdot \tilde{\mathbf{w}} = J_{\tilde{\eta}_0}\varphi_2 + ((I - A_{\tilde{\eta}_0})\nabla)^\perp \cdot \mathbf{w} & \text{in } \Sigma, \\ \tilde{w}_1 = \vartheta_1 & \text{on } \{y_2 = 0\}, \\ \tilde{w}_2 = \sqrt{1 + b'^2}\nu - w_1 b' & \text{on } \{y_2 = -h\}. \end{cases}$$

By virtue of (2.6.3), (2.6.7) and Lemma 2.2.1 we see that

$$\begin{aligned} \|\Phi(\mathbf{w})\|_{s+1,\lambda_1,\lambda_2} \le & \{(C_9 + \lambda_1^{-1}C_{10})(C_{12}(\kappa_1 + \kappa_2) + \lambda_2^{-\gamma}C_{13}C_{11}(d_0 + d')) + \lambda_2^s C_{10}(\kappa_2 + d')\} \\ & \times \|\mathbf{w}\|_{s+1,\lambda_1,\lambda_2} + \lambda_2^s C_{14}(\|\varphi\|_{H^s(\Sigma)} + \|\vartheta_1\|_{H^{s+1/2}(\mathbf{R}^1)} + \|\nu\|_{H^{s+1/2}(\mathbf{R}^1)}), \end{aligned}$$

where $C_{13} = C_{13}(s, s_0, \lambda_1) > 0$, $C_{14} = C_{14}(s, s_0, \lambda_1, \kappa_1, \kappa_2, d_0, d') > 0$ and $\gamma > 0$. If we take $\lambda_1, \lambda_2, d', \kappa_1, \kappa_2$ appropriately, Φ is a contraction mapping with respect to the norm $\|\cdot\|_{s+1,\lambda_1,\lambda_2}$. This shows the first estimate. For the second estimate, use Theorem 2.6.1. $\square$

From Theorem 2.6.2 it follows

Theorem 2.6.3. *Under* Assumption 2.2, *if*

$$\begin{cases} \phi \in C^j([0,T]; H^{s+1/2-j/2}(\Omega)), \\ \theta_1 \in C^j([0,T]; H^{s+1-j/2}(\Gamma_s)), \\ \mu \in C^j([0,T]; H^{s+1-j/2}(\Gamma_b)), & j = 0,1,2 \end{cases}$$

with $1/2 < s \le s_0 - 1/2$, $0 < T < \infty$, *then* (2.6.5) *has a unique solution* $\mathbf{u}$ *satisfying*

$$\begin{cases} \mathbf{u} \in C^j([0,T]; H^{s+3/2-j/2}(\Omega)), \\ \mathbf{u}|_{\Gamma_s \cup \Gamma_b} \in C^j([0,T]; H^{s+1-j/2}(\Gamma_s \cup \Gamma_b)) \quad \text{for } j = 0,1,2. \end{cases}$$

Moreover, the solution $\mathbf{u}$ *satisfies*

$$\begin{aligned} |\partial_t^j \mathbf{u}(t)|_{s+1-j/2,\Omega} \le C_{15}(&\|\partial_t^j \phi(t)\|_{H^{s+1/2-j/2}(\Omega)} + \|\partial_t^j \theta_1(t)\|_{H^{s+1-j/2}(\Gamma_s)} \\ &+ \|\partial_t^j \mu(t)\|_{H^{s+1-j/2}(\Gamma_b)}) \end{aligned} \tag{2.6.9}$$

for $0 \le t \le T$ *and* $j = 0,1,2$, *where* $C_{15} = C_{15}(s, s_0, d_0) > 0$. *Here we used the notation*

$$|\mathbf{u}|_{s,\Omega} = \|\mathbf{u}\|_{H^{s+1/2}(\Omega)} + \|\mathbf{u}|_{\Gamma_s}\|_{H^s(\Gamma_s)} + \|\mathbf{u}|_{\Gamma_b}\|_{H^s(\Gamma_b)}.$$

Now problem (2.6.1) is rewritten as

$$\begin{cases} \nabla \cdot \mathbf{u} = ((I - A_\mathbf{u})\nabla) \cdot \mathbf{u} =: h_1(\mathbf{u}; t) & \text{in } \Omega,\ 0 \le t \le T, \\ \nabla^\perp \cdot \mathbf{u} = \omega_0 + ((I - A_\mathbf{u})\nabla)^\perp \cdot \mathbf{u} =: \omega_0 + h_2(\mathbf{u}; t) & \text{in } \Omega,\ 0 \le t \le T, \\ u_1 = \bar{X}_{1t} & \text{on } \Gamma_s,\ 0 \le t \le T, \\ \mathbf{u} \cdot \mathbf{n}(x) = \mathbf{u} \cdot \left\{\mathbf{n}(x) - \mathbf{n}(x + \int_0^t \mathbf{u}(\tau, x)d\tau)\right\} & \text{on } \Gamma_b,\ 0 \le t \le T. \end{cases}$$

Assumption 2.3. *There exists* $\mathbf{v}_0 \in H^{s+3/2}(\Omega)$ *such that*

$$\omega_0 = \nabla^\perp \cdot \mathbf{v}_0, \quad \nabla \cdot \mathbf{v}_0 = 0 \qquad \text{in } \Omega.$$

Let $T_1 > 0$, $\bar{X}$ *satisfy*

$$\begin{cases} \bar{X}_{1t} \in C^j([0,T_1]; H^{s+1-j/2}(\Gamma_s)), \quad j = 0,1,2, \\ \|\bar{X}_{1t}(t)\|_{H^s(\Gamma_s)} + \|\bar{X}_{1tt}(t)\|_{H^{s+1/2}(\Gamma_s)} + \|\bar{X}_{1ttt}(t)\|_{H^s(\Gamma_s)} \le d_0, \\ \|\bar{X}_{1t}(t)\|_{H^{s+1}(\Gamma_s)} \le (1+C_4)\{J\exp(C_7 t) + J_2 C_7 t \exp(C_7 t)\} + C_4 J_3 \\ \qquad\qquad \le (1+C_4)d_0 + C_4 d_3 \end{cases}$$

and

$$\nabla \mathbf{n} \in H^{s+1}(\Gamma_b). \tag{2.6.10}$$

Theorem 2.6.4. *Under* Assumptions 2.2, 2.3 *with* $s_0 = s + 1/2$ *there exists* $T \in (0, T_1]$ *such that problem* (2.6.1) *has a unique solution* $\mathbf{u}$ *satisfying*

$$\begin{cases} \mathbf{u} \in C^j([0,T]; H^{s+3/2-j/2}(\Omega)), \\ \mathbf{u}|_{\Gamma_s \cup \Gamma_b} \in C^j([0,T]; H^{s+1-j/2}(\Gamma_s \cup \Gamma_b)) \quad \text{for } j = 0,1,2. \end{cases} \tag{2.6.11}$$

Proof. We denote by $\mathcal{S}_2$ the totality of $\mathbf{u}$ satisfying (2.6.11) and

$$\begin{cases} |\mathbf{u}(t)|_{s+1,\Omega} \le 2C_{15}((1+C_4)d_0 + C_4 d_3) + 2d_4 =: e_1, \\ |\mathbf{u}_t(t)|_{s+1/2,\Omega} \le 2(C_{16} + C_{15}\|\nabla \mathbf{n}\|_{H^{s+1/2}(\Gamma_b)})e_1^2 + 2C_{15}d_0 =: e_2, \\ |\mathbf{u}_{tt}(t)|_{s,\Omega} \le 2(C_{16} + 3C_{15}\|\nabla \mathbf{n}\|_{H^s(\Gamma_b)})e_1 e_2 + 2C_{15}d_0 =: e_3, \\ |\mathbf{u}(t)|_{s+1,\Omega} \le 2C_{15}\|\bar{X}_{1t}(t)\|_{H^{s+1}(\Gamma_s)} + 2J_4 \end{cases} \tag{2.6.12}$$

for $0 \le t \le T$, where $d_4 = \max\{1, J_4\}$, $C_{16} = C_{16}(s, d_0) > 1$ and

$$J_4 = C_{15}\|\mathbf{v}_0\|_{H^{s+1}(\Gamma_s)} + C_{15}\|\mathbf{v}_0 \cdot \mathbf{n}\|_{H^{s+1}(\Gamma_b)} + |\mathbf{v}_0|_{s+1,\Omega}. \tag{2.6.13}$$

For $\mathbf{u} \in \mathcal{S}_2$, Theorem 2.6.3 shows that the boundary value problem

$$\begin{cases} \nabla \cdot \mathbf{U} = h_1(\mathbf{u}; t) & \text{in } \Omega,\ 0 \le t \le T, \\ \nabla^\perp \cdot \mathbf{U} = \omega_0 + h_2(\mathbf{u}; t) & \text{in } \Omega,\ 0 \le t \le T, \\ U_1 = \bar{X}_{1t} & \text{on } \Gamma_s,\ 0 \le t \le T, \\ \mathbf{U} \cdot \mathbf{n}(x) = \mathbf{u} \cdot \left\{\mathbf{n}(x) - \mathbf{n}(x + \int_0^t \mathbf{u}(\tau, x)\mathrm{d}\tau)\right\} & \text{on } \Gamma_b,\ 0 \le t \le T, \end{cases}$$

has a unique solution $\mathbf{U} = M_2(\mathbf{u})$ satisfying

$$\begin{cases} |\mathbf{U}(t)|_{s+1,\Omega} \le C_{15}(\|h(\mathbf{u};t)\|_{H^{s+1/2}(\Omega)} + \|\bar{X}_{1t}(t)\|_{H^{s+1}(\Gamma_s)} \\ \qquad + \|\mathbf{v}_0|_{\Gamma_s}\|_{H^{s+1}(\Gamma_s)} + \|\mathbf{u}\cdot\{\mathbf{n}(x) - \mathbf{n}(x+\int_0^t \mathbf{u})\}\|_{H^{s+1}(\Gamma_b)} \\ \qquad + \|\mathbf{v}_0\cdot\mathbf{n}\|_{H^{s+1}(\Gamma_b)}) + |\mathbf{v}_0|_{s+1,\Omega} \\ \quad \le (C_{16} + C_{15}\|\nabla\mathbf{n}\|_{H^{s+1}(\Gamma_b)})|\mathbf{u}(t)|_{s+1,\Omega}\int_0^t |\mathbf{u}(\tau)|_{s+1,\Omega}\mathrm{d}\tau \\ \qquad + C_{15}\|\bar{X}_{1t}(t)\|_{H^{s+1}(\Gamma_s)} + J_4, \\ |\mathbf{U}_t(t)|_{s+1/2,\Omega} \le (C_{16} + C_{15}\|\nabla\mathbf{n}\|_{H^{s+1/2}(\Gamma_b)})|\mathbf{u}_t(t)|_{s+1/2,\Omega}\int_0^t |\mathbf{u}(\tau)|_{s+1/2,\Omega}\mathrm{d}\tau \\ \qquad + (C_{16} + C_{15}\|\nabla\mathbf{n}\|_{H^{s+1/2}(\Gamma_b)})|\mathbf{u}(t)|^2_{s+1/2,\Omega} + C_{15}\|\bar{X}_{1tt}(t)\|_{H^{s+1/2}(\Gamma_s)}, \\ |\mathbf{U}_{tt}(t)|_{s,\Omega} \le (C_{16} + C_{15}\|\nabla\mathbf{n}\|_{H^s(\Gamma_b)})|\mathbf{u}_{tt}(t)|_{s,\Omega}\int_0^t |\mathbf{u}(\tau)|_{s,\Omega}\mathrm{d}\tau \\ \qquad + (C_{16} + 3C_{15}\|\nabla\mathbf{n}\|_{H^s(\Gamma_b)})|\mathbf{u}_t(t)|_{s,\Omega}|\mathbf{u}(t)|_{s,\Omega} + C_{15}\|\bar{X}_{1ttt}(t)\|_{H^s(\Gamma_s)}. \end{cases}$$

If we put

$$T = \min\left\{\varphi_1^{-1}\left(\frac{1}{32(C_{15}(1+C_4)+1)(C_{16} + C_{15}\|\nabla\mathbf{n}\|_{H^{s+1}(\Gamma_b)})(J + J_3 + J_4)}\right),\right.$$
$$\left.\varphi_2^{-1}\left(\frac{1}{32C_{15}(1+C_4)C_7(C_{16} + C_{15}\|\nabla\mathbf{n}\|_{H^{s+1}(\Gamma_b)})J_2}\right),\ T_1\right\}, \quad (2.6.14)$$

where

$$\varphi_1(t) = t\exp(C_7 t), \quad \varphi_2(t) = t^2\exp(C_7 t),$$

the last estimate of (2.6.12) implies that

$$C_{16}\sup_{0\le t\le T}\int_0^t |\mathbf{u}(t)|_{s+1,\Omega}\mathrm{d}\tau \le \frac{1}{8}, \quad C_{15}\|\nabla\mathbf{n}\|_{H^{s+1}(\Gamma_b)}\sup_{0\le t\le T}\int_0^t |\mathbf{u}(t)|_{s+1,\Omega}\mathrm{d}\tau \le \frac{1}{8}.$$

Therefore $\mathbf{U}$ satisfies (2.6.12) and M_2 maps $\mathcal{S}_2$ to itself.

We introduce a new norm

$$|||\mathbf{u}|||_{s,T,\lambda} = \sup_{0\le t\le T}(|\mathbf{u}(t)|_{s+1,\Omega} + \lambda^{-1}|\mathbf{u}_t(t)|_{s+1/2,\Omega} + \lambda^{-2}|\mathbf{u}_{tt}(t)|_{s,\Omega}),$$

where $\lambda \ge 1$ is a parameter to be determined later. For $\mathbf{u}^{(1)}, \mathbf{u}^{(2)} \in \mathcal{S}_2$ we set $\mathbf{U}^{(j)} = M_2(\mathbf{u}^{(j)})$, $j = 1, 2$. Then

$$\begin{cases} \nabla\cdot(\mathbf{U}^{(1)} - \mathbf{U}^{(2)}) = h_1^{(1)} - h_1^{(2)} & \text{in } \Omega,\ 0 \le t \le T, \\ \nabla^\perp\cdot(\mathbf{U}^{(1)} - \mathbf{U}^{(2)}) = h_2^{(1)} - h_2^{(2)} & \text{in } \Omega,\ 0 \le t \le T, \\ U_1^{(1)} - U_1^{(2)} = 0 & \text{on } \Gamma_s,\ 0 \le t \le T, \\ (\mathbf{U}^{(1)} - \mathbf{U}^{(2)})\cdot\mathbf{n}(x) = \mathbf{u}^{(1)}\cdot\left\{\mathbf{n}(x) - \mathbf{n}(x + \int_0^t \mathbf{u}^{(1)}(\tau,x)\mathrm{d}\tau)\right\} & \\ \qquad - \mathbf{u}^{(2)}\cdot\left\{\mathbf{n}(x) - \mathbf{n}(x + \int_0^t \mathbf{u}^{(2)}(\tau,x)\mathrm{d}\tau)\right\} & \text{on } \Gamma_b,\ 0 \le t \le T, \end{cases}$$

where $h_k^{(j)} = h_k(\mathbf{u}^{(j)}; t),\ k, j = 1, 2$. It follows from (2.6.9) and (2.6.12) that

$$|||\mathbf{U}^{(1)} - \mathbf{U}^{(2)}|||_{s,T,\lambda} \leq \frac{1}{2}|||\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|||_{s,T,\lambda} + \lambda^{-1}\{C_{16}(2(e_1 + e_2) + (e_2 + e_3)T) + C_{15}\|\nabla\mathbf{n}\|_{H^{s+1/2}(\Gamma_b)}(3(e_1 + e_2) + (e_2 + e_3)T)\}|||\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|||_{s,T,\lambda}.$$

If we put

$$\lambda = 4\{C_{16}(2(e_1 + e_2) + (e_2 + e_3)T) + C_{15}\|\nabla\mathbf{n}\|_{H^{s+1/2}(\Gamma_b)}(3(e_1 + e_2) + (e_2 + e_3)T)\} + 1,$$

we get

$$|||M_2(\mathbf{u}^{(1)}) - M_2(\mathbf{u}^{(2)})|||_{s,T,\lambda} \leq \frac{3}{4}|||\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|||_{s,T,\lambda}.$$

Hence the desired solution is obtained. ∎

By the same way we have the following lemmas.

Lemma 2.6.2. *Let* $\mathbf{u}$ *be the solution of* (2.6.1) *obtained in* Theorem 2.6.4. *Then it holds that*

$$\begin{cases} |\mathbf{u}(t)|_{s,\Omega} \leq 2C_{15}d_0 + 2d_4 =: e_4, \\ |\mathbf{u}_t(t)|_{s,\Omega} \leq 2(C_{16} + \|\nabla\mathbf{n}\|_{H^{s+1/2}(\Gamma_b)})e_4^2 + 2C_{15}d_0 =: e_5 \end{cases}$$

for $0 \leq t \leq T$.

Note that e_1, e_2 and e_3 depend on d_3, but e_4 and e_5 do not.

Proposition 2.6.1. *Let* $\mathbf{u}$ *be the solution of* (2.6.1) *obtained in* Theorem 2.6.4 *and* $\mathbf{u}^0$ *the solution of* (2.6.1) *with* $\bar{X}$ *replaced by* $\bar{X}^0$, *which satisfies* Assumption 2.3. *Then we have*

$$\sum_{j=0}^{2} \sup_{0\leq\tau\leq t} |\partial_\tau^j \mathbf{u}(\tau) - \partial_\tau^j \mathbf{u}^0(\tau)|_{s+1/2-j/2,\Omega} \leq C_{17}\sum_{j=0}^{2} \sup_{0\leq\tau\leq t} \|\partial_\tau^{j+1}\bar{X}_1(\tau) - \partial_\tau^{j+1}\bar{X}_1^0(\tau)\|_{H^{s+1/2-j/2}(\Gamma_s)}$$

for $s > 2,\ 0 \leq t \leq T$, *where* $C_{17} = C_{17}(e_1, e_2, e_3, C_{15}, C_{16}) > 0$.

Let us consider the second relation of (2.1.4).

Lemma 2.6.3. *Suppose that the same assumptions of* Theorem 2.6.4 *are satisfied. Let* c_0 *be the constant chosen in* Assumption 2.1 *and* $\mathbf{u}$ *the solution of* (2.6.1) *obtained in* Theorem 2.6.4. *There exist positive constants* $\varepsilon_0 = \varepsilon_0(c_0)$ *and* $T_0(\leq T)$ *such that if*

$$\|\eta_0\|_{H^3(\mathbf{R}^1)} + \|b\|_{H^3(\mathbf{R}^1)} \leq \varepsilon_0, \tag{2.6.15}$$

then $X = X(t, y)$ *defined by* (2.1.4) *satisfies* (2.4.2) *with* T *replaced by* T_0.

Proof. Lemma 2.6.1 implies

$$\begin{cases} |||\partial_t^j X(t)|||_{s+3/2-j/2} \le C_{18}|\partial_t^{j-1}\mathbf{u}(t)|_{s+3/2-j/2,\Omega}, \quad j=1,2,3, \\ |||\partial_t^j X(t)|||_s \le C_{18}|\partial_t^{j-1}\mathbf{u}(t)|_{s,\Omega}, \quad j=1,2, \\ |||X(t)|||_{s+1} \le C_{11}(\|\eta_0\|_{H^{s+1}(\mathbf{R}^1)} + \|b\|_{H^{s+1}(\mathbf{R}^1)}) + C_{18}t \sup_{0\le\tau\le t} |\mathbf{u}(\tau)|_{s+1,\Omega}, \\ |||X(t)|||_3 \le C_{12}(\|\eta_0\|_{H^3(\mathbf{R}^1)} + \|b\|_{H^3(\mathbf{R}^1)}) + C_{18}t \sup_{0\le\tau\le t} |\mathbf{u}(\tau)|_{s+1,\Omega} \end{cases}$$

for $0 \le t \le T$, where $C_{18} = C_{18}(s, d_0, d') > 0$. We define $T_0, \varepsilon_0, d, l_j (j=1,2,\dots,5)$ as

$$\begin{cases} T_0 = \min\left\{T, \varphi_1^{-1}\left(\dfrac{c_0}{8(C_{15}(1+C_4)+1)C_{18}(J+J_3+J_4)}\right),\right. \\ \qquad\qquad\qquad\qquad \left.\varphi_2^{-1}\left(\dfrac{c_0}{8C_{15}(1+C_4)C_7C_{18}J_2}\right)\right\}, \\ \varepsilon_0 = (2C_{12})^{-1}c_0, \quad d = C_{11}(d_0+d') + \dfrac{c_0}{2}, \\ l_j = C_{18}e_j, \ j=1,2,\dots,5, \end{cases} \tag{2.6.16}$$

then the desired result follows from (2.1.4) and Lemma 2.6.1. ∎

Proposition 2.6.2. *Suppose that* X^0 *and* $\mathbf{u}^0$ *also satisfy* (2.1.4). *Then we have*

$$\begin{cases} |||\partial_t^{j+1}X(t) - \partial_t^{j+1}X^0(t)|||_{s+1/2-j/2} \le C_{18}|\partial_t^j\mathbf{u}(t) - \partial_t^j\mathbf{u}^0(t)|_{s+1/2-j/2,\Omega}, \quad j=0,1,2, \\ |||X(t) - X^0(t)|||_{s+1/2} \le C_{18}\displaystyle\int_0^t |\mathbf{u}(\tau) - \mathbf{u}^0(\tau)|_{s+1/2,\Omega}\mathrm{d}\tau \quad \text{for} \quad 0\le t\le T. \end{cases}$$

2.7. Proof of Theorem 2.1

In the same way as in Section 2.4, we can prove

Lemma 2.7.1. *Let* $\varepsilon_1 = \varepsilon_1(g)$ *be the constant chosen in* Lemma 2.5.3. *There exists a positive constant* $\varepsilon_2 = \varepsilon_2(g)$ *such that if* $X|_{t=0} = (0, \widetilde{\eta_0})$, $\partial_t X|_{t=0} = \mathbf{u}_0$ *and*

$$|||\widetilde{\eta_0}|||_3 + \|\mathbf{u}_0\|_{H^{3+1/2}(\Sigma)} + \|\omega_1\|_{H^{3+1/2}(\Sigma)} \le \varepsilon_2, \tag{2.7.1}$$

then we have

$$\|H(X)|_{t=0}\|_{H^3(\mathbf{R}^1)} + \|\partial_t H(X)|_{t=0}\|_{H^3(\mathbf{R}^1)} \le \varepsilon_1/2.$$

From Lemma 2.6.1 we see that if (2.1.1) is satisfied, (2.5.21), (2.6.10), (2.6.15), (2.7.1) and Assumption 2.2 with $s_0 = s + 1/2$ are valid. About the constants we take

$$J_2 = C_2\mu_s, \quad J_3 = C_1\mu_s \tag{2.7.2}$$

from (2.5.18), (2.5.24) and (2.4.3). J, that is, $\widetilde{W}$ and $\widetilde{W_t'}$ are determined by η_0 and $\mathbf{v}_0$.

