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2003

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by

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A dissertation
submitted to Keio University
in partial fulfillment of the requirements
for the degree of Ph.D. of Science

2003

Acknowledgements

My deepest appreciation goes to my supervisor Professor Masakazu Jimbo, Keio University, for leading me to the field of combinatorial design theory and giving me strong support. His patience, great knowledge, excellent guidance and continuous encouragement have led me to the successful completion of this thesis.

I would like to thank Professor Katsuhiro Ota, Professor Kunio Shimizu and Professor Yasubumi Sakakibara, who are referees of my thesis in Keio University, for their careful reading and useful comments concerning my thesis.

I am thankful to Professor Hung-Lin Fu in National Chiao Tung University, Taiwan, for providing me combinatorial problems. Also, I would like to appreciate Professor Sanpei Kageyama in Hiroshima University and Professor Ryoh Fuji-Hara in University of Tsukuba for their continuous support and instructive advice. I am indebted Professor Shinji Kuriki in Osaka Prefecture University, Professor Ying Miao in University of Tsukuba, Professor Miwako Mishima in Gifu University and Dr. Meinard Müller in Bonn University, for their knowledge and valuable advice. Moreover, I would like to convey my thanks to Dr. Takaaki Hishida in Aichi Institute of Technology, Dr. Kazuhiro Ozawa in Gifu College of Nursing, Dr. Nobuko Miyamoto in Tokyo University of Science, Dr. Satoshi Shinohara in Meisei University, Dr. Tomoko Adachi in Toho University and my colleagues in Jimbo Laboratory for their encouragement. It has been a great pleasure to meet all of them.

Lastly, I wish to express my gratitude to my parents, sister and friends who gave me a chance to study and provided me continuous warm support.

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Contents

Chapter 1

Introduction

Kirkman's schoolgirls problem: Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two walk twice abreast.

Combinatorial designs (or graph decompositions) have their roots in the work of Euler, who in 1782 introduced the *36 officers problem*. In the mid-19th century, Kirkman, Steiner and Cayley worked on combinatorial designs. The modern history of design theory is originated in the statistical design of experiments found by R. A. Fisher and F. Yates in 1920s. Stimulated by the statistical application, combinatorial design theory has been developed extensively by many researchers including Bose, Ryser, Hanani, Hall and others. The fundamental problems related to combinatorial designs are their *existence*, *construction* and *classification of non-isomorphic designs*. In 1970s, Wilson proved asymptotic existence of a BIB design and the technique was generalized to the case of simple graph decompositions of complete graphs.

Many authors proposed many useful designs. In 1979, Singh and Dey introduced a *balanced incomplete block design with nested rows and columns* (BIBRC for short), which is posed from the statistical point of view. Meanwhile, Raghavarao constructed square lattice designs. Recently, these designs are generalized to *grid-block designs* by Fu, Hwang, Jimbo, Mutoh and Shiue (2004) to utilize them for a pooling design in DNA library screening. These designs are classified into so-called "array type" designs, which is one of the main theme of this thesis.

An "edge-colored graph decomposition" is equivalent to some combinatorial design. That is, the existence of a combinatorial design is shown by applying a corresponding edge-colored graph decomposition of complete graph. In fact, array type designs can be represented by the terms of edge-colored graph decompositions. Such approach can provide more general results not only for array type designs.

In this thesis, we will discuss constructions of array type combinatorial designs and show existence of the designs and will determine the existence of array type designs for some specific parameters. The method of Lamken and Wilson (2000) is useful to show asymptotic existence of combinatorial designs, which correspond to that of simple edge-colored graph decompositions of complete graphs. However, their method may not be applied to the existence of some kinds of array type designs like BIBRCs. We will generalize their notion to the case of "colorwise simple graphs" and show asymptotic existence of such graph decompositions of complete graphs. Moreover, the results are applied to the existence problem of BIBRCs.

In this chapter, we briefly describe some backgrounds of combinatorial designs related to this thesis.

1.1 Background of combinatorial designs

The combinatorial designs were started by Euler who introduced 36 officers problem in 1782 and began the search for pairs of *orthogonal Latin squares* (or *mutually orthogonal Latin squares*). Euler went on to conjecture that such an $n \times n$ array does not exist for $n = 6$, nor does one exist whenever $n \equiv 2 \pmod 4$. This was known as the *Euler conjecture* until its disproof in 1960 by Bose, Shrikhande and Parker [18].

In the mid-19th century, Kirkman [59, 60] and Steiner [92] proved the existence of *Steiner triple systems*. Kirkman introduced the 15 schoolgirls problem in his paper [60]. The existence of *Kirkman triple systems* was a celebrated open problem throughout the period 1850-1970. The first published solution was given by Ray-Chaudhuri and Wilson [84]. The first record of a solution appears to be that of Lu Jiaxi [66] in Mongolia in 1965.

Thus, combinatorial design theory has been started from these problems. Since then, the first papers dealing directly with decompositions of graphs due to Petersen, Kempe, Tait, Heawood, König and others. Some of them are closely related to combinatorial designs.

In the early part of the 20th century, design of experiments has been built up by two founders, R. A. Fisher and F. Yates. In 1925, Fisher introduced the three basic principles for planning experiments, i.e., (i) replication, (ii) randomization, (iii) local control (or blocking), in his famous books [42, 43]. In 1936, Yates proposed the use of *balanced incomplete block designs* for some agricultural experiments in his paper [106].

At the earliest stage of the study, there was no other practical application than agricultural field experiment. Later, block designs have played an important role in industrial experiment. Indeed, it is known that orthogonal array, factorial designs, etc., which are often used for production management in industries, have a deep relation with some kinds of block designs (see, for instance, [14, 29, 38]). Also, we can find a significant role of block designs also in the filed of information science, for example, coding theory [10, 68], cryptography [93], computer science [31], etc.

In the mid-20th century, Bose [16], Rao [82, 83] studied systematic constructions for combinatorial designs by using finite fields and finite geometries. Those techniques have been further developed and contributed to the investigation of various kinds of block designs. Subsequently, Hanani [46, 47, 48] proved the existence of BIB designs with block sizes 3, 4 and 5 and gave partial results for 6 and 7. Recently, we obtained the partial results of the existence of BIB designs with block sizes smaller than 10. Also, Bose [17] introduced the term of *resolvable designs*, which was initially posed by Kirkman [60].

Another attractive property for block designs would be *automorphisms* (for example, *cyclic*, *abelian* and *rotational* property). The reason why cyclic (or abelian, rotational) property is so attractive is that we can generate designs easily from a set of blocks called *base blocks* (*initial* or *starter blocks*) without knowing all blocks of the designs.

The concept of a *nested design* was introduced by Preece [80] in 1967 as a generalization of a resolvable design. In 1979, Singh and Dey [90] introduced a *balanced incomplete block design with nested rows and columns* (BIBRC for short), and they gave a construction with some examples. Several constructions are obtained in many papers. However, most constructions give *completely balanced BIBRCs* (or *criss-cross nested BIBDs*), which were introduced by Morgan [72] and Preece [80], respectively. Uddin and Morgan [98] gave constructions for non-completely balanced BIBRC. As far as the author knows, these are only direct constructions for non-completely balanced BIBRC.

Moreover, combinatorial designs were used as an efficient way of group testing such as medical science and pharmaceutical science (see, for example, Du and Hwang [40]). Recently, Hwang [52] proposed array type designs for DNA library screenings (see, for example, [12, 13]). Since then, Fu, Hwang, Jimbo, Mutoh and Shiue [44] introduced *grid-block designs* for the application to the DNA library screenings. Berger, Mandell and Subrahmanya [13] showed that array type designs are useful for DNA library screenings from information theoretical point of view.

In 1970's, Wilson [99, 100, 101] showed that *PBD-closed sets* are eventually periodic. Three years later, he [103] showed that the necessary conditions for existence of BIB designs are sufficient for all sufficiently large positive integers by utilizing the previous result. Afterward, he [102, 104] also proved that the necessary conditions for existence of simple graph decompositions of complete graphs are sufficient for all sufficiently large integers by the same result.

Since then, Colbourn and Stinson [32] and Caro, Roditty and Schönheim [22, 23, 24] worked on some edge-colored designs (or edge-colored graph decompositions). Lamken and Wilson [63] proved that necessary conditions for simple edge-colored graph decompositions of complete graphs are sufficient for all sufficiently large integers. And they mentioned that these graph decompositions are equivalent to some combinatorial designs, for example, resolvable BIB designs, nested BIB designs, reverse triple systems, skew room squares, etc.

Again, the fundamental problems related to designs are their "existence," "construction" and "classification of non-isomorphic designs" from a combinatorial (or mathematical) point of view. In this thesis, we will mention constructions of array type combinatorial designs and show existence of the designs. And we will generalize the technique of Lamken and Wilson to *colorwise simple* edge-colored graphs and show the asymptotic existence of such graph decompositions of complete graphs and BIBRCs.

1.2 BIB designs and other combinatorial designs

Let V be a set of v elements, called *points* or *treatments*, and $\mathcal{B}$ be a collection of k-subsets, called *blocks*, of V, where $|\mathcal{B}| = b$. A pair $(V, \mathcal{B})$ is called a *balanced incomplete block* (BIB) *design* or 2-*design*, if the following conditions are satisfied:

(i) Every point occurs at most once in each block of $\mathcal{B}$.

(ii) Every pair of two distinct points of V occurs in exactly λ blocks of $\mathcal{B}$.

It is easy to see that the number r of blocks containing a given point x is a constant not depending on the choice of x and that the relations

$$vr = bk \quad \text{and} \quad \lambda(v-1) = r(k-1) \tag{1.2.1}$$

hold among the five parameters v, k, r, b, and λ of a BIB design. Since the parameters satisfies the relations (1.2.1), a BIB design is often denoted by B(v, k, λ) by omitting b and r.

Example 1.2.1 A B(7, 3, 1) is given by $V = \{0, 1, \ldots, 6\}$ and

$$\mathcal{B} = \left\{ \begin{array}{llll} \{0\ \ 1\ \ 3\}, & \{1\ \ 2\ \ 4\}, & \{2\ \ 3\ \ 5\}, & \{3\ \ 4\ \ 6\}, \\ \{4\ \ 5\ \ 0\}, & \{5\ \ 6\ \ 1\}, & \{6\ \ 0\ \ 2\} & \end{array} \right\}.$$

It can be readily checked that each pair of distinct points occurs together in exactly one block, i.e., $\lambda = 1$.

Here, we define an isomorphic BIB design as follows. Let $(V_1, \mathcal{B}_1)$ and $(V_2, \mathcal{B}_2)$ be a $\mathrm{B}(v, k, \lambda)$. $(V_1, \mathcal{B}_1)$ and $(V_2, \mathcal{B}_2)$ are isomorphic $\mathrm{B}(v, k, \lambda)$'s if there exists a bijection $\sigma : V_1 \to V_2$ such that B_1^σ belongs to $\mathcal{B}_2$ for any $B_1 \in \mathcal{B}_1$, where $B^\sigma = \{b_1^\sigma, b_2^\sigma, \ldots, b_k^\sigma\}$.

By the equations (1.2.1), the following lemma is obtained.

Lemma 1.2.1 *Necessary conditions for the existence of a* $\mathrm{B}(v, k, \lambda)$ *are*

$$\lambda(v-1) \equiv 0 \pmod{k-1} \quad \textit{and} \quad \lambda v(v-1) \equiv 0 \pmod{k(k-1)}.$$

When $k = 3$, 4 and 5, it has been proved by Hanani [46, 47, 48] that the conditions of Lemma 1.2.1 are also sufficient for the existence of a BIB design except for the non-existence of B(15, 5, 2). For $k \geq 6$, the conditions in Lemma 1.2.1 may not be sufficient in general. For $k = 6$, 7 and 8, partial results were established for some specified λ by Abel, Bluskov and Greig [1], Abel, Finizio, Greig and Lewis [2, 3], Abel and Greig [6], Hanani [47, 48], etc.

Let M and K be finite or infinite sets of positive integers. Again, assume that V is a finite set of v points and $\mathcal{B}$ is a collection of blocks of V, size of each block from a set K, i.e., $K = \{|B| : B \in \mathcal{B}\}$. Further let $\mathcal{G}$ be a partition of V into subsets called *groups* whose sizes belong to M. Then a triple $(V, \mathcal{G}, \mathcal{B})$ is called a *group divisible design*, denoted by $\mathrm{GD}(v, K, M, \lambda)$, if the following conditions are satisfied:

(i) For each group $G \in \mathcal{G}$ and each block $B \in \mathcal{B}$, $|G \cap B| \leq 1$ holds.

(ii) Every pair of points from distinct groups occurs in exactly λ blocks.

The *type* of a group divisible design $(V, \mathcal{G}, \mathcal{B})$ is the multiset of $\{|G| : G \in \mathcal{G}\}$ and an exponential notation is used to describe types: a type $g_1^{u_1} g_2^{u_2} \cdots g_n^{u_n}$ denotes u_i occurrences of g_i, $1 \leq i \leq n$.

When $M = \{1\}$, that is, the type of a group divisible design is 1^v, then a pair $(V, \mathcal{B})$ is called a *pairwise balanced block design* (PBD), denoted by $\mathrm{B}(v, K, \lambda)$. When $M = \{1\}$ and $K = \{k\}$ for an integer k, a group divisible design $(V, \mathcal{G}, \mathcal{B})$ is naturally a BIB design $(V, \mathcal{B})$. While, when $M = \{n\}$, $K = \{k\}$ and the type of a group divisible design is n^k, a triple $(V, \mathcal{G}, \mathcal{B})$ is called a *transversal design*.

Example 1.2.2 A GD(20, {5}, {4}, 1) is given by $V = \{0, 1, \ldots, 19\}$,

$$\mathcal{G} = \left\{ \begin{array}{l} \{0\ \ 1\ \ 2\ \ 3\},\ \{4\ \ 5\ \ 6\ \ 7\},\ \{8\ \ 9\ \ 10\ \ 11\}, \\ \{12\ \ 13\ \ 14\ \ 15\},\ \{16\ \ 17\ \ 18\ \ 19\} \end{array} \right\}$$

and

$$\mathcal{B} = \left\{ \begin{array}{l} \{0\ 4\ 8\ 12\ 16\},\ \{0\ 5\ 9\ 13\ 17\},\ \{0\ 6\ 10\ 14\ 18\},\ \{0\ 7\ 11\ 15\ 19\}, \\ \{1\ 4\ 9\ 14\ 19\},\ \{1\ 5\ 8\ 15\ 18\},\ \{1\ 6\ 11\ 12\ 17\},\ \{1\ 7\ 10\ 13\ 16\}, \\ \{2\ 4\ 10\ 15\ 17\},\ \{2\ 5\ 11\ 14\ 16\},\ \{2\ 6\ 8\ 13\ 19\},\ \{2\ 7\ 9\ 12\ 18\}, \\ \{3\ 4\ 11\ 13\ 18\},\ \{3\ 5\ 10\ 12\ 19\},\ \{3\ 6\ 9\ 15\ 16\},\ \{3\ 7\ 8\ 14\ 17\} \end{array} \right\},$$

which is a transversal design.

Group divisible designs, transversal designs and pairwise balanced block designs are useful to construct combinatorial designs recursively and to show the existence of combinatorial designs. For a finite or infinite set K of positive integers, $B(K)$ be the set of integers v such that there exists a B(v, K, 1). Then, K is called a *PBD-closed set* if $B(K) = K$ holds. This notion is the most useful tool for showing the asymptotic existence of combinatorial designs.

In 1960, Chowla, Erdös, and Straus [28] showed that transversal designs always exist for sufficiently large positive integers, where they use the term of the maximal number of pairwise orthogonal Latin squares. This proof was based on the result of Bose, *et al.* [18]. As far as the author knows, this result was the first asymptotic existence of the combinatorial designs.

In 1972, Wilson [100, 101] showed that PBD-closed sets are eventually periodic by combining his result [99] and Chowla, *et al.* [28]. Three years later, he [103] showed that the necessary conditions for the existence of BIB designs are sufficient for all sufficiently large positive integers by utilizing the property of PBD-closed sets.

Next, we define a resolvable BIB design. Let $(V, \mathcal{B})$ be a BIB design. For a subclass $\mathcal{B}' \subseteq \mathcal{B}$, if $\{B : B \in \mathcal{B}'\}$ is a partition of V, then $\mathcal{B}'$ is called a *resolution class* (or a *parallel class*). A pair $(V, \mathcal{B})$ is called a *resolvable* BIB design if the collection $\mathcal{B}$ of blocks can be partitioned into resolution classes.

Example 1.2.3 A resolvable B(15, 3, 1) is given by $V = \{\infty\} \cup \{0_0, 1_0, \ldots, 6_0\} \cup \{0_1, 1_1, \ldots, 6_1\}$ and

$$\mathcal{B} = \left\{ \begin{array}{l} \{\infty\ 0_0\ 0_1\},\ \{1_0\ 2_0\ 4_0\},\ \{2_1\ 3_0\ 5_1\},\ \{3_1\ 4_1\ 6_0\},\ \{5_0\ 6_1\ 1_1\}, \\ \{\infty\ 1_0\ 1_1\},\ \{2_0\ 3_0\ 5_0\},\ \{3_1\ 4_0\ 6_1\},\ \{4_1\ 5_1\ 0_0\},\ \{6_0\ 0_1\ 2_1\}, \\ \{\infty\ 2_0\ 2_1\},\ \{3_0\ 4_0\ 6_0\},\ \{4_1\ 5_0\ 0_1\},\ \{5_1\ 6_1\ 1_0\},\ \{0_0\ 1_1\ 3_1\}, \\ \{\infty\ 3_0\ 3_1\},\ \{4_0\ 5_0\ 0_0\},\ \{5_1\ 6_0\ 1_1\},\ \{6_1\ 0_1\ 2_0\},\ \{1_0\ 2_1\ 4_1\}, \\ \{\infty\ 4_0\ 4_1\},\ \{5_0\ 6_0\ 1_0\},\ \{6_1\ 0_0\ 2_1\},\ \{0_1\ 1_1\ 3_0\},\ \{2_0\ 3_1\ 5_1\}, \\ \{\infty\ 5_0\ 5_1\},\ \{6_0\ 0_0\ 2_0\},\ \{0_1\ 1_0\ 3_1\},\ \{1_1\ 2_1\ 4_0\},\ \{3_0\ 4_1\ 6_1\}, \\ \{\infty\ 6_0\ 6_1\},\ \{0_0\ 1_0\ 3_0\},\ \{1_1\ 2_0\ 4_1\},\ \{2_1\ 3_1\ 5_0\},\ \{4_0\ 5_1\ 0_1\} \end{array} \right\}.$$

This is a solution of Kirkman's fifteen schoolgirls problem.

It is obvious that the following result holds.

Lemma 1.2.2 *Necessary conditions for the existence of a resolvable* $(V, \mathcal{B})$ *are*

$$\lambda v \equiv 0 \pmod{k} \quad \textit{and} \quad \lambda(v-1) \equiv 0 \pmod{k-1}.$$

In fact, the first question on a resolvable design was to find a resolvable B(15, 3, 1) posed by Kirkman [60] in 1850, though the concept of resolvability was introduced much later, in 1942, by Bose [17]. In Mathon and Rosa [71], it can be found that there are exactly seven nonisomorphic resolvable B(15, 3, 1).

Solutions of the existence of a resolvable B(v, 3, 1) were given by Lu [66] in 1965 and Ray-Chaudhuri and Wilson [84] in 1971, independently. One year later, Hanani, Ray-Chaudhuri and Wilson [49] derived a necessary and sufficient condition for the existence of a resolvable B(v, 4, 1). Next year, Ray-Chaudhuri and Wilson [85] showed that the necessary conditions of resolvable B(v, k, 1) are sufficient for all sufficiently large integers by utilizing the property of PBD-closed set. In 1984, for any positive integer λ, Lu [67] proved that the necessary conditions of resolvable B(v, k, λ) are sufficient for all sufficiently large integers. His paper was written in Chinese. In 1995, Lee and Furino [64] translated his paper into English. Abel and Greig [5] constructed resolvable B(v, 5, 1) for all but six possible exceptions $v \in \{45, 185, 225, 345, 465, 645\}$. The existence of a B(185, 5, 1) was shown by Abel, Ge, Greig and Zhu [4].

1.3 Grid-block designs, packings and resolvability

Let V be a set of v points and $\mathcal{A}$ be a collection of $k_1 \times k_2$ arrays with elements in V. Each array in $\mathcal{A}$ is called a *grid-block*. A pair $(V, \mathcal{A})$ is called a *grid-block design*, denoted by GB(v, k_1, k_2), if the following conditions are satisfied:

(i) Every point occurs at most once in each grid-block of $\mathcal{A}$.

(ii) Every pair of two distinct points of V occurs exactly once in the same row or in the same column of a grid-block.

Especially, when $v = k_1 \times k_2$ and $k_1 = k_2$ hold, then a pair $(V, \mathcal{A})$ is called *square lattice design*.

Example 1.3.1 A GB(10, 2, 3) is given by $V = \{0, 1, \ldots, 9\}$ and

$$\mathcal{A} = \left\{ \boxed{\begin{array}{ccc} 1 & 2 & 4 \\ 7 & 6 & 9 \end{array}}, \boxed{\begin{array}{ccc} 2 & 3 & 5 \\ 8 & 7 & 0 \end{array}}, \boxed{\begin{array}{ccc} 3 & 4 & 6 \\ 9 & 8 & 1 \end{array}}, \boxed{\begin{array}{ccc} 4 & 5 & 7 \\ 0 & 9 & 2 \end{array}}, \boxed{\begin{array}{ccc} 5 & 6 & 8 \\ 1 & 0 & 3 \end{array}} \right\}.$$

Example 1.3.2 A GB(9, 3, 3) is given by $V = \{1, 2, \ldots, 8\}$ and

$$\mathcal{A} = \left\{ \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}}, \boxed{\begin{array}{ccc} 1 & 6 & 8 \\ 9 & 2 & 4 \\ 5 & 7 & 3 \end{array}} \right\},$$

which is a square lattice design.