In view of (2.5.19), (2.6.14) and (2.6.16), we take

$$
\begin{aligned}
T = \min\Biggl\{ & \frac{c_1}{2(J_0+J_1)},\ \frac{1}{C_7}\log\frac{J_0}{2J},\ \varphi_1^{-1}\left(\frac{J_0}{2C_7J_2}\right), \\
& \varphi_1^{-1}\left(\frac{\min(1,c_0)}{32(C_{15}(1+C_4)+1)(C_{16}+C_{15}\|\nabla \mathbf{n}\|_{H^{s+1}(\Gamma_b)}+C_{18})(J+J_3+J_4)}\right), \\
& \varphi_2^{-1}\left(\frac{\min(1,c_0)}{32C_{15}(1+C_4)C_7(C_{16}+C_{15}\|\nabla \mathbf{n}\|_{H^{s+1}(\Gamma_b)}+C_{18})J_2}\right)\Biggr\}.
\end{aligned}
\tag{2.7.3}
$$

Now define the sets $\mathcal{S}_3, \mathcal{S}_4$ and $\mathcal{S}_5$ as

$$
\begin{aligned}
\mathcal{S}_3 = \{\bar{X}; \bar{X} & \text{ satisfies (2.5.23) and} \\
& \bar{X}|_{t=0} = \widetilde{X}, \quad X_t|_{t=0} = \widetilde{X_t}, \quad \bar{X}_{tt}|_{t=0} = \widetilde{Y}, \\
& \|\bar{X}_{1t}(t)\|_{H^s(\mathbf{R}^1)} + \|\bar{X}_{1tt}(t)\|_{H^{s+1/2}(\mathbf{R}^1)} + \|\bar{X}_{1ttt}(t)\|_{H^s(\mathbf{R}^1)} \le d_0, \\
& \|\bar{X}_{1t}(t)\|_{H^{s+1}(\mathbf{R}^1)} \le (1+C_4)\{J\exp(C_7t) + J_2C_7t\exp(C_7t)\} + C_4J_3 \\
& \qquad \le (1+C_4)d_0 + C_4d_3 \\
& \text{for } \ 0 \le t \le T\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_4 = \{\mathbf{u};\ & \mathbf{u} \text{ satisfies (2.6.11) and} \\
& \mathbf{u}|_{t=0} = \mathbf{v}_0, \quad \mathbf{u}_t|_{t=0} = \mathbf{w}_0, \\
& |\partial_t^j \mathbf{u}(t)|_{s+1-j/2,\Omega} \le e_{j+1}, \quad j = 0,1,2, \\
& |\partial_t^j \mathbf{u}(t)|_{s,\Omega} \le e_{j+4}, \quad j = 0,1, \\
& |\mathbf{u}(t)|_{s+1,\Omega} \le 2C_{15}(1+C_4)\{J\exp(C_7t) + J_2C_7t\exp(C_7t)\} + 2C_{15}C_4J_3 \\
& \qquad + 2J_4 \\
& \text{for } \ 0 \le t \le T\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_5 = \{X; X & \text{ satisfies (2.4.2) and} \\
& X|_{t=0} = (0, \widetilde{\eta_0}), \quad X_t|_{t=0} = \mathbf{u}_0, \quad X_{tt}|_{t=0} = \mathbf{w}_0 \circ \Psi\},
\end{aligned}
$$

where $\mathbf{w}_0$ is the solution of

$$
\begin{cases}
\nabla \cdot \mathbf{w}_0 = 2\left(\dfrac{\partial v_{01}}{\partial x_2}\dfrac{\partial v_{02}}{\partial x_1} - \dfrac{\partial v_{01}}{\partial x_1}\dfrac{\partial v_{02}}{\partial x_2}\right), \quad \nabla^\perp \cdot \mathbf{w}_0 = 0 & \text{in } \Omega, \\
w_{01} = \widetilde{Y}_1 & \text{on } \Gamma_s, \\
\mathbf{w}_0 \cdot \mathbf{n}(x) = \mathbf{w}_0 \cdot \left\{\mathbf{n}(x) - \mathbf{n}(x + \int_0^t \mathbf{u})\right\} - \mathbf{v}_0 \cdot \{(\mathbf{u}_0 \cdot \nabla)\mathbf{n}(x)\} & \text{on } \Gamma_b.
\end{cases}
\tag{2.7.4}
$$

For $X^0 \in \mathcal{S}_5$ we denote by $\bar{X} = M_3(X^0)$ the solution of problem (2.5.1) – (2.5.3) with H replaced by $H(X^0)$. Then Proposition 2.4.1 and the arguments in Section 2.5 show that M_3 is a mapping from $\mathcal{S}_5$ to $\mathcal{S}_3$. For $\bar{X} \in \mathcal{S}_3$, let $\mathbf{u} = M_4(\bar{X})$ be the solution of (2.6.1).

Noting that the solution to (2.6.5) is unique, we see that $\mathbf{u}|_{t=0} = \mathbf{v}_0$ follows from (2.6.1) at $t = 0$ and that $\mathbf{u}_t|_{t=0} = \mathbf{w}_0$ since $\mathbf{u}_t|_{t=0}$ satisfies the same equations as (2.7.4). Therefore M_4 is a mapping from $\mathcal{S}_3$ to $\mathcal{S}_4$ according to the results in Section 2.6. For $\mathbf{u} \in \mathcal{S}_4$ define X by (2.1.4) and set $M_5(\mathbf{u}) = X$. We see that M_5 is a mapping from $\mathcal{S}_4$ to $\mathcal{S}_5$.

Let us define the approximate solutions $\{\bar{X}^n, \mathbf{u}^n, X^n\}$, $n = 1, 2, 3, \ldots$, as

$$\begin{cases} X^0(t,y) = (0, \widetilde{\eta_0}(y)) + t\mathbf{u}_0(y) \quad \text{for } y \in \Sigma, \ {}^\forall t \geq 0, \\ \bar{X}^n = M_3(X^{n-1}), \quad \mathbf{u}^n = M_4(\bar{X}^n), \quad X^n = M_5(\mathbf{u}^n) \quad \text{for } n = 1, 2, 3, \ldots. \end{cases}$$

Since X^0 satisfies (2.4.2), $\bar{X}^1 = M_3(X^0)$ is well-defined and belongs to $\mathcal{S}_3$ with T replaced by some T', which is of a similar form to (2.7.3). Here we denote T' by T again. Repeating this argument, we conclude that $\{\bar{X}^n, \mathbf{u}^n, X^n\}$ are well-defined and $\bar{X}^n \in \mathcal{S}_3$, $\mathbf{u}^n \in \mathcal{S}_4$, $X^n \in \mathcal{S}_5$, $n = 1, 2, 3, \ldots$. Propositions 2.5.1, 2.6.1, 2.6.2 and 2.4.1 show that $\bar{X}^n, \mathbf{u}^n, X^n$ are Cauchy sequences in the corresponding spaces. Hence there exist $\bar{X}, X$ and $\mathbf{u}$ such that

$$\begin{aligned} \sup_{0 \leq \tau \leq t} \Bigg\{ & \|\bar{X}^n(\tau) - \bar{X}(\tau)\|_{H^{s+1/2}(\mathbf{R}^1)} + \sum_{j=0}^{2} \|\partial_\tau^{j+1}\bar{X}^n(\tau) - \partial_\tau^{j+1}\bar{X}(\tau)\|_{H^{s+1/2-j/2}(\mathbf{R}^1)} \\ & + |||X^n(\tau) - X(\tau)|||_{s+1/2} + \sum_{j=0}^{2} |||\partial_\tau^{j+1}X^n(\tau) - \partial_\tau^{j+1}X(\tau)|||_{s+1/2-j/2} \\ & + \sum_{j=0}^{2} |\partial_\tau^j \mathbf{u}^n(\tau) - \partial_\tau^j \mathbf{u}(\tau)|_{s+1/2-j/2,\Omega} \Bigg\} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

We see that $\bar{X}, \mathbf{u}$ and X are solutions of problem (2.5.1) – (2.5.3), problem (2.6.1), (12) and problem (2.1.4), respectively. Moreover $\bar{X} \in \mathcal{S}_3, \mathbf{u} \in \mathcal{S}_4, X \in \mathcal{S}_5$.

For the proof of (2.1.5) it is sufficient to set

$$\mathbf{v}(t,z) = \mathbf{u}(t, \Phi_\mathbf{u}^{-1}(z;t)), \quad \omega(t,z) = \omega_0(t, \Phi_\mathbf{u}^{-1}(z;t)), \quad \Omega(t) = \Phi_\mathbf{u}(\Omega;t).$$

The uniqueness of the solution is proved in the same way.

Finally we define q as a solution of the boundary value problem

$$\begin{cases} \Delta q = -\nabla \cdot (A_\mathbf{u}^{-1}\mathbf{u}_t) & \text{in } \Omega, \ t \geq 0, \\ q = g\left(x_2 + \int_0^t u_2(\tau, x)\mathrm{d}\tau\right) & \text{on } \Gamma_s, \ t \geq 0, \\ \dfrac{\partial q}{\partial \mathbf{n}(\Phi_\mathbf{u})} = -(\mathbf{u} \cdot \nabla_\mathbf{u})\mathbf{u} \cdot \mathbf{n}(\Phi_\mathbf{u}) & \text{on } \Gamma_b, \ t \geq 0, \end{cases} \tag{2.7.5}$$

where the last condition on Γ_b is derived from applying $(\partial/\partial\mathbf{n}(\Phi_\mathbf{u}))$ to both sides of (1). If we take T sufficiently small, which is of the same form as (2.7.3), the results in Section 2.6 imply the unique existence of the solution q of (2.7.5) satisfying $q \in C^j([0,T]; H^{s+3/2-j/2}(\Omega)), j = 0, 1$. Further, if we put $\mathbf{V} = A_\mathbf{u}^{-1}\mathbf{u}_t + \nabla q$, it holds that

$$\nabla \cdot \mathbf{V} = 0, \quad \nabla^\perp \cdot \mathbf{V} = ((A_\mathbf{u}^{-1}\nabla)^\perp \cdot \mathbf{u})_t = 0 \quad \text{in } \Omega, \ 0 \leq t \leq T,$$

$$V_1|_{\Gamma_s} = 0, \quad \mathbf{V}|_{\Gamma_b} \cdot \mathbf{n}(x) = 0, \quad 0 \le t \le T.$$

Again uniqueness of the solution to problem (2.6.2) implies $\mathbf{V} \equiv 0$, hence (8).

We see that $(\mathbf{u}, q)$ satisfy (8) – (12) and (2.1.2). The uniqueness of the solution to problem (8) – (12) comes from that of problem (2.5.1) – (2.5.3), (2.6.1), (12), (2.1.4), and (2.1.5). The proof is complete.

In conclusion, if

$$\|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)} + \|\mathbf{v}_0\|_{H^{s+3/2}(\Omega)} + \|\omega_0\|_{H^{s+3/2}(\Omega)} \to 0,$$

then by Lemma 2.5.3 and (2.4.1), (2.6.13), (2.7.2), $J, J_2, J_3, J_4, \mu_s \to 0$. Putting $J_0 = \max\{\sqrt{2J}, \sqrt{\mu_s}\}$, we have $T \to \infty$.

Chapter 3. Problem with Surface Tension

In this chapter, we are concerned with the free boundary problem when surface tension is effective. We prove the unique existence of the solution, locally in time. Furthermore, it is shown that this solution converges to the solution of the problem without surface tension as the coefficient of surface tension tends to zero.

In Chapter 2, gravity has to work downward for the existence of solution of the problem without surface tension. However, if the surface tension is effective, we see that the problem is well-posed irrespective of the direction of gravity.

3.1. Main results

The unique existence theorem for problem (8) – (12) is the following.

Theorem 3.1. *Let* $\sigma > 0$ *and* $s \geq 5 + 1/2$. *There exist positive constants* $\delta_1 = \delta_1(g, s)$ *and* $\delta_2 = \delta_2(s)$ *such that if*

$$\begin{cases} \eta_0 \in H^{s+9/2}(\mathbf{R}^1), \quad b \in H^{s+5}(\mathbf{R}^1), \quad \mathbf{v}_0 \in H^{s+9/2}(\Omega), \\ \|\eta_0\|_{H^5(\mathbf{R}^1)} + \|b\|_{H^3(\mathbf{R}^1)} + \|\mathbf{v}_0\|_{H^{3+1/2}(\Omega)} + \|\omega_0\|_{H^{3+1/2}(\Omega)} \leq \delta_1, \quad \|b\|_{H^{s+4}(\mathbf{R}^1)} \leq \delta_2, \end{cases} \tag{3.1.1}$$

where $\omega_0 = \nabla_x^\perp \cdot \mathbf{v}_0$, $\nabla_x^\perp = (-\partial/\partial x_2, \partial/\partial x_1)$, *and* $\mathbf{v}_0$ *satisfies the compatibility conditions, then problem* (8) – (12) *has a unique solution* $(\mathbf{u}, q)$ *on some time interval* $[0, T]$ *satisfying*

$$\begin{cases} \mathbf{u} \in C^j([0, T]; H^{s+7/2-3j/2}(\Omega)), \quad j = 0, 1, 2, \\ q \in C^j([0, T]; H^{s+7/2-3j/2}(\Omega)), \quad j = 0, 1. \end{cases}$$

Remark. T *can be taken such that* $T \to \infty$ *as* $\|\eta_0\|_{H^{s+9/2}(\mathbf{R}^1)} + \|\mathbf{v}_0\|_{H^{s+9/2}(\Omega)} \to 0$.

The next theorem is about the convergence of the solution for the surface tension.

Theorem 3.2. *Assume that the conditions of* Theorem 3.1 *are satisfied and* $0 < \sigma \leq 1$. *If* $g > 0$, *we can take the existence time* T *in* Theorem 3.1 *so that* T *is independent of* σ. *Moreover if* $s \geq 5 + 1/2 + s_0$, $0 < s_0 < 2$, *then the solution* $(\mathbf{u}^\sigma, q^\sigma)$ *of problem* (8) – (12) *converges to the solution* $(\mathbf{u}, q)$ *of problem* (8) – (12) *with* $\sigma = 0$:

$$\begin{cases} \mathbf{u}^\sigma \to \mathbf{u} & \text{in } C^1([0, T]; H^{s-1-s_0}(\Omega)), \\ q^\sigma \to q & \text{in } C([0, T]; H^{s-1-s_0}(\Omega)). \end{cases}$$

Notations. Let j be a nonnegative integer, $0 < T < \infty$ and B a Banach space. We say that $u \in C^j([0,T]; B)$ if u is a j-times continuously differentiable function on $[0,T]$ with values in B. Let D be a domain in $\mathbf{R}^n$ and $s > 0$. By $H^s(D)$ we denote the Sobolev-Slobodetskiĭ space. Moreover we use the commutator $[A,B] = AB - BA$ for operators A and B.

3.2. Problem on the surface

At first, we introduce the coordinate transformation mapping $x = y + (0, \widetilde{\eta_0}(y)) \equiv \Psi(y)$ from Ω onto the strip region

$$\Sigma = \{y = (y_1, y_2);\ -h < y_2 < 0,\ y_1 \in \mathbf{R}^1\},$$

where $\widetilde{\eta_0}$ is a function such that $\widetilde{\eta_0}(\cdot, 0) = \eta_0(\cdot)$ and $\widetilde{\eta_0}(\cdot, -h) = b(\cdot)$. Then from (7) it follows

$$z = \Phi_{\mathbf{u}}(\Psi(y); t) \equiv y + X(t,y), \quad X(t,y) = (0, \widetilde{\eta_0}(y)) + \int_0^t \mathbf{u}(\tau, \Psi(y)) \mathrm{d}\tau. \tag{3.2.1}$$

By putting

$$\bar{X}(t, y_1) = X(t, y_1, 0), \tag{3.2.2}$$

it is derived from (8), (10) that

$$(1 + \bar{X}_{1y_1})\bar{X}_{1tt} + \bar{X}_{2y_1}(g + \bar{X}_{2tt}) = \mu R + \mu S \quad \text{for} \quad t \geq 0, \tag{3.2.3}$$

where

$$\begin{cases} \mu = \rho^{-1}\sigma, \\ Q = Q(\bar{X}_{y_1}) = \{(1 + \bar{X}_{1y_1})^2 + \bar{X}_{2y_1}{}^2\}^{1/2}, \\ R = R(\bar{X}_{y_1}, \bar{X}_{y_1 y_1}) = -3Q(\bar{X}_{y_1})^{-5} \left\{(1 + \bar{X}_{1y_1})\bar{X}_{1y_1y_1} + \bar{X}_{2y_1}\bar{X}_{2y_1y_1}\right\} \\ \qquad\qquad \times \left\{-\bar{X}_{2y_1}\bar{X}_{1y_1y_1} + (1 + \bar{X}_{1y_1})\bar{X}_{2y_1y_1}\right\}, \\ S = S(\bar{X}_{y_1}, \bar{X}_{y_1y_1y_1}) = Q(\bar{X}_{y_1})^{-3} \left\{-\bar{X}_{2y_1}\bar{X}_{1y_1y_1y_1} + (1 + \bar{X}_{1y_1})\bar{X}_{2y_1y_1y_1}\right\}. \end{cases}$$

Since vorticity $\nabla^{\perp} \cdot \mathbf{v} = \omega$ can be written as

$$\nabla_{\mathbf{u}}^{\perp} \cdot \mathbf{u} = \omega_0$$

in the Lagrangian coordinates, it follows from (9), (11) that

$$\bar{X}_{2t} = K\bar{X}_{1t} + H \quad \text{for} \quad t \geq 0 \tag{3.2.4}$$

with

$$K = -i \tanh(h\mathrm{D}) + K_1, \quad H = H(X, \omega_1).$$

Here $\mathrm{D} = -i\partial/\partial y_1$, $K_1 = K_1(\bar{X}, b)$ is a smoothing operator and $\omega_1(y) = \omega_0(\Psi(y))$. The explicit forms of K and H are given in Section 2.3.

Assuming that an X in H is given, we solve the Cauchy problem (3.2.3), (3.2.4) for $\bar{X}$ with the initial conditions

$$\bar{X} = (0, \eta_0), \quad \bar{X}_{1t} = u_{01}|_{y_2=0} \equiv v_{01}|_{y_2=0} \quad \text{for} \quad t = 0. \tag{3.2.5}$$

For that we put

$$Y = \bar{X}_{tt}, \quad Z = \bar{X}_{y_1}, \quad W = (\bar{X}, Y, Z), \quad W' = (\bar{X}, Y_1)$$

and reduce problem (3.2.3) – (3.2.5) to the initial value problem for quasi-linear equations. In the remaining of this section, for simplicity we use X and y instead of $\bar{X}$ and y_1.

Differentiating (3.2.4) leads to

$$\partial_t^j X_{2t} = K(X)\partial_t^j X_{1t} + F_{j0}(X, \ldots, \partial_t^j X) + \partial_t^j H, \tag{3.2.6}$$
$$\partial_t^j \partial_y^k X_{2t} = -i\mathrm{sgn}\mathrm{D}\partial_t^j \partial_y^k X_{1t} + F_{jk0} + \partial_t^j \partial_y^k H, \tag{3.2.7}$$
$$F_{jk0} = \{i\mathrm{sgn}\mathrm{D} + K(X)\}\partial_t^j \partial_y^k X_{1t} + F_{jk}(X, \ldots, \partial_t^j \partial_y^k X, \partial_t^j X_{1t}),$$

where $j = 0, 1, 2, \ldots, k = 1, 2, 3, \ldots$ and $F_{jk} = [\partial_t^j \partial_y^k, K_1]X_{1t}$. By (3.2.6) with $j = 2$, we obtain

$$Y_{2t} = K(X)Y_{1t} + F_{20}(X, X_t, Y) + H_{tt} =: f_2(W, W_t', H).$$

It follows from (3.2.7) with $j = 0,\ k = 3$ and $j = 0,\ k = 1$ that

$$Z_{2t} = -i\mathrm{sgn}\mathrm{D}Z_{1t} + (1 + \mu\mathrm{D}^2)^{-1}(F_{010} - \mu F_{030}) + H_y, \tag{3.2.8}$$

where

$$\begin{aligned} F_{0k0} &= F_{0k0}(X, Z, \ldots, \partial_y^{k-1} Z, X_{1t}) \\ &= \{i\mathrm{sgn}\mathrm{D} + K(X)\}(i\mathrm{D})^k X_{1t} + F_{0k}(X, Z, \ldots, \partial_y^{k-1} Z, X_{1t}), \quad k = 1, 3. \end{aligned}$$

Since equation (3.2.3) implies that

$$(1 + Z_1)Y_1 + Z_2(g + Y_2) = \mu R(Z, Z_y) + \mu S(Z, Z_{yy}) + g_0(1 + \mu\mathrm{D}^2)^{-1} i\mathrm{sgn}\mathrm{D}(Z_1 - i\mathrm{D}X_1) \tag{3.2.9}$$

with the constant g_0 being determined later, differentiating (3.2.9) with respect to t and (3.2.8) give

$$\begin{aligned} &\{g_0(1 + \mu\mathrm{D}^2)^{-1} i\mathrm{sgn}\mathrm{D} + P_1\}\, Z_{1t} \\ &+(-g - \mu\mathrm{D}^2 + P_2)\,\{-i\mathrm{sgn}\mathrm{D}Z_{1t} + (1 + \mu\mathrm{D}^2)^{-1}(F_{010} - \mu F_{030})\} \\ &-(1 + Z_1)Y_{1t} - Z_2 Y_{2t} + g_0(1 + \mu\mathrm{D}^2)^{-1}\mathrm{D}\mathrm{sgn}\mathrm{D}X_{1t} + (-g - \mu\mathrm{D}^2 + P_2)H_y = 0 \end{aligned}$$

with

$$\begin{aligned} P_j &= P_j(Y, Z, Z_y, Z_{yy}) \\ &= -Y_j + \mu\frac{\partial R}{\partial Z_j}(Z, Z_y) + \mu\frac{\partial S}{\partial Z_j}(Z, Z_{yy}) + \mu\frac{\partial R}{\partial Z_{jy}}(Z, Z_y) i\mathrm{D} + \mu\frac{\partial S}{\partial Z_{jyy}}(Z, Z_{yy})(i\mathrm{D})^2 \\ &\quad + (j-1)\mu\mathrm{D}^2, \quad j = 1, 2. \end{aligned}$$

Therefore we get

$$
\begin{aligned}
Z_{1t} &= i\mathrm{sgnD}\left\{g + g_0(1+\mu\mathrm{D}^2)^{-1} + \mu\mathrm{D}^2\right\}^{-1}(1+P_3)^{-1} \\
&\quad \times \Big\{-(1+Z_1)Y_{1t} - Z_2 f_2 + g_0(1+\mu\mathrm{D}^2)^{-1}|\mathrm{D}|X_{1t} \\
&\qquad +(-g-\mu\mathrm{D}^2+P_2)(1+\mu\mathrm{D}^2)^{-1}(F_{010}-\mu F_{030}) + (-g-\mu\mathrm{D}^2+P_2)H_y\Big\} \\
&=: f_3(W, W'_t, H), \\
Z_{2t} &= -i\mathrm{sgnD} f_3 + (1+\mu\mathrm{D}^2)^{-1}(F_{010}-\mu F_{030}) + H_y =: f_4(W, W'_t, H),
\end{aligned}
$$

where

$$P_3 = -(P_1 i\mathrm{sgnD} + P_2)\left\{g + g_0(1+\mu\mathrm{D}^2)^{-1} + \mu\mathrm{D}^2\right\}^{-1}.$$

Notice that if we put

$$g_0 = \begin{cases} \frac{1}{4}(2-g)^2 & \text{if } g \le 0, \\ 0 & \text{if } g > 0, \end{cases}$$

then the operator

$$\left\{g + g_0(1+\mu\mathrm{D}^2)^{-1} + \mu\mathrm{D}^2\right\}^{-1}$$

is well-defined.