In a $k_1 \times k_2$ grid-block design $(V, \mathcal{A})$, each point x of V has $v-1$ distinct points which occur together with x in the same row or in the same column, while each entry of a $k_1 \times k_2$ grid-block has $k_1 + k_2 - 2$ entries in the same row or in the same column. That is, the number r of grid-blocks containing a given point x is

$$r = \frac{v-1}{k_1 + k_2 - 2}, \tag{1.3.1}$$

which is a constant not depending on the choice of x. Also, there are $v(v-1)/2$ pairs which occur once in a grid-block of $\mathcal{A}$ while each grid-block generates $k_1k_2(k_1 + k_2 - 2)/2$ pairs. Thus, the number b of grid-blocks is

$$b = \frac{v(v-1)}{k_1k_2(k_1 + k_2 - 2)}. \tag{1.3.2}$$

Since r and b must be integers, we obtain the following lemma from equations (1.3.1) and (1.3.2).

Lemma 1.3.1 *Necessary conditions for the existence of a* GB(v, k_1, k_2) *are*

$$\begin{aligned} v - 1 &\equiv 0 \pmod{k_1 + k_2 - 2} \text{ and} \\ v(v-1) &\equiv 0 \pmod{k_1k_2(k_1 + k_2 - 2)}. \end{aligned} \tag{1.3.3}$$

A grid-block designs was introduced by Fu *et al.* [44] to apply it to DNA library screening. When $k_1 = k_2 = 2$, it is known that the condition of Lemma 1.3.1 is also sufficient for the existence of a GB$(v, 2, 2)$ in terms of "4-cycle systems." When $k_1 = 2$ and $k_2 = 3$, it has been proved by Carter [25] that the condition of Lemma 1.3.1 is also sufficient for the existence of a GB$(v, 2, 3)$. He utilized the notion of 3-regular graph decompositions.

Next, we define a packing and a grid-block packing. For a set V of v points, let $\mathcal{B}$ be a collection of k-subsets. A pair $(V, \mathcal{B})$ is called a *packing*, denoted by P(v, k, λ), if the following condition (ii)' is satisfied instead of the condition (ii) in the definition of a BIB design:

(ii)' Every pair of two distinct points of V occurs in at most λ blocks $\mathcal{B}$.

Similarly, let $\mathcal{A}$ be a collection of grid-blocks. A pair $(V, \mathcal{A})$ is called a *grid-block packing*, denoted by GBP(v, k_1, k_2), if the following condition (ii)' is satisfied instead of the condition (ii) in the definition of a $k_1 \times k_2$ grid-block design:

(ii)' Every pair of two distinct points of V occurs at most once in the same row or in the same column of a grid-block.

Similarly, for a packing $(V, \mathcal{B})$, it is called a resolvable packing if the collection of blocks can be partitioned into resolution classes. And for a grid-block design (or grid-block packing) $(V, \mathcal{A})$ is also called resolvable if the collection of grid-blocks can be partitioned into resolution classes. Example 1.3.2 is a resolvable grid-block design.

Example 1.3.3 A resolvable grid-block packing GBP(8, 2, 2) is given by $V = \{\infty_0, 0_0, 1_0, 2_0\} \cup \{\infty_1, 0_1, 1_1, 2_1\}$ and

$$\mathcal{A} = \left\{ \begin{array}{ccc} \boxed{\begin{array}{cc} \infty_0 & 0_0 \\ 0_1 & \infty_1 \end{array}}, & \boxed{\begin{array}{cc} \infty_0 & 1_0 \\ 1_1 & \infty_1 \end{array}}, & \boxed{\begin{array}{cc} \infty_0 & 2_0 \\ 2_1 & \infty_1 \end{array}}, \\ \\ \boxed{\begin{array}{cc} 1_0 & 2_0 \\ 2_1 & 1_1 \end{array}}, & \boxed{\begin{array}{cc} 2_0 & 0_0 \\ 0_1 & 2_1 \end{array}}, & \boxed{\begin{array}{cc} 0_0 & 1_0 \\ 1_1 & 0_1 \end{array}} \end{array} \right\}.$$

Example 1.3.4 A resolvable grid-block packing GBP(18, 3, 3) is given by $V = \{0, 1, \ldots, 17\}$ and

$$\mathcal{A} = \left\{ \begin{array}{ccc} \boxed{\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{array}}, & \boxed{\begin{array}{ccc} 0 & 4 & 8 \\ 5 & 6 & 9 \\ 7 & 10 & 14 \end{array}}, & \boxed{\begin{array}{ccc} 0 & 9 & 13 \\ 10 & 17 & 8 \\ 12 & 7 & 3 \end{array}}, \\ \\ \boxed{\begin{array}{ccc} 9 & 10 & 11 \\ 12 & 13 & 14 \\ 15 & 16 & 17 \end{array}}, & \boxed{\begin{array}{ccc} 1 & 3 & 15 \\ 12 & 16 & 11 \\ 17 & 2 & 13 \end{array}}, & \boxed{\begin{array}{ccc} 1 & 5 & 11 \\ 6 & 15 & 2 \\ 16 & 14 & 4 \end{array}} \end{array} \right\}.$$

For a grid-block packing $(V, \mathcal{A})$ with v points, let r_x be the number of grid-blocks containing a point x. Then,

$$r_x \leq \left\lfloor \frac{v-1}{k_1 + k_2 - 2} \right\rfloor$$

holds, where $\lfloor a \rfloor$ be the largest integers not exceeding a. If a grid-block packing is resolvable, then v is divisible by k_1k_2, r_x is a constant $(= r)$ and the number of grid-blocks is

$$b = r\frac{v}{k_1k_2} \leq \frac{v}{k_1k_2}\left\lfloor\frac{v-1}{k_1+k_2-2}\right\rfloor.$$

A resolvable grid-block packing attaining this bound is said to be *maximal*. In Example 1.3.3, the resolvable GBP(8, 2, 2) is maximal. On the other hand, the resolvable GBP(18, 3, 3) in Example 1.3.4 is not maximal since the upper bound of the number of resolution classes is 4.

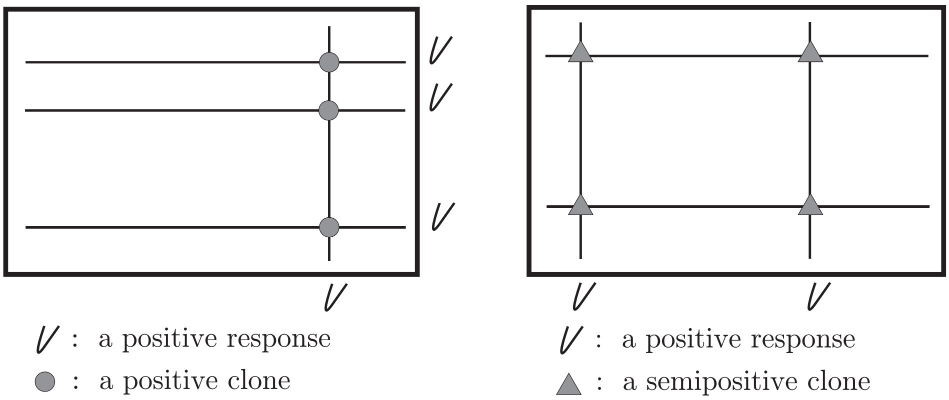
1.4 DNA library screening: an application of grid-block designs

In DNA library screening, there are many oligonucleotides (clones) to be tested whether they are positive or negative. An oligonucleotide is a short string of nucleotides *adenine* (A), *cytosine* (C), *guanine* (G) and *thymine* (T). The goal of a DNA library screening is to identify all positive clones. Economy of time and costs requires that the clones be assayed in groups. Each group is called a *pool*. If a pool gives a negative outcome, all clones contained in it are found to be negative. In this case, we can save numbers of tests. On the other hand, if the pool is positive, at the second stage we test each clone individually. This screening method is called a two-stage test, which is a popular group testing.

In such screening, a microtiter plate, which is an array with size 8×12 or 16×24, etc. is utilized and different clones are settled in each spot, called *well*, of the plate.

In this method, every row and every column in a microtiter plate is tested at the same time as a pool in the first stage, and each clone with positive response is tested individually in the second stage. This method is called the *basic matrix method* (BMM). In this method each clone is tested twice. If the array contains only a single positive clone, or more generally, if there is only one row (or column) of positive then we can determine the positive clones without individual tests. However, it does not always occur, that is, arrays often contain several positive clones. For example if two rows and two columns are positive as we see in Figure 1.4.1 (b), we can not determine whether the four clones settled at the crossing spots of positives are really positive or not.

Thus, if it is allowed to test more than twice for each clone, then, it is desired that every two distinct clones occur at most once in the same row or the same column, which is called the *unique collinearity condition*. The efficiency of the unique collinearity condition was shown by Barillot, Lacroix and Cohen [12] by simulation and was also proved theoretically by Berger, Mandell and Subrahmanya [13].



(a) One column is positive.

(b) Two rows and columns are positive.

Figure 1.4.1: Results of the first stage group tests in DNA library screening.

We consider the case when there is a single positive clone within the set of v clones and we place those clones on t $k_1 \times k_2$ microtiter plates at random allowing repetition, where $n = tk_1k_2 \geq v$ holds. Then, the expectation of the total number of different clones, which occur in at least one microtiter plate, is

$$\frac{1}{v^n}\sum_{k=1}^{v} k\binom{v}{k}\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}i^n = v - \frac{(v-1)^n}{v^{n-1}} = v - \left(1-\frac{1}{v}\right)^n v.$$

In this case, the expectation of the number of individual tests we need is at least $(1-\frac{1}{v})^n v$. However, if $n = v$ and each clone is settled exactly once on the microtiter plates, then we can decide the positive/negative only by the first stage group tests since there is only one positive clone and we can reduce about $(1-\frac{1}{v})^v v$ tests comparing with the randomly allocated test. In the case when the probability p of positive clones are given, we can also show it by a simulation shown in Figure 1.4.2. In this figure, there are $v = 1000$ clones. The vertical line is the number of tests for (i) the case of the same replication number and (ii) the case when the replication numbers are not constant. From Figure 1.4.2, we can see that, in case of the constant replications, the number of tests can be reduced comparing with the case of non-constant replications.

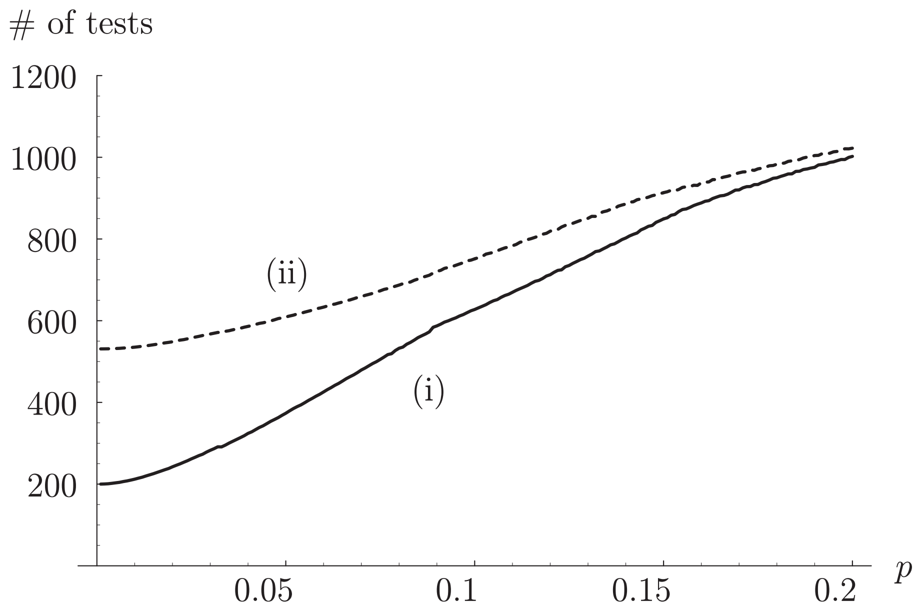


Figure 1.4.2: A simulation result of a comparison between (i) constant replications and (ii) random replications.

It is a favorite property that the number of replications for each clone should be almost the same in the first stage. This condition is called *the equal replication number of tests*.

A $k_1 \times k_2$ grid-block packing defined by Section 1.3 satisfies "the unique collinearity condition," besides, a resolvable $k_1 \times k_2$ grid-block packing satisfies also "the equal replication number of tests."

Berger *et al.* [13] gave the optimal size of the array and the optimal replication number according as the probability (ratio) p of positive clones under the implicit condition of the equal replication number of tests. Though they utilized the terminology of "n-dimensional array," it implies that the replication numbers are equal $(= 2n)$. Knill, Bruno and Torney [96] considered non-adaptive group testing problems with some errors.

1.5 Nested BIB designs and BIB designs with nested rows and columns

For a set V of v points, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be collections of k_1-subsets (called *superblocks*) and k_2-subsets (called *subblocks*) of V, respectively, where k_2 divides k_1. A triple $(V, \mathcal{B}_1, \mathcal{B}_2)$ is called a *nested balanced incomplete block* *design* (nested BIB deign) and is denoted by nested B(v; k_1, λ_1; k_2, λ_2) if the triple satisfies the following conditions:

(i) (V, $\mathcal{B}_1$) is a B(v, k_1, λ_1),

(ii) (V, $\mathcal{B}_2$) is a B(v, k_2, λ_2) and

(iii) Each block of $\mathcal{B}_1$ can be partitioned into k_1/k_2 subblocks having k_2 elements each such that the resulting collection of subblocks coincides with the collection $\mathcal{B}_2$.

For a nested BIB design (V, $\mathcal{B}_1$, $\mathcal{B}_2$), we say that the blocks $\mathcal{B}_2$ are *nested within* those in $\mathcal{B}_1$.

The concept of this "nested BIB design" was first introduced in the statistical literature in 1967 by Preece [80] as a generalization of a resolvable design in which a resolution class and a block are considered as a nesting block and a subblock of a nested BIB design, respectively. Independently of Preece, in 1972, Federer [41] brought another concept under the name of a "nested BIB design." Kageyama and Miao [56, 57] unified the two concepts of nested designs.

Example 1.5.1 A nested B(7; 6, 5; 3, 2) is given by $V = \{0, 1, \ldots, 6\}$ and

$$\mathcal{B} = \left\{ \begin{array}{ll} (\{0\ \ 1\ \ 3\}\ \ \{5\ \ 4\ \ 2\}), & (\{1\ \ 2\ \ 4\}\ \ \{6\ \ 5\ \ 3\}), \\ (\{2\ \ 3\ \ 5\}\ \ \{0\ \ 6\ \ 4\}), & (\{3\ \ 4\ \ 6\}\ \ \{1\ \ 0\ \ 5\}), \\ (\{4\ \ 5\ \ 0\}\ \ \{2\ \ 1\ \ 6\}), & (\{5\ \ 6\ \ 1\}\ \ \{3\ \ 2\ \ 0\}), \\ (\{6\ \ 0\ \ 2\}\ \ \{4\ \ 3\ \ 1\}) & \end{array} \right\}.$$

Each part enclosed by parentheses is a superblock. In a superblock, there are two subblocks of size 3 which are enclosed by the braces.

Constructions for nested BIB designs have been studied by Bailey, Goldrei and Holt [11], Dey, Das and Banerjee [37], Jimbo and Kuriki [54], Kageyama and Miao [58] and other people. Morgan [72] and Morgan, Preece and Rees [73] gave some constructions of nested BIB designs and listed known results on the existence of nested BIB designs for $v \leq 36$ and $r \geq v - 1$. The uses and statistical analysis of nested designs are available in the literature (see, for example, [21, 41, 72, 80]).

Next, we give a definition of a "BIBRC." For a set V of v points, let $\mathcal{A}$ be a collection of b arrays of size $k_1 \times k_2$ (called *blocks*) whose entries are elements of V. We denote the numbers of blocks of $\mathcal{A}$ in which two distinct points x and y occur in the same row, in the same column and in the same block by $\lambda_R\{x, y\}$, $\lambda_C\{x, y\}$ and $\lambda_B\{x, y\}$, respectively. We often use $\lambda_E\{x, y\} = \lambda_B\{x, y\} - \lambda_R\{x, y\} - \lambda_C\{x, y\}$ instead of $\lambda_B\{x, y\}$. A pair $(V, \mathcal{A})$ is called a *balanced incomplete block design with nested rows and columns* (BIBRC for short), denoted by BIBRC(v, k_1, k_2, λ), if the following conditions are satisfied:

(i) Every point occurs at most once in each block of $\mathcal{A}$.

(ii) Every point occurs in exactly r blocks of $\mathcal{A}$.

(iii) For any pair of distinct points x and y,

$$\begin{aligned}\lambda &= k_1\lambda_R\{x, y\} + k_2\lambda_C\{x, y\} - \lambda_B\{x, y\}\\ &= (k_1 - 1)\lambda_R\{x, y\} + (k_2 - 1)\lambda_C\{x, y\} - \lambda_E\{x, y\}\end{aligned}$$

is a constant independent of the pair of points x and y.

A BIBRC was introduced by Singh and Dey [90]. Moreover, if the following stronger condition (iii)' holds instead of (iii), then a pair $(V, \mathcal{A})$ is called a *criss-cross nested BIBD* or a *completely balanced BIBRC* which were introduced by Preece [80] (see also [72]).

(iii)' For any pair of distinct points x and y, $\lambda_R\{x, y\}$, $\lambda_C\{x, y\}$ and $\lambda_B\{x, y\}$ (or $\lambda_E\{x, y\}$) are constants, say λ_R, λ_C and λ_B (or λ_E), independent of the pair of points x and y.

In this case, we call the constants λ_R, λ_C, λ_B and λ_E *indices* of a completely balanced BIBRC. For a completely balanced BIBRC, it is easy to show that the indices are uniquely determined by k_1, k_2 and λ as follows:

$$\lambda_R = \frac{\lambda}{k_1 - 1},\ \lambda_C = \frac{\lambda}{k_2 - 1},\ \lambda_B = \frac{(k_1k_2 - 1)\lambda}{(k_1 - 1)(k_2 - 1)},\ \text{and } \lambda_E = \lambda. \tag{1.5.1}$$

Example 1.5.2 A BIBRC(5, 2, 2, 1) is given by $V = \{0, 1, \ldots, 4\}$ and

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \right\}.$$

In this case, λ is a constant 1, but $\lambda_R\{x, y\}$, $\lambda_C\{x, y\}$ and $\lambda_B\{x, y\}$ are not. For example, the pair $\{0, 2\}$ is contained once in the same row, once in the same column and three times in blocks, which gives

$$\lambda = k_1\lambda_R\{0, 2\} + k_2\lambda_C\{0, 2\} - \lambda_B\{0, 2\} = 1.$$

On the other hand, the pair $\{3, 4\}$ is contained twice in the same row, three times in blocks, but not contained in any columns, which also gives

$$\lambda = k_1\lambda_R\{3, 4\} + k_2\lambda_C\{3, 4\} - \lambda_B\{3, 4\} = 1.$$

Example 1.5.3 A BIBRC(7, 2, 3, 2) is given by $V = \{0, 1, \ldots, 6\}$ and

$$\mathcal{A} = \left\{ \begin{array}{llll} \begin{pmatrix} 0 & 1 & 3 \\ 5 & 4 & 2 \end{pmatrix}, & \begin{pmatrix} 1 & 2 & 4 \\ 6 & 5 & 3 \end{pmatrix}, & \begin{pmatrix} 2 & 3 & 5 \\ 0 & 6 & 4 \end{pmatrix}, & \begin{pmatrix} 3 & 4 & 6 \\ 1 & 0 & 5 \end{pmatrix}, \\ \\ \begin{pmatrix} 4 & 5 & 0 \\ 2 & 1 & 6 \end{pmatrix}, & \begin{pmatrix} 5 & 6 & 1 \\ 3 & 2 & 0 \end{pmatrix}, & \begin{pmatrix} 6 & 0 & 2 \\ 4 & 3 & 1 \end{pmatrix} & \end{array} \right\}.$$

This BIBRC is a completely balanced BIBRC(7, 2, 3, 2). In this case, λ_R, λ_C and λ_B are constants, i.e., $\lambda_R = 2$, $\lambda_C = 1$ and $\lambda_B = 5$, independent of the choice of two distinct points.

For the existence of a BIBRC(v, k_1, k_2, λ), Singh and Dey [90] showed as the following:

Lemma 1.5.1 *Necessary conditions for the existence of a* BIBRC(v, k_1, k_2, λ) *are*

$$\begin{aligned} \lambda(v-1) &\equiv 0 \pmod{(k_1-1)(k_2-1)} \text{ and} \\ \lambda v(v-1) &\equiv 0 \pmod{k_1 k_2 (k_1-1)(k_2-1)}. \end{aligned} \tag{1.5.2}$$

For a BIBRC, several constructions were given by Agrawal and Prasad [7, 8, 9], Cheng [26], Hishida and Jimbo [51], Jimbo and Kuriki [54], Morgan [72], Mukerjee and Gupta [75], Street [94], Uddin [97], Uddin and Morgan [98], etc. The existence of BIBRC(v, 2, 2, λ) was completely solved by Srivastav and Morgan [91].

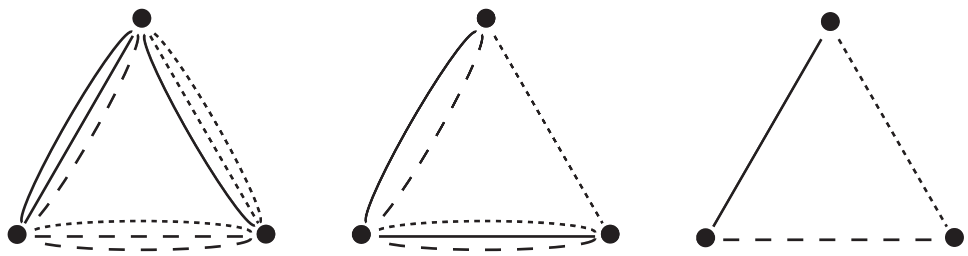
1.6 Graph decompositions of complete graphs

Let C be a set of colors $\{1, 2, \ldots, c\}$. An *edge-c-colored graph* G is an ordered 4-tuple $(X(G), E(G), \theta_G, \psi_G)$ consisting of a nonempty set $X(G)$ of *vertices*, a set $E(G)$, disjoint from $X(G)$, of *edges*, a *color function* θ_G assigned from $E(G)$ to C and an *incidence function* ψ_G that associates each edge of G with an unordered pair of distinct vertices of G. If i is a color and e is an edge such that $\theta_G(e) = i$, it is said that e has the color i. If e is an edge and x and y are vertices such that $\psi_G(e) = \{x, y\}$, then e is said to *join* x and y; the vertices x and y are called the *ends* of e.