In virtue of (3.2.6) with $j = 3$, we have

$$Y_{2tt} = KY_{1tt} + F_{30}(X, X_t, Y, Y_t) + H_{ttt}. \tag{3.2.10}$$

Hence, by differentiating (3.2.3) twice with respect to t, it follows from (3.2.10) that

$$
\begin{aligned}
&(1 + X_{1y} + X_{2y}K)Y_{1tt} \\
&= \mu(R+S)_{tt} - Y_1 X_{1tty} - (g+Y_2)X_{2tty} - 2X_{ty}\cdot Y_t - X_{2y}F_{30} - X_{2y}H_{ttt} \\
&\equiv P_1(Y,\ X_y,\ X_{yy},\ X_{yyy})X_{1tty} + \{-g-\mu\mathrm{D}^2 + P_2(Y,\ X_y,\ X_{yy},\ X_{yyy})\}X_{2tty} \\
&\quad +\mu I_1(X_y,\ X_{ty},\ X_{yy},\ X_{tyy},\ X_{yyy},\ X_{tyyy}) - 2X_{ty}\cdot Y_t - X_{2y}F_{30} - X_{2y}H_{ttt}.
\end{aligned} \tag{3.2.11}
$$

On the other hand, by (3.2.7) with $j = k = 1$ and $j = 1,\ k = 3$, it holds that

$$
\begin{aligned}
Y_{2y} &= -i\mathrm{sgnD}Y_{1y} + (1+\mu\mathrm{D}^2)^{-1}(F_{110} - \mu F_{130}) + H_{ty}, \qquad (3.2.12)\\
F_{1k0} &= F_{1k0}(X, X_t, Z, Z_t, \ldots, \partial_y^{k-1}Z, \partial_y^{k-1}Z_t, Y_1) \\
&= \{i\mathrm{sgnD} + K(X)\}(i\mathrm{D})^k Y_1 + F_{1k}(X, X_t, Z, Z_t, \ldots, \partial_y^{k-1}Z, \partial_y^{k-1}Z_t, Y_1),\ k = 1, 3.
\end{aligned}
$$

Therefore putting (3.2.12) into (3.2.11) leads to

$$
\begin{aligned}
Y_{1tt} &= Q^{-2}(1 + Z_1 + Z_2 i\mathrm{sgnD})\{P_1 - (-g-\mu\mathrm{D}^2+P_2)i\mathrm{sgnD}\}\, i\mathrm{D}Y_1 \\
&\quad + P_4\{P_1 - (-g-\mu\mathrm{D}^2+P_2)i\mathrm{sgnD}\}\, i\mathrm{D}Y_1 + (1+Z_1+Z_2K)^{-1}I_2 \\
&\quad + (1+Z_1+Z_2K)^{-1}\{(-g-\mu\mathrm{D}^2+P_2)(1+\mu\mathrm{D}^2)^{-1}H_{ty} - Z_2H_{ttt}\},
\end{aligned}
$$

where

$$\begin{aligned}P_4 &= P_4(X, Z)\\ &= -Q^{-2}Z_2(i\mathrm{sgn}\mathrm{D} + K(X)) + Q^{-2}Z_2\{[K, Z_1] + [K, Z_2]K + Z_2(1 + K^2)\}\\ &\quad \times (1 + Z_1 + Z_2K)^{-1},\\ I_2 &= I_2(X,\ X_t,\ Y,\ Y_t,\ Z,\ Z_t,\ Z_y,\ Z_{ty},\ Z_{yy},\ Z_{tyy})\\ &= \mu I_1(Z,\ Z_t,\ Z_y,\ Z_{ty},\ Z_{yy},\ Z_{tyy}) + \{-g - \mu\mathrm{D}^2 + P_2(Y,\ Z,\ Z_y,\ Z_{yy})\}\\ &\quad \times (1 + \mu\mathrm{D}^2)^{-1}(F_{110} - \mu F_{130}) - 2Z_t \cdot Y_t - Z_2F_{30}(X,\ X_t,\ Y,\ Y_t).\end{aligned}$$

Using the identity

$$\begin{aligned}&(1 + Z_1 + Z_2i\mathrm{sgn}\mathrm{D})\{P_1 - (-g - \mu\mathrm{D}^2 + P_2)i\mathrm{sgn}\mathrm{D}\}\\ &= (1 + Z_1)P_1 + Z_2(-g - \mu\mathrm{D}^2 + P_2) + \{Z_2P_1 - (1 + Z_1)(-g - \mu\mathrm{D}^2 + P_2)\}\, i\mathrm{sgn}\mathrm{D}\\ &\quad + iZ_2\left\{[\mathrm{sgn}\mathrm{D}, P_1] - [\mathrm{sgn}\mathrm{D}, P_2]i\mathrm{sgn}\mathrm{D}\right\},\end{aligned}$$

we get the equation for Y_1:

$$Y_{1tt} + (M + L)Y_1 = f_1,$$

where

$$\begin{cases} M = M(W, \mu) = \mu\{Q(Z)^{-3}|\mathrm{D}|^3 - i(Q(Z)^{-3})_y\mathrm{D}|\mathrm{D}| + A_1\mathrm{D}^2\},\\ L = L(W, \mu) = i\mu A_2\mathrm{D} + (\mu A_3 + A_4)|\mathrm{D}|,\\ Q = Q(Z) = \{(1 + Z_1)^2 + Z_2^2\}^{1/2},\\ A_1 = A_1(Z, Z_y) = -3Q(Z)^{-5}\{-Z_2Z_{1y} + (1 + Z_1)Z_{2y}\},\\ A_2 = A_2(Z, Z_y, Z_{yy}) = Q(Z)^{-2}\{4R(Z, Z_y) + 3S(Z, Z_{yy})\},\\ A_3 = A_3(Z, Z_y, Z_{yy}) = 3Q(Z)^{-7}\{(-Z_2Z_{1y} + (1 + Z_1)Z_{2y})^2 - ((1 + Z_1)Z_{1y} + Z_2Z_{2y})^2\}\\ \qquad\qquad\qquad\qquad + Q(Z)^{-5}\{(1 + Z_1)Z_{1yy} + Z_2Z_{2yy}\},\\ A_4 = A_4(Y, Z) = Q(Z)^{-2}\{(1 + Z_1)(g + Y_2) - Z_2Y_1\},\\ f_1 = iZ_2Q(Z)^{-2}\{[\mathrm{sgn}\mathrm{D}, P_1] - [\mathrm{sgn}\mathrm{D}, P_2]i\mathrm{sgn}\mathrm{D}\}i\mathrm{D}Y_1\\ \qquad + P_4\{P_1 - (-g - \mu\mathrm{D}^2 + P_2)i\mathrm{sgn}\mathrm{D}\}i\mathrm{D}Y_1 + (1 + Z_1 + Z_2K)^{-1}I_2\\ \qquad + (1 + Z_1 + Z_2K)^{-1}\{(-g - \mu\mathrm{D}^2 + P_2)(1 + \mu\mathrm{D}^2)^{-1}H_{ty} - Z_2H_{ttt}\} \end{cases}$$

with Y_{2t}, $\partial_y^k Z_t$ in I_2 are replaced by f_2, $\partial_y^k(f_3, f_4)$, respectively.

Thus we obtain the quasi-linear equations for W of the form

$$\begin{cases} X_{tt} = Y, \quad Y_{1tt} + (M + L)Y_1 = f_1(W, W_t', H, \mu),\\ Y_{2t} = f_2(W, W_t', H, \mu), \quad Z_{1t} = f_3(W, W_t', H, \mu), \quad Z_{2t} = f_4(W, W_t', H, \mu). \end{cases} \tag{3.2.13}$$

Here the new notations are introduced:

$$\begin{cases} \|W, W_t'\|_{s,\mu} \\ \quad = (\|X\|^2_{H^s(\mathbf{R}^1)} + \|X_t\|^2_{H^s(\mathbf{R}^1)})^{1/2} + (\mu\|X_{1t}\|^2_{H^{s+3/2}(\mathbf{R}^1)} + g_0^2\mu^{-1}\|X_{1t}\|^2_{H^s(\mathbf{R}^1)})^{1/2} \\ \qquad + (\|Y_{1t}\|^2_{H^s(\mathbf{R}^1)} + \mu\|Y_1\|^2_{H^{s+3/2}(\mathbf{R}^1)} + \|Y_1\|^2_{H^s(\mathbf{R}^1)})^{1/2} + \|Y_2\|_{H^s(\mathbf{R}^1)} \\ \qquad + \|(1+\mu \mathrm{D}^2)Z\|_{H^s(\mathbf{R}^1)}, \\ [H(t)]_{s,\mu} = \|H_{tt}(t)\|_{H^s(\mathbf{R}^1)} + \|(1+\mu\mathrm{D}^2)H(t)\|_{H^{s+1}(\mathbf{R}^1)}, \\ |H(t)|_{s,\mu} = [H(t)]_{s,\mu} + \|H_t(t)\|_{H^{s+1}(\mathbf{R}^1)} + \|H_{ttt}(t)\|_{H^s(\mathbf{R}^1)}, \\ W'' = (Y_2, Z), \\ \Lambda = 1 + |\mathrm{D}|, \\ U(W, W_t') = (\Lambda^{3/2}X_1,\ X_2,\ \Lambda^{3/2}Y_1,\ Y_2,\ \Lambda^2 Z,\ \Lambda^{3/2}X_{1t},\ X_{2t},\ Y_{1t}), \\ V(W_{tt}', W_t'') = (\Lambda^{3/2}X_{1tt},\ X_{2tt},\ \Lambda^{-3/2}Y_{1tt},\ Y_{2t},\ \Lambda^2 Z_t). \end{cases}$$

Lemma 3.2.1. *Let $\mu > 0,\ s \geq 1,\ d_0 > 0$. There exists a positive constant c such that if $W = (0, Y, Z)$ satisfies*

$$Y \in H^s(\mathbf{R}^1),\ Z \in H^{s+2}(\mathbf{R}^1),\ \|Z_1\|_{H^1(\mathbf{R}^1)} \leq c,\ \|Y\|_{H^s(\mathbf{R}^1)} + \|Z\|_{H^s(\mathbf{R}^1)} \leq d_0, \tag{3.2.14}$$

then it holds that

$$\|(M+L)u\|_{H^s(\mathbf{R}^1)} \leq C_1(1 + \|(1+\mu\mathrm{D}^2)Z\|_{H^s(\mathbf{R}^1)})^2\|(1+\mu\mathrm{D}^2)\mathrm{D}u\|_{H^s(\mathbf{R}^1)},$$

where $C_1 = C_1(c, d_0, s, g) > 0$. Moreover if $W^0 = (0, Y^0, Z^0)$ satisfies (3.2.14), *then*

$$\begin{aligned} &\|(M - M^0 + L - L^0)u\|_{H^s(\mathbf{R}^1)} \\ &\quad \leq C_1\left\{\|Y - Y^0\|_{H^s(\mathbf{R}^1)} + (1 + \|(1+\mu\mathrm{D}^2)Z\|_{H^s(\mathbf{R}^1)} + \|(1+\mu\mathrm{D}^2)Z^0\|_{H^s(\mathbf{R}^1)})^2 \right. \\ &\qquad \left. \times\|(1+\mu\mathrm{D}^2)(Z - Z^0)\|_{H^s(\mathbf{R}^1)}\right\} \|(1+\mu\mathrm{D}^2)\mathrm{D}u\|_{H^s(\mathbf{R}^1)} \end{aligned}$$

with $M^0 = M(W^0, \mu),\ L^0 = L(W^0, \mu)$.

Proof. Since Sobolev's embedding theorem leads to

$$|Z_1| \leq C\|Z_1\|_{H^1(\mathbf{R}^1)},$$

where C is a universal constant, the condition $(3.2.14)_3$ shows that the inverse operators in M and L exist. Then the above estimates follow easily. ∎

Lemma 3.2.2. *Let $\mu > 0$, $0 < T < \infty$, $s \geq 4$, $d_0 > 0$, $d_1 > 0$. There exists a positive constant $c = c(g)$ such that if $b \in H^{s+3}(\mathbf{R}^1)$ satisfies $\|b\|_{H^3(\mathbf{R}^1)} \leq c$, $\|b\|_{H^{s+3}(\mathbf{R}^1)} \leq d_0$ and*

$$\begin{cases} W,\ W_t',\ \Lambda^{3/2}Y_1,\ \Lambda^2 Z,\ \Lambda^{3/2}X_{1t} \in C^0([0,T];H^s(\mathbf{R}^1)), \\ H \in C^j([0,T];H^{s+9/2-3j/2}(\mathbf{R}^1)), \quad j=1,3, \\ \|X\|_{H^3(\mathbf{R}^1)} + \|Y\|_{H^1(\mathbf{R}^1)} + \|(1+\mu \mathrm{D}^2)Z\|_{H^1(\mathbf{R}^1)} + \|Z\|_{H^3(\mathbf{R}^1)} \leq c, \\ \|(W,W_t')\|_{H^s(\mathbf{R}^1)} \leq d_0, \\ |H|_{s,\mu} \leq d_1 \qquad \text{for } 0 \leq t \leq T, \end{cases} \tag{3.2.15}$$

then f_1, f_2, $(1+\mu\mathrm{D}^2)f_3$, $(1+\mu\mathrm{D}^2)f_4 \in C^0([0,T];H^s(\mathbf{R}^1))$ and

$$\begin{cases} \|(f_1,f_2)\|_{H^s(\mathbf{R}^1)} + \|(1+\mu\mathrm{D}^2)(f_3,f_4)\|_{H^s(\mathbf{R}^1)} \\ \qquad \leq C_2(1+\mu)(1+\|W,W_t'\|_{s,\mu})^{4s+7}(\|W,W_t'\|_{s,\mu} + |H|_{s,\mu}), \\ \|f_2\|_{H^s(\mathbf{R}^1)} + \|(1+\mu\mathrm{D}^2)(f_3,f_4)\|_{H^s(\mathbf{R}^1)} \\ \qquad \leq C_3(1+\mu^{1/2})(1+\|W,W_t'\|_{s,\mu})^{2s+2}(\|W,W_t'\|_{s,\mu} + [H]_{s,\mu}), \end{cases}$$

where $C_2 = C_2(c,d_0,s,g,d_1) > 0$, $C_3 = C_3(c,d_0,s,g) > 0$ and $f_j = f_j(W,W_t',H,\mu)$, $j=1,\dots,4$. Moreover for $W^0, W_t^{0'}, H^0$ satisfying (3.2.15)*, it holds that*

$$\begin{aligned} &\|f_1 - f_1^0\|_{H^s(\mathbf{R}^1)} + \|f_2 - f_2^0\|_{H^s(\mathbf{R}^1)} + \|(1+\mu\mathrm{D}^2)(f_3 - f_3^0)\|_{H^s(\mathbf{R}^1)} \\ &+\|(1+\mu\mathrm{D}^2)(f_4 - f_4^0)\|_{H^s(\mathbf{R}^1)} \\ &\qquad \leq C_2(1+\mu)(1+\|W,W_t'\|_{s,\mu} + \|W^0,W_t^{0'}\|_{s,\mu})^{6s+10} \\ &\qquad\qquad \times (\|W-W^0, W_t' - W_t^{0'}\|_{s,\mu} + |H-H^0|_{s,\mu}). \end{aligned}$$

Proof. If condition $(3.2.15)_3$ is satisfied for a sufficiently small c, the operator $(1+P_3)^{-1}$ in f_3 exists. Using Lemma 2.4.1, we obtain the estimates. ∎

In order to solve the initial value problem (3.2.13), we need the unique existence theorem for the problem

$$\begin{cases} u_{tt} + (M+L)u = f & \text{for } 0 \leq t \leq T, \\ u = u_0, \quad u_t = u_1 & \text{at } t = 0, \end{cases} \tag{3.2.16}$$

which is obtained in [47, Theorem 4.35].

Theorem 3.2.1. *Let $s \geq 3$ and $d_0 > 0$. There exists a positive constant c such that if Y, Z satisfy*

$$\begin{cases} Y \in C^j([0,T];H^{2-j}(\mathbf{R}^1)), \quad Z \in C^j([0,T];H^{4-2j}(\mathbf{R}^1)), \quad j=0,1, \\ Y(t) \in H^s(\mathbf{R}^1), \quad Z(t) \in H^{s+2}(\mathbf{R}^1), \\ \|Y(t)\|_{H^2(\mathbf{R}^1)} + \|Z(t)\|_{H^3(\mathbf{R}^1)} \leq c, \quad \|Y(t)\|_{H^s(\mathbf{R}^1)} + \|Z(t)\|_{H^s(\mathbf{R}^1)} \leq d_0 \end{cases}$$

and $u_0 \in H^{s+3/2}(\mathbf{R}^1)$, $u_1 \in H^s(\mathbf{R}^1)$, $f \in C^0([0,T];H^0(\mathbf{R}^1))$, $f(t) \in H^s(\mathbf{R}^1)$, *then the initial value problem* (3.2.16) *has the unique solution*

$$u \in C^j([0,T];H^{s+3/2-3j/2}(\mathbf{R}^1)) \cap C^2([0,T];H^0(\mathbf{R}^1)), \quad j=0,1,$$

such that

$$|u(t)|_{s,\mu} \leq C_4 e^{C_5 t}|u(0)|_{s,\mu} + C_4\int_0^t e^{C_5(t-\tau)}\|f(\tau)\|_{H^s(\mathbf{R}^1)}d\tau,$$

where $C_4 = C_4(c,s,g) > 2$, $C_5 = C_5(c,s,g,\mu,d_0,T) > 0$,

$$|u(t)|_{s,\mu} = \|u_t(t)\|_{H^s(\mathbf{R}^1)} + \mu\|u(t)\|_{H^{s+3/2}(\mathbf{R}^1)} + \|u(t)\|_{H^{s+1/2}(\mathbf{R}^1)}.$$

In addition, if Y, $f \in C^0([0,T];H^s(\mathbf{R}^1))$, $Z \in C^0([0,T];H^{s+2}(\mathbf{R}^1))$, *then we have* $u \in C^2([0,T];H^{s-3/2}(\mathbf{R}^1))$.

Now we consider the initial value problem (3.2.13) with

$$W(0) = \widetilde{W} = (\widetilde{X},\widetilde{Y},\widetilde{Z}), \quad W_t'(0) = \widetilde{W_t'} = (\widetilde{X_t},\widetilde{Y_{1t}}). \tag{3.2.17}$$

By Theorem 3.2.1 together with Lemmas 3.2.1, 3.2.2, we obtain

Theorem 3.2.2. *Let* $s \geq 5+1/2$, $0 < T_1 < \infty$. *There exists a positive constant* $c = c(g)$ *such that if* $H \in C^j([0,T_1];H^{s+9/2-3j/2}(\mathbf{R}^1))$, $j=1,3$, $b \in H^{s+3}(\mathbf{R}^1)$, $\|b\|_{H^3(\mathbf{R}^1)} \leq c$,

$$U(\widetilde{W},\widetilde{W_t'}) \in H^s(\mathbf{R}^1), \quad \|\widetilde{X}\|_{H^3(\mathbf{R}^1)} + \|\widetilde{Y}\|_{H^2(\mathbf{R}^1)} + \|(1+\mu D^2)\widetilde{Z}\|_{H^1(\mathbf{R}^1)} + \|\widetilde{Z}\|_{H^3(\mathbf{R}^1)} \leq c,$$

then for some $T \in (0,T_1]$ *problem* (3.2.13), (3.2.17) *has a unique solution* W *satisfying*

$$\begin{cases} U(W,W_t'),\ V(W_{tt}',W_t'') \in C^0([0,T];H^s(\mathbf{R}^1)), \\ \|X(t)\|_{H^3(\mathbf{R}^1)} + \|Y(t)\|_{H^2(\mathbf{R}^1)} + \|(1+\mu D^2)Z(t)\|_{H^1(\mathbf{R}^1)} + \|Z(t)\|_{H^3(\mathbf{R}^1)} \leq c, \\ \qquad\qquad 0 \leq t \leq T. \end{cases}$$

Let us specify the initial data $\widetilde{W},\widetilde{W_t'}$ as the value of W, W_t' at $t=0$ in order that the solution of problem (3.2.13), (3.2.17) becomes the solution of (3.2.3) – (3.2.5):

$$\begin{cases} \widetilde{X} = (0,\eta_0),\ \widetilde{Z} = \widetilde{X}_y,\ \widetilde{X_{1t}} = u_{01}(\cdot,0),\ \widetilde{X_{2t}} = K(\widetilde{X})\widetilde{X_{1t}} + H(0), \\ \widetilde{Y_1} = (1+\widetilde{Z_1}+\widetilde{Z_2}K(\widetilde{X}))^{-1} \\ \qquad \times\left\{\mu R(\widetilde{Z},\widetilde{Z}_y) + \mu S(\widetilde{Z},\widetilde{Z}_{yy}) - \widetilde{Z_2}(g + F_{10}(\widetilde{X},\widetilde{X_t}) + H_t(0))\right\}, \\ \widetilde{Y_2} = K(\widetilde{X})\widetilde{Y_1} + F_{10}(\widetilde{X},\widetilde{X_t}) + H_t(0), \\ \widetilde{Y_{1t}} = (1+\widetilde{Z_1}+\widetilde{Z_2}K(\widetilde{X}))^{-1} \\ \qquad \times\left\{-\widetilde{Z_2}(F_{20}(\widetilde{X},\widetilde{X_t},\widetilde{Y}) + H_{tt}(0)) + P_1(\widetilde{Y},\widetilde{Z},\widetilde{Z}_y,\widetilde{Z}_{yy})\widetilde{X_{1ty}}\right. \\ \qquad\quad \left. +(-g-\mu D^2 + P_2(\widetilde{Y},\widetilde{Z},\widetilde{Z}_y,\widetilde{Z}_{yy}))\widetilde{X_{2ty}}\right\}. \end{cases} \tag{3.2.18}$$

Then Theorem 3.2.2 and (3.2.18) lead us to

Theorem 3.2.3. *Let* $s \geq 5+1/2,\ 0 < T_1 < \infty$. *There exists a positive constant* $\varepsilon_1 = \varepsilon_1(g)$ *such that if* $b \in H^{s+9/2}(\mathbf{R}^1),\ \|b\|_{H^3(\mathbf{R}^1)} \leq c$ *and* $\eta_0,\ u_{01}(\cdot,0),\ H$ *satisfy the conditions*

$$\begin{cases} \eta_0 \in H^{s+9/2}(\mathbf{R}^1), \quad u_{01}(\cdot,0) \in H^{s+3}(\mathbf{R}^1), \\ \|\eta_0\|_{H^4(\mathbf{R}^1)} + \|(1+\mu \mathrm{D}^2)\eta_0\|_{H^3(\mathbf{R}^1)} + \|u_{01}(\cdot,0)\|_{H^3(\mathbf{R}^1)} \leq \varepsilon_1/2, \end{cases}$$

$$\begin{cases} H \in C^j([0,T_1]; H^{s+9/2-3j/2}(\mathbf{R}^1)), \quad j=1,3, \\ \|H(0)\|_{H^3(\mathbf{R}^1)} + \|H_t(0)\|_{H^2(\mathbf{R}^1)} \leq \varepsilon_1/2, \end{cases} \tag{3.2.19}$$

then for some $T \in (0,T_1]$ *problem* (3.2.3) – (3.2.5) *has a unique solution* $\bar{X}$ *satisfying conditions*

$$\begin{cases} \bar{X} \in C^j([0,T]; H^{s+9/2-3j/2}(\mathbf{R}^1)), \quad j=0,1,2,3, \\ \|\bar{X}(t)\|_{H^3(\mathbf{R}^1)} \leq c, \quad 0 \leq t \leq T. \end{cases} \tag{3.2.20}$$

We need an additional estimate for the solution $\bar{X}$ of problem (3.2.3) – (3.2.5).

Proposition 3.2.1. *Suppose that* H^0 *satisfies the conditions in* (3.2.19) *and* $\bar{X}^0$ *is the solution of* (3.2.3), (3.2.4) *with* H *replaced by* H^0 *and* (3.2.5). *Then we have*

$$\begin{aligned} &\sum_{j=1}^{3} \|\partial_t^j(\bar{X}(t) - \bar{X}^0(t))\|_{H^{s+3-3j/2}(\mathbf{R}^1)} \\ &\quad \leq C_6\left([H(0)-H^0(0)]_{s-3/2,\mu} + [H(t)-H^0(t)]_{s-3/2,\mu} + \int_0^t |H(\tau)-H^0(\tau)|_{s-3/2,\mu}\mathrm{d}\tau\right) \end{aligned}$$

for $0 \leq t \leq T$, *where* $C_6 = C_6(c, d_0, s, g, d_1, \mu, T) > 0$.

3.3. Proof of Theorem 3.1

We apply the successive approximation to problem (3.2.3) – (3.2.5), problem

$$\begin{cases} \nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 & \text{in } \Omega,\ t \geq 0, \\ \nabla_{\mathbf{u}}^{\perp} \cdot \mathbf{u} = \omega_0 & \text{in } \Omega,\ t \geq 0, \\ u_1 = \bar{X}_{1t} & \text{on } \Gamma_s,\ t \geq 0, \\ \mathbf{u}\cdot\mathbf{n}(\Phi_{\mathbf{u}}) = 0 & \text{on } \Gamma_b,\ t \geq 0 \end{cases} \tag{3.3.1}$$

and problem $(3.2.1)_2$. We remark that if $X|_{t=0} = (0,\widetilde{\eta_0}),\ \partial_t X|_{t=0} = \mathbf{u}_0$ and (3.1.1) are satisfied, the assumptions in Theorem 3.2.3 are also satisfied.

Let us first define X^0 as

$$X^0(t,y) = (0,\widetilde{\eta_0}(y)) + t\mathbf{u}_0(\Psi(y)) \quad \text{for } y \in \Sigma,\ {}^{\forall}t \geq 0.$$

Putting X^0 into H, we see that (3.2.19) holds. Hence Theorem 3.2.3 guarantees that there exists the unique solution $\bar{X}^1$ on some time interval $[0,T]$ for problem (3.2.3) – (3.2.5) which satisfies (3.2.20). Next, for a given $\bar{X}^1$, we find $\mathbf{u}^1$ on some time interval

$0 \le t \le T_1$, as the solution of (3.3.1) under condition (3.1.1). We denote $T_1(\le T)$ by T again. Then X^1 is defined through $(3.2.1)_2$. Clearly $X^1|_{t=0} = (0, \widetilde{\eta_0})$, $\partial_t X^1|_{t=0} = \mathbf{u}_0$.

Take T, in the form similar as (2.7.3), sufficiently small. Then repeating the above procedure difines the approximate solutions $\{\bar{X}^n, \mathbf{u}^n, X^n\}$, $n = 1, 2, 3, \dots$, on $[0, T]$. Moreover it follows from Section 2.4 that

$$\begin{cases} [H(X^n) - H(X^{n-1})]_{s-3/2} \le C \sum_{j=0}^{2} |||\partial_t^j X^n(t) - \partial_t^j X^{n-1}(t)|||_{s+3/2-3j/2}, \\ |H(X^n) - H(X^{n-1})|_{s-3/2} \le C(|||X^n(t) - X^{n-1}(t)|||_{s+3/2} \\ \qquad\qquad + \sum_{j=1}^{3} |||\partial_t^j X^n(t) - \partial_t^j X^{n-1}(t)|||_{s+3-3j/2}) \end{cases} \tag{3.3.2}$$

for $0 \le t \le T$, where

$$|||X|||_s = \|X\|_{H^{s+1/2}(\Sigma)} + \|X(\cdot, 0)\|_{H^s(\mathbf{R}^1)} + \|X(\cdot, -h)\|_{H^s(\mathbf{R}^1)}.$$

Here and in what follows in this section, C means the positive constant independent of n and t. For the solution $\mathbf{u}^n$ of problem (3.3.1) with $\bar{X}_{1t}$ replaced by $\bar{X}_{1t}^n$, it holds that

$$\begin{aligned} &\sum_{j=0}^{2} \sup_{0 \le \tau \le t} |\partial_\tau^j \mathbf{u}^{n+1}(\tau) - \partial_\tau^j \mathbf{u}^n(\tau)|_{s+3/2-3j/2,\Omega} \\ &\qquad \le C \sum_{j=0}^{2} \sup_{0 \le \tau \le t} \|\partial_\tau^{j+1} \bar{X}_1^{n+1}(\tau) - \partial_\tau^{j+1} \bar{X}_1^n(\tau)\|_{H^{s+3/2-3j/2}(\mathbf{R}^1)}, \quad 0 \le t \le T \end{aligned} \tag{3.3.3}$$

with

$$|\mathbf{u}|_{s,\Omega} = \|\mathbf{u}\|_{H^{s+1/2}(\Omega)} + \|\mathbf{u}|_{\Gamma_s}\|_{H^s(\mathbf{R}^1)} + \|\mathbf{u}|_{\Gamma_b}\|_{H^s(\mathbf{R}^1)}.$$

By $(3.2.1)_2$ we have

$$\begin{cases} |||\partial_t^j X^{n+1}(t) - \partial_t^j X^n(t)|||_{s+3/2-3j/2} \le C|\partial_t^j \mathbf{u}^{n+1}(t) - \partial_t^j \mathbf{u}^n(t)|_{s+3/2-3j/2,\Omega}, \quad j = 0, 1, 2, \\ |||X^{n+1}(t) - X^n(t)|||_{s+3/2} \le C \int_0^t |\mathbf{u}^{n+1}(\tau) - \mathbf{u}^n(\tau)|_{s+3/2,\Omega} d\tau, \quad 0 \le t \le T. \end{cases} \tag{3.3.4}$$

Then Proposition 3.2.1 and (3.3.2) – (3.3.4) imply that there exists a limit function $\{\bar{X}, \mathbf{u}, X\}$ on $[0, T]$ of $\{\bar{X}^n, \mathbf{u}^n, X^n\}$ in the sense that for any $t \in [0, T]$ and $s \ge 5 + 1/2$,

$$\begin{aligned} \sup_{0 \le \tau \le t} \Big\{ &\|\bar{X}^n(\tau) - \bar{X}(\tau)\|_{H^{s+3/2}(\mathbf{R}^1)} + \sum_{j=0}^{2} \|\partial_\tau^{j+1}(\bar{X}^n(\tau) - \bar{X}(\tau))\|_{H^{s+3/2-3j/2}(\mathbf{R}^1)} \\ &+ |||X^n(\tau) - X(\tau)|||_{s+3/2} + \sum_{j=0}^{2} |||\partial_\tau^{j+1}(X^n(\tau) - X(\tau))|||_{s+3/2-3j/2} \\ &+ \sum_{j=0}^{2} |\partial_\tau^j(\mathbf{u}^n(\tau) - \mathbf{u}(\tau))|_{s+3/2-3j/2,\Omega} \Big\} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Obviously $\bar{X}, \mathbf{u}$ and X are the solutions of problem (3.2.3) – (3.2.5), problem (3.3.1) and problem $(3.2.1)_2$, respectively, and satisfy

$$\bar{X} \in C^j([0,T]; H^{s+9/2-3j/2}(\mathbf{R}^1)), \quad j = 1,2,3, \tag{3.3.5}$$

$$\begin{cases} \mathbf{u} \in C^j([0,T]; H^{s+7/2-3j/2}(\Omega)), \\ \mathbf{u}|_{\Gamma_s \cup \Gamma_b} \in C^j([0,T]; H^{s+7/2-3j/2}(\Gamma_s \cup \Gamma_b)), \quad j = 0,1,2, \end{cases} \tag{3.3.6}$$

$$\begin{cases} X \in C^j([0,T]; H^{s+5-3j/2}(\Sigma)), \\ X|_{\Gamma_s \cup \Gamma_b} \in C^j([0,T]; H^{s+9/2-3j/2}(\mathbf{R}^1)), \quad j = 1,2,3 \end{cases} \tag{3.3.7}$$

with s replaced by $s - 3/2$. Then applying the arguments in [47, Theorem 5.7] leads to (3.3.5) – (3.3.7). Moreover we see that (3.2.2) holds.

The uniqueness of the solutions to problems (3.2.3) – (3.2.5), (3.3.1), $(3.2.1)_2$ follows in the same way as above.

Now define q as a solution of boundary value problem

$$\begin{cases} \Delta q = -\nabla \cdot (A_{\mathbf{u}}^{-1} \mathbf{u}_t) & \text{in } \Omega, \ t \geq 0, \\ q = g\left(x_2 + \int_0^t u_2(\tau, x) \mathrm{d}\tau\right) - \sigma \mathcal{H}(\Phi_{\mathbf{u}}) & \text{on } \Gamma_s, \ t \geq 0, \\ \dfrac{\partial q}{\partial \mathbf{n}(\Phi_{\mathbf{u}})} = -(\mathbf{u} \cdot \nabla_{\mathbf{u}}) \mathbf{u} \cdot \mathbf{n}(\Phi_{\mathbf{u}}) & \text{on } \Gamma_b, \ t \geq 0. \end{cases} \tag{3.3.8}$$

Then it is easy to show the unique existence of the solution q of problem (3.3.8) satisfying

$$q \in C^j([0,T]; H^{s+7/2-3j/2}(\Omega)), \quad j = 0,1$$

for sufficiently small T of the form similar as above. Furthermore, the uniqueness of solution to problem (8) – (12) is obtained by the uniqueness of the solutions to problems (3.2.3) – (3.2.5), (3.3.1), $(3.2.1)_2$. The proof is complete.

3.4. Proof of Theorem 3.2

Throughout this section we assume that $g > 0$ is a fixed constant. Then as in Section 3.2 we can get the following lemmas.

Lemma 3.4.1. *There exists a positive constant c such that if the assumptions in* Lemma 3.2.1 *are satisfied, then we have*

$$\begin{aligned} &\|\{M(W,\varepsilon) - M(W,\delta) + L(W,\varepsilon) - L(W,\delta)\}u\|_{H^s(\mathbf{R}^1)} \\ &\quad \leq C_7(\delta - \varepsilon)\beta^{-1}(1 + \|(1 + \beta \mathrm{D}^2)Z\|_{H^s(\mathbf{R}^1)})^2 \|(1 + \beta \mathrm{D}^2)\mathrm{D}u\|_{H^s(\mathbf{R}^1)}, \end{aligned}$$

where $0 < \varepsilon < \delta$, $\beta = \delta^{(2-s_0)/2}$, $0 < s_0 < 2$ *and* $C_7 = C_7(c, d_0, s, g) > 0$.

Lemma 3.4.2. *There exists a positive constant $c = c(g)$ such that if the assumptions in* Lemma 3.2.2 *are satisfied with s replaced by $s + s_0$, $0 < s_0 < 2$, then for $0 < \varepsilon < \delta < 1$, it holds that*

$$\begin{aligned}
&\|f_1^\varepsilon - f_1^\delta\|_{H^s(\mathbf{R}^1)} + \|f_2^\varepsilon - f_2^\delta\|_{H^s(\mathbf{R}^1)} + \|(1+\varepsilon \mathrm{D}^2)(f_3^\varepsilon - f_3^\delta)\|_{H^s(\mathbf{R}^1)} \\
&+\|(1+\varepsilon \mathrm{D}^2)(f_4^\varepsilon - f_4^\delta)\|_{H^s(\mathbf{R}^1)} \\
&\quad \le C_8(1 + \|W, W_t'\|_{s+s_0,\delta})^{6s+2s_0+9}(\delta-\varepsilon)^{s_0/2}(\|W, W_t'\|_{s+s_0,\delta} + |H|_{s+s_0,\delta}),
\end{aligned}$$

where $f_k^\mu = f_k(W, W_t', H, \mu)$ and $C_8 = C_8(c, d_0, s, s_0, g, d_1) > 1$.

Lemma 3.4.3. *Let $\mu > 0$, $s \ge 3$, $d_0 > 0$. There exists a positive constant c such that if $b \in H^{s+3}(\mathbf{R}^1)$ satisfies $\|b\|_{H^3(\mathbf{R}^1)} \le c$, $\|b\|_{H^{s+3}(\mathbf{R}^1)} \le d_0$ and*

$$\begin{cases}
\eta_0 \in H^{s+9/2}(\mathbf{R}^1), \quad u_{01}(\cdot, 0) \in H^{s+3}(\mathbf{R}^1), \\
\partial_t^j H(0) \in H^{s+3-3j/2}(\mathbf{R}^1), \quad j = 0, 1, 2, \\
\|\eta_0\|_{H^4(\mathbf{R}^1)} \le c, \quad \|\eta_0\|_{H^{s+9/2}(\mathbf{R}^1)} + \|u_{01}(\cdot, 0)\|_{H^{s+3}(\mathbf{R}^1)} \le d_0,
\end{cases}$$

then we have

$$\|\widetilde{Y^\varepsilon} - \widetilde{Y^\delta}\|_{H^{s+3/2}(\mathbf{R}^1)} + \|\widetilde{Y_{1t}^\varepsilon} - \widetilde{Y_{1t}^\delta}\|_{H^s(\mathbf{R}^1)} \le C_9(\delta - \varepsilon)(\|\eta_0\|_{H^{s+9/2}(\mathbf{R}^1)} + \|u_{01}(\cdot, 0)\|_{H^{s+3}(\mathbf{R}^1)}),$$

where $C_9 = C_9(c, d_0, s) > 0$ and $0 < \varepsilon < \delta$.

Theorem 3.4.1. *Suppose that the assumptions in* Theorem 3.2.3 *hold. Then T in* Theorem 3.2.3 *can be taken in such a way that T depends only on $c, d_0, s, \varepsilon_1, g, d_1$, but does not depend on μ. Moreover if $s \ge 5 + 1/2 + s_0$, $0 < s_0 < 2$, then the solution $\bar{X} = \bar{X}^\mu$ of problem* (3.2.3) – (3.2.5) *with $\mu > 0$ converges to the solution of problem* (3.2.3) – (3.2.5) *with $\mu = 0$ as $\mu \to 0$.*

Proof. The first part of the proof is standard.

In order to prove the second part, it is sufficient to prove

$$\sum_{j=0}^{2} \left\| \partial_t^j (\bar{X}^\varepsilon(t) - \bar{X}^\delta(t)) \right\|_{H^r(\mathbf{R}^1)} \le C_{10}(\delta - \varepsilon)^{s_0/2}(\|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)} + \|u_{01}(\cdot, 0)\|_{H^s(\mathbf{R}^1)} + d_1),$$

where $0 \le t \le T$, $0 < \varepsilon < \delta < 1$, $r = s - 3/2 - s_0$ and $C_{10} = C_{10}(c, d_0, s, \varepsilon_1, g, d_1, s_0, T) > 0$. Let W^μ be the solution of the problem (3.2.13), (3.2.17) corresponding to $\bar{X}^\mu$. Then $W = W^\varepsilon - W^\delta$ is a solution of the problem

$$\begin{cases}
\bar{X}_{tt} + \bar{X} = \bar{X} + Y, \\
Y_{1tt} + (M^\varepsilon + L^\varepsilon)Y_1 = f_1^\varepsilon - f_1^\delta - (M^\varepsilon - M^\delta + L^\varepsilon - L^\delta)Y_1^\delta, \\
Y_{2t} = f_2^\varepsilon - f_2^\delta, \quad Z_{1t} = f_3^\varepsilon - f_3^\delta, \quad Z_{2t} = f_4^\varepsilon - f_4^\delta, \\
W(0) = \widetilde{W^\varepsilon} - \widetilde{W^\delta}, \quad W_t'(0) = \widetilde{W_t^{\varepsilon'}} - \widetilde{W_t^{\delta'}},
\end{cases}$$

where $f_j^\mu = f_j^\mu(W^\mu, W_t^{\mu'}, H, \mu)$, $j = 1, \ldots, 4$, $M^\mu = M(W^\mu, \mu)$, $L^\mu = L(W^\mu, \mu)$. Using Lemmas 3.4.1 – 3.4.3, we estimate W as in Theorem 3.2.2. Then the required result is obtained. ∎

Since the estimate of the solution $\mathbf{u}^\sigma$ of the boundary value problem (3.3.1) with $\bar{X}_{1t}$ replaced by $\bar{X}_{1t}^\sigma$ implies that

$$\|\partial_t^j(\mathbf{u}^\varepsilon - \mathbf{u}^\delta)\|_{H^{r+1/2}(\Omega)} + \|\partial_t^j(\mathbf{u}^\varepsilon - \mathbf{u}^\delta)|_{\Gamma_s\cup\Gamma_b}\|_{H^r(\mathbf{R}^1)}$$
$$\leq C\|\partial_t^{j+1}(\bar{X}^\varepsilon - \bar{X}^\delta)\|_{H^r(\mathbf{R}^1)}, \qquad j = 0, 1,$$

Theorem 3.4.1 leads to

$$\begin{cases} \mathbf{u}^\sigma \to \mathbf{u} & \text{in } C^1([0,T]; H^{r+1/2}(\Omega)), \\ \mathbf{u}^\sigma|_{\Gamma_s\cup\Gamma_b} \to \mathbf{u}|_{\Gamma_s\cup\Gamma_b} & \text{in } C^1([0,T]; H^r(\Gamma_s \cup \Gamma_b)). \end{cases}$$

Then similar arguments in Section 3.3 show that

$$q^\sigma \to q \qquad \text{in } C([0,T]; H^{r+1/2}(\Omega)),$$

which completes the proof.

Chapter 4. Problem Far from Equilibrium

We consider the free boundary problem in the same situation as Chapter 2 except almost flatness of the boundaries, that is, in case that the effect of surface tension is negligible. We show the unique existence of the solution, even if the initial surface and the bottom are uneven.

4.1. Main Result

Theorem 4.1. *Let $\sigma = 0$, $g > 0$ and $s \geq 4$. There exists a positive constant $\delta = \delta(g)$ such that if*

$$\begin{cases} \eta_0 \in H^{s+2}(\mathbf{R}^1), \quad b \in H^{s+3}(\mathbf{R}^1), \quad \mathbf{v}_0 \in H^{s+3/2}(\Omega), \\ \inf\{\eta_0(x_1) - (-h + b(x_1))\} > 0, \\ \|\mathbf{v}_0\|_{H^{2+1/2}(\Omega)} + \|\omega_0\|_{H^{2+1/2}(\Omega)} \leq \delta, \end{cases} \tag{4.1.1}$$

where $\omega_0 = \nabla_x^\perp \cdot \mathbf{v}_0, \nabla_x^\perp = (-\partial/\partial x_2, \partial/\partial x_1)$, and $\mathbf{v}_0$ satisfies the compatibility conditions, then problem (8) – (12) has a unique solution $(\mathbf{u}, q)$ on some time interval $[0, T]$ satisfying

$$\begin{cases} \mathbf{u} \in C^j([0,T]; H^{s+3/2-j/2}(\Omega)), \quad j = 0,1,2,3, \\ q \in C^j([0,T]; H^{s+2-j/2}(\Omega)), \quad j = 1,2. \end{cases} \tag{4.1.2}$$

Our approach is as follows: put

$$X(t,x) = \int_0^t \mathbf{u}(\tau, x) \mathrm{d}\tau \tag{4.1.3}$$

and

$$\bar{X}(t,x_1) = X(t,x_1,\eta_0(x_1)), \qquad \check{X}(t,x_1) = X(t,x_1,-h+b(x_1)). \tag{4.1.4}$$

Then by (8), (10), we get

$$\left(1 + \frac{\partial \bar{X}_1}{\partial x_1}\right) \frac{\partial^2 \bar{X}_1}{\partial t^2} + \left(\frac{\mathrm{d}\eta_0}{\mathrm{d}x_1} + \frac{\partial \bar{X}_2}{\partial x_1}\right) \left(g + \frac{\partial^2 \bar{X}_2}{\partial t^2}\right) = 0 \qquad \text{for } t \geq 0. \tag{4.1.5}$$

On the other hand, since in the Lagrangian coordinates vorticity

$$\nabla^{\perp} \cdot \mathbf{v} = \omega$$

can be written as

$$\nabla_{\mathbf{u}}^{\perp} \cdot \mathbf{u} = \omega_0 \qquad \text{in } \ \Omega, \quad t \geq 0, \tag{4.1.6}$$

it follows from (9), (4.1.6) that

$$\bar{X}_{2t} = K\bar{X}_{1t} + H \qquad \text{for } \ t \geq 0 \tag{4.1.7}$$

with an operator $K = K(\bar{X})$ and a function $H = H(X, \check{X}, \omega_0)$. We will give the explicit form of K and H in Section 4.3. Here the operator K has a simpler form than those in Chapters 2 and 3, or in the previous articles for the free boundary problem in case of finite depth ([14], [46], [47]). In Section 4.4, the properties of K and H will be investigated. To verify the existence of the inverse operators in K and H, we apply the method by Verchota [43] and Kenig [20] as in [14], [45]. Wu and Iguchi assumed that the flow is irrotational, but we will see that this method is applicable to the problem for rotational motion. In Section 4.5, assuming that an H is given, we solve the Cauchy problem (4.1.5), (4.1.7) for $\bar{X}$ with the initial conditions determined by (4.1.3), $(4.1.4)_1$. As in Chapter 2, we convert to the quasi-linear system which contains a weakly hyperbolic equation. Moreover we show that the solution of the quasi-linear system satisfies the initial value problem (4.1.5), (4.1.7). In Section 4.6, for a given $\bar{X}$, we find $\mathbf{u}$ by solving the boundary value problem for (9), (4.1.6). Then X and $\check{X}$ are determined through (4.1.3) and $(4.1.4)_2$, respectively. In Section 4.7 by repeating this procedure, the solution $(\bar{X}, \mathbf{u}, X, \check{X})$ is obtained. Moreover we can obtain q by (8).

4.2. Notations

Let j be a nonnegative integer, $0 < T < \infty$ and B a Banach space. We say that $u \in C^j([0,T]; B)$ if u is a j-times continuously differentiable function on $[0,T]$ with values in B. Let D be a domain in $\mathbf{R}^n$. Then by $H^s(D)$, $-\infty < s < \infty$, we denote the Sobolev space. We use the commutator $[A, B] = AB - BA$ for operators A and B. Moreover the adjoint operator of A is denoted by A^*.

For a Lipschitz continuous function φ on $\mathbf{R}^1$, we define the curve Γ by $\Gamma = \{(x_1, \varphi(x_1));$ $x \in \mathbf{R}^1\}$. Then the non-tangential cones $C^{\pm}(P)$, $P = (y_1, \varphi(y_1)) \in \Gamma$, are given by

$$\begin{cases} C^{+}(P) = \{(x_1, x_2) \in \mathbf{R}^2; x_2 - \varphi(x_1) > M|x_1 - y_1|\}, \\ C^{-}(P) = \{(x_1, x_2) \in \mathbf{R}^2; x_2 - \varphi(x_1) < -M|x_1 - y_1|\}, \end{cases} \tag{4.2.1}$$

where $\|\varphi'\|_{L^\infty(\mathbf{R}^1)} < M$. For a function v on $\mathbf{R}^2 \setminus \Gamma$, the non-tangential maximal functions and the non-tangential limits of v are defined by

$$v_*^\pm(P) = \sup_{X \in C^\pm(P)} |v(X)| \quad \text{for } P \in \Gamma,$$

$$v^\pm(P) = \lim_{X \to P, X \in C^\pm(P)} v(X) \quad \text{for } P \in \Gamma,$$

respectively.