Let E_i be the subset of E assigned color i, that is, $E_i = \theta_G^{-1}(i)$. Then E can be divided into disjoint sets $\{E_1, E_2, \ldots, E_c\}$ and we define $\mathcal{E}$ as the partition $\{E_1, E_2, \ldots, E_c\}$. As long as there are no confusion, we often omit θ_G and ψ_G and use a pair $G = (X, \mathcal{E})$ as an edge-c-colored graph instead of $(X(G), E(G), \theta_G, \psi_G)$. If C consists of a single color, an edge-1-colored graph $G = (X(G), E(G), \psi_G)$, or (X, E), is simply called *graph* which is the usual graph.

If $(X, E_1 \cup E_2 \cup \cdots \cup E_c)$ is a *simple graph*, that is $E_1 \cup E_2 \cup \cdots \cup E_c$ does not include multiple edges nor loops, then $(X, \mathcal{E})$ is called a *simple edge-c-colored graph*. And a pair $(X, \mathcal{E})$ is called a *colorwise simple graph with c colors* if (X, E_i) is a simple graph for each color i. That is, there are no loops, there is at most one edge $\{x, y\}$ between any two vertices x and y in each (X, E_i). When dealing with colorwise simple graph with c colors, it is often convenient to refer to the edge of color i with ends x and y as "the edge $\{x, y\}$ of color i."

Example 1.6.1 Three graphs G_1, G_2 and G_3 in Figure 1.6.1 are edge-3-colored, colorwise simple edge-3-colored and simple edge-3-colored graphs, respectively.



(a) an edge-3-colored graph

(b) a colorwise simple edge-3-colored graph

(c) a simple edge-3-colored graph

Figure 1.6.1: Examples of edge-3-colored graphs.

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_c)$ be a vector of positive integers. An edge-c-colored graph $G = (X, \mathcal{E})$ is called an *edge-c-colored complete graph of multiplicity* $\boldsymbol{\lambda}$, denoted by $K_v^{\boldsymbol{\lambda}}$, if the graph on v vertices has exactly λ_i edges of color i between any two distinct vertices x and y. When the greatest common divisor λ of λ_i's is greater than 1, we often use $\lambda K_v^{\boldsymbol{\lambda}/\lambda}$ instead of $K_v^{\boldsymbol{\lambda}}$. Especially in the case when $\boldsymbol{\lambda} = (1, 1, \ldots, 1)$, $K_v^{\boldsymbol{\lambda}}$ is a *colorwise simple edge-c-colored complete graph* and denoted by $K_v^{[c]}$ instead of $K_v^{\boldsymbol{\lambda}}$. Moreover, in the case of $c = 1$, $K_v^{[1]}$ is the usual complete graph, denoted by K_v.

Let $G = (X(G), E(G), \theta_G, \psi_G)$ and $G' = (X(G'), E(G'), \theta_{G'}, \psi_{G'})$ be edge-c-colored graphs with the same color set C. G is said to be *isomorphic* to G' if there exist bijections Φ_X from $X(G)$ to $X(G')$ and Φ_E from $E(G)$ to

$E(G')$ such that $\theta_G(e) = i$ and $\psi_G(e) = \{x, y\}$ if and only if $\theta_{G'}(\Phi_E(e)) = i$ and $\psi_{G'}(\Phi_E(e)) = \{\Phi_X(x), \Phi_X(y)\}$.

Let $\mathcal{F}$ be a family of subgraphs of a graph K. $\mathcal{F}$ is called a *decomposition* of K if every edge in $E(K)$ belongs to exactly one member of $\mathcal{F}$. Given a family $\mathcal{G}$ of edge-c-colored graphs, a $\mathcal{G}$*-decomposition* of K is a decomposition $\mathcal{F}$, denoted by $\mathrm{D}(K, \mathcal{G})$, such that every graph F in $\mathcal{F}$ is isomorphic to some graph G in $\mathcal{G}$. If $\mathcal{G}$ consists of a single graph G, then $\mathcal{G}$-decomposition is simply called a G*-decomposition* denoted by $\mathrm{D}(K, G)$.

Example 1.6.2 Let G_4 be a colorwise simple edge-2-colored graph shown in Figure 1.6.2. A $\mathrm{D}(K_7^{[2]}, G_4)$ with vertex set $V = \{0, 1, \ldots, 6\}$ is given by $\mathcal{F}$ in Figure 1.6.3.

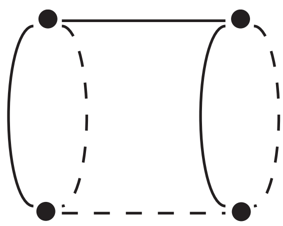


Figure 1.6.2: A colorwise simple edge-2-colored graph G_4.

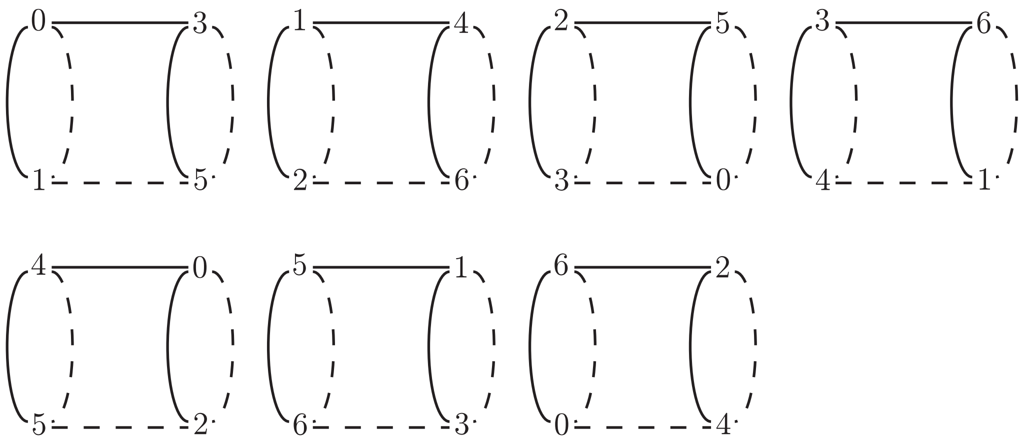


Figure 1.6.3: A G_4-decomposition of $K_7^{[2]}$.

There are a number of examples of decompositions of K_v into graphs with a single color. For example, a $\mathrm{B}(v, k, \lambda)$ is equivalent to a $\mathrm{D}(\lambda K_v, K_k)$ and a $\mathrm{B}(v,\, K,\, \lambda)$ is equivalent to a $\mathrm{D}(\lambda K_v,\, \mathcal{G})$, where $\mathcal{G}$ is a family of complete graphs with k vertices for $k \in K$.

The *cartesian product* of graphs $G = (X,\, E)$ and $G' = (X',\, E')$, denoted by $G \times G'$, is defined by a graph on the vertex set $X \times X'$ such that two vertices $\boldsymbol{x} = (x,\, x')$ and $\boldsymbol{y} = (y,\, y')$ are adjacent whenever $x = y$ and x' is adjacent to y' in G' or symmetrically if $x' = y'$ and x is adjacent to y in G. Then, a $k_1 \times k_2$ grid-block is equivalent to a graph $K_{k_1} \times K_{k_2}$ and a $\mathrm{GB}(v,\, k_1,\, k_2)$ is equivalent to a $\mathrm{D}(K_v,\, K_{k_1} \times K_{k_2})$.

Other decompositions of K_v into cycles and other small graphs have also been investigated and surveys of these results can be found in [15, 50, 65].

There are a few examples of decompositions of $K_v^{\boldsymbol{\lambda}}$ by graphs with more than one color (see [22, 23, 24, 32]). For a color set $C = \{1,\, 2\}$ and a vertex set X of sk_2 vertices, let $G_5 = (X,\, \mathcal{E})$ be the following simple edge-2-colored graph:

(i) $X_1,\, X_2,\, \ldots,\, X_s$ is a partition of X such that each group X_i has k_2 points.

(ii) There is an edge of color 1 between every two vertices from distinct groups.

(iii) There is an edge of color 2 between every two vertices from the same group.

Then, a nested $\mathrm{B}(v;\, k_1,\, \lambda_1;\, k_2,\, \lambda_2)$ is equivalent to a $\mathrm{D}(K_v^{(\lambda_1 - \lambda_2,\, \lambda_2)},\, G_5)$, where $k_1 = sk_2$.

Similarly, for a color set $C = \{1,\, 2,\, 3\}$ and for vertex sets X_1 and X_2 with k_1 and k_2 vertices each, let $G_6 = (X_1 \times X_2,\, \mathcal{E})$ be the following simple edge-3-colored graph:

(i) Every edge between vertices $(x_1,\, x_2)$ and $(x_1,\, x_2')$ has the color 1 for $x_1 \in X_1$ and $x_2 \neq x_2' \in X_2$.

(ii) Every edge between vertices $(x_1,\, x_2)$ and $(x_1',\, x_2)$ has the color 2 for $x_1 \neq x_1' \in X_1$ and $x_2 \in X_2$.

(iii) Every edge between vertices $(x_1,\, x_2)$ and $(x_1',\, x_2')$ has the color 3 for $x_1 \neq x_1' \in X_1$ and $x_2 \neq x_2' \in X_2$.

By identifying G_6 with a $k_1 \times k_2$ array, a completely balanced $\mathrm{BIBRC}(v,\, k_1,\, k_2,\, \lambda)$ is equivalent a $\mathrm{D}(K_v^{(\lambda_R,\, \lambda_C,\, \lambda_E)},\, G_6)$, where $\lambda_R = \lambda/(k_1 - 1)$, $\lambda_C = \lambda/(k_2 - 1)$ and $\lambda_E = \lambda$. Other decompositions of $K_v^{\boldsymbol{\lambda}}$ into some simple edge-r-colored graphs have been studied and such decompositions were applied to show the asymptotic existence of combinatorial designs by Lamken and Wilson [63].

We define a notion of "admissibility" to show necessary conditions for the existence of $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$. Let $\mathcal{G}$ be a family of edge-c-colored graphs G and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_c)$ be a vector of positive integers. For a vertex x of an edge-c-colored graph $G = (X, \mathcal{E})$, the *degree-vector* of x is defined by

$$\tau_G(x) = (\deg_1(x), \deg_2(x), \ldots, \deg_c(x)),$$

where $\deg_i(x)$ denotes the *degree* of vertex x in the subgraph (X, E_i) of G determined by the number of edges of color i with end x, $1 \le i \le c$. We denote by $\alpha(\mathcal{G}; \boldsymbol{\lambda})$ the greatest common divisor of the integers t satisfying

$$\sum_{G\in\mathcal{G}} \sum_{x\in V(G)} a_{G,x}\tau_G(x) = t(\lambda_1, \lambda_2, \ldots, \lambda_c)$$

for integers $a_{G,x}$. If there is no such t, we define $\alpha(\mathcal{G}; \boldsymbol{\lambda}) = 0$. Equivalently, $\alpha(\mathcal{G}; \boldsymbol{\lambda})$ is the least positive integer t_0 such that $t_0\boldsymbol{\lambda}$ is an integral linear combination of the degree-vectors $\tau_G(x)$. When $\mathcal{G}$ consists of a single edge-c-colored graph, $\alpha(\{G\}; \boldsymbol{\lambda})$ is simply denoted by $\alpha(G; \boldsymbol{\lambda})$.

For each G, let $\mu(G) = (m_1, m_2, \ldots, m_c)$, where m_i is the number of edges of color i in G. Then it follows that $\mu(G) = \frac{1}{2}\sum_{x\in V(G)} \tau_G(x)$. We denote by $\beta(\mathcal{G}; \boldsymbol{\lambda})$ the greatest common divisor of integers m satisfying

$$\sum_{G\in\mathcal{G}} b_G\mu(G) = m(\lambda_1, \lambda_2, \ldots, \lambda_c)$$

for integers b_G. If there is no such m, we define $\beta(\mathcal{G}; \boldsymbol{\lambda}) = 0$. Equivalently, $\beta(\mathcal{G}; \boldsymbol{\lambda})$, if not zero, is the least positive integer m_0 such that $m_0\boldsymbol{\lambda}$ is an integral linear combination of the vectors $\mu(G)$. When a family $\mathcal{G}$ consists of a single graph G, assume that the greatest common divisor of λ_i's is 1. If G has $m\lambda_i$ edges of each color i, then $\beta(\{G\}; \boldsymbol{\lambda})$ (or simply $\beta(G; \boldsymbol{\lambda})$) is m and is zero otherwise.

We remark that $\alpha(\mathcal{G}; \boldsymbol{\lambda})$ is always a divisor of $2\beta(\mathcal{G}; \boldsymbol{\lambda})$ since

$$2\beta(\mathcal{G}; \boldsymbol{\lambda}) \cdot \boldsymbol{\lambda} = \sum_{G\in\mathcal{G}} b_G \cdot 2\mu(G) = \sum_{G\in\mathcal{G}} \sum_{x\in V(G)} b_G\tau_G(x),$$

which is a scalar multiple of $\alpha(\mathcal{G}; \boldsymbol{\lambda}) \cdot \boldsymbol{\lambda}$.

If a $\mathcal{G}$-decomposition of $K_v^{\boldsymbol{\lambda}}$ exists, then the set of $\lambda_i(v-1)$ edges of each color i incident with some fixed point x of $K_v^{\boldsymbol{\lambda}}$ are partitioned by the isomorphic copies of $G \in \mathcal{G}$ so that vector $(v-1)(\lambda_1, \lambda_2, \ldots, \lambda_c)$ is a nonnegative integral linear combination of the vectors $\tau_G(x)$, $x \in V(G)$ and $G \in \mathcal{G}$. Thus $\alpha(\mathcal{G}; \boldsymbol{\lambda})$ divides $v-1$ whenever a decomposition exists. And it is obvious that the vector $v(v-1)\boldsymbol{\lambda}/2$ is a nonnegative integral linear combination of the vectors $\mu(G)$, hence $2\beta(\mathcal{G};\,\boldsymbol{\lambda})$ divides $v(v-1)$.

We say that a graph G_0 is *useless* in $\mathcal{G}$ when in any nonnegative rational linear relation

$$(\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_c) = \sum_{G\in\mathcal{G}} \overline{b_G}\mu(G) \quad \text{with all} \quad \overline{b_G} \geq 0, \tag{1.6.1}$$

we have $\overline{b_{G_0}} = 0$. Such graphs can not occur in any $\mathcal{G}$-decomposition of a graph $K_v^{\boldsymbol{\lambda}}$. We say that $\mathcal{G}$ is $\boldsymbol{\lambda}$-*admissible* when there exists a nonnegative rational linear relation (1.6.1) and when no member of $\mathcal{G}$ is useless in $\mathcal{G}$. Then, the following lemma is obtained.

Lemma 1.6.1 *Let $\mathcal{G}$ be a $\boldsymbol{\lambda}$-admissible family of edge-c-colored graphs. Then, necessary conditions for the existence of $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ are*

$$\begin{aligned} v-1 &\equiv 0 \pmod{\alpha(\mathcal{G};\,\boldsymbol{\lambda})} \text{ and} \\ v(v-1) &\equiv 0 \pmod{2\beta(\mathcal{G};\,\boldsymbol{\lambda})}. \end{aligned} \tag{1.6.2}$$

If $\alpha(\mathcal{G};\,\boldsymbol{\lambda}) = 0$ or $\beta(\mathcal{G};\,\boldsymbol{\lambda}) = 0$, there do not exist any $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$. When $\boldsymbol{\lambda} = (1,\, 1,\, \ldots,\, 1)$ is the all-one vector, $\alpha(\mathcal{G};\,\boldsymbol{\lambda})$ and $\beta(\mathcal{G};\,\boldsymbol{\lambda})$ are simply denoted by $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$, respectively. Also $\boldsymbol{\lambda}$-admissible is simply called *admissible*. If $\mathcal{G}$ consists of only one edge-r-colored graph, $\alpha(\{G\})$ and $\beta(\{G\})$ are similarly denoted by $\alpha(G)$ and $\beta(G)$. Then Lemma 1.6.1 is rewritten as follows:

Lemma 1.6.2 *Let $\mathcal{G}$ be an admissible family of edge-c-colored graphs. Then, necessary conditions for the existence of $\mathcal{G}$-decompositions of $K_v^{[c]}$ are*

$$\begin{aligned} v-1 &\equiv 0 \pmod{\alpha(\mathcal{G})} \text{ and} \\ v(v-1) &\equiv 0 \pmod{2\beta(\mathcal{G})}. \end{aligned} \tag{1.6.3}$$

When $\mathcal{G}$ consists of a single simple edge-1-colored graph G, it was shown by Wilson [104] that necessary conditions for the existence of G-decomposition of K_v is asymptotically sufficient. In this thesis, the term "asymptotically sufficient" means that there exists a constant $v_0 = v_0(\mathcal{G})$ such that $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ exist for all integers $v \geq v_0$ satisfying the necessary conditions. Meanwhile, when $\mathcal{G}$ consists of only simple edge-c-colored graphs, necessary conditions for the existence of $\mathcal{G}$-decomposition of $K_v^{\boldsymbol{\lambda}}$ is asymptotically sufficient by Lamken and Wilson [63]. Note that they proved the asymptotic existence not only in the case of unordered edges but also in the case of directed edges that are not described in this thesis.

1.7 Cyclic and rotational combinatorial designs

Let $(V, \mathcal{B})$ be a B(v, k, λ). For a BIB design $(V, \mathcal{B})$, let σ be a permutation on V. If $\mathcal{B}^\sigma = \{B^\sigma : B \in \mathcal{B}\} = \mathcal{B}$, then σ is called an *automorphism* of $(V, \mathcal{B})$, where $B^\sigma = \{b_1^\sigma, b_2^\sigma, \ldots, b_k^\sigma\}$ for any $B = \{b_1, b_2, \ldots, b_k\} \in \mathcal{B}$. If there exits an automorphism with a single orbit of length v, then the BIB design is said to be *cyclic* and the point set V can be identified with $\mathbb{Z}_v$, i.e., the additive group of residues modulo v. In this case, the automorphism is represented by $\sigma : i \mapsto i+1 \pmod v$, the block orbit of B is defined by a set of distinct blocks

$$B^{\sigma^i} = B + i = \{b_1 + i, b_2 + i, \ldots, b_k + i\} \pmod v$$

for $i \in \mathbb{Z}_v$ and the length of a block orbit is the minimum positive integer t such that $B+t = B$ for an arbitrary block B in the block orbit. A block orbit of length v is said to be *full*, otherwise *short*. We fix one block arbitrarily in each block orbit and call it a *base block*.

It is easy to show that if a cyclic B$(v, k, 1)$ exists, then $v \equiv 1, k \pmod{k(k-1)}$. When $v \equiv 1 \pmod{k(k-1)}$, the design is developed only from base blocks with full block orbits and the family of base blocks is called a *cyclic difference family*, while if $v \equiv k \pmod{k(k-1)}$, then it consists of full block orbits and a single short block orbit developed from

$$\left\{0, \frac{v}{k}, \frac{2v}{k}, \ldots, \frac{(k-1)v}{k}\right\},$$

which is called a *regular short base block*.

Example 1.7.1 Let $V = \mathbb{Z}_{15}$ be a point set and

$$\mathcal{B} = \{\{0, 1, 4\}, \{0, 2, 8\}, \{0, 5, 10\} \pmod{15}\}$$

be a collection of blocks. Then $(V, \mathcal{B})$ is a cyclic B$(15, 3, 1)$. The base block $\{0, 5, 10\}$ has regular short block orbit length of 5.

In Example 1.2.1, the B$(7, 3, 1)$ is also cyclic.

Cyclic designs have a simple structure and are related to interesting algebraic properties. In fact, the "method of differences" introduced by Bose [16] is an algebraic technique to construct BIB designs effectively, say, cyclically. There have been many methods of constructing cyclic BIB designs (see, for example, [34, 55, 70]). The spectrum of cyclic B$(v, 3, \lambda)$ was determined by

Colbourn and Colbourn [33]. But for $k \geq 4$, the existence problem has not been solved yet in general.

For a BIB design $(V, \mathcal{B})$, if there exists an automorphism consisting of a single fixed point and l cycles of length $(v-1)/l$, then the BIB design is said to be *l-rotational*. The automorphism can be represented by

$$\pi = (\infty)(0_0,\, 1_0,\, \ldots,\, (n-1)_0) \cdots (0_{l-1},\, 1_{l-1},\, \ldots,\, (n-1)_{l-1})$$

on the point set $V = \{\infty\} \cup (\mathbb{Z}_n \times \{0, 1, \ldots, l-1\})$, where $n = (v-1)/l$ and x_i denotes the element $(x, i) \in \mathbb{Z}_n \times \{0, 1, \ldots, l-1\}$. A block orbit of an l-rotational BIB design is defined similarly to that of a cyclic BIB design. In Example 1.2.3, the B(15, 3, 1) is 2-rotational.

The terminology of "l-rotational" was initially introduced by Phelps and Rosa [79]. There are several results on rotational BIB designs. For example, necessary and sufficient conditions for the existence of a 1-rotational BIB design with block size 3 were derived by Cho [27] and Kuriki and Jimbo [62], independently. Colbourn and Jiang [30] solved the existence problem of l-rotational BIBD designs with block size 3 completely by use of recursive constructions together with some results due to Cho [27], Doyen [39], Phelps and Rosa [79], Rosa [87] and Teirlinck [95].

We give a notion before we define cyclic or l-rotational grid-block designs. Two $k_1 \times k_2$ grid-blocks A and A' are said to be *equivalent* if there exist permutation matrices P and Q such that $PAQ = A'$. For a $k_1 \times k_2$ grid-block design $(V, \mathcal{A})$, let σ be a permutation on V. If there is a permutation σ such that an equivalent $k_1 \times k_2$ grid-block to A^σ belongs to $\mathcal{A}$ for any $A \in \mathcal{A}$, then σ is called an automorphism of the $k_1 \times k_2$ grid-block designs, where $A^\sigma = (a_{ij}^\sigma)$ for any $A = (a_{ij}) \in \mathcal{A}$. Thus, if an automorphism σ of $k_1 \times k_2$ grid-block design $(V, \mathcal{A})$ has a cycle of length v, the design is said to be cyclic. If a $k_1 \times k_2$ grid-block design $(V, \mathcal{A})$ has an automorphism π consisting of a single fixed point and l cycles of length $(v-1)/l$ each, the design is said to be l-rotational. A grid-block orbit of a $k_1 \times k_2$ grid-block design is defined similarly to that of BIB designs. The 2×3 grid-block design with 10 points in Example 1.3.1 is cyclic.

1.8 Finite geometries and cyclotomic cosets

For a prime power q, let AG(n, q) denote the affine geometry of dimension n over the finite field GF(q) with q element. Each point of AG(n, q) is represented by x where x is an element of GF(q^n). And AG$_t(n, q)$ denotes the set of t-dimensional subspaces and their cosets of AG(n, q). Specifically, AG$_0(n, q)$ denotes the set of points of AG(n, q). Each element of AG$_t(n, q)$ is called *t-flat*. Let $\mathrm{AG}_t^*(n,\, q)$ be the set of t-flats passing through the origin $\mathbf{0}$. It is well-known that the numbers of the t-flats of $\mathrm{AG}_t(n,\, q)$ and $\mathrm{AG}_t^*(n,\, q)$ are $q^{n-t}\phi(n,\, t,\, q)$ and $\phi(n,\, t,\, q)$, respectively, where

$$\phi(n,\, t,\, q) = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-t+1}-1)}{(q^t-1)(q^{t-1}-1)\cdots(q-1)}.$$

A pair $(\mathrm{GF}(q^n),\, \mathrm{AG}_t(n,\, q))$ is a $\mathrm{B}(q^n,\, q^t,\, \phi(n-1,\, t-1,\, q))$.

For a t-flat U of $\mathrm{AG}_t(n,\, q)$, we define a *parallel class* $\mathcal{P}(U)$ containing U as the set of all t-flats which are parallel to the t-flat U. Here U is said to be *parallel* to U' if there exists some element x in $\mathrm{GF}(q^n)$ such that $U = U' + x$ holds. A parallel class $\mathcal{P}(U)$ has q^{n-t} t-flats. Clearly, $\mathrm{AG}_t(n,\, q)$ is partitioned into parallel classes and each parallel class contains exactly one t-flat passing through the origin $\mathbf{0}$. Thus, the pair $(\mathrm{GF}(q^n),\, \mathrm{AG}_t(n,\, q))$ is a resolvable BIB design.

Let $\mathrm{PG}(n-1,\, q)$ denote the projective geometry of dimension $n-1$ over $\mathrm{GF}(q)$. We introduce an equivalence relation $x \sim y$ on $\mathrm{GF}(q)^n$ if and only if there exists an element u $(\neq 0)$ in $\mathrm{GF}(q)$ such that $y = ux$ holds. An equivalence class containing x is denoted by (x) and the set of all points in (x) is a 1-flat of $\mathrm{AG}(n,\, q)$ passing through the origin $\mathbf{0}$. Thus, each point of $\mathrm{PG}(n-1,\, q)$ is represented by a 1-flat of $\mathrm{AG}_1^*(n,\, q)$. And $\mathrm{PG}_{t-1}(n-1,\, q)$ is the set of $U/_\sim$ for all U in $\mathrm{AG}_t^*(n,\, q)$, where $U/_\sim = \{(x) : x \in U\}$. Each element of $\mathrm{PG}_{t-1}(n-1,\, q)$ is called a *$(t-1)$-flat*. The number of $(t-1)$-flat of $\mathrm{PG}_{t-1}(n-1,\, q)$ is $\phi(n,\, t,\, q)$ since the number of t-flat of $\mathrm{AG}_t^*(n,\, q)$ is $\phi(n,\, t,\, q)$. A pair $(V,\, \mathrm{PG}_{t-1}(n-1,\, q))$ is a $\mathrm{BIBD}((q^n-1)/(q-1),\, (q^t-1)/(q-1),\, \phi(n-2,\, t-2,\, q))$, where $V = \mathrm{PG}_0(n-1,\, q)$ is the set of all points in $\mathrm{PG}(n-1,\, q)$.

Next, we define the notions of the sum, the scalar multiplication and the product over additive groups for lists. For a finite set V, a formal sum $L = \sum_{x\in V} m_x\{x\}$ is called a *list*, where the nonnegative integer m_x is the multiplicity of x in the list L. Also we use the notation $L = (x_i : i \in I)$ to indicate the list of x_i's, where I is an index set. We identify a subset S of V with a list whose multiplicities x_i are 1 or 0 depending on whether x belongs to S or not.

We define the addition and the scalar multiplication for lists $L = \sum_{x\in V} l_x$ $\{x\}$ and $M = \sum_{x\in V} m_x\{x\}$ by $L + M = \sum_{x\in V}(l_x + m_x)\{x\}$ and $\lambda L = \sum_{x\in V} \lambda l_x\{x\}$ for a nonnegative integer λ. Moreover, if $l_x \leq m_x$ holds for each $x \in V$, then we write $L \leq M$.

In the case when V is an additive group of order v, the product of two lists $L = \sum_{x \in V} l_x\{x\}$ and $M = \sum_{x \in v} m_x\{x\}$ is defined by

$$L \circ M = \sum_{z \in V} \left(\sum_{\substack{x,\, y \\ xy = z}} l_x m_x \right) \{z\}.$$

List multiplication is commutative, associative and distributive over the addition of lists. For any subset S of V and for any element y of V, let $S + y = \{s + y : s \in S\}$ and $Sy = \{sy : s \in S\}$.

Now, we define difference families. For an abelian group Γ of order v, let $\Delta B = (b_j - b_i : 1 \leq i \neq j \leq k)$ be the list of differences of a k-set $B = \{b_1,\, b_2,\, \ldots,\, b_k\}$ with elements in Γ. For a family $\boldsymbol{B} = \{B_i : i \in I\}$ of k-subsets of Γ, we define $\Delta \boldsymbol{B} = \sum_{i \in I} \Delta B$. The family $\boldsymbol{B}$ is called a $(v,\, k,\, \lambda)$-*difference family* in Γ, denoted by $(v,\, k,\, \lambda)$-DF, if $\Delta \boldsymbol{B} = \lambda(\Gamma \setminus \{0\})$.

We generalize such notion to a $k_1 \times k_2$ grid-block. We introduced the list of differences of a $k_1 \times k_2$ grid-block $A = (a_{ij})$ with elements in Γ as follows:

$$\begin{aligned} \partial A = {} & (a_{ij'} - a_{ij} : 1 \leq i \leq k_1,\, 1 \leq j \neq j' \leq k_2) \\ & + (a_{i'j} - a_{ij} : 1 \leq i \neq i' \leq k_1,\, 1 \leq j \leq k_2). \end{aligned}$$

For a family of grid-blocks $\boldsymbol{A} = \{A_i : i \in I\}$ with elements in Γ, we define $\partial \boldsymbol{A} = \sum_{i \in I} \partial A_i$. Then, the family $\boldsymbol{A}$ is called a *gird-block difference family*, denoted by $(v,\, k_1,\, k_2)$-GBDF, if $\partial \boldsymbol{A} = \Gamma \setminus \{0\}$. For a $(v,\, k_1,\, k_2)$-GBDF $\boldsymbol{A}$, let $\mathcal{A} = \{A_i + x : A_i \in \boldsymbol{A},\, x \in \Gamma\}$, then $(\Gamma,\, \mathcal{A})$ is a GB$(v,\, k_1,\, k_2)$.

Next, we give a notion of the *method of mixed differences* introduced by Bose [16]. For an additive group Γ and an index set $L = \{0,\, 1,\, \ldots,\, l - 1\}$, let $V = \Gamma \times L$ be a set of v points. For $g \in \Gamma$ and $B = \{(b_1,\, l_1),\, (b_2,\, l_2),\, \ldots,\, (b_k,\, l_k)\} \subseteq V$, we define the addition $B + g$ by

$$B + g = \{(b_1 + g,\, l_1),\, (b_2 + g,\, l_2),\, \ldots,\, (b_k + g,\, l_k)\}.$$

For a k-set $B = \{(b_1,\, l_1),\, (b_2,\, l_2),\, \ldots,\, (b_k,\, l_k)\}$, let $\Delta_{ij} B$ be the list of differences $b_{t'} - b_t$ such that $(b_t,\, i)$ and $(b_{t'},\, j)$ occur in B, that is,

$$\Delta_{ij} B = (b_{t'} - b_t : 1 \leq t \neq t' \leq k,\, l_t = i,\, l_{t'} = j).$$

Note that if $i \neq j$ the difference $b_{t'} - b_t = 0$ can occur, but not for $i = j$. Obviously, $\Delta_{ij} B = -\Delta_{ji} B$. For a family $\boldsymbol{B}$ of k-subsets of Γ, we define $\Delta_{ij} \boldsymbol{B} = \sum_{B \in \boldsymbol{B}} \Delta_{ij} B$. $\Delta_{ii} \boldsymbol{B}$ is called the *i-th list of pure differences.* In case of $i \neq j$ the list $\Delta_{ij} \boldsymbol{B}$ is called the *list of mixed differences* for the index pair $(i,\, j)$. Similarly, $\Delta_{ij} \boldsymbol{B} = -\Delta_{ji} \boldsymbol{B}$ holds and the difference 0 is allowed in $\Delta_{ij} \boldsymbol{B}$ if and only if $i \neq j$ holds.

The development of $\boldsymbol{B}$ is defined by $\mathcal{B} = \{B + g : B \in \boldsymbol{B},\, g \in \Gamma\}$. A pair $(V,\, \mathcal{B})$ is a BIB design B$(v,\, k,\, \lambda)$ if and only if

(i) $\Delta_{ii}\boldsymbol{B} = \Gamma \setminus \{0\}$ holds for every $i \in L$ and

(ii) $\Delta_{ij}\boldsymbol{B} = \Gamma$ holds for any pair of two distinct indices i and $j \in L$.

Then, the BIB design is said to be *l-cyclic*.

Similarly, for a $k_1 \times k_2$ grid-block $A = ((a_{st}, l_{st}))$, let $\partial_{ij}A$ be the list of differences $a_{s't'} - a_{st}$ which occur in the same row or in the same column of A, that is,

$$\begin{aligned}\partial_{ij}A = &(a_{st'} - a_{st} : 1 \le s \le k_1,\ 1 \le t \ne t' \le k_2,\ l_{st} = i,\ l_{st'} = j) \\ &+ (a_{s't} - a_{st} : 1 \le s \ne s' \le k_1,\ 1 \le t \le k_2,\ l_{s't} = i,\ l_{st} = j) \\ &.\end{aligned}$$

For a family of grid-blocks $\boldsymbol{A}$ with elements in Γ, we define $\partial_{ij}\boldsymbol{A} = \sum_{A\in\boldsymbol{A}} A$. Obviously, $\partial_{ij}\boldsymbol{A} = -\partial_{ji}\boldsymbol{A}$ and the difference 0 is allowed in $\partial_{ij}\boldsymbol{A}$ if and only if $i \ne j$ holds.

The development of $\boldsymbol{A}$ is defined by $\mathcal{A} = \{A + g : A \in \boldsymbol{A},\ g \in \Gamma\}$. Then a pair $(\Gamma, \mathcal{A})$ is a $k_1 \times k_2$ grid-block design if and only if

(i) $\partial_{ii}\boldsymbol{A} = \Gamma \setminus \{0\}$ holds for every $i \in L$ and

(ii) $\partial_{ij}\boldsymbol{A} = \Gamma$ holds for any pair of two distinct indices i and $j \in L$.

Finally, we define cyclotomic cosets and give a proposition to show several theorems. For a positive integer m, let q be a prime power such that $q \equiv 1 \pmod{m}$. The cyclic multiplicative subgroup $\mathrm{GF}(q)^*$ of nonzero elements in the field of q elements has a unique subgroup H_0^m of index m. The multiplicative cosets $H_0^m, H_1^m, \ldots, H_{m-1}^m$ of H_0^m are called the cyclotomic classes of index m and may be indexed so that $a \in H_i^m$ and $b \in H_j^m$ imply $ab \in H_{i+j}^m$, where the subscripts are read modulo m; if ω is a primitive element in $\mathrm{GF}(q)$, we may take $H_i^m = \{\omega^t : t \equiv i \pmod{m}\}$. We select an element s_i from each H_i^m for $m = 0, 1, \ldots, m-1$ and call the set $S_m = \{s_0, s_1, \ldots, s_{m-1}\}$ a *system of representatives* for cosets modulo H_0^m. Then $\mathrm{GF}(q)^* = H_0^m \circ S_m$ holds.

For an integer $k \ge 2$, let P_k be the set of ordered pairs $\{(i, j) : 1 \le i < j \le k\}$. For $\mathcal{H}^m = \{H_0^m, H_1^m, \ldots, H_{m-1}^m\}$, we define a *choice* to be any map $M : P_k \to \mathcal{H}^m$, assigning each pair $(i, j) \in P_k$ to a coset $M(i, j)$ modulo H_0^m in $\mathrm{GF}(q)$. A k-tuple $(x_1, x_2, \ldots, x_k)$ of elements in $\mathrm{GF}(q)$ is said to be *consistent* with the choice M if and only if $x_j - x_i \in M(i, j)$ for all $1 \le i < j \le k$. The following proposition is proved by Wilson [99].

Proposition 1.8.1 *For given m and k, there exists a constant $q_0 = q_0(m, k)$ such that for all prime power $q \equiv 1 \pmod{m}$ with $q \ge q_0$, and for all choices $M : P_k \to \mathcal{H}^m$, there exists a k-tuple $(x_1, x_2, \ldots, x_k)$ of elements of* $\mathrm{GF}(q)$ *which is consistent with M.*

1.9 Summary of this thesis

In Chapter 2, grid-block designs, resolvable grid-block designs and packings are discussed. In Section 2.1, we list known results and give new direct and recursive constructions. In Sections 2.2 and 2.3, it is shown that the necessary conditions for the existence of 3×3 and 2×4 grid-block designs are sufficient by utilizing direct and recursive constructions listed in Section 2.1. In Section 2.4, the definition of a grid-block design is generalized to n-dimensional case and direct constructions for a $2 \times 2 \times 2$ grid-block design for every parameters satisfying the necessary conditions are given. In Section 2.5, we construct resolvable grid-block designs and show the existence of resolvable grid-block designs for sufficiently large prime powers. In Section 2.6, some constructions of resolvable grid-block packings are given. Some of them are able to construct maximal resolvable grid-block packings.

In Chapter 3, nested BIB designs and BIBRCs are treated. In Section 3.1, a construction of nested BIB designs is given by utilizing finite affine geometries. Some of them are new nested BIB designs which are not found in the tables of Morgan [72] and Morgan, Preece and Rees [73]. In Section 3.2, a construction of BIBRCs is given by the same method in Section 3.1. In Section 3.3, a construction of BIBRCs is given by utilizing finite fields and it is shown that the existence of BIBRCs for sufficiently large prime powers. Moreover, it is listed that a table for existence of BIBRCs with small parameters in Appendix A, which are obtained by the construction in Section 3.3.

In Chapter 4, asymptotic existence of colorwise simple edge-colored graph decompositions of complete graphs is shown. Firstly, in Section 4.1, we give a notion of "treeordered," which plays an important role for the proof of asymptotic existence of colorwise simple edge-colored graph decompositions of complete graphs. In Sections 4.2, 4.3, 4.4 and 4.5, it is shown that there exist such decompositions of $K_v^{[c]}$ for sufficiently large integers v satisfying the congruences (1.6.3). In Section 4.6, we consider the case when the decomposition is "balanced" and an asymptotic existence theorem of balanced graph decompositions of $K_v^{[c]}$. In Section 4.7, we generalize the results given in Sections 4.2, 4.3, 4.4, 4.5 and 4.6 to the case when the graph is $K_v^{\boldsymbol{\lambda}}$.

Finally, in Chapter 5, it is shown that there exist BIBRCs with some λ's for sufficiently large integers satisfying the necessary conditions by applying the asymptotic existence results given in Chapter 4. In Section 5.1, we discuss a relationship between BIBRCs and some balanced edge-colored graph decompositions of complete graphs. In Section 5.2, it is shown that there exist completely balanced BIBRCs for sufficiently large integers satisfying the necessary conditions, which is proved by utilizing the result of Lamken and Wilson [63]. In Sections 5.3 and 5.4, we also show asymptotic existence of BIBRCs with some λ by utilizing our theorem in Chapter 4. These results can not be obtained by the result of Lamken and Wilson [63]. In Section 5.5, it is proved that BIBRCs exist for sufficiently large integers satisfying the necessary conditions in the case of $\lambda \geq k_1k_2(k_1-1)(k_2-1)$ by combining the results in Sections 5.3 and 5.4.

Chapter 2

Existence and construction of grid-block designs

In this chapter, existence and construction of grid-block designs and resolvable grid-block designs are discussed. In Section 2.1, some constructions of grid-block designs are given. In Sections 2.2 and 2.3, it is shown that grid-block designs GB(v, 3, 3) and GB(v, 2, 4) exist for all integers v satisfying the necessary conditions by constructing a few grid-block designs and using the methods in Section 2.1. In Section 2.4, the definition of grid-block designs is generalized to n-dimensional case and cyclic or 3-rotational $2 \times 2 \times 2$ grid-block designs are constructed directly by the "method of differences." In Section 2.5, we construct resolvable grid-block designs by utilizing grid-block difference families and show the existence of resolvable grid-block designs for sufficiently large integers satisfying some conditions. Lastly, in Section 2.6, constructions of resolvable grid-block packings are given. Some of them give maximal resolvable grid-block packings.

2.1 Constructions of grid-block designs

For a set V of v points (or vertices), let $\{V_1, V_2, \ldots, V_s\}$ be a partition of V with $|V_i| = t_i$. Each V_i is called a *partite set*. A pair $G = (V, E)$ is said to be a *complete s-partite graph*, denoted by $K_{t_1, t_2, \ldots, t_s}$, if an edge $\{x, y\}$ belongs to $E(G)$ for all pairs of two points x and y from distinct partite sets. If $t_i = t$ holds for all i, then the complete s-partite graph is *regular* and is also denoted by $K_{s(t)}$. Then, a group divisible design GD(v, K, M, λ) with a group type $t_1^{u_1} t_2^{u_2} \cdots t_n^{u_n}$ is equivalent to a $\mathcal{G}$-decomposition D(λK, $\mathcal{G}$), where K is a complete $(\sum_{i=1}^{n} u_i)$-partite graph having u_i partite sets with t_i vertices for each i, λK means λ copies of K and $\mathcal{G}$ is a family of complete graphs with k vertices belonging to K. In this thesis, we consider mainly regular complete s-partite graphs.

Let G_{k_1,k_2} be a graph $K_{k_1} \times K_{k_2}$ which is equivalent to a $k_1 \times k_2$ grid-block. If there exists a G_{k_1,k_2}-decomposition of $K_{s(t)}$ (or $\mathrm{D}(K_{s(t)}, G_{k_1,k_2})$), then a triple $(V, \mathcal{P}, \mathcal{A})$ is called a *group divisible grid-block design*, where V is the point set of $K_{s(t)}$, $\mathcal{P}$ is the family of the partite sets (called groups) and $\mathcal{A}$ is a family of $k_1 \times k_2$ grid-blocks that are equivalent to the subgraphs of $\mathrm{D}(K_{s(t)}, G_{k_1,k_2})$. It is easy to show that the following lemma holds:

Lemma 2.1.1 *Necessary conditions for existence of a* $\mathrm{D}(K_{s(t)}, G_{k_1,k_2})$ *are*

$$\begin{aligned} (s-1)t &\equiv 0 \pmod{k_1+k_2-2} \text{ and} \\ (s-1)st^2 &\equiv 0 \pmod{k_1k_2(k_1+k_2-2)}. \end{aligned}$$

We list some recursive constructions from the results in Fu *et al.* [44]. We omit the subscript k_1, k_2 in G_{k_1,k_2} in this section.

Proposition 2.1.2 *There exists a* $\mathrm{D}(K_{st+1}, G)$ *if there exist a* $\mathrm{D}(K_{t+1}, G)$ *and a* $\mathrm{D}(K_{s(t)}, G)$.

Proposition 2.1.3 *There exists a* $\mathrm{D}(K_{v(t)}, G)$ *if there exist a* $\mathrm{B}(v, K, 1)$ *and* $\mathrm{D}(K_{k(t)}, G)$*'s for* $k \in K$. *Especially, there exists a* $\mathrm{D}(K_{v(t)}, G)$ *if there exist a* $\mathrm{B}(v, k, 1)$ *and a* $\mathrm{D}(K_{k(t)}, G)$.

Corollary 2.1.4 *There exists a* $\mathrm{D}(K_{vt+1}, G)$ *if there exist a* $\mathrm{B}(v, K, 1)$*, a* $\mathrm{D}(K_{t+1}, G)$ *and* $\mathrm{D}(K_{k(t)}, G)$*'s for* $k \in K$. *Especially, there exists a* $\mathrm{D}(K_{vt+1}, G)$ *if there exist a* $\mathrm{B}(v, k, 1)$ *and a* $\mathrm{D}(K_{k(t)}, G)$.

Proposition 2.1.5 *There exists a* $\mathrm{D}(K_{(v-1)t+1}, G)$ *if there exist a* $\mathrm{B}(v, k, 1)$*, a* $\mathrm{D}(K_{t+1}, G)$*, a* $\mathrm{D}(K_{(k-1)t+1}, G)$ *and a* $\mathrm{D}(K_{k(t)}, G)$.