We often use some integral operators. The layer potentials $\mathcal{L}_1(\varphi; u)$ and $\mathcal{L}_2(\varphi; u)$ are defined by

$$\begin{cases} \mathcal{L}_1(\varphi; u)(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\varphi(y_1) - x_2 - \varphi'(y_1)(y_1 - x_1)}{(y_1 - x_1)^2 + (\varphi(y_1) - x_2)^2} u(y_1) \mathrm{d}y_1, \\ \mathcal{L}_2(\varphi; u)(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \frac{y_1 - x_1 + \varphi'(y_1)(\varphi(y_1) - x_2)}{(y_1 - x_1)^2 + (\varphi(y_1) - x_2)^2} u(y_1) \mathrm{d}y_1, \quad x \in \mathbf{R}^2 \setminus \Gamma. \end{cases}$$

Further the singular integral operators $L_1(\varphi; u)$ and $L_2(\varphi; u)$ are defined by

$$\begin{cases} L_1(\varphi; u)(x_1) = \dfrac{1}{2\pi} \mathrm{v.p.} \displaystyle\int_{-\infty}^{\infty} \frac{\varphi(y_1) - \varphi(x_1) - \varphi'(y_1)(y_1 - x_1)}{(y_1 - x_1)^2 + (\varphi(y_1) - \varphi(x_1))^2} u(y_1) \mathrm{d}y_1, \\ L_2(\varphi; u)(x_1) = \dfrac{1}{2\pi} \mathrm{v.p.} \displaystyle\int_{-\infty}^{\infty} \frac{y_1 - x_1 + \varphi'(y_1)(\varphi(y_1) - \varphi(x_1))}{(y_1 - x_1)^2 + (\varphi(y_1) - \varphi(x_1))^2} u(y_1) \mathrm{d}y_1, \quad x_1 \in \mathbf{R}^1. \end{cases}$$

We also use the layer potential $\mathcal{M}(\varphi; u) = (\mathcal{M}_1(\varphi; u), \mathcal{M}_2(\varphi; u))$,

$$\mathcal{M}(\varphi; u)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(y_1 - x_1, \varphi(y_1) - x_2)}{(y_1 - x_1)^2 + (\varphi(y_1) - x_2)^2} u(y_1) \mathrm{d}y_1, \quad x \in \mathbf{R}^2 \setminus \Gamma.$$

4.3. Representation of K and H

Throughout this section let the time $t \geq 0$ be arbitrarily fixed. We assume that $\mathbf{v}$ and X are smooth and tend to zero as variables tend to infinity. We regard the plane $\mathbf{R}^2_{z_1, z_2}$ as a complex space of $z = z_1 + iz_2$. Hence $\Gamma_s(t)$ and Γ_b are given by

$$\begin{cases}\Gamma_s(t) : w_s(x_1) = x_1 + \bar{X}_1(x_1) + i(\eta_0(x_1) + \bar{X}_2(x_1)), \\ \Gamma_b \quad : w_b(x_1) = x_1 + i(-h + b(x_1)), \qquad -\infty < x_1 < \infty.\end{cases}$$

We suppose that $\mathbf{v}$ satisfies the equations

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla^{\perp} \cdot \mathbf{v} = \omega \quad \text{in } \Omega(t)$$

and put

$$\begin{cases}F = v_1 - iv_2, \\ f(x_1) = f_1(x_1) + if_2(x_1) = F(w_s(x_1)), \\ g(y_1) = g_1(x_1) + ig_2(x_1) = F(w_b(x_1)).\end{cases}$$

Then Cauchy integral formula implies that

$$\begin{aligned}F(z^0) = &-\frac{1}{2\pi i}\int_{\Gamma_s(t)} \frac{f(y_1)}{w_s(y_1) - z^0}\frac{\mathrm{d}w_s(y_1)}{\mathrm{d}y_1}\mathrm{d}y_1 + \frac{1}{2\pi i}\int_{\Gamma_b} \frac{g(y_1)}{w_b(y_1) - z^0}\frac{\mathrm{d}w_b(y_1)}{\mathrm{d}y_1}\mathrm{d}y_1 \\ &+ i\iint_{\Omega(t)} \omega\frac{\partial E(z - z^0)}{\partial z_1}\mathrm{d}z_1\mathrm{d}z_2 - \iint_{\Omega(t)} \omega\frac{\partial E(z - z^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2,\end{aligned} \tag{4.3.1}$$

where $z^0 \in \Omega(t)$ and

$$E(z) = \frac{1}{2\pi}\log|z|.$$

Therefore if we take z^0 to $w_s^0 = w_s(x_1)$ on $\Gamma_s(t)$ non-tangentially, by the relation

$$\begin{aligned}&\frac{1}{w_s(y_1) - w_s(x_1)}\frac{\mathrm{d}w_s(y_1)}{\mathrm{d}y_1} \\ &= \frac{\partial}{\partial y_1}\log(w_s(y_1) - w_s(x_1)) \\ &= \frac{\partial}{\partial y_1}\left\{\log\{y_1 - x_1 + i(\eta_0(y_1) - \eta_0(x_1))\} + \log\frac{w_s(y_1) - w_s(x_1)}{y_1 - x_1 + i(\eta_0(y_1) - \eta_0(x_1))}\right\}\end{aligned}$$

and the imaginary part of (4.3.1), we get the equation

$$f_2 = \frac{1}{2}f_2 + A_2 f_1 + A_1 f_2 - A_4 f_1 + A_3 f_2 - A_6 g_1 - A_5 g_2 + H_1,$$

where

$$\begin{cases}
A_1 u(x_1) = L_1(\eta_0; u)(x_1), \\
A_2 u(x_1) = L_2(\eta_0; u)(x_1), \\
A_3 u(x_1) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \mathrm{Im}\log\{1+ \\
\quad + \Big\{(y_1 - x_1)(\bar{X}_1(y_1) - \bar{X}_1(x_1)) + (\eta_0(y_1) - \eta_0(x_1))(\bar{X}_2(y_1) - \bar{X}_2(x_1)) \\
\qquad -i\{(\eta_0(y_1) - \eta_0(x_1))(\bar{X}_1(y_1) - \bar{X}_1(x_1)) - (y_1 - x_1)(\bar{X}_2(y_1) - \bar{X}_2(x_1))\}\Big\} \\
\quad \times \Big\{(y_1 - x_1)^2 + (\eta_0(y_1) - \eta_0(x_1))^2\Big\}^{-1}\} u'(y_1)\mathrm{d}y_1, \\
A_4 u(x_1) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \mathrm{Re}\log\{1+ \\
\quad + \Big\{(y_1 - x_1)(\bar{X}_1(y_1) - \bar{X}_1(x_1)) + (\eta_0(y_1) - \eta_0(x_1))(\bar{X}_2(y_1) - \bar{X}_2(x_1)) \\
\qquad -i\{(\eta_0(y_1) - \eta_0(x_1))(\bar{X}_1(y_1) - \bar{X}_1(x_1)) - (y_1 - x_1)(\bar{X}_2(y_1) - \bar{X}_2(x_1))\}\Big\} \\
\quad \times \Big\{(y_1 - x_1)^2 + (\eta_0(y_1) - \eta_0(x_1))^2\Big\}^{-1}\} u'(y_1)\mathrm{d}y_1, \\
A_5 u(x_1) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{-h + b(y_1) - \eta_0(x_1) - \bar{X}_2(x_1) - b'(y_1 - x_1 - \bar{X}_1(x_1))}{(y_1 - x_1 - \bar{X}_1(x_1))^2 + (-h + b(y_1) - \eta_0(x_1) - \bar{X}_2(x_1))^2} u(y_1)\mathrm{d}y_1, \\
A_6 u(x_1) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{y_1 - x_1 - \bar{X}_1(x_1) + b'(-h + b(y_1) - \eta_0(x_1) - \bar{X}_2(x_1))}{(y_1 - x_1 - \bar{X}_1(x_1))^2 + (-h + b(y_1) - \eta_0(x_1) - \bar{X}_2(x_1))^2} u(y_1)\mathrm{d}y_1, \\
H_1 = \displaystyle\iint_{\Omega(t)} \omega(z) \dfrac{\partial E(z - w_s^0)}{\partial z_1} \mathrm{d}z_1 \mathrm{d}z_2.
\end{cases}$$

Since $f_1 = v_1|_{\Gamma_s(t)}$, $f_2 = -v_2|_{\Gamma_s(t)}$, $g_1 = v_1|_{\Gamma_b}$ and $g_2 = -v_2|_{\Gamma_b}$, we see that $\bar{X}_{2t} = K\bar{X}_{1t} + H$ with

$$\begin{aligned}
K &= -\left(\frac{1}{2} - A_1 - A_3\right)^{-1} (A_2 - A_4) \\
&= -\left(\frac{1}{2} - A_1 - A_3\right)^{-1} \left(\frac{1}{2} i\mathrm{sgnD} - A_7 - A_4\right) \\
&= -i\mathrm{sgnD} + 2(-A_7 - A_4) \\
&\quad + 2(-A_1 + A_3)\left(\frac{1}{2} - A_1 + A_3\right)^{-1} \left(\frac{1}{2} i\mathrm{sgnD} + A_7 + A_4\right) \\
&=: -i\mathrm{sgnD} + K_1,
\end{aligned} \tag{4.3.2}$$

$$H = -\left(\frac{1}{2} - A_1 - A_3\right)^{-1}(-A_6\check{X}_{1t} + A_5\check{X}_{2t} + H_1),$$

where

$$\mathrm{D} = -i\partial/\partial x_1, \quad A_7 u(x_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \log\{1 + \left(\frac{\eta_0(y_1) - \eta_0(x_1)}{y_1 - x_1}\right)^2\}^{1/2} u'(y_1)\mathrm{d}y_1.$$

4.4. Estimates for K and H

First we investigate the operators A_j $(j = 1, 3, 4, \ldots, 7)$.

Lemma 4.4.1. *Suppose that* $\inf\{\eta_0(x_1) - (-h + b(x_1))\} > 0$.

(1) *Let* $\eta_0 \in H^s(\mathbf{R}^1)$, $s, s_0 > 3/2$. *It holds that*

$$\|A_j u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^{s_0}(\mathbf{R}^1)}, \quad j = 1, 7, \ C = C(s, s_0, \|\eta_0\|_{H^s(\mathbf{R}^1)}) > 0.$$

(2) *Let* η_0 *be the Lipschitz continuous function and* $\eta_0 \in H^{s+3/2}(\mathbf{R}^1)$, $s \geq 0$. *It holds that*

$$\|A_j u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^0(\mathbf{R}^1)}, \quad j = 1, 7, \ C = C(s, \|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)}, \|\eta_0'\|_{L^\infty(\mathbf{R}^1)}) > 0.$$

(3) *There exists a positive constant* c_0 *such that if* $\eta_0'' \in L^\infty(\mathbf{R}^1)$, $\eta_0, \bar{X}, \bar{X}^0 \in H^s(\mathbf{R}^1)$, $s \geq 2$ *and* $\|\bar{X}\|_{H^2(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^2(\mathbf{R}^1)} \leq c_0$, $\|\bar{X}\|_{H^s(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^s(\mathbf{R}^1)} \leq d$ *for some* $d > 0$, *then it holds that*

$$\begin{cases} \|A_j(\bar{X})u\|_{H^s(\mathbf{R}^1)} \leq C\|\bar{X}\|_{H^s(\mathbf{R}^1)}\|u\|_{H^{s_0}(\mathbf{R}^1)}, \\ \|A_j(\bar{X})u - A_j(\bar{X}^0)u\|_{H^s(\mathbf{R}^1)} \leq C\|\bar{X} - \bar{X}^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^{s_0}(\mathbf{R}^1)}, \ j = 3, 4, \ s_0 > 3/2, \end{cases}$$

where $C = C(s, s_0, c_0, d, \|\eta_0\|_{H^s(\mathbf{R}^1)}, \|\eta_0''\|_{L^\infty(\mathbf{R}^1)}) > 0$.

(4) *Let* $\eta_0, \bar{X}, \bar{X}^0 \in H^s(\mathbf{R}^1)$, $s \geq 0$, *b the Lipschitz continuous function and* $\|\bar{X}\|_{H^s(\mathbf{R}^1)}$, $\|\bar{X}^0\|_{H^s(\mathbf{R}^1)} \leq d$ *for some* $d > 0$. *It holds that*

$$\begin{cases} \|A_j(\bar{X})u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^0(\mathbf{R}^1)}, \\ \|A_j(\bar{X})u - A_j(\bar{X}^0)u\|_{H^s(\mathbf{R}^1)} \leq C\|\bar{X} - \bar{X}^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^0(\mathbf{R}^1)}, \quad j = 5, 6, \end{cases}$$

where $C = C(s, d, \|\eta_0\|_{H^s(\mathbf{R}^1)}, \|b'\|_{L^\infty(\mathbf{R}^1)}) > 0$.

Proof. (1) and (2) follow from [46, Section 4] and [6, Section 9], respectively. The proof for (4) is standard. It remains to show (3). If Re $z > -1$, it holds that $\log(1 + z) = z\mathrm{f}(z)$

where $\mathsf{f}(z)$ is holomorphic in z. Hence, if $\bar{X}$ is small in $H^t(\mathbf{R}^1), t > 3/2$, it is sufficient to investigate the function

$$\int_{-\infty}^{\infty} \frac{(y_1 - x_1)^2}{(y_1 - x_1)^2 + (\eta_0(y_1) - \eta_0(x_1))^2}$$

$$\times \left\{ \frac{\bar{X}_1(y_1) - \bar{X}_1(x_1)}{y_1 - x_1} + \frac{\eta_0(y_1) - \eta_0(x_1)}{y_1 - x_1} \frac{\bar{X}_2(y_1) - \bar{X}_2(x_1)}{y_1 - x_1} \right.$$

$$\left. -i \left(\frac{\eta_0(y_1) - \eta_0(x_1)}{y_1 - x_1} \frac{\bar{X}_1(y_1) - \bar{X}_1(x_1)}{y_1 - x_1} - \frac{\bar{X}_2(y_1) - \bar{X}_2(x_1)}{y_1 - x_1} \right) \right\}$$

$$\times \mathsf{F} \left(\frac{\eta_0(y_1) - \eta_0(x_1)}{y_1 - x_1}, \frac{\bar{X}_1(y_1) - \bar{X}_1(x_1)}{y_1 - x_1}, \frac{\bar{X}_2(y_1) - \bar{X}_2(x_1)}{y_1 - x_1} \right) u(y_1) \mathrm{d}y_1,$$

where F is a smooth function. Then the arguments in [46, Section 4] show (3). ∎

Now we will show the operator $\frac{1}{2} - A_1 - A_3$ is invertible. To see this fact, we use the following proposition.

Proposition 4.4.1. *Suppose that A is a bounded linear operator in $L^2(\mathbf{R}^1)$ and satisfies*

$$\|Au\|_{L^2(\mathbf{R}^1)} \geq C\|u\|_{L^2(\mathbf{R}^1)}, \quad \|A^*u\|_{L^2(\mathbf{R}^1)} \geq C\|u\|_{L^2(\mathbf{R}^1)} \tag{4.4.1}$$

for any $u \in L^2(\mathbf{R}^1)$, where $C > 0$. Then the operator A is invertible in $L^2(\mathbf{R}^1)$.

Proof. By the first estimate of (4.4.1), we see that the operator A is injective. The second estimate implies that the adjoint operator A^* has the closed range. Since the operator A is bounded, A is surjective on $L^2(\mathbf{R}^1)$. ∎

In the same way as [14, Lemma 5.2], we can obtain

Lemma 4.4.2. *Let η_0 be the Lipschitz continuous function and $C^\pm(P), P \in \Gamma_s$, the cones defined by* (4.2.1) *with φ replaced by η_0. Suppose that*

(1) *$\dot{\mathbf{v}} = (\dot{v}_1, \dot{v}_2)$ satisfies $\nabla \cdot \dot{\mathbf{v}} = 0$ and $\nabla^\perp \cdot \dot{\mathbf{v}} = 0$ in $\mathbf{R}^2 \setminus \Gamma_s$,*

(2) *The non-tangential maximal functions $\dot{\mathbf{v}}_*^\pm = \sup_{X \in C^\pm(P)} |\dot{\mathbf{v}}(X)|, P \in \Gamma_s$, belong to $L^2(\mathbf{R}^1)$,*

(3) *The non-tangential limits $\dot{\mathbf{V}}^\pm = (\dot{V}_1^\pm, \dot{V}_2^\pm) = \lim_{X \to P, X \in C^\pm(P)} \dot{\mathbf{v}}(X), P \in \Gamma_s$, exist for almost every P,*

(4) *$\dot{\mathbf{v}}(x) = O(|x|^{-1})$ as $|x| \to \infty$.*

If we denote the normal vector and the tangential vector to Γ_s by $\mathbf{N} = (N_1, N_2)$, $\mathbf{T} = (N_2, -N_1)$, respectively, then the norms $\|\dot{V}_1\|_{L^2(\mathbf{R}^1)}, \|\dot{V}_2\|_{L^2(\mathbf{R}^1)}, \|\mathbf{N} \cdot \dot{\mathbf{V}}\|_{L^2(\mathbf{R}^1)}$ and $\|\mathbf{T} \cdot \dot{\mathbf{V}}\|_{L^2(\mathbf{R}^1)}$ are equivalent, where $\dot{\mathbf{V}} = \dot{\mathbf{V}}^+$ or $\dot{\mathbf{V}}^-$.

Lemma 4.4.3. *Suppose that η_0 is the Lipschitz continuous function. Then the operator $\frac{1}{2} - A_1 : L^2(\mathbf{R}^1) \to L^2(\mathbf{R}^1)$ is invertible. Moreover, it holds that*

$$\|(\frac{1}{2} - A_1)^{-1} u\|_{L^2(\mathbf{R}^1)} \leq C\|u\|_{L^2(\mathbf{R}^1)}$$

with $C = C(\|\eta_0'\|_{L^\infty(\mathbf{R}^1)}) > 0$.

Proof. Let us first consider the layer potentials

$$\dot{v}_1 = \mathcal{L}_1(\eta_0; u), \quad \dot{v}_2 = -\mathcal{L}_2(\eta_0; u)$$

for $u \in L^2(\mathbf{R}^1)$. In the same way as [9, Theorem 1.3], we see that

$$\dot{V}_1^{\pm} = \mp\frac{1}{2}u + A_1 u, \quad \dot{V}_2^{\pm} = -A_2 u.$$

Moreover, $\dot{\mathbf{v}}$ satisfies $\nabla \cdot \dot{\mathbf{v}} = 0$ and $\nabla^{\perp} \cdot \dot{\mathbf{v}} = 0$. Hence it follows from Lemma 4.4.2 that

$$\|(\frac{1}{2} + A_1)u\|_{L^2(\mathbf{R}^1)} \leq C\|(\frac{1}{2} - A_1)u\|_{L^2(\mathbf{R}^1)}.$$

Therefore we see that

$$\|u\|_{L^2(\mathbf{R}^1)} \leq C\|(\frac{1}{2} - A_1)u\|_{L^2(\mathbf{R}^1)}. \tag{4.4.2}$$

Secondly, we consider the layer potentials

$$\ddot{v}_1 = \mathcal{M}_1(\eta_0; u), \quad \ddot{v}_2 = \mathcal{M}_2(\eta_0; u)$$

for $u \in L^2(\mathbf{R}^1)$. Then for the non-tangential limits $\ddot{\mathbf{V}}^{\pm}$ of $\ddot{\mathbf{v}}$ we have

$$\mathbf{N} \cdot \ddot{\mathbf{V}}^{\pm} = N_2(\mp\frac{1}{2}u - A_1^* u), \quad \mathbf{T} \cdot \ddot{\mathbf{V}}^{\pm} = -N_2 A_2^* u,$$

which lead to

$$\|(\frac{1}{2} + A_1^*)u\|_{L^2(\mathbf{R}^1)} \leq C\|(\frac{1}{2} - A_1^*)u\|_{L^2(\mathbf{R}^1)}.$$

Again by Lemma 4.4.2, we see that

$$\|u\|_{L^2(\mathbf{R}^1)} \leq C\|(\frac{1}{2} - A_1^*)u\|_{L^2(\mathbf{R}^1)}. \tag{4.4.3}$$

Therefore the estimates (4.4.2) and (4.4.3) give our assertion. ∎

Lemma 4.4.4.

(1) *Suppose that $\eta_0 \in H^{s+3/2}(\mathbf{R}^1)$, $\bar{X}, \bar{X}^0 \in H^s(\mathbf{R}^1)$, $\|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)} \leq \kappa$ and $s \geq 2$. There exists a positive constant c_0 such that if $\|\bar{X}\|_{H^2(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^2(\mathbf{R}^1)} \leq c_0$, then the operator $\frac{1}{2} - A_1 - A_3 : H^s(\mathbf{R}^1) \to H^s(\mathbf{R}^1)$ is invertible. Moreover it holds that*

$$
\begin{cases}
\|(\frac{1}{2} - A_1 - A_3)^{-1}u\|_{H^s(\mathbf{R}^1)} \le C\|u\|_{H^s(\mathbf{R}^1)}, \\
\|(\frac{1}{2} - A_1 - A_3)^{-1}(\bar{X})u - (\frac{1}{2} - A_1 - A_3)^{-1}(\bar{X}^0)u\|_{H^s(\mathbf{R}^1)} \\
\qquad \le C\|\bar{X} - \bar{X}^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^s(\mathbf{R}^1)},
\end{cases}
\tag{4.4.4}
$$

where $C = C(s, \kappa) > 0$.

(2) *Suppose that* $\eta_0, \bar{X}, \bar{X}^0 \in H^s(\mathbf{R}^1)$, $\|\eta_0\|_{H^s(\mathbf{R}^1)} \le \kappa$ *and* $s \ge 4$. *There exists a positive constant* c_0 *such that if* $\|\bar{X}\|_{H^2(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^2(\mathbf{R}^1)} \le c_0$, *then the operator* $\frac{1}{2} - A_1 - A_3 : H^s(\mathbf{R}^1) \to H^s(\mathbf{R}^1)$ *is invertible. Moreover* (4.4.4) *holds.*

Proof. We only show the proof for (1). At first, we show that the operator $\frac{1}{2} - A_1$ is invertible in $H^s(\mathbf{R}^1), s \ge 0$. Note that for any positive integer m

$$
\begin{aligned}
\partial_{x_1}^m \left(\frac{1}{2} - A_1\right)^{-1} &= \partial_{x_1}^{m-1}\left[\partial_{x_1}, \left(\frac{1}{2} - A_1\right)^{-1}\right] + \partial_{x_1}^{m-1}\left(\frac{1}{2} - A_1\right)^{-1}\partial_{x_1} \\
&= \partial_{x_1}^{m-1}\left(\frac{1}{2} - A_1\right)^{-1}[\partial_{x_1}, A_1]\left(\frac{1}{2} - A_1\right)^{-1} + \partial_{x_1}^{m-1}\left(\frac{1}{2} - A_1\right)^{-1}\partial_{x_1},
\end{aligned}
\tag{4.4.5}
$$

where $\partial_{x_1} = \partial/\partial x_1$. Then Lemma 4.4.3 lead that $\frac{1}{2} - A_1$ is invertible in $H^m(\mathbf{R}^1), m = 1, 2, \ldots, [s]$, inductively. Here $[s]$ is the largest integer no more than s. Hence by interpolation, it holds that

$$
\|(\frac{1}{2} - A_1)^{-1}u\|_{H^t(\mathbf{R}^1)} \le C\|u\|_{H^t(\mathbf{R}^1)}, \quad 0 \le t \le [s],\ C = C(t, \|\eta_0'\|_{L^\infty(\mathbf{R}^1)}, \kappa) > 0.
$$

From (4.4.5) replaced $\partial_{x_1}^m$ by $\partial_{x_1}^{[s]}(1 + |D|)^r$, $0 \le r \le s - [s]$, it follows that $\frac{1}{2} - A_1$ is invertible in $H^s(\mathbf{R}^1)$. Then by the proof for Lemma 2.2.2(4), the above assertions are obtained. ∎

It follows from (4.3.2) and Lemmas 4.4.1, 4.4.4(1) that

Lemma 4.4.5. *There exists a positive constant* c_0 *such that if* $\eta_0 \in H^s(\mathbf{R}^1)$, $\eta_0 \in H^{s_1+3/2}(\mathbf{R}^1)$, $\bar{X}, \bar{X}^0 \in H^s(\mathbf{R}^1)$, $s \ge 2$, $s_0, s_1 > 3/2$ *and* $\|\eta_0\|_{H^s(\mathbf{R}^1)} \le \kappa, \|\eta_0\|_{H^{s_1+3/2}(\mathbf{R}^1)} \le \kappa'$, $\|\bar{X}\|_{H^2(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^2(\mathbf{R}^1)} \le c_0$, $\|\bar{X}\|_{H^s(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^s(\mathbf{R}^1)} \le d$ *for some* $d > 0$, *then it holds that*

$$
\begin{cases}
\|K_1(\bar{X})u\|_{H^s(\mathbf{R}^1)} \le C\|u\|_{H^{s_0}(\mathbf{R}^1)}, \\
\|K_1(\bar{X})u - K_1(\bar{X}^0)u\|_{H^s(\mathbf{R}^1)} \le C\|\bar{X} - \bar{X}^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^{s_0}(\mathbf{R}^1)},
\end{cases}
$$

where $C = C(s, s_0, c_0, d, \kappa, \kappa') > 0$.