Proposition 2.1.6 *There exists a* $\mathrm{D}(K_{(v+i)t+1}, G)$ *if there exist a resolvable* $\mathrm{B}(v, K, 1)$ *with at least i resolution classes, a* $\mathrm{D}(K_{t+1}, G)$*,* $\mathrm{D}(K_{it+1}, G)$*'s, a* $\mathrm{D}(K_{k(t)}, G)$ *and a* $\mathrm{D}(K_{k+1(t)}, G)$.

Proposition 2.1.7 *There exists a* $\mathrm{D}(K_{s(mt)+1}, G)$ *if there exist a* $\mathrm{D}(K_{s(t)}, G)$ *and $s-2$ mutually orthogonal Latin squares of order m for $s \geq 3$.*

We give a recursive construction generalized from these results in Propositions 2.1.2, 2.1.5 and 2.1.6 and Corollary 2.1.4.

Theorem 2.1.8 *There exists a* $\mathrm{D}(K_{vt+1}, G)$ *if there exist a* $\mathrm{GD}(v, K, M, 1)$*,* $\mathrm{D}(K_{mt+1}, G)$*'s and* $\mathrm{D}(K_{k(t)}, G)$*'s for any $m \in M$ and $k \in K$.*

Proof. For a set V of v points, let a triple $(V, \mathcal{P}, \mathcal{B})$ be a GD$(v, K, M, 1)$. Let $T = \{0, 1, \dots, t-1\}$ and $V^* = (V \times T) \cup \{\infty\}$. For each block $B \in \mathcal{B}$ of size $k \in K$, let $(B \times T, \mathcal{P}(B), \mathcal{A}(B))$ be the ingredient design $\mathrm{D}(K_{k(t)}, G)$, where $\mathcal{A}(B)$ is a collection of grid-blocks and $\mathcal{P}(B)$ is a family of groups $\{b_i \times T\}$ for each $b_i \in B$. We define a collection of grid-blocks $\mathcal{A}_1^* = \bigcup_{B \in \mathcal{B}} \mathcal{A}(B)$. Also, for each group $P \in \mathcal{P}$ of size $m \in M$, let $((P \times T) \cup \{\infty\}, \mathcal{A}'(P))$ be the ingredient design $\mathrm{D}(K_{mt+1}, G)$, where $\mathcal{A}'(P)$ is a collection of grid-blocks. We define another collection of grid-blocks $\mathcal{A}_2^* = \bigcup_{P \in \mathcal{P}} \mathcal{A}'(P)$ and let $\mathcal{A}^* = \mathcal{A}_1^* \cup \mathcal{A}_2^*$. Then a pair $(V^*, \mathcal{A}^*)$ is the desired $\mathrm{D}(K_{vt+1}, G)$.

In fact, if two distinct points x and y in V are not contained in the same group P, then x and y occur together exactly once in a block $B \in \mathcal{B}$. Hence (x, i) and (y, j) occur exactly once in the same row or in the same column of a grid-block in $\mathcal{A}_1^*$ and do not occur in $\mathcal{A}_2^*$ since they occur once in the same row or in the same column in the ingredient design $(B \times T, \mathcal{P}(B), \mathcal{A}(B))$. Otherwise two points x and y in V are contained in the same group P including the case of $x = y$, then x and y does not occur together in any $B \in \mathcal{B}$. In this case, (x, i) and (y, j) except for $x = y$ and $i = j$ occur exactly once in the same row or in the same column of a grid-block in $\mathcal{A}_2^*$ and do not occur in $\mathcal{A}_1^*$ since they occur once in the same row or in the same column in the ingredient design $((P \times T) \cup \{\infty\}, \mathcal{A}'(P))$. Lastly, ∞ and (x, i) for any $x \in V$ and $i \in T$ occur exactly once in the same row or in the same column of a grid-block in $\mathcal{A}_2^*$ since they occur once in the same row or in the same column in the ingredient design $((P \times T) \cup \{\infty\}, \mathcal{A}'(P))$ and x belongs to a group P which is a partition of V. $\square$

In the case of $k_1 = k_2$, we give two direct constructions in Fu *et al.* [44]. Firstly, we give a construction by utilizing affine geometries.

Theorem 2.1.9 *For an even integer n and an odd prime power q, there exists a* $\mathrm{GB}(q^n, q, q)$ *(or* $\mathrm{D}(K_{q^n}, G_{q,q})$*).*

Proof. Let ω be a primitive element of $\mathrm{GF}(q^n)$. Then each point of $\mathrm{AG}(n, q)$ is represented by ω^i. For convenience, let $\omega^\infty = 0 (= \mathbf{0})$. Let A be a $q \times q$ grid-block as follows:

$$\boxed{\begin{array}{lllll} \omega^\infty & \omega^0 & \omega^{2u} & \cdots & \omega^{(2q-4)u} \\ \omega^u & \omega^0 + \omega^u & \omega^{2u} + \omega^u & \cdots & \omega^{(2q-4)u} + \omega^u \\ \omega^{3u} & \omega^0 + \omega^{3u} & \omega^{2u} + \omega^{3u} & \cdots & \omega^{(2q-4)u} + \omega^{3u} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{(2q-3)u} & \omega^0 + \omega^{(2q-3)u} & & \cdots & \omega^{(2q-4)u} + \omega^{(2q-3)u} \end{array}}, \qquad (2.1.1)$$

where

$$u = \frac{q^n - 1}{2(q-1)}.$$

Then A is a 2-flat and rows and columns are 1-flats in $\mathrm{AG}(n,\,q)$. Let $\mathcal{P}(A)$ be a parallel class containing A. The development of $\mathcal{P}(A)$ is defined by $\mathcal{A} = \{\omega^i A' : A' \in \mathcal{P}(A),\, 0 \le i \le u-1\}$. A pair $(\mathrm{GF}(q^n),\,\mathcal{A})$ is the desired $\mathrm{GB}(q^n,\,q,\,q)$.

In fact, let ω^i and ω^j be two distinct points in $\mathrm{AG}(n,\,q)$. To count the number of rows and columns of $q \times q$ grid-blocks containing ω^i and ω^j simultaneously, we have only to count the number of rows and columns such that the origin $\mathbf{0}(=\omega^\infty)$ and $\omega^j - \omega^i$ occur together. We can represent $\omega^l = \omega^j - \omega^i$ for some integer l. There is a 1-flat passing through the origin $\mathbf{0}$ and ω^l, which proves the theorem. $\square$

Secondly, we give a construction by combining base blocks of a cyclic BIBD.

Theorem 2.1.10 *Let p be an odd prime and $v \equiv p \pmod{2p(p-1)}$. Then there exists a $\mathrm{GB}(pv,\,p,\,p)$ if there exists a cyclic $\mathrm{B}(v,\,p,\,1)$.*

Proof. It is known that a cyclic $\mathrm{B}(pv,\,p,\,1)$ can be constructed from a cyclic $\mathrm{B}(v,\,p,\,1)$ for a prime p, which was obtained by Colbourn and Colbourn [35], Grannell and Griggs [45] and Jimbo and Kuriki [55], independently. We use the similar method to construct a $\mathrm{GB}(pv,\,p,\,p)$.

Let a pair $(V,\,\mathcal{B})$ be a cyclic $\mathrm{B}(v,\,p,\,1)$, where V is identified with $\mathbb{Z}_v$. Then, the cyclic $\mathrm{B}(v,\,p,\,1)$ has $2t$ base blocks with cycle length v and a base block with regular short block orbit u, where $t = (v-p)/2p(p-1)$ and $u = v/p$. Let $\boldsymbol{B} = \{B_1,\,B_2,\,\ldots,\,B_{2t}\}$ be a family of base blocks with cycle length v. Without loss of generality, we assume that B_i includes 0 for each i. It is obvious that

$$\mathcal{B} = \{B_m + x : m = 1,\,2,\,\ldots,\,2t,\; x \in \mathbb{Z}_v\} \cup \{B_0 + x : x = 0,\,1,\,\ldots,\,u-1\}$$

and $\Delta\boldsymbol{B} = \mathbb{Z}_v \setminus \{0,\,u,\,2u,\,\ldots,\,(p-1)u\}$ hold.

Let $V^* = \mathbb{Z}_{pv}$. By combining B_{2m-1} and B_{2m} for $m = 1,\,2,\,\ldots,\,t$, we obtain the following $p \times p$ base grid-blocks:

$$\begin{aligned} A_m &= (b_{2m-1,\,i} + b_{2m,\,j} + ijv), \\ &= \boxed{\begin{array}{llll} 0 & b_{2m-1,\,1} & \cdots & b_{2m-1,\,p-1} \\ b_{2m,\,1} & & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ b_{2m,\,p-1} & & \cdots & b_{2m-1,\,p-1} + b_{2m,\,p-1} + (p-1)^2 v \end{array}}, \end{aligned} \qquad (2.1.2)$$

where $B_{m'} = \{0, b_{m',1}, b_{m',2}, \ldots, b_{m',p-1}\}$. Then all elements of each A_m are distinct. In fact, elements in the same row or in the same column are obviously distinct. We check the other cases as follows. Firstly, $b_{2m-1,1}$, $b_{2m-1,2}, \ldots, b_{2m-1,p-1}$ and $b_{2m,1}, b_{2m,2}, \ldots, b_{2m,p-1}$ in A_m are distinct. In fact, if $b_{2m-1,i} = b_{2m,j}$ holds, the same difference occurs in B_{2m-1} and B_{2m}. Secondly, we assume that there exists indices $i \neq i'$ and $j \neq j'$ such that $b_{2m-1,i} + b_{2m,j} + ijv = b_{2m-1,i'} + b_{2m,j'} + i'j'v$ holds. Let $d = b_{2m-1,i} - b_{2m-1,i'}$ and $d' = b_{2m,j} - b_{2m,j'}$. Then $d + d'$ is not multiple of v. If $d + d'$ is a multiple of v, $d' = -d$ holds in $\mathbb{Z}_v$. This means that d and d' are the same difference in $\mathbb{Z}_v$. Thus, $b_{2m-1,i} + b_{2m,j} + ijv$ and $b_{2m-1,i'} + b_{2m,j'} + i'j'v$ are distinct.

Let $\boldsymbol{A} = \{A_1, A_2, \ldots, A_t\}$ be a family of base grid-blocks obtained by the equation (2.1.2) and $\mathcal{A}_1 = \{A_m + x : m = 1, 2, \ldots, t, x \in \mathbb{Z}_{pv}\}$.

By Theorem 2.1.9, there exists a GB(p^2, p, p). Let a pair $(U, \mathcal{A}_2)$ be the GB(p^2, p, p), where $U = \{0, u, 2u, \ldots, (p^2-1)u\}$. We define another collection of grid-blocks $\mathcal{A}'_2 = \{A + x : A \in \mathcal{A}_2, x = 0, 1, \ldots, u-1\}$ and $\mathcal{A}^* = \mathcal{A}_1 \cup \mathcal{A}'_2$. Then a pair $(V^*, \mathcal{A}^*)$ is the desired GB(pv, p, p).

In fact, let x and y be two distinct points in $\mathbb{Z}_{pv}$. To count the number of rows and columns of grid-blocks containing x and y simultaneously, we have only to count the number of rows and columns such that 0 and $z = y - x$ occur together. For a p-subset in $\mathbb{Z}_{pv}$, it is obvious that $\Delta B = \Delta(B + x)$ holds for $x \in \mathbb{Z}_{pv}$ by the definition. Thus, we obtain the following equations:

$$
\begin{aligned}
\partial \boldsymbol{A} &= \sum_{m=1}^{t} \partial A_t \\
&= \sum_{m=1}^{t} \Bigg(\sum_{j=0}^{p-1} \Delta\{0, b_{2m-1,1} + jv, \ldots, b_{2m-1,p-1} + (p-1)jv\} \\
&\qquad + \sum_{i=0}^{p-1} \Delta\{0, b_{2m,1} + iv, \ldots, b_{2m,p-1} + (p-1)iv\} \Bigg) \\
&= \sum_{m=1}^{2t} \sum_{i=0}^{p-1} \Delta\{0, b_{m,1} + iv, b_{m,2} + 2v, \ldots, b_{m,p-1} + (p-1)iv\} = \mathbb{Z}_{pv} \setminus U,
\end{aligned}
$$

since p is a prime. That is, if z is not a multiple of u, 0 and z occur exactly once in the same row or same column in a grid-block in $\mathcal{A}_1$ and do not occur in $\mathcal{A}'_2$. Otherwise, z is a multiple of u. They occur exactly once in the same row or same column in a grid-block in $\mathcal{A}'_2$ and do not occur in $\mathcal{A}_1$. $\square$

2.2 Existence of 3×3 grid-block designs

In this section, we show the existence theorem of 3×3 grid-block designs GB(v, 3, 3). By Lemma 1.3.1, the necessary conditions for the existence of a GB(v, 3, 3) are $v \equiv 1, 9 \pmod{36}$. We show the following theorem by constructing these designs directly.

Theorem 2.2.1 *The necessary conditions* $v \equiv 1, 9 \pmod{36}$ *for the existence of a* GB(v, 3, 3) *are also sufficient.*

Note that the existence of a GB(9, 3, 3) is shown in Example 1.3.2. By utilizing the GB(9, 3, 3) and a B(v, 9, 1), we can obtain a GB(v, 3, 3). That is, if there exists a B(v, 9, 1) for $v \equiv 1, 9 \pmod{72}$, then there exists a GB(v, 3, 3). Unfortunately the existence problem for a B(v, 9, 1) is not completely solved yet. Thus, we construct a GB(v, 3, 3) for all $v \equiv 1, 9 \pmod{36}$ directly. Firstly, we need the following proposition (see, for example, [29]).

Proposition 2.2.2 *If* $v \equiv 1, 3 \pmod{6}$ *and* $v \neq 9$*, then there exists a cyclic* B(v, 3, 1).

By virtue of Theorem 2.1.10 and Proposition 2.2.2, we obtain the following lemma.

Lemma 2.2.3 *If* $v \equiv 9 \pmod{36}$*, then there exists a* GB(v, 3, 3).

Secondly, we obtain the following lemma by utilizing a computer.

Lemma 2.2.4 *If* $v \equiv 1 \pmod{36}$*, then there exists a* GB(v, 3, 3).

Proof. Firstly, in the case of $v = 72t + 1$, Peltesohn [78] showed that there exists a cyclic B(v, 3, 1) (see also Beth, *et al.* [14, pp. 483–484]). According to his result,

$$(0,\ 1+2m,\ 33t+1+m); \qquad m = 0, 1, \ldots, 3t-1; \qquad (2.2.1)$$
$$(0,\ 2+2m,\ 24t+2+m); \qquad m = 0, 1, \ldots, 3t-2; \qquad (2.2.2)$$
$$(0,\ 9t+1+2m,\ 27t+1+m); \qquad m = 0, 1, \ldots, 3t-1; \qquad (2.2.3)$$
$$(0,\ 9t+2+2m,\ 18t+2+m); \qquad m = 0, 1, \ldots, 3t-1; \qquad (2.2.4)$$
$$(0,\ 6t,\ 24t+1); \qquad (2.2.5)$$

are base blocks of a cyclic B(v, 3, 1).

By adding some constants for these base blocks and arranging them in 3×3 grid-blocks as follows, we obtain base grid-blocks for a GB($72t+1$, 3, 3).

$$A_m = \boxed{\begin{array}{lll} 0 & 3+2m & 33t+2+m \\ 9t+3+2m & 9t+4+4m & 42t+4+3m \\ 27t+2+m & 27t+4+3m & 51t+4+2m \end{array}},$$

$$B_m = \boxed{\begin{array}{lll} 0 & 6+2m & 24t+4+m \\ 9t+6+2m & 9t+10+4m & 33t+9+3m \\ 18t+4+m & 18t+9+3m & 51t+7+2m \end{array}}$$

for $m = 0, 3, \ldots, 3t-6$, and

$$C_1 = \boxed{\begin{array}{lll} 0 & 6t-3 & 36t-1 \\ 15t-3 & 21t-8 & 51t-5 \\ 30t-1 & 36t-5 & 57t-2 \end{array}}, \quad C_2 = \boxed{\begin{array}{lll} 0 & 15t & 66t \\ 15t-1 & 21t-1 & 39t \\ 30t & 51t & 45t-2 \end{array}}.$$

We define

$$\boldsymbol{A} = \{A_m : m = 0, 1, \ldots, 3t-6\} \cup \{B_m : m = 0, 1, \ldots, 3t-6\} \cup \{C_1, C_2\}$$

and $\mathcal{A} = \{A + x : A \in \boldsymbol{A},\ x \in V\}$, where $V = \mathbb{Z}_{72t+1}$. Then, $(V, \mathcal{A})$ is the desired GB($72t+1$, 3, 3).

In fact the rows in A_m are obtained by adding 0, $9t+3+2m$ and $27t+2+m$ to (2.2.1) for $m = 1$, (2.2.1) for $m = 0$ and (2.2.2) for $m = 0$ in Table 2.2.1. And the columns in A_m are obtained by adding 0, $3+2m$ and $33t+2+m$ to (2.2.1) for $m = 1$, (2.2.1) for $m = 0$ and (2.2.2) for $m = 0$.

Similarly, for B_m, C_1, and R_2, the rows and columns are constructed by (2.2.1) to (2.2.5). Moreover, note that $m \equiv 0$, 1, and 2 (mod 3) occurs exactly once for each of (2.2.1) to (2.2.5) in A_m and B_m of Table 2.2.1. Thus by considering A_m, B_m for $m = 0, 3, 6, \ldots, 3t-6$ and C_1 and C_2, the base blocks in (2.2.1) to (2.2.5) occur exactly once.

Similarly, in the case of $v = 72t+37$, the following 3×3 grid-blocks generate a GB(v, 3, 3) for $m = 0, 3, \ldots, 3t-3$:

$$A_m = \boxed{\begin{array}{lll} 0 & 33t+16-m & 33t+17+m \\ 9t+7+2m & 42t+22+m & 42t+25+3m \\ 27t+15+m & 51t+26 & 51t+28+2m \end{array}},$$

$$B_m = \boxed{\begin{array}{lll} 0 & 5+2m & 33t+19+m \\ 18t+7-m & 18t+11+m & 42t+21 \\ 27t+16+m & 27t+22+3m & 51t+31+2m \end{array}},$$

Table 2.2.1: The correspondence of the base grid-blocks and base blocks

	base block $\sharp$	m	adding constants
rows in A_m	(2.2.1)	1	0
	(2.2.1)	0	$9t+3+2m$
	(2.2.2)	0	$27t+2+m$
columns in A_m	(2.2.3)	1	0
	(2.2.3)	0	$3+2m$
	(2.2.4)	0	$33t+2+m$
rows in B_m	(2.2.2)	2	2
	(2.2.2)	1	$9t+6+2m$
	(2.2.1)	2	$18t+4+m$
columns in B_m	(2.2.4)	2	0
	(2.2.4)	1	$6+2m$
	(2.2.3)	2	$24t+4+m$
rows in C_1	(2.2.1)	$3t-2$	0
	(2.2.1)	$3t-3$	$15t-3$
	(2.2.2)	$3t-3$	$30t-1$
columns in C_1	(2.2.3)	$3t-2$	0
	(2.2.3)	$3t-3$	$6t-3$
	(2.2.4)	$3t-3$	$36t-1$
rows in C_2	(2.2.4)	$3t-1$	$66t$
	(2.2.5)	–	$15t-1$
	(2.2.4)	$3t-2$	$30t$
columns in C_2	(2.2.3)	$3t-1$	0
	(2.2.1)	$3t-1$	$15t$
	(2.2.2)	$3t-2$	$39t$

$$C = \boxed{\begin{array}{lll} 0 & 18t+9 & 24t+12 \\ 15t+7 & 45t+23 & 9t+6 \\ 30t+15 & 9t+4 & 3t+2 \end{array}}\,.$$

Thus, the lemma is proved. □

2.3 Existence of 2×4 grid-block designs

In this section we apply the results in Section 2.1 to prove the following theorem.

Theorem 2.3.1 *The necessary conditions* $v \equiv 1 \pmod{32}$ *for the existence of a* GB(v, 2, 4) *are also sufficient.*

The existence theorem is shown by utilizing a recursive construction. Firstly, we give an existence of a group divisible design.

Lemma 2.3.2 *For any integer* $v \geq 12$*, there exists a* GD(v, K, M, 1)*, where* $K = \{4, 5\}$ *and* $M = \{1, 2, \ldots, 7\}$.

Proof. According to Brouwer [19], Brouwer, Schrijver and H. Hanani [20] and Beth *et al.* [14], we know the existence of a GD(v, K, M, 1) for any $v \geq 12$ except for $v = 18$ and 19 as is listed in Table 2.3.1 (see also [61] and [74]).

Table 2.3.1: Table of the existence of group divisible designs

v	K	group type	u	exceptions	ref.
$0, 1 \pmod 4$	$\{4, 5\}$	1^u	$0, 1 \pmod 4$	12	[14]
12	4	3^4	–	–	[19]
$2 \pmod{12}$	4	2^u	$1 \pmod 3$	–	[20]
$3 \pmod{12}$	4	3^u	$1 \pmod 4$	–	[20]
$6 \pmod{12}$	4	6^u	Anything	18	[20]
$7 \pmod{12}$	4	$7^1 1^u$	$0 \pmod{12}$	19	[19]
$10 \pmod{12}$	4	$7^1 1^u$	$3 \pmod{12}$	–	[19]
$11 \pmod{12}$	4	$5^1 2^u$	$0 \pmod 3$	–	[19]

Moreover, it is known that there exists a GD(20, {5}, {4}, 1), which was listed in Example 1.2.2. By deleting a single point of the GD(20, {5}, {4}, 1), we can show the existence of a GD(19, {4, 5}, {3, 4}, 1). Similarly, by deleting two points from the same group of the GD(20, {5}, {4}, 1), we obtain a GD(18, {4, 5}, {2, 4}, 1), which prove the case of $v = 18$ and 19. Thus, the lemma is proved. $\square$

Secondly, we give two group divisible grid-block designs which are obtained by computer.