Assuming that $\bar{X}$ depends on x_1 and t, we define $K_{1,k,l}(\bar{X},\dots,\partial_t^k\partial_{x_1}^l\bar{X}),\ \partial_t=\partial/\partial t,\ k,l = 0,1,2,\dots,$ by

$$\begin{cases} K_{1,0,0}=K_1, \qquad K_{1,0,l}=\left[\dfrac{\partial}{\partial x_1},K_{1,0,l-1}\right], \quad l=1,2,3,\dots, \\ K_{1,k,l}=\left[\dfrac{\partial}{\partial t},K_{1,k-1,l}\right], \quad k=1,2,3,\dots, \quad l=0,1,2,\dots. \end{cases}$$

Moreover we replace $\partial_t^p\partial_{x_1}^q\bar{X}$ by $\bar{X}^{pq}$. Then we have

Lemma 4.4.6. *There exists a positive constant c_0 such that if $\eta_0\in H^{s+l}(\mathbf{R}^1)$, $\eta_0\in H^{s_1+l+3/2}(\mathbf{R}^1)$, $s\geq 2$, $s_1>3/2$, $\|\bar{X}^{00}\|_{H^2(\mathbf{R}^1)},\|\bar{X}'^{\,00}\|_{H^2(\mathbf{R}^1)}\leq c_0$ and $\|\eta_0\|_{H^{s+l}(\mathbf{R}^1)}\leq\kappa$, $\|\eta_0\|_{H^{s_1+l+3/2}(\mathbf{R}^1)}\leq\kappa'$, $\|(\bar{X}^{00},\dots,\bar{X}^{kl})\|_{H^s(\mathbf{R}^1)},\|(\bar{X}'^{\,00},\dots,\bar{X}'^{\,kl})\|_{H^s(\mathbf{R}^1)}\leq d$ for some $d>0$, then for any $u\in H^{s_0}(\mathbf{R}^1), s_0>3/2$, it holds that*

$$\begin{cases} \|K_{1,k,l}(\bar{X}^{00},\dots,\bar{X}^{kl})u\|_{H^s(\mathbf{R}^1)}\leq C\|u\|_{H^{s_0}(\mathbf{R}^1)}, \\ \|K_{1,k,l}(\bar{X}^{00},\dots,\bar{X}^{kl})u-K_{1,k,l}(\bar{X}'^{\,00},\dots,\bar{X}'^{\,kl})u\|_{H^s(\mathbf{R}^1)} \\ \qquad\qquad \leq C\|(\bar{X}^{00}-\bar{X}'^{\,00},\dots,\bar{X}^{kl}-\bar{X}'^{\,kl})\|_{H^s(\mathbf{R}^1)}\|u\|_{H^{s_0}(\mathbf{R}^1)}, \end{cases}$$

where $C=C(s,s_0,c_0,d,\kappa,\kappa')>0$.

Let us introduce the new norm

$$|||X|||_s\equiv\|X\|_{H^{s+1/2}(\Omega)}+\|X(\cdot,\eta_0(\cdot))\|_{H^s(\mathbf{R}^1)}.$$

Lemma 4.4.7. *Suppose that $\eta_0\in H^s(\mathbf{R}^1)$, $b\in H^s(\mathbf{R}^1)$, $s\geq 2$. There exists a positive constant c_0 such that if*

$$|||X|||_2\leq c_0,\ |||X|||_s\leq d,\ d>0,$$

then we have

$$\begin{cases} \|H_1\|_{H^s(\mathbf{R}^1)}\leq C\|\omega_0\|_{H^{s+1/2}(\Omega)}, \\ \|H_1(X^1)-H_1(X^2)\|_{H^s(\mathbf{R}^1)}\leq C|||X^1-X^2|||_s\|\omega_0\|_{H^{s+1/2}(\Omega)}, \end{cases}$$

where $C=C(s,c_0,d)>0$.

Proof. We introduce the coverings $\{\Omega^{(i)}\}_{i=1}^{n}$ of $\overline{\Omega}$ and associated smooth cut-off functions $\{\zeta_i\}_{i=1}^{n}$ on Ω which has following properties:

$$\begin{aligned}
&\bigcup_i \Omega^{(i)} = \overline{\Omega}, \\
&\Omega^{(1)} = \{x \in \Omega;\ x_1 < -\alpha_1\}, \quad \Omega^{(2)} = \{x \in \Omega;\ x_1 > \alpha_2\}, \\
&\text{diameter}(\Omega^{(i)}) < \infty, \quad i = 3, \dots, n, \\
&(\Omega^{(i)} \cap \Gamma_s) \equiv \Gamma_{si} \neq \phi, \quad i = 3, \dots, m, \\
&(\Omega^{(i)} \cap \Gamma_s) = \phi, \quad i = m+1, \dots, n, \\
&0 \le \zeta^{(i)}(x) \le 1, \quad \text{supp}\zeta^{(i)} = \Omega^{(i)}, \quad \sum_i \zeta^{(i)} \equiv 1
\end{aligned}$$

for $\alpha_1 > 0, \alpha_2 > 0$. Take m, n, α_1, α_2 sufficiently large and diameter$(\Omega^{(i)})$, $i = 3, \dots, m$, sufficiently small. Then it is enough to estimate $H_1(\zeta^{(i)}\omega_0)$ for all i.

(1) Proof for $i = 1, 2$.

Let us define a function $\beta(x_1) \in H^s(\mathbf{R}^1)$ such that

$$\begin{cases} \beta(x_1) = b(x_1) & \text{if } x_1 \le -\alpha_1,\ x_1 \ge \alpha_2, \\ \|\beta\|_2 < \varepsilon, \end{cases}$$

where $\varepsilon > 0$ is a sufficient small constant. Then by the mapping

$$x = y + (0, \widetilde{\eta_0}(y)),$$

$\Omega^{(i)}$ is transformed onto a part of the horizontal strip

$$\Sigma = \{y = (y_1, y_2);\ -h < y_2 < 0,\ y_1 \in \mathbf{R}^1\}.$$

Here $\widetilde{\eta_0}$ is the function

$$\begin{aligned}
&\widetilde{\eta_0}(y_1, y_2) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-\mathrm{e}^{iy_1\xi}(\mathrm{e}^{|\xi|(y_2+2h)} - \mathrm{e}^{|\xi|y_2})\hat{\eta_0}(\xi) + \mathrm{e}^{iy_1\xi}(\mathrm{e}^{|\xi|(y_2+h)} - \mathrm{e}^{|\xi|(h-y_2)})\hat{\beta}(\xi)}{1 - \mathrm{e}^{2|\xi|h}} \mathrm{d}\xi.
\end{aligned}$$

By Lemma 2.6.1, similar arguments as in Section 2.4 show the desired estimates for $H_1(\zeta^{(i)}\omega_0)$.

(2) Proof for $i = 3, \dots, m$.

Notice that

$$\|H_1(\zeta^{(i)}\omega_0)\|_{H^s(\Gamma_s)} \le \|H_1(\zeta^{(i)}\omega_0)\|_{H^s(\Gamma_{si})} + \|H_1(\zeta^{(i)}\omega_0)\|_{H^s(\Gamma_s \setminus \Gamma_{si})}.$$

The first term on the right hand side is estimated as in (1). The estimate for the second term is easily obtained.

(3) Proof for $i = m+1, \dots, n$.

In this case the proof is standard. $\square$

Assumption 4.1. *The functions* η_0 *and* b *satisfy*

$$\eta_0 \in H^{s+2}(\mathbf{R}^1), \quad b \in H^{s+1}(\mathbf{R}^1).$$

There exist $c_0 > 0,\ d > 0,\ l_j > 0\ (j = 1, 2, \ldots, 6),\ \kappa > 0$ *such that for* $s \geq 3,\ 0 < T < \infty$, $X, \check{X}$ *and* η_0 *satisfy*

$$\begin{cases} X \in C^j([0,T]; H^{s+2-j/2}(\Omega)), \qquad j = 1,2,3,4, \\ X|_{\Gamma_s} \in C^j([0,T]; H^{s+3/2-j/2}(\mathbf{R}^1)), \qquad j = 1,2,3,4, \\ \check{X} = X|_{\Gamma_b} \in C^j([0,T]; H^{s+3/2-j/2}(\mathbf{R}^1)), \qquad j = 1,2,3,4, \\ |||X(t)|||_2 \leq c_0, \quad |||X(t)|||_{s+1} \leq d, \\ |||\partial_t^j X(t)|||_{s+3/2-j/2} + \|\partial_t^j \check{X}(t)\|_{H^{s+3/2-j/2}(\mathbf{R}^1)} \leq l_j, \qquad j = 1,2,3,4, \\ |||\partial_t^j X(t)|||_s + \|\partial_t^j \check{X}(t)\|_{H^s(\mathbf{R}^1)} \leq l_{j+4}, \qquad j = 1,2, \\ \|\eta_0\|_{H^{s+2}(\mathbf{R}^1)} \leq \kappa. \end{cases} \tag{4.4.6}$$

We use the following notations:

$$\begin{cases} [H(t)]_s = \|H(t)\|_{H^{s+1}(\mathbf{R}^1)} + \|\partial_t H(t)\|_{H^{s+1/2}(\mathbf{R}^1)} + \|\partial_t^2 H(t)\|_{H^s(\mathbf{R}^1)}, \\ |H(t)|_s = \|H(t)\|_{H^{s+1}(\mathbf{R}^1)} + \|\partial_t H(t)\|_{H^{s+1}(\mathbf{R}^1)} + \|\partial_t^2 H(t)\|_{H^s(\mathbf{R}^1)} + \|\partial_t^3 H(t)\|_{H^s(\mathbf{R}^1)}, \\ \mu_s = 1 + \|\omega_0\|_{H^{s+3/2}(\Omega)}. \end{cases}$$

Then it follows from Lemmas 4.4.1, 4.4.4, 4.4.7 that

Proposition 4.4.2. *Under* Assumption 4.1 *we have*

$$H = H(X, \check{X}) \in C^j([0,T]; H^{s+3/2-j/2}(\mathbf{R}^1)), \qquad j = 1,3,$$

$$[H]_s \leq C_1 \mu_s, \quad |H|_s \leq C_2 \mu_s, \qquad 0 \leq t \leq T. \tag{4.4.7}$$

Moreover, for $X^1, \check{X}^1$ *and* $X^2, \check{X}^2$ *satisfying* (4.4.6), *we have*

$$\begin{cases} [H(X^1, \check{X}^1) - H(X^2, \check{X}^2)]_s \\ \qquad \leq C_1 \mu_s \sum_{j=0}^{2} \left(|||\partial_t^j X^1(t) - \partial_t^j X^2(t)|||_{s+1-j/2} + \|\partial_t^{j+1} \check{X}^1 - \partial_t^{j+1} \check{X}^2\|_{H^0(\mathbf{R}^1)} \right), \\ |H(X^1, \check{X}^1) - H(X^2, \check{X}^2)|_s \\ \qquad \leq C_2 \mu_s \{ |||X^1(t) - X^2(t)|||_{s+1} + \|\check{X}_t^1 - \check{X}_t^2\|_{H^0(\mathbf{R}^1)} \\ \qquad\qquad + \sum_{j=1}^{3} \left(|||\partial_t^j X^1(t) - \partial_t^j X^2(t)|||_{s+3/2-j/2} + \|\partial_t^{j+1} \check{X}^1 - \partial_t^{j+1} \check{X}^2\|_{H^0(\mathbf{R}^1)} \right) \}, \end{cases}$$

where $0 \leq t \leq T, C_1 = C_1(s, c_0, d, l_5, l_6, \kappa) > 0$ *and* $C_2 = C_2(s, c_0, d, l_1, l_2, l_3, l_4, \kappa) > 0$.

4.5. Problem on the surface

In this section, for a given H, we solve the initial value problem

$$\left(1+\frac{\partial \bar{X}_1}{\partial x_1}\right)\frac{\partial^2 \bar{X}_1}{\partial t^2}+\left(\frac{d\eta_0}{dx_1}+\frac{\partial \bar{X}_2}{\partial x_1}\right)\left(g+\frac{\partial^2 \bar{X}_2}{\partial t^2}\right)=0 \qquad \text{for } t\geq 0, \tag{4.5.1}$$

$$\bar{X}_{2t}=K\bar{X}_{1t}+H \qquad \text{for } t\geq 0, \tag{4.5.2}$$

$$\bar{X}|_{t=0}=(0,0), \quad \bar{X}_{1t}|_{t=0}=u_{01}|_{\Gamma_s}. \tag{4.5.3}$$

We first reduce the problem (4.5.1) – (4.5.3) to the initial value problem for a quasi-linear system. Then the existence and uniqueness of solution to the quasi-linear system is proved. Moreover, we show that the solution of this quasi-linear system satisfies the problem (4.5.1) – (4.5.3). In the remaining of this section, for simplicity, we use X and x instead of $\bar{X}$ and x_1, respectively.

From (4.3.2) and (4.5.2) it follows that

$$\partial_t^k X_{2t}=K(X)\partial_t^k X_{1t}+F_{k0}(X,\dots,\partial_t^k X)+\partial_t^k H, \tag{4.5.4}$$

$$\partial_t^k\partial_x^l X_{2t}=K(X)\partial_t^k\partial_x^l X_{1t}+F_{kl}(X,\dots,\partial_t^k\partial_x^l X,\partial_t^{k+1}X_1)+\partial_t^k\partial_x^l H, \tag{4.5.5}$$

where $k=0,1,2,\dots,\ l=1,2,3,\dots$, and $F_{kl}=[\partial_t^k\partial_x^l, K_1]X_{1t}$. We put

$$Y=X_{tt}, \quad Z=X_x, \quad W=(X,Y,Z), \quad W'=(X,Y_1).$$

In virtue of (4.5.4) with $k=2$ we have

$$Y_{2t}=K(X)Y_{1t}+F_{20}(X,X_t,Y)+H_{tt}=: f_2(W,W_t',H). \tag{4.5.6}$$

Differentiating (4.5.1) with respect to t and using (4.5.5) with $k=0,\ l=1$, we obtain

$$\begin{aligned} Z_{1t} = &-\{(g+Y_2)(-i\mathrm{sgnD})+Y_1\}^{-1}\{(g+Y_2)(K_1(X)\partial_x X_{1t}+F_{01}(X,Z,X_{1t})+H_x)\\ &+(1+Z_1)Y_{1t}+(\eta_{0x}+Z_2)f_2(W,W_t',H)\}\\ =:&\ f_3(W,W_t',H). \end{aligned} \tag{4.5.7}$$

Putting (4.5.7) into (4.5.5) with $k=0,\ l=1$ leads to

$$Z_{2t}=-i\mathrm{sgnD} f_3(W,W_t',H)+K_1(X)\partial_x X_{1t}+F_{01}+H_x=: f_4(W,W_t',H). \tag{4.5.8}$$

Next we proceed to the equation for Y_1. Differentiating (4.5.1) twice with respect to t implies

$$(1+Z_1)Y_{1tt}+(\eta_{0x}+Z_2)Y_{2tt}+Y_1Y_{1x}+(g+Y_2)Y_{2x}+2Z_t\cdot Y_t=0. \tag{4.5.9}$$

By (4.5.4) with $k=3$ and (4.5.5) with $k=l=1$, it holds that

$$\begin{cases} Y_{2tt}=K(X)Y_{1tt}+F_{30}(X,X_t,Y,Y_t)+H_{ttt},\\ Y_{2x}=K(X)Y_{1x}+F_{11}(X,X_t,Z,Z_t,Y_1)+H_{tx}. \end{cases}$$

Hence it follows from (4.5.9) that

$$\begin{aligned} &Y_{1tt} + \{1 + Z_1 + (\eta_{0x} + Z_2)K\}^{-1}\{Y_1 + (g + Y_2)K\}\partial_x Y_1 \\ &= -\{1 + Z_1 + (\eta_{0x} + Z_2)K\}^{-1} \\ &\quad \times \{2Z_t \cdot Y_t + (\eta_{0x} + Z_2)(F_{30} + H_{ttt}) + (g + Y_2)(F_{11} + H_{tx})\}. \end{aligned} \tag{4.5.10}$$

The identity

$$\begin{aligned} &\{1 + Z_1 + (\eta_{0x} + Z_2)K\}^{-1}\{Y_1 + (g + Y_2)K\} \\ &\quad = \{(1 + Z_1)^2 + (\eta_{0x} + Z_2)^2\}^{-1}\{(1 + Z_1)Y_1 + (\eta_{0x} + Z_2)(g + Y_2)\} \\ &\qquad + \{(1 + Z_1)^2 + (\eta_{0x} + Z_2)^2\}^{-1}\{(1 + Z_1)(g + Y_2) - (\eta_{0x} + Z_2)Y_1\}(-i\operatorname{sgn}\mathrm{D}) + P_1, \end{aligned}$$

$$\begin{aligned} P_1 =& P_1(W; X, Z) \\ =& \{(1 + Z_1)^2 + (\eta_{0x} + Z_2)^2\}^{-1}\{(1 + Z_1)(g + Y_2) - (\eta_{0x} + Z_2)Y_1\}K_1 \\ &- \{(1 + Z_1)^2 + (\eta_{0x} + Z_2)^2\}^{-1}(\eta_{0x} + Z_2)\{[K, Y_1] + [K, Y_2]K + (g + Y_2)(1 + K^2)\} \\ &+ \{(1 + Z_1)^2 + (\eta_{0x} + Z_2)^2\}^{-1}(\eta_{0x} + Z_2) \\ &\times \{[K, Z_1] + [K, (\eta_{0x} + Z_2)]K + (\eta_{0x} + Z_2)(1 + K^2)\} \\ &\times \{1 + Z_1 + (\eta_{0x} + Z_2)K\}^{-1}\{Y_1 + (g + Y_2)K\}, \end{aligned}$$

and (4.5.6) – (4.5.8) lead the equivalent equation to (4.5.10)

$$Y_{1tt} + a(W)|\mathrm{D}|Y_1 = f_1(W, W_t', H)$$

with

$$\begin{cases} a(W) = \{(1 + Z_1)^2 + (\eta_{0x} + Z_2)^2\}^{-1}\{(1 + Z_1)(g + Y_2) - (\eta_{0x} + Z_2)Y_1\}, \\ f_1 = -P_1\partial_x Y_1 - \{1 + Z_1 + (\eta_{0x} + Z_2)K\}^{-1}\{2Z_t \cdot Y_t \\ \qquad + (\eta_{0x} + Z_2)(F_{30}(X, X_t, Y, Y_t) + H_{ttt}) + (g + Y_2)(F_{11}(X, X_t, Z, Z_t, Y_1) + H_{tx})\}. \end{cases}$$

Thus the required quasi-linear system has the form

$$\begin{cases} X_{tt} = Y, \quad Y_{1tt} + a(W)|\mathrm{D}|Y_1 = f_1(W, W_t', H), \\ Y_{2t} = f_2(W, W_t', H), \quad Z_{1t} = f_3(W, W_t', H), \quad Z_{2t} = f_4(W, W_t', H). \end{cases} \tag{4.5.11}$$

We show the estimates for the inverse operators in (4.5.11). The following lemma is obtained by Lemma 2.2.2(4).

Lemma 4.5.1.

(1) *There exists a positive constant* $c_1 = c_1(g)$ *such that if* $Y, Y^0 \in H^s(\mathbf{R}^1)$, $s \geq 2$ *and* $\|Y\|_{H^2(\mathbf{R}^1)}, \|Y^0\|_{H^2(\mathbf{R}^1)} \leq c_1$, $\|Y\|_{H^s(\mathbf{R}^1)}, \|Y^0\|_{H^s(\mathbf{R}^1)} \leq d_1$, $d_1 > 0$, *then it holds that*

$$\begin{cases} \|\{Y_1 + (g+Y_2)(-i\mathrm{sgnD})\}^{-1}u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^s(\mathbf{R}^1)}, \\ \|\{Y_1 + (g+Y_2)(-i\mathrm{sgnD})\}^{-1}u - \{Y_1^0 + (g+Y_2^0)(-i\mathrm{sgnD})\}^{-1}u\|_{H^s(\mathbf{R}^1)} \\ \qquad \leq C\|Y - Y^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^s(\mathbf{R}^1)}, \qquad C = C(s, c_1, d_1, g) > 0. \end{cases}$$

(2) *There exists a positive constant* c_1 *such that if* $\eta_0 \in H^{s+1}(\mathbf{R}^1)$, $Z, Z^0 \in H^s(\mathbf{R}^1)$, $s \geq 1$ *and* $\|Z_1\|_{H^1(\mathbf{R}^1)}, \|Z_1^0\|_{H^1(\mathbf{R}^1)} \leq c_1$, $\|Z\|_{H^s(\mathbf{R}^1)}, \|Z^0\|_{H^s(\mathbf{R}^1)} \leq d_1$, $d_1 > 0$, *then it holds that*

$$\begin{cases} \|\{(1+Z_1)^2 + (\eta_{0x} + Z_2)^2\}^{-1}u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^s(\mathbf{R}^1)}, \\ \|\{(1+Z_1)^2 + (\eta_{0x} + Z_2)^2\}^{-1}u - \{(1+Z_1^0)^2 + (\eta_{0x} + Z_2^0)^2\}^{-1}u\|_{H^s(\mathbf{R}^1)} \\ \qquad \leq C\|Z - Z^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^s(\mathbf{R}^1)}, \qquad C = C(s, c_1, d_1) > 0. \end{cases}$$

In order to define the operator $\{1 + Z_1 + (\eta_{0x} + Z_2)K\}^{-1}$, we consider the operator $\{1 - \eta_{0x}(\frac{1}{2} - A_1)^{-1}A_2\}^{-1}$.

Lemma 4.5.2. *The operator* $1 - \eta_{0x}(\frac{1}{2} - A_1)^{-1}A_2 : L^2(\mathbf{R}^1) \to L^2(\mathbf{R}^1)$ *is invertible. Moreover it holds that*

$$\left\| \left\{1 - \eta_{0x}(\frac{1}{2} - A_1)^{-1}A_2\right\}^{-1} u \right\|_{L^2(\mathbf{R}^1)} \leq C\|u\|_{L^2(\mathbf{R}^1)}$$

with $C = C(\|\eta_{0x}\|_{L^\infty(\mathbf{R}^1)}) > 0$.

Proof. Suppose that the function $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2)$ satisfies

$$\begin{cases} \nabla \cdot \tilde{\mathbf{v}} = 0, \quad \nabla^\perp \cdot \tilde{\mathbf{v}} = 0 & \text{in } \Omega_\infty, \\ \tilde{v}_1 = \theta & \text{on } \Gamma_s, \end{cases} \tag{4.5.12}$$

where

$$\Omega_\infty = \{x = (x_1, x_2);\ x_2 < \eta_0(x_1),\ x_1 \in \mathbf{R}^1\}.$$

Then similar arguments as in Section 4.3 conclude that the non-tangential limit $\widetilde{\mathbf{V}}^- = (\widetilde{V}_1^-, \widetilde{V}_2^-)$ of $\tilde{\mathbf{v}}$ satisfies $\widetilde{V}_2^- = -(\frac{1}{2} - A_1)^{-1}A_2\widetilde{V}_1^-$. Moreover, it holds that

$$\mathbf{T} \cdot \widetilde{\mathbf{V}}^- = N_2\left\{1 - \eta_{0x}(\frac{1}{2} - A_1)^{-1}A_2\right\}\theta, \tag{4.5.13}$$

where $\mathbf{T} = (N_2, -N_1)$ is the tangential vector to Γ_s. On the other hand, the layer potentials

$$\tilde{v}_1 = \mathcal{L}_1\left(\eta_0; (\frac{1}{2} + A_1)^{-1}\theta\right), \quad \tilde{v}_2 = -\mathcal{L}_2\left(\eta_0; (\frac{1}{2} + A_1)^{-1}\theta\right)$$

are the unique solution of the problem (4.5.12) and it holds that

$$\mathbf{T}\cdot\widetilde{\mathbf{V}}^{-}=N_2\left\{1-\eta_{0x}A_2(\frac{1}{2}+A_1)^{-1}\right\}\theta. \tag{4.5.14}$$

Here we can show that the operator $\frac{1}{2}+A_1$ is invertible in the same way as Lemma 4.4.3. Therefore, by virtue of (4.5.13) and (4.5.14), it is sufficient to show that the operator $1-\eta_{0x}A_2(\frac{1}{2}+A_1)^{-1}=(\frac{1}{2}+A_1-\eta_{0x}A_2)(\frac{1}{2}+A_1)^{-1}$ is invertible.