Lemma 2.3.3 *There exists a* $\mathrm{D}(K_{k(32)},\, G_{2,4})$ *for* $k = 4$ *and* 5, *where* $K_{k(32)}$ *is the complete* k*-partite graph and* $G_{2,4}$ *is the graph* $K_2 \times K_4$.

Proof. For $V = \mathbb{Z}_{128}$, let

$$A_1 = \boxed{\begin{array}{cccc} 0 & 1 & 6 & 15 \\ 13 & 30 & 3 & 48 \end{array}}, \quad A_2 = \boxed{\begin{array}{cccc} 0 & 21 & 58 & 47 \\ 22 & 63 & 20 & 97 \end{array}} \text{ and}$$

$$A_3 = \boxed{\begin{array}{cccc} 0 & 25 & 74 & 55 \\ 63 & 56 & 17 & 122 \end{array}},$$

be base grid-blocks which are listed in Table 2.3.2. Now we define $\mathcal{A} = \{A_i + x : i = 1,\, 2,\, 3,\, x \in \mathbb{Z}_{128}\}$. then $(V,\, \mathcal{A})$ is the desired $\mathrm{D}(K_{4(32)},\, G_{2,4})$. In fact, by calculating $\sum_{i=1}^{3} \partial A_i$, any difference except for multiples of 4 occurs exactly once.

Similarly, for $V = \mathbb{Z}_{160}$, by utilizing four base grid-blocks A_1, A_2, A_3 and A_4 in Table 2.3.2, we obtain a $\mathrm{D}(K_{5(32)},\, G_{2,4})$. In fact, by calculating $\sum_{i=1}^{4} \partial A_i$, any difference except for multiples of 5 occurs exactly once. $\square$

Table 2.3.2: Table of the base grid-blocks of group divisible grid-block designs

k	base grid-blocks											
4	0	1	6	15	0	21	58	47	0	25	74	55
	13	30	3	48	22	63	20	97	63	56	17	122
5	0	1	7	3	0	31	17	63	0	66	47	133
	11	27	48	39	22	73	129	30	13	149	105	51
	0	111	52	23								
	84	15	141	102								

Thirdly, we give some grid-block designs which are also obtained by computer.

Lemma 2.3.4 *There exists a* GB$(32m+1,\,2,\,4)$ *for any* $m = 1,\,2,\,\ldots,\,11$.

Proof. By utilizing the base grid-blocks in Tables 2.3.3 and 2.3.4, we obtain the desired GB$(32m+1,\,2,\,4)$'s for $m = 1,\,2,\,3,\,6,\,7,\,\ldots,\,11$. By applying Proposition 2.1.2 to a D$(K_{4(32)},\,G_{2,4})$ and D$(K_{5(32)},\,G_{2,4})$ in Lemma 2.3.3 and a GB(33, 2, 4), GB$(32m+1,\,2,\,4)$'s are obtained for $m = 4,\,5$. □

Table 2.3.3: Table of the base grid-blocks of grid-block designs

v	base grid-blocks											
33	0	1	3	9								
	12	5	23	28								
65	0	1	3	7	0	10	21	45				
	5	13	22	38	47	32	60	9				
97	0	1	3	7	0	10	23	41	0	15	37	61
	5	13	22	33	33	65	86	3	39	55	84	12
193	0	36	65	60	0	46	180	153	0	55	108	73
	89	155	152	153	186	23	71	169	114	77	133	81
	0	14	97	165	0	105	54	44	0	76	67	148
	102	52	40	134	75	34	178	55	39	189	73	174
225	0	104	76	167	0	189	223	92	0	221	77	194
	67	137	121	209	156	74	167	199	41	94	42	16
	0	122	177	140	0	15	220	111	0	7	10	24
	212	190	106	67	95	76	55	46	38	82	206	32
	0	87	161	99								
	79	192	102	13								

Now, we will show the existence theorem.

Proof of Theorem 2.3.1. It is sufficient to show that the necessary conditions $v \equiv 1 \pmod{32}$ for the existence of a GB$(v,\,2,\,4)$ are sufficient. Now we write $v = 32m+1$, then there exists a GB$(32m+1,\,2,\,4)$ for $m \leq 11$ by Lemma 2.3.4. By Lemma 2.3.2, a GD$(m,\,K,\,M,\,1)$ exists for $m \geq 12$, where $K = \{4,\,5\}$ and $M = \{1,\,2,\,\ldots,\,7\}$. And a D$(K_{k(32)},\,G_{2,4})$ exists for $k = 4$ and 5 by Lemma 2.3.3. Thus, by Theorem 2.1.8 there exists a GB$(32m+1,\,2,\,4)$ for any $m \geq 12$, which prove the existence theorem. □

Table 2.3.4: Table of the base grid-blocks of grid-block designs (continued)

v	base grid-blocks											
257	0	51	168	216	0	22	230	37	0	58	61	234
	148	147	81	37	30	211	187	193	200	118	101	154
	0	107	73	14	0	169	42	98	0	132	246	124
	50	79	202	176	63	61	96	216	20	41	72	162
	0	171	210	65	0	75	178	247				
	202	190	197	206	72	255	210	185				
289	0	217	34	207	0	199	54	19	0	228	8	13
	28	188	253	168	105	282	236	183	86	35	165	189
	0	179	122	4	0	241	47	244	0	27	256	218
	209	37	211	284	124	191	110	98	248	182	225	98
	0	185	148	163	0	133	271	227	0	25	32	213
	128	186	216	180	166	14	150	206	77	255	266	164
321	0	235	247	257	0	3	101	281	0	35	186	37
	310	101	228	133	76	105	212	309	244	138	264	16
	0	160	1	265	0	26	317	9	0	7	157	25
	158	66	291	221	269	178	228	315	23	205	143	74
	0	146	61	16	0	315	211	33	0	279	200	255
	283	288	174	115	206	78	146	254	34	105	272	308
	0	240	165	294								
	313	59	255	175								
353	0	286	267	129	0	133	95	248	0	81	72	26
	198	149	219	118	22	20	275	113	82	257	147	261
	0	294	142	15	0	88	76	2476	0	337	109	217
	34	173	198	1	71	222	144	194	66	150	2	211
	0	340	7	343	0	169	254	122	0	193	8	44
	195	5	234	264	316	229	17	59	352	103	127	76
	0	52	23	154	0	186	40	83				
	45	192	134	4	236	298	201	293				

2.4 Existence of $2 \times 2 \times 2$ grid-block designs

In this section, we generalize the definition of a grid-block design to the n-dimensional case. Let V be a set of v points and $\mathcal{A}$ be a collection of $k_1 \times k_2 \times \cdots \times k_n$ arrays with elements in V. Each array in $\mathcal{A}$ is called an n-dimensional grid-block. Let $a_{i_1 i_2 \ldots i_n}$ be the $(i_1, i_2, \ldots, i_n)$-element of an n-dimensional grid-block. We call $L = \{a_{i_1 \ldots i_m \ldots i_n} : 1 \leq i_m \leq k_m\}$ a *grid-block line*. A pair $(V, \mathcal{A})$ is called a $k_1 \times k_2 \times \cdots \times k_n$ *grid-block design*, if the following conditions are satisfied:

(i) Every point occurs at most once in each grid-block of $\mathcal{A}$.

(ii) Every pair of two distinct points of V occurs exactly once in the same grid-block line.

Example 2.4.1 A $2 \times 2 \times 2$ grid-block design with 16 points is given by $V = \{\infty\} \cup \{0_0, 1_0, \ldots, 4_0\} \cup \{0_1, 1_1, \ldots, 4_1\} \cup \{0_2, 1_2, \ldots, 4_2\}$ and $\mathcal{A}$ shown in Figure 2.4.1.

For $k_1 \times k_2 \times \cdots \times k_n$ grid-block design $(V, \mathcal{A})$ with v points, each point x of V has $v-1$ distinct points which occur together with x in the same grid-block line, while each entry of a $k_1 \times k_2 \times \cdots \times k_n$ grid-block has $k_1 + k_2 + \cdots + k_n - n$ entries in the same grid-block line. That is, the number r of grid-blocks containing a given point x is

$$r = \frac{v-1}{k_1 + k_2 + \cdots + k_n - n}, \tag{2.4.1}$$

which is a constant not depending on the choice of x. Also, there are $v(v-1)/2$ pairs which occur once in a grid-block of $\mathcal{A}$ while each grid-block generates $k_1 k_2 \cdots k_n (k_1 + k_2 + \cdots + k_n - n)/2$ pairs. Thus, the number b of grid-blocks is

$$b = \frac{v(v-1)}{k_1 k_2 \cdots k_n (k_1 + k_2 + \cdots + k_n - n)}. \tag{2.4.2}$$

Since r and b must be integers, we obtain the following lemma by the equations (2.4.1) and (2.4.2).

Lemma 2.4.1 *Necessary conditions for the existence of a $k_1 \times k_2 \times \cdots \times k_n$ grid block design with v points are*

$$\begin{aligned} v - 1 &\equiv 0 \pmod{k_1 + k_2 + \cdots + k_n - n} \text{ and} \\ v(v-1) &\equiv 0 \pmod{k_1 k_2 \cdots k_n (k_1 + k_2 + \cdots + k_n - n)}. \end{aligned}$$

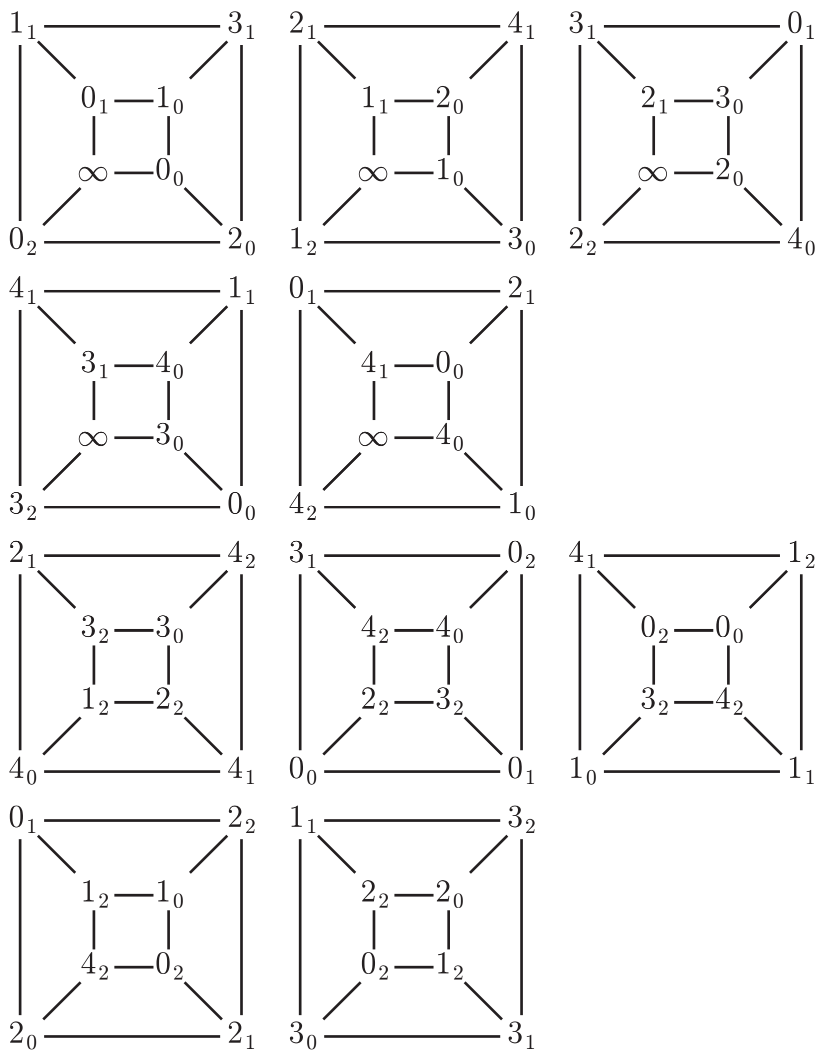


Figure 2.4.1: An example of a $2 \times 2 \times 2$ grid-block design with 16 points.

For graphs G, G' and G'', $G \times G' \times G''$ is defined by $(G \times G') \times G''$. Then, a $k_1 \times k_2 \times \cdots \times k_n$ grid-block is equivalent to the graph $K_{k_1} \times K_{k_2} \times \cdots \times K_{k_n}$. Let G be the graph $K_{k_1} \times K_{k_2} \times \cdots \times K_{k_n}$. Then, Propositions 2.1.2, 2.1.3, 2.1.5 and 2.1.6 and Theorem 2.1.8 hold in term of $k_1 \times k_2 \times \cdots \times k_n$ grid-block design.

By Lemma 2.4.1, necessary conditions for the existence of a $2 \times 2 \times 2$ grid-block design with v points are $v \equiv 1,\ 16 \pmod{24}$. Maheo [69] proved the following proposition by utilizing recursive constructions.

Proposition 2.4.2 *The necessary conditions* $v \equiv 1,\ 16 \pmod{24}$ *for the existence of a* $2 \times 2 \times 2$ *grid-block design with* v *points are also sufficient.*

We define the cyclic or l-rotational grid-block designs in the similar way in Section 1.7. Then, we give another proof of Proposition 2.4.2 by constructing cyclic and 3-rotatoinal grid-block designs with $24t+1$ and $24t+16$ points, respectively. Firstly, we define the $2 \times 2 \times 2$ grid-block difference families in Γ. For a $2 \times 2 \times 2$ grid-block $A = (a_{i_1 i_2 i_3})$ with elements in Γ as follows:

$$\begin{aligned}\partial A = &(a_{i_1 i_2 i'_3} - a_{i_1 i_2 i_3} : 1 \le i_1,\ i_2 \le 2,\ 1 \le i_3 \ne i'_3 \le 2)\\ &+ (a_{i_1 i'_2 i_3} - a_{i_1 i_2 i_3} : 1 \le i_1,\ i_3 \le 2,\ 1 \le i_2 \ne i'_2 \le 2)\\ &+ (a_{i'_1 i_2 i_3} - a_{i_1 i_2 i_3} : 1 \le i_2,\ i_3 \le 2,\ 1 \le i_1 \ne i'_1 \le 2).\end{aligned}$$

For a family of grid-blocks $\boldsymbol{A}$, we define $\partial \boldsymbol{A} = \sum_{A \in \boldsymbol{A}} \partial A$.

Similarly, for a $2 \times 2 \times 2$ grid-block $A = ((a_{t_1 t_2 t_3},\ l_{t_1 t_2 t_3}))$, let $\partial_{ij} A$ be the list of differences $a_{t'_1 t'_2 t'_3} - a_{t_1 t_2 t_3}$ occur in the same line of A, that is,

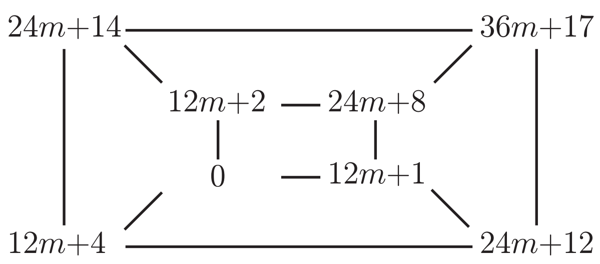
$$\begin{aligned}\partial_{ij} A = &(a_{t_1 t_2 t'_3} - a_{t_1 t_2 t_3} : 1 \le t_1,\ t_2 \le 2,\ 1 \le t_3 \ne t'_3 \le 2,\ l_{t_1 t_2 t_3} = i,\ l_{t_1 t_2 t'_3} = j)\\ &+ (a_{t_1 t'_2 t_3} - a_{t_1 t_2 t_3} : 1 \le t_1,\ t_3 \le 2,\ 1 \le t_2 \ne t'_2 \le 2,\ l_{t_1 t_2 t_3} = i,\ l_{t_1 t'_2 t_3} = j)\\ &+ (a_{t'_1 t_2 t_3} - a_{t_1 t_2 t_3} : 1 \le t_2,\ t_3 \le 2,\ 1 \le t_1 \ne t'_1 \le 2,\ l_{t_1 t_2 t_3} = i,\ l_{t'_1 t_2 t_3} = j).\end{aligned}$$

For a family of grid-blocks $\boldsymbol{A}$, we define $\partial_{ij} \boldsymbol{A} = \sum_{A \in \boldsymbol{A}} \partial_{ij} A$. Obviously, $\partial_{ij} \boldsymbol{A} = -\partial_{ji} \boldsymbol{A}$ and the difference 0 is allowed in $\partial_{ij} \boldsymbol{A}$ if and only if $i \ne j$ holds.

Lemma 2.4.3 *For any* $v \equiv 1 \pmod{24}$*, there exists a cyclic* $2 \times 2 \times 2$ *grid-block design with* v *points.*

Proof. Let $v = 24t+1$ for $t \ge 1$, $V = \mathbb{Z}_v$ and

$A_m =$



be $2 \times 2 \times 2$ base grid-blocks for $m = 0, 1, \ldots, t-1$. In fact, by identifying A_m as a block,

$$\begin{aligned}\Delta A_m = (&\pm 1, \pm 2, \pm 2, \pm 3, \pm 4, \pm 6, \pm(12m+4), \pm(12m+10),\\ &\pm (12m+13), \pm(24m+8), \pm(24m+12), \pm(24m+13),\\ &\pm (24m+14), \pm(24m+15), \pm(24m+16), \pm(36m+17))\\ \cup\, (&\pm(12m+1), \pm(12m+2), \ldots, \pm(12m+12))\end{aligned}$$

It is obvious that each ΔA_m does not have $0 \in \mathbb{Z}_v$. That is, all elements of A_m are distinct for each $m = 0, 1, \ldots, t-1$.

Now we define $\mathcal{A} = \{A_m + x : m = 0, 1, \ldots, m-1, x \in V\}$, then $(V, \mathcal{A})$ is the desired $2 \times 2 \times 2$ grid-block design. In fact, for each A_m,

$$\partial A_m = (\pm(12m+1), \pm(12m+2), \ldots, \pm(12m+12))$$

holds. Thus,

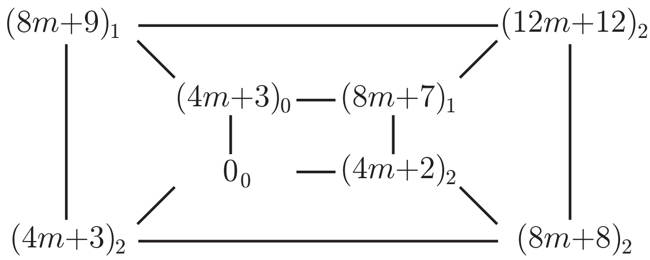
$$\sum_{m=0}^{t-1} \partial A_m = \mathbb{Z}_v \setminus \{0\}$$

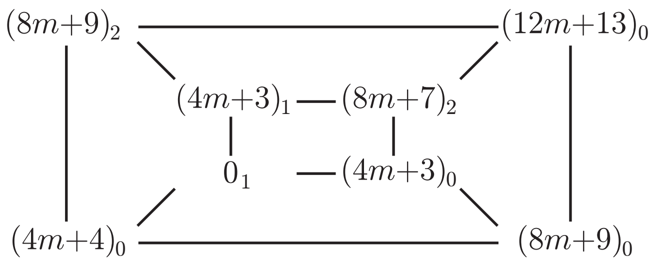
holds. □

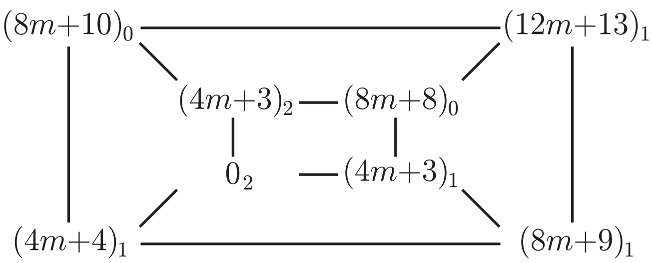
Lemma 2.4.4 *For any $v \equiv 16 \pmod{24}$, there exists a 3-rotational $2 \times 2 \times 2$ grid-block design with v points.*

Proof. In the case of $v = 16$, there exists a $2 \times 2 \times 2$ grid-block design listed in Example 2.4.1. Let $v = 24t + 16$ for $t \geq 1$, $V = (\mathbb{Z}_{8t+5} \times \{0, 1, 2\}) \cup \{\infty\}$

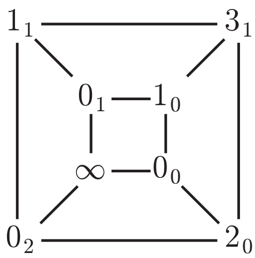
and

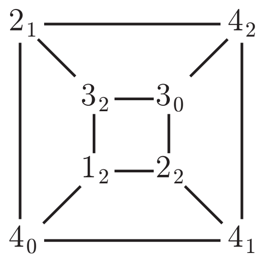
$A_{m,0} =$ 

$A_{m,1} =$ 

$A_{m,2} =$ 

be $2 \times 2 \times 2$ base grid-blocks for $m = 0, 1, \dots, t-1$. Moreover, let

$B_0 =$ 

and $B_1 =$ 

be $2 \times 2 \times 2$ base grid-blocks.

Firstly, we check that all elements of each $2 \times 2 \times 2$ base grid-block are distinct. By identifying $A_{m,s}$ as a block,

$$\begin{aligned}
\Delta_{s,s} A_{m,s} &= (\pm(4m+3)), \qquad \Delta_{1+s,1+s} A_{m,s} = (\pm 2), \\
\Delta_{2+s,2+s} A_{m,s} &= (\pm 1, \pm(4m+3), \pm(4m+4), \\
&\qquad \pm(4m+5), \pm(4m+6), \pm(8m+9), \pm(8m+10))
\end{aligned}$$

hold for each $m = 0, 1, \ldots, t-1$ and $s = 0, 1, 2$. Note that the indices are calculated by modulo 3 for $s = 0, 1, 2$. Each $\Delta_{ii} A_{m,s}$ does not have $0 \in \mathbb{Z}_{8t+5}$ for $i = 0, 1, 2$. Thus, all elements of $A_{m,s}$ are distinct. It is obvious that all elements of B_0 or B_1 are distinct.