Now we consider the layer potentials

$$\dot{v}_1=\mathcal{L}_1(\eta_0;u_1)+\mathcal{L}_2(\eta_0;u_2),\quad \dot{v}_2=-\mathcal{L}_2(\eta_0;u_1)+\mathcal{L}_1(\eta_0;u_2)$$

for $u_1,u_2\in L^2(\mathbf{R}^1)$. Then it holds that

$$\dot{V}_1^{\pm}=\mp\frac{1}{2}u_1+A_1u_1+A_2u_2,\quad \dot{V}_2^{\pm}=-A_2u_1\mp\frac{1}{2}u_2+A_1u_2.$$

Furthermore, we can apply Lemma 4.4.2. Putting $u_1=0$ and $u_2=u$, we see that

$$\|u\|_{L^2(\mathbf{R}^1)}\le C\|(\frac{1}{2}+A_1-\eta_{0x}A_2)u\|_{L^2(\mathbf{R}^1)}. \tag{4.5.15}$$

Next, for $u_1,u_2\in L^2(\mathbf{R}^1)$, consider the layer potentials

$$\ddot{v}_1=\mathcal{M}_1(\eta_0;u_1)+\mathcal{M}_2(\eta_0;u_2),\quad \ddot{v}_2=\mathcal{M}_2(\eta_0;u_1)-\mathcal{M}_1(\eta_0;u_2).$$

Again by Lemma 4.4.2, taking $u_1=u$ and $u_2=\eta_{0x}u$ gives

$$\|u\|_{L^2(\mathbf{R}^1)}\le C\|(\frac{1}{2}+A_1^*-A_2^*\eta_{0x})u\|_{L^2(\mathbf{R}^1)}. \tag{4.5.16}$$

Therefore the estimates (4.5.15) and (4.5.16) give the desired assertions. □

Lemma 4.5.3. *Suppose that $\eta_0\in H^{s+2}(\mathbf{R}^1)$, $\|\eta_0\|_{H^{s+2}(\mathbf{R}^1)}\le\kappa$, $s\ge 2$. There exists a positive constant c_1 such that if $\|X\|_{H^2(\mathbf{R}^1)},\|Z\|_{H^2(\mathbf{R}^1)},\|X^0\|_{H^2(\mathbf{R}^1)},\|Z^0\|_{H^2(\mathbf{R}^1)}\le c_1$, $\|X\|_{H^s(\mathbf{R}^1)},\|X^0\|_{H^s(\mathbf{R}^1)}\le d_1$, $d_1>0$, then the operator $1+Z_1+(\eta_{0x}+Z_2)K:H^s(\mathbf{R}^1)\to H^s(\mathbf{R}^1)$ is invertible and satisfies*

$$\begin{cases}\|\{1+Z_1+(\eta_{0x}+Z_2)K(X)\}^{-1}u\|_{H^s(\mathbf{R}^1)}\le C\|(X,Z)\|_{H^s(\mathbf{R}^1)}\|u\|_{H^s(\mathbf{R}^1)},\\ \|\{1+Z_1+(\eta_{0x}+Z_2)K(X)\}^{-1}u-\{1+Z_1^0+(\eta_{0x}+Z_2^0)K(X^0)\}^{-1}u\|_{H^s(\mathbf{R}^1)}\\ \qquad\qquad\le C\|(X-X^0,Z-Z^0)\|_{H^s(\mathbf{R}^1)}\|u\|_{H^s(\mathbf{R}^1)},\end{cases}$$

where $C=C(s,c_1,d_1,\kappa)>0$.

Proof. As the proof for Lemma 4.4.4, we can show that the operator $1-\eta_{0x}(\frac{1}{2}-A_1)^{-1}A_2:H^s(\mathbf{R}^1)\to H^s(\mathbf{R}^1), s\ge 0$, is invertible and satisfies

$$\left\|\left\{1-\eta_{0x}(\frac{1}{2}-A_1)^{-1}A_2\right\}^{-1}u\right\|_{H^s(\mathbf{R}^1)}\le C\|u\|_{H^s(\mathbf{R}^1)},$$

where $C = C(s, \kappa) > 0$. By the proof for Lemma 2.2.2(4), it holds that the operator $1 + \eta_{0x}K = 1 - \eta_{0x}(\frac{1}{2} - A_1 - A_3)^{-1}(A_2 - A_4)$ is invertible in $H^s(\mathbf{R}^1), s \geq 2$, and

$$\|(1 + \eta_{0x}K)^{-1}u\|_{H^s(\mathbf{R}^1)} \leq C'\|u\|_{H^s(\mathbf{R}^1)},$$

where $C' = C'(s, c_1, d_1, \kappa) > 0$. Again, Lemma 2.2.2(4) implies that the operator $1 + Z_1 + (\eta_{0x} + Z_2)K$ is invertible and satisfies the first estimate. The second estimate is also obtained in the same way. ∎

Now we consider the initial value problem (4.5.11) with

$$W(0) = \widetilde{W} = (\widetilde{X}, \widetilde{Y}, \widetilde{Z}), \quad W_t'(0) = \widetilde{W_t'} = (\widetilde{X}_t, \widetilde{Y_{1t}}). \tag{4.5.17}$$

Assumption 4.2. *Let $T_1 > 0$. There exist positive constants J and J' such that*

$$\begin{cases} |H(t)|_s \leq J, \\ [H(t)]_s \leq J', \quad 0 \leq t \leq T_1. \end{cases}$$

By Lemmas 4.5.1, 4.5.3, similar arguments as in Theorem 2.5.2 lead to the following.

Theorem 4.5.1. *There exists a positive constant $c_1 = c_1(g)$ such that if $H \in C^j([0, T_1]; H^{s+3/2-j/2}(\mathbf{R}^1))$, $j = 1, 3$, $s \geq 3 + 1/2$, $0 < T_1 < \infty$, $\eta_0 \in H^{s+2}(\mathbf{R}^1)$ and*

$$\widetilde{X}, \widetilde{Z}, \widetilde{W_t'} \in H^s(\mathbf{R}^1), \quad \widetilde{Y_1} \in H^{s+1/2}(\mathbf{R}^1), \quad \|\widetilde{W}\|_{H^2(\mathbf{R}^1)} \leq c_1/2,$$

then there exists $T \in (0, T_1]$ such that problem (4.5.11), (4.5.17) has a unique solution $W = (X, Y, Z)$ satisfying

$$\begin{cases} X \in C^2([0, T]; H^s(\mathbf{R}^1)), \quad Y_2, Z \in C^1([0, T]; H^s(\mathbf{R}^1)), \\ Y_1 \in C^j([0, T]; H^{s+1/2-j/2}(\mathbf{R}^1)), \quad j = 0, 1, 2, \\ \|W(t)\|_{H^2(\mathbf{R}^1)} \leq c_1 \quad for \ 0 \leq t \leq T. \end{cases}$$

In view of the original problem, we set the initial data as follows:

$$\begin{cases} \widetilde{X} = (0, 0), \ \widetilde{Z} = \widetilde{X}_x = (0, 0), \ \widetilde{X_{1t}} = u_{01}(\cdot, \eta_0(\cdot)), \ \widetilde{X_{2t}} = K(0, 0)\widetilde{X_{1t}} + H(0), \\ \widetilde{Y_1} = -(1 + \eta_{0x}K(0, 0))^{-1}\eta_{0x}(g + F_{10}(\widetilde{X}, \widetilde{X_t}) + H_t(0)), \\ \widetilde{Y_2} = K(0, 0)\widetilde{Y_1} + F_{10}(\widetilde{X}, \widetilde{X_t}) + H_t(0), \\ \widetilde{Y_{1t}} = -(1 + \eta_{0x}K(0, 0))^{-1} \\ \qquad \times \{\eta_{0x}(F_{20}(\widetilde{X}, \widetilde{X_t}, \widetilde{Y}) + H_{tt}(0)) + \widetilde{Y_1}\partial_x\widetilde{X_{1t}} + (g + \widetilde{Y_2})\partial_x\widetilde{X_{2t}}\}. \end{cases}$$

Then Theorem 4.5.1 yields

Theorem 4.5.2. *There exists a positive constant* $\varepsilon_1 = \varepsilon_1(g)$ *such that if* $s \geq 3 + 1/2$, $0 < T_1 < \infty$ *and* $\eta_0, u_{01}|_{\Gamma_s}$, H *satisfy the conditions*

$$\begin{cases} \eta_0 \in H^{s+2}(\mathbf{R}^1), \quad u_{01}|_{\Gamma_s} \in H^{s+1}(\mathbf{R}^1), \\ \|u_{01}|_{\Gamma_s}\|_{H^2(\mathbf{R}^1)} \leq \varepsilon_1/2, \end{cases} \tag{4.5.18}$$

$$\begin{cases} H \in C^j([0, T_1]; H^{s+3/2-j/2}(\mathbf{R}^1)), \quad j = 1, 3, \\ \|H(0)\|_{H^2(\mathbf{R}^1)} + \|H_t(0)\|_{H^2(\mathbf{R}^1)} \leq \varepsilon_1/2, \end{cases} \tag{4.5.19}$$

then there exists $T \in (0, T_1]$ *such that problem* (4.5.1) – (4.5.3) *has a unique solution*

$$\bar{X} \in C^j([0, T]; H^{s+3/2-j/2}(\mathbf{R}^1)), \quad j = 1, 2, 3, 4. \tag{4.5.20}$$

Here we put

$$\begin{cases} d_2 = \max\{1, J\}, \quad d_3 = \max\{1, J'\}, \\ d_4 = \|\eta_0\|_{H^{s+2}(\mathbf{R}^1)} + \|u_{01}|_{\Gamma_s}\|_{H^{s+1}(\mathbf{R}^1)} + \sum_{j=0}^{2} \|\partial_t^j H(0)\|_{H^{s+1-j/2}(\mathbf{R}^1)}. \end{cases}$$

Lemma 4.5.4. *Let* $\bar{X}$ *be the solution of* (4.5.1) – (4.5.3) *obtained in* Theorem 4.5.2. *Then there exist a positive constant* $d_0 = d_0(c_1, g, s, \varepsilon_1, d_4)$, *which is independent of* t, *and a monotone increasing function* $d_5(t)$ *such that*

$$\begin{cases} \|\bar{X}_t(t)\|_{H^s(\mathbf{R}^1)} + \|\bar{X}_{1tt}(t)\|_{H^{s+1/2}(\mathbf{R}^1)} + \|\bar{X}_{1ttt}(t)\|_{H^s(\mathbf{R}^1)} \leq d_0, \\ \|\bar{X}_t(t)\|_{H^{s+1}(\mathbf{R}^1)} \leq d_5(t), \quad 0 \leq t \leq T. \end{cases} \tag{4.5.21}$$

Furthermore we obtain

Proposition 4.5.1. *Let* $s \geq 4$. *Suppose that* H^0 *satisfies the conditions in* (4.5.19) *and* $\bar{X}^0$ *is the solution of* (4.5.1), (4.5.2) *with* H *replaced by* H^0, *where* $H(0) = H^0(0)$, *and* (4.5.3). *Then it holds that*

$$\begin{cases} \displaystyle\sum_{j=0}^{2} \|\partial_t^{j+1}\bar{X}(t) - \partial_t^{j+1}\bar{X}^0(t)\|_{H^{s+1/2-j/2}(\mathbf{R}^1)} \\ \qquad \leq C_3\Big([H(t) - H^0(t)]_{s-1/2} + \displaystyle\int_0^t |H(\tau) - H^0(\tau)|_{s-1/2} \mathrm{d}\tau\Big), \\ \|\bar{X}_{1tttt}(t) - \bar{X}^0_{1tttt}(t)\|_{H^{s-1}(\mathbf{R}^1)} \leq C_3\left(|H(t) - H^0(t)|_{s-1} + \displaystyle\int_0^t |H(\tau) - H^0(\tau)|_{s-1} \mathrm{d}\tau\right) \end{cases}$$

for $0 \leq t \leq T$, *where* $C_3 = C_3(c_1, g, d_0, d_2, s, T) > 0$.

4.6. Problem in the interior

Suppose that an $\bar{X}$ is given. We consider the boundary value problem

$$\begin{cases} \nabla_{\mathbf{u}} \cdot \mathbf{u} = 0, \quad \nabla_{\mathbf{u}}^{\perp} \cdot \mathbf{u} = \omega_0 & \text{in } \Omega,\ t \geq 0, \\ u_1 = \bar{X}_{1t} & \text{on } \Gamma_s,\ t \geq 0, \\ \mathbf{u} \cdot \mathbf{n}(\Phi_{\mathbf{u}}(x;t)) = 0 & \text{on } \Gamma_b,\ t \geq 0. \end{cases} \tag{4.6.1}$$

At first, let us investigate problem

$$\begin{cases} \nabla \cdot \mathbf{u} = \phi_1, \quad \nabla^{\perp} \cdot \mathbf{u} = \phi_2 & \text{in } \Omega, \\ u_1 = \theta_1 & \text{on } \Gamma_s, \\ \mathbf{u} \cdot \mathbf{n} = \theta_2 & \text{on } \Gamma_b. \end{cases} \tag{4.6.2}$$

In order to solve this problem we use the identification as in Section 4.3:

$$x = x_1 + ix_2.$$

Put

$$\begin{cases} \mathcal{F}(x) = u_1(x) - iu_2(x), \\ \mathfrak{f}(x_1) = \mathcal{F}(x_1 + i\eta_0(x_1)), \\ \mathfrak{g}(x_1) = \mathcal{F}(x_1 + i(-h + b(x_1))). \end{cases}$$

Then it follows that

$$\begin{aligned} \mathcal{F}(x^0) = {} & \mathcal{L}_1(\eta_0;\mathfrak{f})(x^0) + i\mathcal{L}_2(\eta_0;\mathfrak{f})(x^0) - \mathcal{L}_1(-h+b;\mathfrak{g})(x^0) - i\mathcal{L}_2(-h+b;\mathfrak{g})(x^0) \\ & + i\mathcal{H}_1(x^0) + \mathcal{H}_2(x^0), \qquad x^0 \in \Omega, \end{aligned} \tag{4.6.3}$$

where

$$\begin{cases} \mathcal{H}_1(x_1,x_2) = \dfrac{1}{2\pi} \displaystyle\iint_{\Omega} \frac{\phi_2(y_1-x_1) - \phi_1(y_2-x_2)}{(y_1-x_1)^2 + (y_2-x_2)^2} \mathrm{d}y_1 \mathrm{d}y_2, \\ \mathcal{H}_2(x_1,x_2) = \dfrac{1}{2\pi} \displaystyle\iint_{\Omega} \frac{\phi_1(y_1-x_1) + \phi_2(y_2-x_2)}{(y_1-x_1)^2 + (y_2-x_2)^2} \mathrm{d}y_1 \mathrm{d}y_2. \end{cases}$$

If x^0 tends to points on Γ_s and Γ_b non-tangentially, it holds that

$$\begin{cases} \mathfrak{f} = (\frac{1}{2} + \mathcal{A}_1)\mathfrak{f} + i\mathcal{A}_2\mathfrak{f} - \mathcal{A}_5\mathfrak{g} - i\mathcal{A}_6\mathfrak{g} + i\mathcal{H}_1|_{\Gamma_s} + \mathcal{H}_2|_{\Gamma_s}, \\ \mathfrak{g} = \mathcal{A}_7\mathfrak{f} + i\mathcal{A}_8\mathfrak{f} - (-\frac{1}{2} + \mathcal{A}_3)\mathfrak{g} - i\mathcal{A}_4\mathfrak{g} + i\mathcal{H}_1|_{\Gamma_b} + \mathcal{H}_2|_{\Gamma_b}, \end{cases} \tag{4.6.4}$$

where

$$\begin{cases} \mathcal{A}_1u(x_1) = L_1(\eta_0;u)(x_1),\ \mathcal{A}_2u(x_1) = L_2(\eta_0;u)(x_1), \\ \mathcal{A}_3u(x_1) = L_1(-h+b;u)(x_1),\ \mathcal{A}_4u(x_1) = L_2(-h+b;u)(x_1), \\ \mathcal{A}_5u(x_1) = \mathcal{L}_1(-h+b;u)(x_1,\eta_0(x_1)),\ \mathcal{A}_6u(x_1) = \mathcal{L}_2(-h+b;u)(x_1,\eta_0(x_1)), \\ \mathcal{A}_7u(x_1) = \mathcal{L}_1(\eta_0;u)(x_1,-h+b(x_1)),\ \mathcal{A}_8u(x_1) = \mathcal{L}_2(\eta_0;u)(x_1,-h+b(x_1)). \end{cases}$$

Since $\mathfrak{g}_2 = -b'\mathfrak{g}_1 - \theta_2$, the imaginary part of the first relation of (4.6.4) and the real part of the second lead to

$$u_2|_{\Gamma_s} = \mathcal{K}\theta_1 + \mathcal{H}, \tag{4.6.5}$$

where

$$\begin{cases} \mathcal{K} = -\left(\frac{1}{2} - \mathcal{B}_2\right)^{-1}\mathcal{B}_1, \\ \mathcal{H} = -\left(\frac{1}{2} - \mathcal{B}_2\right)^{-1} \\ \qquad \times\left\{(-\mathcal{A}_5b' + \mathcal{A}_6)\left(\frac{1}{2} + \mathcal{A}_3 + \mathcal{A}_4b'\right)^{-1}(\mathcal{A}_4\theta_2 - \mathcal{H}_2|_{\Gamma_b}) + \mathcal{A}_5\theta_2 + \mathcal{H}_1|_{\Gamma_s}\right\}, \\ \mathcal{B}_1 = \mathcal{A}_2 - (-\mathcal{A}_5b' + \mathcal{A}_6)\left(\frac{1}{2} + \mathcal{A}_3 + \mathcal{A}_4b'\right)^{-1}\mathcal{A}_7, \\ \mathcal{B}_2 = \mathcal{A}_1 + (-\mathcal{A}_5b' + \mathcal{A}_6)\left(\frac{1}{2} + \mathcal{A}_3 + \mathcal{A}_4b'\right)^{-1}\mathcal{A}_8 \end{cases}$$

and

$$\begin{cases} u_1|_{\Gamma_b} = \left(\frac{1}{2} + \mathcal{A}_3 + \mathcal{A}_4b'\right)^{-1}(\mathcal{A}_7\theta_1 - \mathcal{A}_8u_2|_{\Gamma_s} - \mathcal{A}_4\theta_2 + \mathcal{H}_2|_{\Gamma_b}), \\ u_2|_{\Gamma_b} = b'u_1|_{\Gamma_b} + \theta_2. \end{cases} \tag{4.6.6}$$

The arguments similar as in Lemma 4.5.2 show the following lemma.

Lemma 4.6.1. *The operator* $\frac{1}{2} + \mathcal{A}_3 + \mathcal{A}_4b' : L^2(\mathbf{R}^1) \to L^2(\mathbf{R}^1)$ *is invertible. Moreover, it holds that*

$$\|(\frac{1}{2} + \mathcal{A}_3 + \mathcal{A}_4b')^{-1}u\|_{L^2(\mathbf{R}^1)} \le C\|u\|_{L^2(\mathbf{R}^1)},$$

where $C = C(\|b'\|_{L^\infty(\mathbf{R}^1)}) > 0$.

Lemma 4.6.2 ([14, Lemma 5.5]). *Let* η_0, b *be the Lipschitz continuous functions,* inf $\{\eta_0(x_1) - (-h+b(x_1))\} > 0$ *and* $C^\pm(P), P \in \Gamma_s \cup \Gamma_b$, *the cones defined by* (4.2.1) *with* φ *replaced by* η_0 *or* b. *Suppose that*

(1) $\dot{\mathbf{v}} = (\dot{v}_1, \dot{v}_2)$ *satisfies* $\nabla\cdot\dot{\mathbf{v}} = 0$ *and* $\nabla^\perp\cdot\dot{\mathbf{v}} = 0$ *in* $\mathbf{R}^2 \setminus (\Gamma_s \cup \Gamma_b)$,

(2) *The non-tangential maximal functions* $\dot{\mathbf{v}}_*^\pm = \sup_{X\in C^\pm(P)} |\dot{\mathbf{v}}(X)|, P \in \Gamma_s\cup\Gamma_b$, *belong to* $L^2(\mathbf{R}^2)$,

(3) *The non-tangential limits* $\dot{\mathbf{V}}^{\pm} = (\dot{V}_1^{\pm}, \dot{V}_2^{\pm}) = \lim_{X\to P, X\in C^{\pm}(P)} \dot{\mathbf{v}}(X), P \in \Gamma_s$, *and* $\dot{\mathbf{W}}^{\pm} = (\dot{W}_1^{\pm}, \dot{W}_2^{\pm}) = \lim_{X\to P, X\in C^{\pm}(P)} \dot{\mathbf{v}}(X), P \in \Gamma_b$, *exist for almost every* P,

(4) $\dot{\mathbf{v}}(x) = O(|x|^{-1})$ *as* $|x| \to \infty$.

If we denote the normal vector and the tangential vector to Γ_s *and* Γ_b *by* $\mathbf{N}, \mathbf{T}$, *respectively, then it holds that*

(1) *The norms* $\|\dot{V}_1^{+}\|_{L^2(\mathbf{R}^1)}, \|\dot{V}_2^{+}\|_{L^2(\mathbf{R}^1)}, \|\mathbf{N}\cdot\dot{\mathbf{V}}^{+}\|_{L^2(\mathbf{R}^1)}$ *and* $\|\mathbf{T}\cdot\dot{\mathbf{V}}^{+}\|_{L^2(\mathbf{R}^1)}$ *are equivalent,*

(2) *The norms* $\|\dot{W}_1^{-}\|_{L^2(\mathbf{R}^1)}, \|\dot{W}_2^{-}\|_{L^2(\mathbf{R}^1)}, \|\mathbf{N}\cdot\dot{\mathbf{W}}^{-}\|_{L^2(\mathbf{R}^1)}$ *and* $\|\mathbf{T}\cdot\dot{\mathbf{W}}^{-}\|_{L^2(\mathbf{R}^1)}$ *are equivalent,*

(3) *If, in addition,* $\mathbf{N}\cdot\dot{\mathbf{W}}^{+} = 0$ *or* $\dot{W}_2^{+} = 0$, *then* $\|\dot{V}_2^{-}\|_{L^2(\mathbf{R}^1)} \le C\|\dot{V}_1^{-}\|_{L^2(\mathbf{R}^1)}$ *and* $\|\mathbf{N}\cdot\dot{\mathbf{V}}^{-}\|_{L^2(\mathbf{R}^1)} \le C\|\mathbf{T}\cdot\dot{\mathbf{V}}^{-}\|_{L^2(\mathbf{R}^1)}$ *with* $C = C(\|\eta_0'\|_{L^\infty(\mathbf{R}^1)}) > 0$.

Lemma 4.6.3. *The operator* $\frac{1}{2} - \mathcal{B}_2 : L^2(\mathbf{R}^1) \to L^2(\mathbf{R}^1)$ *is invertible. Moreover, it holds that*

$$\|(\frac{1}{2} - \mathcal{B}_2)^{-1}u\|_{L^2(\mathbf{R}^1)} \le C\|u\|_{L^2(\mathbf{R}^1)},$$

where $C = C(\|b'\|_{L^\infty(\mathbf{R}^1)}) > 0$.

Proof. For $u_i \in L^2(\mathbf{R}^1)$, $i = 1, \ldots, 4$, let us consider the layer potentials

$$\begin{cases} \dot{v}_1 = \mathcal{L}_1(\eta_0; u_1) + \mathcal{L}_2(\eta_0; u_2) + \mathcal{L}_1(-h+b; u_3) + \mathcal{L}_2(-h+b; u_4), \\ \dot{v}_2 = -\mathcal{L}_2(\eta_0; u_1) + \mathcal{L}_1(\eta_0; u_2) - \mathcal{L}_2(-h+b; u_3) + \mathcal{L}_1(-h+b; u_4). \end{cases}$$

Note that $\dot{\mathbf{v}}$ satisfies $\nabla\cdot\dot{\mathbf{v}} = 0$ and $\nabla^{\perp}\cdot\dot{\mathbf{v}} = 0$. Putting $u_1 = 0$, $u_2 = u$, $u_3 = -(\frac{1}{2} + \mathcal{A}_3 + \mathcal{A}_4 b')^{-1}\mathcal{A}_8 u$, $u_4 = b'u_3$, by Lemma 4.6.2 we see that

$$\|u\|_{L^2(\mathbf{R}^1)} \le C\|(\frac{1}{2} - \mathcal{B}_2)u\|_{L^2(\mathbf{R}^1)} \tag{4.6.7}$$

with $C = C(\|b'\|_{L^\infty(\mathbf{R}^1)}) > 0$.