Now we define $\boldsymbol{A}_m = \{A_{m,0}, A_{m,1}, A_{m,2}\}$, $\boldsymbol{A} = \cup_{m=0}^{t-1} \boldsymbol{A}_m \cup \{B_0, B_1\}$ and $\mathcal{A} = \{A + x : A \in \boldsymbol{A}, x \in \mathbb{Z}_{8t+5}\}$. Then $(V, \mathcal{A})$ is the desired $2 \times 2 \times 2$ grid-block design.

Firstly,

$$\partial_{ii} B_0 + \partial_{ii} B_1 = (\pm 1, \pm 2)$$

and

$$\begin{aligned}
&\partial_{01} B_0 + \partial_{01} B_1 = \partial_{12} B_0 + \partial_{12} B_1 = (0, \pm 1, \pm 2) \\
&\partial_{02} B_0 + \partial_{02} B_1 = (0, \pm 1, -2, -3)
\end{aligned}$$

hold for $i = 0, 1, 2$. For $m = 0, 1, \ldots, t-1$,

$$\begin{aligned}
\partial_{ii} \boldsymbol{A}_m &= (\pm(4m+3), \pm(4m+4), \pm(4m+5), \pm(4m+6)) \\
\partial_{01} \boldsymbol{A}_m &= \partial_{12} \boldsymbol{A}_m = (\pm(4m+3), \pm(4m+4), \pm(4m+5), \pm(4m+6)) \\
\partial_{02} \boldsymbol{A}_m &= (4m+2, 4m+3, \pm(4m+4), \pm(4m+5), -(4m+6), -(4m+7))
\end{aligned}$$

hold. Thus, we obtain the following:

$$\partial_{ii} \boldsymbol{A} = \partial_{ii} B_0 + \partial_{ii} B_1 + \sum_{m=0}^{t-1} \partial_{ii} \boldsymbol{A}_m = \mathbb{Z}_{8t+5} \setminus \{0\}$$

for $i = 0, 1, 2$. Similarly,

$$\begin{aligned}
\partial_{01} \boldsymbol{A} &= \partial_{12} \boldsymbol{A} = \partial_{01} B_0 + \partial_{01} B_1 + \sum_{m=0}^{t-1} \partial_{01} \boldsymbol{A}_m = \mathbb{Z}_{8t+5}, \\
\partial_{02} \boldsymbol{A} &= (0, \pm 1, -2, -3) + \sum_{m=0}^{t-2} \partial_{02} \boldsymbol{A}_m \\
&\qquad + (4t-2, 4t-1, \pm 4t, \pm(4t+1), -(4t+2), -(4t+3)) \\
&= \mathbb{Z}_{8t+5}
\end{aligned}$$

hold since $-(4t+3) = 4t+2$ holds.

Thus, for any two distinct points x and y in $V \setminus \{\infty\}$, they occur exactly once in the same grid-block line. Lastly, it is easy to show that ∞ and x_i occur exactly once in the same grid-block line in the grid-block $B_0 + x$ for $x \in \mathbb{Z}_{8t+5}$ and $i = 0, 1, 2$. □

2.5 An asymptotic existence of resolvable grid-block designs

In this section, we give constructions of resolvable grid-block designs. Firstly, we give a recursive construction. We utilize a resolvable group grid-block design $\mathrm{D}(K_{s(t)}, G_{k_1,k_2})$ which is defined by the similar way in Section 1.3. Now, we define an OA. For $N = \{0, 1, \ldots, n-1\}$, an *orthogonal array of order n, degree k and index λ*, denoted by $\mathrm{OA}(n, k, \lambda)$, is an $(n^2\lambda \times k)$-matrix with entries from N such that each $(n^2\lambda \times 2)$-submatrix contains every ordered pair of N precisely λ times.

Example 2.5.1 The following (9×4)-matrix forms an $\mathrm{OA}(3, 4, 1)$:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \end{pmatrix}^T,$$

where T is a transpose of a matrix.

We give a recursive construction of a resolvable $\mathrm{GB}(v, k_1, k_2)$.

Theorem 2.5.1 *Assume that $k_1 \leq k_2$. If there exist a resolvable $\mathrm{D}(K_{s(t)}, G_{k_1,k_2})$, an $\mathrm{OA}(n, k_2+1, 1)$ and a resolvable $\mathrm{GB}(nt, k_1, k_2)$, then there exists a resolvable $\mathrm{GB}(nst, k_1, k_2)$.*

Proof. For an st-set V, let a triple $(V, \mathcal{M}, \mathcal{A})$ be a resolvable $\mathrm{D}(K_{s(t)}, G_{k_1,k_2})$. The number b_V of the grid-blocks is $t^2s(s-1)/k_1k_2(k_1+k_2-2)$. Let $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{r_V}\}$ be a resolution of the resolvable $\mathrm{D}(K_{s(t)}, G_{k_1,k_2})$, the number r_V of the resolution classes is $t(s-1)/(k_1+k_2-2)$.

Similarly, for an nt-set W, let a pair $(W, \mathcal{B})$ be a resolvable $\mathrm{GB}(k_1, k_2, nt)$. The number b_W of the grid-blocks is $nt(nt-1)/k_1k_2(k_1+k_2-2)$. Let $\{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_{r_W}\}$ be a resolution of the resolvable $\mathrm{GB}(nt, k_1, k_2)$, the number r_W of the resolution classes is $(nt-1)/(k_1+k_2-2)$.

For $N = \{0, 1, \ldots, n-1\}$, let $(n^2 \times (k_2+1))$-matrix $L = (\rho_{ij})$, for $i = 0, 1, \ldots, n^2-1$ and $j = 0, 1, \ldots, k_2$, be an $\mathrm{OA}(n, k_2+1, 1)$. By applying a permutation to rows of L, we assume that the (k_2+1)-th column as follows:

$$\begin{array}{llll} \rho_{0,k_2} = 0, & \rho_{1,k_2} = 0, & \ldots, & \rho_{n-1,k_2} = 0, \\ \rho_{n,k_2} = 1, & \rho_{n+1,k_2} = 1, & \ldots, & \rho_{2n-1,k_2} = 1, \\ \vdots & \vdots & & \vdots \\ \rho_{(n-1)n,k_2} = n-1, & \rho_{(n-1)n+1,k_2} = n-1, & \ldots, & \rho_{n^2-1,k_2} = n-1. \end{array}$$

Moreover, each $n \times 1$-column vector $(\rho_{un,j}, \rho_{un+1,j}, \ldots, \rho_{un+n-1,j})^T$ for $u = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, k_2 - 1$ contains every element of N precisely once.

Let $V^* = V \times N = \{(a, \rho) : a \in V, \rho \in N\}$ and for each grid-blocks $A = (a_{l,m})$ of $(V, \mathcal{M}, \mathcal{A})$, we define

$$
\begin{aligned}
C_i(A) &= ((a_{l,m}, \rho_{i,l+m})) \\
&= \boxed{\begin{array}{llll}
(a_{00}, \rho_{i,0}) & (a_{01}, \rho_{i,1}) & \cdots & (a_{0,k_2-1}, \rho_{i,k_2-1}) \\
(a_{10}, \rho_{i,1}) & (a_{11}, \rho_{i,2}) & \cdots & (a_{1,k_2-1}, \rho_{i,0}) \\
\vdots & \vdots & \ddots & \vdots \\
(a_{k_1-1,0}, \rho_{i,k_1-1}) & (a_{k_1-1,1}, \rho_{i,k_1}) & \cdots & (a_{k_1-1,k_2-1}, \rho_{i,k_2+k_2-2})
\end{array}}
\end{aligned}
$$

for $A \in \mathcal{A}$ and $i = 0, 1, \ldots, n^2 - 1$. Note that in the second subscript of ρ, $l + m$ means $l + m \pmod{k_2}$. We define $\mathcal{A}_1^*$ as $\{C_i(A) : A \in \mathcal{A}, i = 0, 1, \ldots, n^s - 1\}$. Up to now we have $n^2 b_V = n^2 t^2 s(s-1)/k_1 k_2 (k_1 + k_2 - 2)$ gird-blocks, but we need $nts(nts-1)/k_1 k_2(k_1 + k_2 - 2)$ grid-blocks in total.

In order to get further grid-blocks, let $W = M \times N$ and let $(W(M), \mathcal{B}(M))$ be the ingredient resolvable design GB(nt, k_1, k_2) for each partite set $M \in \mathcal{M}$. Now, we define $\mathcal{A}_2^* = \bigcup_{M \in \mathcal{M}} \mathcal{B}(M)$. Then we obtain $sb_W = snt(nt - 1)/k_1 k_2 (k_1 + k_2 - 2)$ new grid-blocks, in total

$$n^2 b_V + sb_W = \frac{nst(nss-1)}{k_1 k_2 (k_1 + k_2 - 2)}$$

grid-blocks. Let $\mathcal{A}^* = \mathcal{A}_1^* \cup \mathcal{A}_2^*$, then a pair $(V^*, \mathcal{A}^*)$ is the desired GB(k_1, k_2, nst).

In fact, if two distinct points a_1 and a_2 in V are not contained in the same partite set $M \in \mathcal{M}$, then a_1 and a_2 occur together exactly once in $A \in \mathcal{A}$ and the pair (ρ_1, ρ_2) occur exactly once in the OA$(n, k_2 + 1, 1)$. Hence each pair (a_1, ρ_1) and (a_2, ρ_2) for any $\rho_1, \rho_2 \in N$ occurs exactly once in the same row or in the same column of a grid-block in $\mathcal{A}_1^*$ and does not occur in $\mathcal{A}_2^*$. Otherwise two distinct points a_1 and a_2 in V are in the same partite set M, then each pair (a_1, ρ_1) and (a_2, ρ_2) for all ρ_1, ρ_2 occurs exactly once in the same row or in the same column of a grid-block in $(W(M), \mathcal{B}(M))$ and does not occur in $\mathcal{A}_1^*$. That is, $(V^*, \mathcal{A}^*)$ is a GB(nst, k_1, k_2). It remains to show that $(V^*, \mathcal{A}^*)$ is resolvable.

We partition the grid-blocks into $r^* = (nst-1)/k_1 k_2 (k_1 + k_2 - 2)$ resolution classes. At first, let $\mathcal{R}_m^u$ be as follows:

$$\mathcal{R}_m^u = \{C_i(A) : i = us, us+1, \ldots, us+s-1, A \in \mathcal{P}_l\}$$

for $m = 1, 2, \ldots, r_V$ and $u = 1, 2, \ldots, n-1$. Then, $\mathcal{R}_m^u$ is a resolution class.

We construct still more resolution classes. For resolution classes $\{\mathcal{Q}_1(M), \mathcal{Q}_2(M), \ldots, \mathcal{Q}_{r_W}(M)\}$ in $(W(M), \mathcal{B}(M))$, let $\mathcal{O}_l = \bigcup_{M \in \mathcal{M}} \mathcal{Q}_l(M)$. Obviously, $\mathcal{O}_l$ is a resolution class. The total number of resolution classes $\mathcal{R}_m^u$ and $\mathcal{O}_l$ is $nr_V + r_W = (nst-1)/(k_1+k_2-2)$ as desired. □

If each partite set has a single point, then we obtain the following corollary.

Corollary 2.5.2 *Assume that $k_1 \leq k_2$. If there exist a resolvable* GB(s, k_1, k_2)*, an* OA$(n, k_2+1, 1)$ *and a resolvable* GB(n, k_1, k_2)*, then there exists a resolvable* GB(ns, k_1, k_2).

The following construction for a resolvable BIB design is obtained by Ray-Chaudhuri and Wilson [85].

Proposition 2.5.3 *For a prime power q, if there exists a mutually disjoint difference family* $(q, k, 1)$-DF *in* GF(q)*, then there exists a resolvable* B$(kq, k, 1)$.

By combining a resolvable grid-block design with a resolvable BIB designs, we obtain the following corollary.

Corollary 2.5.4 *Let q be a prime power, if there exists a mutually disjoint* $(q, v, 1)$-DF *in* GF(q) *and a resolvable* GB(v, k_1, k_2)*, then there exists a resolvable* GB(vq, k_1, k_2).

Similarly, we obtain the following theorem by utilizing a mutually disjoint (q, k_1, k_2)-GBDF.

Theorem 2.5.5 *For a prime power q, assume that $k_1k_2(k_1+k_2-2)$ divides $q-1$. If there exists a mutually disjoint* (q, k_1, k_2)-GBDF *in* GF(q) *and a* GB(k_1k_2, k_1, k_2)*, then there exists a resolvable* GB(k_1k_2q, k_1, k_2).

Proof. Let $(W, \mathcal{F})$ be a GB(k_1k_2, k_1, k_2), where $\mathcal{F} = \{F_0, F_1, \ldots, F_{b_{k_1k_2}-1}\}$ and $b_{k_1k_2} = (k_1k_2-1)/(k_1+k_2-2)$. And let $\{A_1, A_2, \ldots, A_t\}$ be a mutually disjoint (q, k_2, k_1)-DF, where the number t of base grid-blocks is $(q-1)/k_1k_2(k_1+k_2-2)$. Hence,

$$\sum_{i=1}^{t} = tk_1k_2 < q,$$

and without loss of generality, we assume that $0 \notin A_i$ for $i = 1, 2, \ldots, t$. For $N = \{0, 1, \ldots, k_1k_2 - 1\}$, let $V = \mathrm{GF}(q) \times N$. For a k_1k_2-set let $B_0 = \{(0, 0), (0, 1), \ldots, (0, k_1k_2 - 1)\}$, $(B_0, \mathcal{F}(B_0))$ be the $\mathrm{GB}(k_1k_2, k_1, k_2)$ and

$$B_i^j = A_i \times \{j\} = ((a_{l,m}^i, j))$$

for $i = 1, 2, \ldots, t$ and $j \in N$ and $A_i = (a_{l,m}^i)$. Up to now we have $b_{k_1k_2} + k_1k_2t = (k_1k_2 + q - 2)/(k_1 + k_2 - 2)$ base grid-blocks.

In order to get further base grid-blocks, we choose arbitrary k_1k_2 distinct points $u_0 = 1, u_1, \ldots, u_{k_1k_2-1}$ of $\mathrm{GF}(q) \setminus \{0\}$, and let

$$C_x = \{(u_0x, 0), (u_1x, 1), \ldots, (u_{k_1k_2-1}x, k_1k_2 - 1)\}$$

for $x \in \mathrm{GF}(q) \setminus \{0\}$. And we define $(C_x, \mathcal{F}(C_x))$ as the $\mathrm{GB}(k_1k_2, k_1, k_2)$. Then, we have $b_{k_1k_2}(q - 1)$ new base grid-blocks, in total

$$b_{k_1k_2} + k_1k_2t + b_{k_1k_2}(q - 1) = (k_1k_2q - 1)/(k_1 + k_2 - 2)$$

base grid-blocks $F_h(B_0)$, B_i^j and $F_h(C_x)$ are obtained, where $F_h(B_0)$ and $F_h(C_x)$ are grid-blocks in $(B_0, \mathcal{F}(B_0))$ and $(C_x, \mathcal{F}(C_x))$, respectively. Now we replace the base grid-blocks B_i^j by $u_jB_i^j$ to satisfy the condition of resolvability and we define $\boldsymbol{A}$ of new grid-blocks by

$$\begin{aligned}\boldsymbol{A} = &\{F_h(B_0) : h = 0, 1, \ldots, b_{k_1k_2} - 1\}\\ &\cup \{u_jB_i^j : i = 1, 2, \ldots, t, \ j \in N\}\\ &\cup \{F_h(C_x) : h = 0, 1, \ldots, b_{k_1k_2} - 1, \ x \in \mathrm{GF}(q) \setminus \{0\}\}.\end{aligned}$$

The pure differences arise from the $u_jB_i^j$, and the mixed differences come from the $F_h(C_x)$ and $F_h(B_0)$. Since $\{A_1, A_2, \ldots, A_t\}$ is a (q, k_1, k_2)-GBDF,

$$\sum_{i=1}^{t} \partial_{jj}(u_jB_i^j) = \sum_{i=1}^{t} u_j\partial_{jj}B_i^j = u_j(\mathrm{GF}(q) \setminus \{0\}) = \mathrm{GF}(q) \setminus \{0\}$$

holds. Furthermore, for $i < j$,

$$\sum_{h=0}^{b_{k_1k_2}-1} \partial_{ij}F_h(B_0) = \Delta_{ij}B_0 = \{0\}$$

and

$$\begin{aligned}\sum_{x\in\mathrm{GF}(q)\setminus\{0\}} \left(\sum_{h=0}^{b_{k_1k_2}-1} \partial_{ij}F_h(C_x) \right) &= \sum_{x\in\mathrm{GF}(q)\setminus\{0\}} \Delta_{ij}C_x\\ &= (u_j - u_i)(\mathrm{GF}(q) \setminus \{0\})\\ &= \mathrm{GF}(q) \setminus \{0\}\end{aligned}$$

hold.

Hence $\partial_{ii}\boldsymbol{A} = \mathrm{GF}(q)\setminus\{0\}$ and $\partial_{ij}\boldsymbol{A} = \mathrm{GF}(q)$ hold for $i \neq j$, which implies that the pair $(V, \mathcal{A})$ is a $\mathrm{GB}(k_1k_2q, k_1, k_2)$ for $\mathcal{A} = \{A + x : A \in \boldsymbol{A}, x \in \mathrm{GF}(q)\}$. It remains to show that $\mathcal{A}$ is resolvable.

We partition the grid-blocks into $r = (k_1k_2q - 1)/(k_1 + k_2 - 2)$ resolution classes. We identify the set $\{a^i_{l,m} : l = 1, 2, \ldots, k_1, m = 1, 2, \ldots, k_2\}$ with the grid-block A_i. Let $\mathcal{P}_0$ be as follows:

$$\begin{aligned}\mathcal{P}_0 = \{F_0(B_0)\} &\cup \{u_jB^j_i : i = 1, 2, \ldots, t, j \in N\} \\ &\cup \{F_0(C_x) : x \notin A_i \text{ for any } i\}.\end{aligned}$$

Then the number of grid-blocks in $\mathcal{P}_0$ is $1 + k_1k_2t + (q - 1 - k_1k_2t) = q$. The point in these grid-blocks are

$$\begin{aligned}&(0, 0), (0, 1), \ldots, (0, k_1k_2 - 1), \\ &(u_ja^i_{11}, j), \ldots, (u_ja^i_{21}, j), \ldots, (u_ja^i_{k_1,k_2}, j) \text{ for } i = 1, 2, \ldots, t \text{ and } j \in N, \\ &(u_0x, 0), (u_1x, 1), \ldots, (u_{k_1k_2-1}x, k_1k_2 - 1) \text{ for all } x \text{ except for } x \in A_i.\end{aligned}$$

Obviously every point V occurs exactly once, i.e. $\mathcal{P}_0$ is a resolution class.

We define a map $\pi_g : (x, j) \mapsto (x + g, j)$ for all $g \in \mathrm{GF}(q)$ and $\mathcal{P}_g = \{\pi_g(A) : A \in \mathcal{P}_0\}$. Then it is obvious that $\mathcal{P}_g$'s are resolution classes. It is easy to see that $\mathcal{Q}_x = \{\pi_g(F(C_x)) : g \in \mathrm{GF}(q)\}$ is a resolution class for each $x \in A_1 \cup A_2 \cup \cdots \cup A_t$. Similarly, we construct still more classes $\mathcal{R}^h_x = \{\pi_g(F_h(C_x)) : g \in \mathrm{GF}(q)\}$ and $\mathcal{R}^h_0 = \{\pi_g(F_h(B_0)) : g \in \mathrm{GF}(q)\}$ for $h = 1, 2, \ldots, b_{k_1k_2} - 1$. Each $\mathcal{R}^h_x$ is also a resolution class. The total number of resolution classes $\mathcal{P}_g$, $\mathcal{Q}_x$ and $\mathcal{R}^h_x$ is

$$q + k_1k_2t + (b_{k_1k_2} - 1)q = \frac{k_1k_2q - 1}{k_1 + k_2 - 2} = r.$$

Hence the theorem is proved. $\square$

From Corollary 2.5.4 with $v = k_1k_2$, we obtain a resolvable $\mathrm{GB}(k_1^2k_2^2(k_1k_2 - 1)t + 1, k_1, k_2)$ when $k_1k_2(k_1k_2 - 1)t + 1$ is a prime power. But from Theorem 2.5.5, we obtain a resolvable $\mathrm{GB}(k_1^2k_2^2(k_1 + k_2 - 2)t + 1, k_1, k_2)$ when $k_1k_2(k_1+k_2-2)t+1$ is a prime power. By the existence of a $\mathrm{GB}(k_1k_2, k_1, k_2)$, $k_1 + k_2 - 2$ divides $k_1k_2 - 1$. That is, Corollary 2.5.4 is included in Theorem 2.5.5. For example, in the case of $k_1 = k_2 = 3$ and $q = 37$, there exists a mutually distinct (37, 3, 3)-GBDF but a B(37, 9, 1) does not exists. Hence we can not find the existence by Corollary 2.5.4, while we can claim the existence by Theorem 2.5.5.

Finally, we show the existence of a resolvable GB(k_1k_2q, k_1, k_2) when q is sufficiently large prime powers satisfying $q \equiv 1 \pmod{k_1k_2(k_1+k_2-2)}$ and there exists a resolvable GB(k_1k_2, k_1, k_2). First, we give a lemma to show the existence of resolvable grid-block designs.

Lemma 2.5.6 *For a prime power $q = 1 \pmod{k_1k_2(k_1+k_2-2)}$, let $m = k_1k_2(k_1+k_2-2)/2$. If there exists a $k_1 \times k_2$ array $A = (a_{ij})$ over* GF(q) *such that two differences of ∂A lie in each coset modulo $\mathcal{H}^m$, or equivalently, such that*

$$\begin{aligned}\overrightarrow{\partial} A = &(a_{jl} - a_{il} : 1 \le i < j \le k_1,\ 1 \le l \le k_2)\\ &\cup (a_{lj} - a_{li} : 1 \le l \le k_1,\ 1 \le i < j \le k_2).\end{aligned}$$

*are a system of representative for the cosets $\mathcal{H}^m$, then there exists a (q, k_1, k_2)-*GBDF *in* GF(q).