Next we consider the layer potentials

$$\begin{cases} \ddot{v}_1 = \mathcal{M}_1(\eta_0; u_1) + \mathcal{M}_2(\eta_0; u_2) + \mathcal{M}_1(-h+b; u_3) + \mathcal{M}_2(-h+b; u_4), \\ \ddot{v}_2 = \mathcal{M}_2(\eta_0; u_1) - \mathcal{M}_1(\eta_0; u_2) + \mathcal{M}_2(-h+b; u_3) - \mathcal{M}_1(-h+b; u_4). \end{cases}$$

Then taking $u_1 = 0$, $u_2 = u$, $u_3 = -(\frac{1}{2} + \mathcal{A}_3^* + b'\mathcal{A}_4^*)^{-1}(b'\mathcal{A}_5^* - \mathcal{A}_6^*)u$, $u_4 = 0$ gives

$$\|u\|_{L^2(\mathbf{R}^1)} \le C\|(\frac{1}{2} - \mathcal{B}_2^*)u\|_{L^2(\mathbf{R}^1)}. \tag{4.6.8}$$

Therefore the estimates (4.6.7) and (4.6.8) imply the desired assertion. ∎

Let us introduce the operator $\mathcal{A}_9$ by

$$\mathcal{A}_9 u = \frac{1}{2\pi}\int_{\mathbf{R}^1} \log\{1 + \left(\frac{\eta_0(y_1) - \eta_0(x_1)}{y_1 - x_1}\right)^2\}^{1/2} u'(y_1)\mathrm{d}y_1.$$

Then we have

$$\mathcal{A}_2 = \frac{1}{2}i\mathrm{sgnD} - \mathcal{A}_9. \tag{4.6.9}$$

Lemma 4.6.4. *Suppose that* $\inf\{\eta_0(x_1) - (-h + b(x_1))\} > 0$. *Then we have*

(1) $\|\mathcal{A}_j u\|_{H^s(\mathbf{R}^1)} \le C\|u\|_{H^{s_0}(\mathbf{R}^1)}, \quad j = 1, 9,\ s, s_0 > 3/2, C = C(s, s_0, \|\eta_0\|_{H^s(\mathbf{R}^1)}) > 0,$

(2) $\|\mathcal{A}_j u\|_{H^s(\mathbf{R}^1)} \le C\|u\|_{H^0(\mathbf{R}^1)}, \quad j = 1, 9,\ s \ge 0,$
$C = C(s, \|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)}, \|\eta_0'\|_{L^\infty(\mathbf{R}^1)}) > 0,$

(3) $\|\mathcal{A}_j u\|_{H^0(\mathbf{R}^1)} \le C\|u\|_{H^0(\mathbf{R}^1)}, \quad j = 3, 4,\ C = C(\|b'\|_{L^\infty(\mathbf{R}^1)}) > 0,$

(4) $\|\mathcal{A}_j u\|_{H^s(\mathbf{R}^1)} \le C\|u\|_{H^0(\mathbf{R}^1)}, \quad j = 5, 6,\ s \ge 0, C = C(\|b\|_{H^s(\mathbf{R}^1)}, \|b'\|_{L^\infty(\mathbf{R}^1)}) > 0,$

(5) $\|\mathcal{A}_j u\|_{H^0(\mathbf{R}^1)} \le C\|u\|_{H^0(\mathbf{R}^1)}, \quad j = 7, 8,\ C = C(\|\eta_0'\|_{L^\infty(\mathbf{R}^1)}) > 0,$

(6) $\|\mathcal{H}_1|_{\Gamma_s}\|_{H^s(\mathbf{R}^1)} + \|\mathcal{H}_2|_{\Gamma_b}\|_{H^s(\mathbf{R}^1)} \le C\|\phi\|_{H^{s-1/2}(\Omega)}, \quad s \ge 1/2, C = C(s) > 0.$

Proof. (1) and (2) follow from [46, Section 4] and [6, Section 9], respectively. The proofs for (3) – (5) are standard. Similar arguments as in Lemma 4.4.7 show (6). ∎

Theorem 4.6.1. *Suppose that* $\phi = (\phi_1, \phi_2) \in H^{s-1/2}(\Omega)$, $\theta = (\theta_1, \theta_2) \in H^s(\mathbf{R}^1)$, $\eta_0, b \in H^s(\mathbf{R}^1)$ *and* $\|\eta_0\|_{H^s(\mathbf{R}^1)}, \|b\|_{H^s(\mathbf{R}^1)} \le \kappa$ *with* $s > 3$. *Then the boundary value problem* (4.6.2) *has a unique solution* $\mathbf{u} = (u_1, u_2)$ *such that*

$$\begin{cases} \mathbf{u} \in H^{s+1/2}(\Omega), \\ \mathbf{u}(\cdot, \eta_0(\cdot)), \mathbf{u}(\cdot, -h + b(\cdot)) \in H^s(\mathbf{R}^1), \end{cases}$$

$$\begin{cases} \|\mathbf{u}\|_{H^{s+1/2}(\Omega)} \le C_4(\|\phi\|_{H^{s-1/2}(\Omega)} + \|\theta\|_{H^s(\mathbf{R}^1)}), \\ \|\mathbf{u}(\cdot, \eta_0(\cdot))\|_{H^s(\mathbf{R}^1)} + \|\mathbf{u}(\cdot, -h + b(\cdot))\|_{H^s(\mathbf{R}^1)} \le C_4(\|\phi\|_{H^{s-1/2}(\Omega)} + \|\theta\|_{H^s(\mathbf{R}^1)}), \end{cases} \tag{4.6.10}$$

where $C_4 = C_4(s, \kappa) > 0$.

Proof. It is sufficient to show the proof for estimates. Since

$$-\left(\frac{1}{2} - \mathcal{B}_2\right)^{-1} \mathcal{B}_1 = -2\mathcal{B}_1 - 4\left\{\mathcal{B}_2 + \mathcal{B}_2^2\left(\frac{1}{2} - \mathcal{B}_2\right)^{-1}\right\}\mathcal{B}_1, \tag{4.6.11}$$

by (4.6.9) and Lemma 4.6.4, it holds that

$$\|\mathcal{K}\theta_1\|_{H^s(\mathbf{R}^1)} \le C\|\theta_1\|_{H^s(\mathbf{R}^1)},$$

where $C = C(s, \kappa) > 0$. Using the relation similar to (4.6.11), we get

$$\|\mathcal{H}\|_{H^s(\mathbf{R}^1)} \le C(\|\phi\|_{H^{s-1/2}(\Omega)} + \|\theta\|_{H^s(\mathbf{R}^1)}).$$

Hence the required estimate for $u_2|_{\Gamma_s}$ follows from (4.6.5). The estimate for $\mathbf{u}|_{\Gamma_b}$ is also obtained by (4.6.6). Moreover slight improvement of the proof in Lemma 4.4.7 shows that

$$\begin{aligned}
\|\mathcal{L}_i(\eta_0; \mathfrak{f})\|_{H^s(\Omega)} + \|\mathcal{L}_i(-h+b; \mathfrak{g})\|_{H^s(\Omega)} &\le C(\|\mathfrak{f}\|_{H^{s-1/2}(\mathbf{R}^1)} + \|\mathfrak{g}\|_{H^{s-1/2}(\mathbf{R}^1)}) \\
&\le C(\|\phi\|_{H^{s-1/2}(\Omega)} + \|\theta\|_{H^s(\mathbf{R}^1)}), \\
\|\mathcal{H}_i\|_{H^s(\Omega)} \le C\|\phi\|_{H^{s-1}(\Omega)},
\end{aligned}$$

where $i = 1, 2,\ C = C(s) > 0$. Then the first estimate of (4.6.10) follows from (4.6.3). $\square$

Now problem (4.6.1) is written as

$$\begin{cases}
\nabla \cdot \mathbf{u} = ((I - A_{\mathbf{u}})\nabla) \cdot \mathbf{u} & \text{in } \Omega,\ 0 \le t \le T, \\
\nabla^{\perp} \cdot \mathbf{u} = \omega_0 + ((I - A_{\mathbf{u}})\nabla)^{\perp} \cdot \mathbf{u} & \text{in } \Omega,\ 0 \le t \le T, \\
u_1 = \bar{X}_{1t} & \text{on } \Gamma_s,\ 0 \le t \le T, \\
\mathbf{u} \cdot \mathbf{n}(x) = \mathbf{u} \cdot \left\{\mathbf{n}(x) - \mathbf{n}(x + \int_0^t \mathbf{u}(\tau, x) d\tau)\right\} & \text{on } \Gamma_b,\ 0 \le t \le T.
\end{cases}$$

Assumption 4.3. *There exists* $\mathbf{v}_0 \in H^{s+3/2}(\Omega)$ *such that*

$$\nabla \cdot \mathbf{v}_0 = 0, \quad \omega_0 = \nabla^{\perp} \cdot \mathbf{v}_0 \qquad \textit{in } \Omega.$$

Let $T_1 > 0,\ \bar{X}$ *satisfy*

$$\begin{cases}
\bar{X}_{1t} \in C^j([0, T_1]; H^{s+1-j/2}(\mathbf{R}^1)), \quad j = 0, 1, 2, 3, \\
\|\bar{X}_{1t}(t)\|_{H^s(\mathbf{R}^1)} + \|\bar{X}_{1tt}(t)\|_{H^{s+1/2}(\mathbf{R}^1)} + \|\bar{X}_{1ttt}(t)\|_{H^s(\mathbf{R}^1)} \le d_0, \\
\|\bar{X}_{1t}(t)\|_{H^{s+1}(\mathbf{R}^1)} \le d_5(t)
\end{cases}$$

and

$$\eta_0 \in H^{s+1}(\mathbf{R}^1), \quad \mathbf{n} \in H^{s+2}(\mathbf{R}^1). \tag{4.6.12}$$

Here we introduce the notation

$$|\mathbf{u}|_{s,\Omega} = \|\mathbf{u}\|_{H^{s+1/2}(\Omega)} + \|\mathbf{u}(\cdot, \eta_0(\cdot))\|_{H^s(\mathbf{R}^1)} + \|\mathbf{u}(\cdot, -h + b(\cdot))\|_{H^s(\mathbf{R}^1)}.$$

Theorem 4.6.2. *Under* Assumption 4.3 *there exists* $T \in (0, T_1]$ *such that problem* (4.6.1) *has a unique solution* $\mathbf{u}$ *satisfying*

$$\begin{cases}
\mathbf{u} \in C^j([0, T]; H^{s+3/2-j/2}(\Omega)), \\
\mathbf{u}|_{\Gamma_s}, \mathbf{u}|_{\Gamma_b} \in C^j([0, T]; H^{s+1-j/2}(\mathbf{R}^1)) \quad \textit{for } j = 0, 1, 2, 3.
\end{cases} \tag{4.6.13}$$

Proof. The similar arguments as in Theorem 2.6.4 lead to the result. $\square$

Lemma 4.6.5. *Let* $\mathbf{u}$ *be the solution of* (4.6.1) *obtained in* Theorem 4.6.2. *There exist positive constants* e_j, $j = 1, \dots, 5, 7$, *which are independent of* t, *and a monotone increasing function* $e_6(t)$ *such that*

$$\begin{cases} |\partial_t^j \mathbf{u}(t)|_{s+1-j/2,\Omega} \le e_j, \quad j = 1,2,3, \\ |\partial_t^j \mathbf{u}(t)|_{s,\Omega} \le e_{j+4}, \quad j = 0,1, \\ |\mathbf{u}(t)|_{s+1,\Omega} \le e_6(t) \le e_7, \qquad 0 \le t \le T. \end{cases} \tag{4.6.14}$$

Here e_4 *and* e_5 *are independent of* d_3.

Proposition 4.6.1. *Let* $\mathbf{u}$ *be the solution of* (4.6.1) *obtained in* Theorem 4.6.2 *and* $\mathbf{u}^0$ *the solution of* (4.6.1) *with* $\bar{X}$ *replaced by* $\bar{X}^0$, *which satisfies* Assumption 4.3. *Then we have*

$$\sum_{j=0}^{3} \sup_{0 \le \tau \le t} |\partial_\tau^j \mathbf{u}(\tau) - \partial_\tau^j \mathbf{u}^0(\tau)|_{s+1/2-j/2,\Omega}$$

$$\le C_5 \sum_{j=0}^{3} \sup_{0 \le \tau \le t} \|\partial_\tau^{j+1} \bar{X}_1(\tau) - \partial_\tau^{j+1} \bar{X}_1^0(\tau)\|_{H^{s+1/2-j/2}(\mathbf{R}^1)}$$

for $s > 2$, $0 \le t \le T$, *where* $C_5 = C_5(e_1, e_2, e_3, e_7, s, d_0) > 0$.

Let us investigate equation (4.1.3).

Lemma 4.6.6. *Suppose that* Assumption 4.3 *with* $s \ge 3/2$ *is satisfied. Let* c_0 *be the constant chosen in* Assumption 4.1 *and* $\mathbf{u}$ *the solution of* (4.6.1) *obtained in* Theorem 4.6.2. *There exists a positive constant* $T_0 (\le T)$ *such that* $X = X(t,x)$, $\check{X} = \check{X}(t,x)$ *defined by* (4.1.3), $(4.1.4)_2$ *satisfy* (4.4.6) *with* T *replaced by* T_0.

Proof. It is obvious that

$$\begin{cases} |||\partial_t^j X(t)|||_{s+3/2-j/2} + \|\partial_t^j \check{X}(t)\|_{H^{s+3/2-j/2}(\mathbf{R}^1)} \le |\partial_t^{j-1} \mathbf{u}(t)|_{s+3/2-j/2,\Omega}, \quad j = 1,2,3,4, \\ |||\partial_t^j X(t)|||_s + \|\partial_t^j \check{X}(t)\|_{H^s(\mathbf{R}^1)} \le |\partial_t^{j-1} \mathbf{u}(t)|_{s,\Omega}, \quad j = 1,2, \\ |||X(t)|||_{s+1} \le t \sup_{0 \le \tau \le t} |\mathbf{u}(\tau)|_{s+1,\Omega}, \\ |||X(t)|||_2 \le t \sup_{0 \le \tau \le t} |\mathbf{u}(\tau)|_{s+1,\Omega} \end{cases}$$

for $0 \le t \le T$. If we define $T_0, d, l_j (j = 1, 2, \dots, 6)$ appropriately, then the desired result follows. ∎

Proposition 4.6.2. *Suppose that* $X, \check{X}$ *and* $X^0, \check{X}^0$ *satisfy* $(4.1.3), (4.1.4)_2$ *for* $\mathbf{u}, \mathbf{u}^0$, *respectively. Then we have*

$$\begin{cases} |||\partial_t^{j+1}(X(t) - X^0(t))|||_{s+1/2-j/2} + \|\partial_t^{j+1}(\check{X}(t) - \check{X}^0(t))\|_{H^{s+1/2-j/2}(\mathbf{R}^1)} \\ \qquad \le C_6 |\partial_t^j(\mathbf{u}(t) - \mathbf{u}^0(t))|_{s+1/2-j/2,\Omega}, \quad j = 0,1,2,3, \\ |||X(t) - X^0(t)|||_{s+1/2} \le C_6 \int_0^t |\mathbf{u}(\tau) - \mathbf{u}^0(\tau)|_{s+1/2,\Omega} \mathrm{d}\tau, \end{cases}$$

where $C_6 = C_6(s, d_0) > 0,\ 0 \le t \le T$.

4.7. Proof of Theorem 4.1

In this section we prove Theorem 4.1. Let us define the sets $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ as

$$\begin{aligned} \mathcal{S}_1 = \{&\bar{X};\ \bar{X} \text{ satisfies } (4.5.20), (4.5.21) \text{ and} \\ &\bar{X}|_{t=0} = \widetilde{X},\ \bar{X}_t|_{t=0} = \widetilde{X_t},\ \bar{X}_{tt}|_{t=0} = \widetilde{Y}\}, \\ \mathcal{S}_2 = \{&\mathbf{u};\ \mathbf{u} \text{ satisfies } (4.6.13), (4.6.14) \text{ and} \\ &\mathbf{u}|_{t=0} = \mathbf{v}_0,\ \mathbf{u}_t|_{t=0} = \mathbf{w}_0\}, \\ \mathcal{S}_3 = \{&(X, \check{X});\ X \text{ and } \check{X} \text{ satisfy } (4.4.6) \text{ and} \\ &X|_{t=0} = (0,0),\ X_t|_{t=0} = \mathbf{u}_0,\ X_{tt}|_{t=0} = \mathbf{w}_0, \\ &\check{X}|_{t=0} = (0,0),\ \check{X}_t|_{t=0} = \mathbf{u}_0|_{\Gamma_b},\ \check{X}_{tt}|_{t=0} = \mathbf{w}_0|_{\Gamma_b}\}, \end{aligned}$$

where $\mathbf{w}_0$ is the solution of

$$\begin{cases} \nabla \cdot \mathbf{w}_0 = 2\left(\dfrac{\partial v_{01}}{\partial x_2}\dfrac{\partial v_{02}}{\partial x_1} - \dfrac{\partial v_{01}}{\partial x_1}\dfrac{\partial v_{02}}{\partial x_2}\right), \quad \nabla^\perp \cdot \mathbf{w}_0 = 0 & \text{in } \Omega, \\ w_{01} = \widetilde{Y}_1 & \text{on } \Gamma_s, \\ \mathbf{w}_0 \cdot \mathbf{n}(x) = \mathbf{w}_0 \cdot \left\{\mathbf{n}(x) - \mathbf{n}(x + \int_0^t \mathbf{u})\right\} - \mathbf{v}_0 \cdot \{(\mathbf{u}_0 \cdot \nabla)\mathbf{n}(x)\} & \text{on } \Gamma_b. \end{cases} \tag{4.7.1}$$

First, notice that if (4.1.1) is satisfied, (4.5.18), (4.6.12) are valid. For $(X^0, \check{X}^0) \in \mathcal{S}_3$ we denote by $\bar{X} = M_1(X^0, \check{X}^0)$ the solution of problem (4.5.1) – (4.5.3) with H replaced by $H(X^0, \check{X}^0)$. In view of (4.4.7), J and J' in Assumption 4.2 are taken as follows:

$$J = C_2\mu_s, \quad J' = C_1\mu_s.$$

There exists a positive constant $\varepsilon_2 = \varepsilon_2(g)$ such that if

$$\|\mathbf{u}_0\|_{H^{2+1/2}(\Omega)} + \|\omega_0\|_{H^{2+1/2}(\Omega)} \le \varepsilon_2,$$

then the second estimate of (4.5.19) is satisfied. Therefore Proposition 4.4.2 and the arguments in Section 4.5 show that M_1 maps from $\mathcal{S}_3$ to $\mathcal{S}_1$ for a sufficiently small T. For

$\bar{X} \in \mathcal{S}_1$, let $\mathbf{u} = M_2(\bar{X})$ be the solution of (4.6.1). We see that $\mathbf{u}|_{t=0} = \mathbf{v}_0$ follows from (4.6.1) at $t = 0$ and that $\mathbf{u}_t|_{t=0} = \mathbf{w}_0$ since $\mathbf{u}_t|_{t=0}$ satisfies the same equations as (4.7.1). Therefore, M_2 becomes a mapping from $\mathcal{S}_1$ to $\mathcal{S}_2$ on some time interval $[0, T_1]$, owing to the results in Section 4.6. Here we denote $T_1(\leq T)$ by T again. For $\mathbf{u} \in \mathcal{S}_2$ define X by (4.1.3), $\check{X}$ by $(4.1.4)_2$ and set $M_3(\mathbf{u}) = (X, \check{X})$. We see that M_3 is a mapping from $\mathcal{S}_2$ to $\mathcal{S}_3$.

Here we define the approximate solutions $\{\bar{X}^n, \mathbf{u}^n, X^n, \check{X}^n\}$, $n = 1, 2, 3, \ldots$, as

$$\begin{cases} X^0(t) = t\mathbf{u}_0, \quad \check{X}^0(t) = t\mathbf{u}_0|_{\Gamma_b} \quad \text{for } t \geq 0, \\ \bar{X}^n = M_1(X^{n-1}, \check{X}^{n-1}), \quad \mathbf{u}^n = M_2(\bar{X}^n), \quad (X^n, \check{X}^n) = M_3(\mathbf{u}^n) \quad \text{for } n = 1, 2, 3, \ldots. \end{cases}$$

If we take T sufficiently small, $\bar{X}^1 = M_1(X^0, \check{X}^0)$ is well-defined and belongs to $\mathcal{S}_1$ because X^0 and $\check{X}^0$ satisfy (4.4.6). Repeating this argument, we conclude that $\{\bar{X}^n, \mathbf{u}^n, X^n, \check{X}^n\}$ are well-defined and $\bar{X}^n \in \mathcal{S}_1$, $\mathbf{u}^n \in \mathcal{S}_2$, $(X^n, \check{X}^n) \in \mathcal{S}_3$, $n = 1, 2, 3, \ldots$. Propositions 4.5.1, 4.6.1, 4.6.2 and 4.4.2 show that there exist $\bar{X}, \mathbf{u}, X$ and $\check{X}$ such that

$$\begin{aligned} \sup_{0 \leq \tau \leq t} \Bigg\{ & \|\bar{X}^n(\tau) - \bar{X}(\tau)\|_{H^{s+1/2}(\mathbf{R}^1)} + \sum_{j=0}^{3} \|\partial_\tau^{j+1} \bar{X}_1^n(\tau) - \partial_\tau^{j+1} \bar{X}_1(\tau)\|_{H^{s+1/2-j/2}(\mathbf{R}^1)} \\ & + \sum_{j=0}^{2} \|\partial_\tau^{j+1} \bar{X}_2^n(\tau) - \partial_\tau^{j+1} \bar{X}_2(\tau)\|_{H^{s+1/2-j/2}(\mathbf{R}^1)} + |||X^n(\tau) - X(\tau)|||_{s+1/2} \\ & + \sum_{j=0}^{3} \Big(|||\partial_\tau^{j+1} X^n(\tau) - \partial_\tau^{j+1} X(\tau)|||_{s+1/2-j/2} + \|\partial_\tau^{j+1} \check{X}^n(\tau) - \partial_\tau^{j+1} \check{X}(\tau)\|_{H^{s+1/2-j/2}(\mathbf{R}^1)} \Big) \\ & + \sum_{j=0}^{3} |\partial_\tau^j \mathbf{u}^n(\tau) - \partial_\tau^j \mathbf{u}(\tau)|_{s+1/2-j/2,\Omega} \Bigg\} \to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

We see that $\bar{X}, \mathbf{u}, X$ and $\check{X}$ are solutions of problem (4.5.1) – (4.5.3), problem (4.6.1), (12), problem (4.1.3) and problem $(4.1.4)_2$, respectively. Moreover $\bar{X} \in \mathcal{S}_1$, $\mathbf{u} \in \mathcal{S}_2$, $(X, \check{X}) \in \mathcal{S}_3$.

If we set

$$\mathbf{v}(t, z) = \mathbf{u}(t, \Phi_{\mathbf{u}}^{-1}(z; t)), \quad \omega(t, z) = \omega_0(t, \Phi_{\mathbf{u}}^{-1}(z; t)), \quad \Omega(t) = \Phi_{\mathbf{u}}(\Omega; t),$$

then $(4.1.4)_1$ holds.

The uniqueness of the solutions to problem (4.5.1) – (4.5.3), problem (4.6.1), (12), problem (4.1.3) and problem (4.1.4) is proved in the same way.

Finally we define q as a solution of the boundary value problem

$$\begin{cases} \Delta q = -\nabla \cdot (A_{\mathbf{u}}^{-1}\mathbf{u}_t) & \text{in } \Omega,\ t \geq 0, \\ q = g\left(x_2 + \int_0^t u_2(\tau, x)\mathrm{d}\tau\right) & \text{on } \Gamma_s,\ t \geq 0, \\ \dfrac{\partial q}{\partial \mathbf{n}(\Phi_{\mathbf{u}})} = -(\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{u} \cdot \mathbf{n}(\Phi_{\mathbf{u}}) & \text{on } \Gamma_b,\ t \geq 0. \end{cases} \tag{4.7.2}$$

Then the results in Section 4.6 imply the unique existence of the solution q of (4.7.2) satisfying $q \in C^j([0,T]; H^{s+2-j/2}(\Omega))$, $j = 1, 2$, for a sufficiently small T.

We see that $(\mathbf{u}, q)$ satisfy (8) – (12) and (4.1.2). The uniqueness of the solution to problem (8) – (12) comes from that of problem (4.5.1) – (4.5.3), (4.6.1), (12), (4.1.3), and (4.1.4). The proof is complete.

Acknowledgement.

The author would like to express his deepest gratitude to Professor A. Tani for important suggestions and fruitful discussions during this reseach.

The author also would like to thank sincerely to Professors S. Ishikawa, N. Kikuchi, S. Ukai and M. Yoshizawa for their helpful comments and advice.

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