Proof. Since $2m$ divides $q-1$, we have $-1 \ne 1 \in H_0^m$. By the assumption, $\overrightarrow{\partial} A$ must have precisely one entry in each coset H_i^m for $0 \le i \le m-1$, and $\partial A = (1, -1) \circ \overrightarrow{\partial} A$ holds. Let S be a system of representatives for the cosets of the quotient group $H_0^m/\{1, -1\}$, so that $H_0^m = S \circ (1, -1)$. Let $\boldsymbol{A} = \{sA : s \in S\}$. Then,

$$\partial \boldsymbol{A} = S \circ \partial A = S \circ (1, -1) \circ \overrightarrow{\partial} A = \mathrm{GF}(q) \setminus \{0\},$$

i.e. $\boldsymbol{A}$ is a (q, k_1, k_2)-GBDF in GF(q). □

By Proposition 1.8.1 and Lemma 2.5.6, we obtain the following theorem.

Theorem 2.5.7 *If there exists a* GB(k_1k_2, k_1, k_2), *then there exists a constant $q_0 = q_0(k_1, k_2)$ such that a resolvable* GB(k_1k_2q, k_1, k_2) *exists for all prime powers $q \ge q_0$ satisfying the congruence $q \equiv 1 \pmod{k_1k_2(k_1+k_2-2)}$.*

Proof. It is sufficient that there exists a mutually disjoint (q, k_1, k_2)-GBDF in GF(q). Let $I = \{(i, j) : 1 \le i \le k_1,\ 1 \le j \le k_2\}$ and let $P_{k_1 \times k_2 + 1}$ be the set of the following ordered pairs of $I \cup \{0\}$, that is,

$$\begin{aligned}P_{k_1 \times k_2 + 1} = &\{((i, j), (i', j')) : 1 \le i \le i' \le k_1,\ 1 \le j, j' \le k_2,\\ &\quad \text{except for } i = i' \text{ and } j = j'\}\\ &\cup \{(0, (i, j)) : 1 \le i \le k_1,\ 1 \le j \le k_2\}.\end{aligned}$$

We divide $P_{k_1\times k_2+1}$ into three subsets as follows:

$$\begin{aligned}
P^R_{k_1\times k_2} &= \{((i,\, l),\, (i',\, l)) : 1 \le i < i' \le k_1,\, 1 \le l \le k_2\}\\
P^C_{k_1\times k_2} &= \{((l,\, j),\, (l,\, j')) : 1 \le l \le k_1,\, 1 \le j < j' \le k_2\}\\
P^E_{k_1\times k_2} &= \{((i,\, j),\, (i',\, j')) : 1 \le i < i' \le k_1,\, 1 \le j \ne j' \le k_2\}\\
P^S_{k_1\times k_2} &= \{(0,\, (i,\, j)) : 1 \le i \le k_1,\, 1 \le j \le k_2\}.
\end{aligned} \tag{2.5.1}$$

By considering a pair $(i,\, j) \in I$ as $k_2 j + i$, the set of pairs in $P_{k_1\times k_2+1}$ can be identified with $P_{k_1k_2+1} = \{(i'',\, j'') : 1 \le i'' < j'' \le k_1k_2 + 1\}$.

Let $m = k_1k_2(k_1 + k_2 - 2)/2$ and $M : P_{k_1\times k_2+1} \to \mathcal{H}^m$ be a choice such that (i) M is an injection from $P^R_{k_1\times k_2} \cup P^C_{k_1\times k_2}$ to $\mathcal{H}^m$, (ii) it maps $P^E_{k_1\times k_2}$ into $\mathcal{H}^m$ arbitrarily and (iii) each ordered pair $(0,\, (i,\, j)) \in P^S_{k_1\times k_2}$ into mutually distinct cosets H^m_l. Then by Proposition 1.8.1, we can find an element $x_0 \in \mathrm{GF}(q)$ and a $k_1 \times k_2$ array (x_{ij}) over $\mathrm{GF}(q)$ consistent with the choice M.

Let $A = (a_{ij}) = (x_{ij} - x_0)$, then the elements of ∂A occur exactly twice in each coset of $\mathcal{H}^m$. Then $\boldsymbol{A} = \{hA : h \in H^m_0/\{1,\, -1\}\}$ is a $(q,\, k_1,\, k_2)$-GBDF by Lemma 2.5.6. Moreover, a_{ij}'s lie in distinct cosets modulo H^m_0. Thus, all points contained in all $hA \in \boldsymbol{A}$ are distinct, that is, the sets hA for $h \in H^m_0/\{1,\, -1\}$ are disjoint, i.e. $\boldsymbol{A}$ is also a mutually disjoint $(q,\, k_1,\, k_2)$-GBDF, which prove the theorem by Theorem 2.5.5. $\square$

2.6 Constructions of resolvable grid-block packings

In this section, we construct maximal resolvable 2×2 grid-block packings and $q \times q$ grid-block packings for a prime power q and give some recursive constructions. Firstly, we give the following theorem by constructing directly.

Theorem 2.6.1 *There exists a maximal resolvable* $\mathrm{GBP}(v,\, 2,\, 2)$ *for any* $v \equiv 0 \pmod 4$.

Proof. Let $V = (\mathbb{Z}_{2t-1} \cup \{\infty\}) \times \{0,\, 1\}$ for any $t \ge 1$. We define base gird-blocks

$$A_\infty = \boxed{\begin{array}{cc} \infty_0 & 0_0 \\ 0_1 & \infty_1 \end{array}}$$

and

$$A_m = \boxed{\begin{array}{cc} m_0 & (2t-m-1)_0 \\ (2t-m-1)_1 & m_1 \end{array}},$$

for $m = 1, 2, \ldots, t-1$. Also we define the family $\boldsymbol{A}$ of base grid-blocks by

$$\boldsymbol{A} = \{A_m : m = \infty, 1, 2, \ldots, t-1\}.$$

Moreover, we define a map $\pi_g : (m, i) \mapsto (m+g, i)$ for $m \in \mathbb{Z}_{2t-1} \cup \{\infty\}$, $g \in \mathbb{Z}_{2t-1}$ and $i = 0, 1$. Note that $\infty + g = \infty$. Let $\mathcal{A} = \{\pi_g(A_m) : m = \infty, 1, 2, \ldots, t-1, g \in \mathbb{Z}_{2t-1}\}$. Then $(V, \mathcal{A})$ is a GB$(4t, 2, 2)$ since

$$\partial_{00}\boldsymbol{A} = \partial_{01}\boldsymbol{A} = \partial_{10}\boldsymbol{A} = \partial_{11}\boldsymbol{A} = \mathbb{Z}_{2t-1} \setminus \{0\}$$

hold and ∞_i and g_j occur exactly once in the same row or in the same column of a grid-block for each $i, j = \{0, 1\}$ and each $g \in \mathbb{Z}_{2t-1}$.

Also, $\pi_g(\boldsymbol{A})$ is obviously a resolution class for each $g \in \mathbb{Z}_{2t-1}$. The number of resolution classes is

$$2t - 1 = \left\lfloor \frac{4t-1}{2} \right\rfloor,$$

which implies that the resolvable grid-block packing is maximal. □

In Theorem 2.1.9, when q is an odd prime power and n is an even integer, then there exists a GB(q^n, q, q). It is easy to show that this grid-block design is resolvable. Next, for any prime power q and a positive integer n, we can construct a maximal resolvable GBP(q^n, q, q) as follows:

Theorem 2.6.2 *For a positive integer n and a prime power q, there exists a maximal resolvable* GBP(q^n, q, q). *Moreover, when n is even and q is odd, the maximal resolvable grid-block packing is a resolvable grid-block design.*

Proof. Let $V = \mathrm{GF}(q)$ and

$$u = \left\lfloor \frac{q^n - 1}{2(q-1)} \right\rfloor.$$

We define A as the same array (2.1.1) and $\mathcal{A}$ by a similar manner to Theorem 2.1.9. Then a pair $(V, \mathcal{A})$ is the desired maximal resolvable GBP(q^n, q, q).

In fact, it is sufficient to count the number of rows and columns of $q \times q$ grid-blocks containing the origin $\mathbf{0}(= \omega^\infty)$ and ω^l. Then, there is at most one line passing through the origin $\mathbf{0}$ and ω^l. Furthermore, we define a class $\mathcal{P}_0$ as a set consisting of A and its parallel 2-flats. Its cyclic shifts $\omega^i \mathcal{P}_0$ for $i = 0, 1, \ldots, u-1$ are obviously resolution classes and it is obvious that there are u resolution classes, which implies that the grid-block packing is maximal. □

Next, we give some recursive constructions of resolvable GBP(v, k_1, k_2)'s. Firstly, we give a construction by generalizing Theorem 2.5.1.

Theorem 2.6.3 *Assume that $k_1 \leq k_2$. If there exist a resolvable* GBP$(v,$ $k_1,$ $k_2)$ *with t resolution classes and an* OA$(n,$ $k_2+1,$ $1)$*, then there exists a resolvable* GBP$(nv,$ $k_1,$ $k_2)$ *with nt resolution classes.*

Proof. A proof is similar to that of Theorem 2.5.1. □

Moreover, we give constructions of resolvable GBP$(v,$ $k_1,$ $k_2)$'s by utilizing a resolvable packing.

Theorem 2.6.4 *In case of $k_1 \leq k_2$, if there exists a resolvable* P$(v,$ $k_2,$ $1)$ *with t resolution classes, then there exists a resolvable* GBP$(k_1 v,$ $k_1,$ $k_2)$ *with t resolution classes.*

Proof. For a v-set V, let a pair $(V,$ $\mathcal{B})$ be a resolvable P$(v,$ $k_2,$ $1)$ with t resolution classes. Let $\{\mathcal{Q}_1,$ $\mathcal{Q}_2,$ $\ldots,$ $\mathcal{Q}_t\}$ be a resolution of the resolvable P$(v,$ $k_2,$ $1)$.

For $N = \{0,$ $1,$ $\ldots,$ $k_1 - 1\}$, let $V^* = V \times N$ and let

$$
\begin{aligned}
A(B) &= ((b_{l+n},\ n)) \\
&= \begin{array}{|llll|}
\hline
(b_0,\ 0) & (b_1,\ 0) & \ldots & (b_{k_2-1},\ 0) \\
(b_1,\ 1) & (b_2,\ 1) & \ldots & (b_0,\ 1) \\
\vdots & \vdots & \ddots & \vdots \\
(b_{k_1-1},\ k_1-1) & (b_{k_1},\ k_1-1) & \ldots & (b_{k_1+k_2-2},\ k_1-1) \\
\hline
\end{array}
\end{aligned}
$$

for each block $B = \{b_i\}$. Note that in the first subscript of b, $l+n$ means $l+n$ (mod k_2). We define the set $\mathcal{A}$ as $\{A(B) : B \in \mathcal{B}\}$.

Since two distinct points b_1 and b_2 in V occur together at most once in a block of the P$(v,$ $k_2,$ $1)$, each pair $(b_1,$ $i)$ and $(b_2,$ $i)$ occurs at most once in the same row of an array in $\mathcal{A}$ for any $i \in N$. And each pair $(b_1,$ $i)$ and $(b_2,$ $j)$ occurs at most once in the same column of an array in for any $i \neq j \in N$. Hence, the pair $(V^*,$ $\mathcal{A})$ is a GBP$(k_1 v,$ $k_1,$ $k_2)$. Moreover, let

$$\mathcal{P}_j = \{A(B) : B \in \mathcal{Q}_j\}$$

for $j = 1,$ $2,$ $\ldots,$ t. Obviously, each $\mathcal{P}_j$ is a resolution class. Thus, the theorem is proved. □

Theorem 2.6.5 *In case of $k_1 \leq k_2$, if there exists a resolvable* P$(v,$ $k_2,$ $1)$ *with t resolution classes and a resolvable* GBP$(k_1 k_2,$ $k_1,$ $k_2)$ *with $s+1$ grid-blocks, then there exists a resolvable* GBP$(k_1 v,$ $k_1,$ $k_2)$ *with $st+1$ resolution classes.*

Proof. For sets $N_1 = \{0, 1, \ldots, k_1\}$ and $N_2 = \{0, 1, \ldots, k_2\}$, let $W = N_1 \times N_2$ and $(W, \mathcal{F})$ be a resolvable GBP(k_1k_2, k_1, k_2), where $\mathcal{F} = \{F_0, F_1, \ldots, F_s\}$. Without loss of generality, we assume

$$F_0 = \boxed{\begin{array}{llll} (0,\, 0) & (0,\, 1) & \ldots & (0,\, k_2 - 1) \\ (1,\, 0) & (1,\, 1) & \ldots & (1,\, k_2 - 1) \\ \vdots & \vdots & \ddots & \vdots \\ (k_1 - 1,\, 0) & (k_1 - 1,\, 1) & \ldots & (k_1 - 1,\, k_2 - 1) \end{array}}. \tag{2.6.1}$$

It is obvious that each point in W occurs exactly once in every F_u. Moreover, (i, j) and (i, j') do not occur in the same row and column of $F_1, F_2, \ldots, F_s$. Similarly, (i, j) and (i', j) do not occur, either.

Now, let $(V, \mathcal{B})$ be a resolvable P$(v, k_2, 1)$ with t resolution classes $\{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_t\}$. Let $V^* = V \times N_1$ and let

$$A^u(B) = ((b_{\rho^u_{lm}},\, \sigma^u_{lm}))$$

for each block $B = \{b_i\}$ and $F_u = ((\rho^u_{lm}, \sigma^u_{lm}))$. We define $\mathcal{P}^u_w = \{A^u(B) : B \in \mathcal{Q}_w\}$. It is obvious that $\mathcal{P}^u_w$ is a resolution class. Now, let

$$\mathcal{A}^* = \mathcal{P}^0_1 \cup \left\{ \bigcup_{u=1}^{s} \bigcup_{w=1}^{t} \mathcal{P}^u_w \right\}.$$

Then, a pair $(V^*, \mathcal{A}^*)$ is the desired resolvable grid-block packing.

In fact, for any two distinct points (b_1, i_1) and (b_2, i_2) in V^*

(i) in the case of $b_1 = b_2$, (b_1, i_1) and (b_2, i_2), $i_1 \neq i_2$, occur exactly once in a column of a grid-block in $\mathcal{P}^0_1$,

(ii) in the case of $b_1 \neq b_2$,

   (a) if there is no block in $(V, \mathcal{B})$ containing b_1 and b_2 simultaneously, then (b_1, i_1) and (b_2, i_2) do not occur in the same grid-block in $\mathcal{A}$,

   (b) if there is a block B containing b_1 and b_2, there is at most one row or one column of a grid-block in $\mathcal{P}^u_w$ which contains (b_1, i_1) and (b_2, i_2), $i_1 \neq i_2$, simultaneously,

   (c) if there is a block $B \in \mathcal{Q}_1$ containing b_1 and b_2, there is exactly one row of a grid-block in $\mathcal{P}^0_1$ which contains (b_1, i) and (b_2, i), simultaneously.

Thus, the theorem is proved. □

By coupling two mutually orthogonal $k \times k$ Latin squares, we can obtain an Euler square. Thus together with F_0 in the array (2.6.1) for $k_1 = k_2 = k$, we obtain a GBP(k_1k_2, k_1, k_2) with two grid-blocks. Since there are two mutually orthogonal Latin squares except for $k = 6$, we obtain the following corollary.

Corollary 2.6.6 *For a positive integer $k \neq 6$, if there exists a resolvable* B(v, k, 1)*, then there exists a resolvable* GBP(kv, k, k) *with* $(v-1)/(k-1)+1$ *resolution classes.*

Moreover, when k is an odd prime power, there exists a resolvable GB(k^2, k, k) by Theorem 2.1.9. Thus, we obtain the following corollary.

Corollary 2.6.7 *For an odd prime power k, if there exists a resolvable* P(v, k, 1) *with t resolution classes, then there exists a resolvable* GBP(kv, k, k) *with $t(k-1)/2+1$ resolution classes.*

For example, in the case of $k_1 = k_2 = 3$, it is well known that there is a resolvable B($6t + 3$, 3, 1) for any positive integer t. By Corollary 2.6.6, we obtain a resolvable GBP($18t + 9$, 3, 3) with $3t + 2$ resolution classes for any t. The number of pairs which occur in the same row or in the same column in a grid-block is $18(3t + 2)(2t + 1)$ and the total number of the pairs of two distinct points is $(18t + 9)(18t + 8)/2$. That is, more than 2/3 of the pairs occur in the same row or in the same column in the grid-block packing.

In addition, it is known that there is a resolvable P($6t$, 3, 1) with $3t - 1$ resolution classes for any $t \geq 2$ (see, for example, [29]). That is, we obtain a resolvable GBP($18t$, 3, 3) with $3t$ resolution classes. Similarly, in this case, about 2/3 of the pairs occur in the same row or in the same column in the grid-block packing.

Chapter 3

Constructions of Nested BIB designs and BIB designs with nested rows and columns

In this chapter, constructions of nested BIB designs and BIBRCs are discussed. In Section 3.1, constructions of BIB designs and nested BIB designs are given by utilizing affine geometries. In Section 3.2, a construction of completely balanced BIBRCs is given by the same method. In the case when a dimension of affine geometry is even and $k_1 = k_2$ holds, BIBRCs, which are not completely balanced, are obtained by the same construction. In Section 3.3, a construction of BIBRCs is given by utilizing finite fields, which are not necessarily completely balanced. And the existence of BIBRCs for sufficiently large prime powers is shown by applying this construction to Proposition 1.8.1. In Appendix A, we list the parameters of BIBRCs with small parameters which are obtained by computer based on this construction.

3.1 A construction of nested BIB designs

Firstly, we give the following results given by Rao [82] and Yamamoto, Fukuda and Hamada [105]. For any m-flat U, $\omega U = \{\omega u : u \in U\}$ is also an m-flat, where ω is a primitive element of $\mathrm{GF}(q^n)$. The minimum positive integer θ satisfying $\omega^\theta U = U$ is called the *minimum cycle length* of the m-flat U. Let θ be the minimum cycle length of an m-flat U passing through the origin $\mathbf{0}$. Let $p = (q^n - 1)/(q - 1)$, then θ divides p and the minimum cycle length of an m-flat passing through the origin $\mathbf{0}$ is a divisor of p. The set $\mathcal{O}(U) = \{\omega^i U : i = 0, 1, \ldots, \theta - 1\}$ is called the *orbit* or *cycle* containing the m-flat U. If $\theta = p$, then the orbit is said to be *full*, otherwise *short*. A necessary condition for the existence of an m-flat having the minimum cycle length $\theta < (q^n - 1)/(q - 1)$ is that $(q^n - 1)/(q - 1)$ and $(q^m - 1)/(q - 1)$ are not relatively prime. An m-flat passing through the origin $\mathbf{0}$ which has the minimum cycle length p always exists. And all m-flat not passing through the origin $\mathbf{0}$ have the minimum cycle length $q^n - 1$.

When d is a divisor of n, let $\theta = (q^n - 1)/(q^d - 1)$ then $q^d - 1$ is the least integer c satisfying $(\omega^\theta)^c = 1$ where ω is a primitive element of $\mathrm{GF}(q^n)$. Thus, ω^θ is one of the primitive elements of $\mathrm{GF}(q^d)$. $\mathrm{GF}(q^d)$ can, therefore, be represented as $\mathrm{GF}(q^d) = \{0, \omega^0, \omega^\theta, \ldots, \omega^{(q^d-2)\theta}\}$. In particular, $\mathrm{GF}(q) = \{0, \omega^0, \omega^\eta, \ldots, \omega^{(q^d-2)\eta}\}$, where $\eta = (q^n - 1)/(q - 1)$. When d is a divisor of n, the set of points in $\mathrm{AG}(n/d, q^d)$ is identified with the set of points in $\mathrm{AG}(n, q)$.

Moreover, when d is a common divisor of n and m, an (m/d)-flat in $\mathrm{AG}_{m/d}(n/d, q^d)$ is also an m-flat of $\mathrm{AG}_m(n, q)$. There always exists an (m/d)-flat U of $\mathrm{AG}^*_{m/d}(n/d, q^d)$ whose cycle length is $\theta = (q^n-1)/(q^d-1)$. All points on (m/d)-flat U are given by $\{\mathbf{0}\} \cup (S \circ T)$, where $S = \{\omega^0, \omega^\theta, \ldots, \omega^{(q^d-2)\theta}\}$, $T \subset \{\omega^0, \omega^1, \ldots, \omega^{\theta-1}\}$ and $S \circ T = \{st : s \in S, t \in T\}$. Therefore, this (m/d)-flat U is equivalent to an m-flat of $\mathrm{AG}^*_m(n, q)$ whose cycle length is θ. Note that T is an $(m/d - 1)$-flat of $\mathrm{PG}_{m/d-1}(n/d - 1, q^d)$.

If T has the minimum cycle length $\theta' < \theta$ in $\mathrm{PG}_{m/d-1}(n/d - 1, q^d)$, then the m-flat $U = \{\mathbf{0}\} \cup (S \circ T)$ also has the minimum cycle length θ' in $\mathrm{AG}^*_m(n, q)$. The following lemma follows from these results.

Lemma 3.1.1 *Let q be a prime power and d be a common divisor of n and m. Then there exists a* $\mathrm{B}(q^n, q^m, (q^m - 1)/(q^d - 1))$.

Proof. Let $V = \mathrm{AG}_0(n, q)$ and $\theta = (q^n - 1)/(q^d - 1)$. Let U be an m-flat passing through the origin $\mathbf{0}$ whose cycle length is θ. We define

$$\mathcal{B} = \bigcup_{U' \in \mathcal{P}(U)} \mathcal{O}(U') = \{\omega^i U' : i = 0, 1, \ldots, \theta - 1, U' \in \mathcal{P}(U)\},$$

where $\mathcal{P}(U)$ is a parallel class containing U. Note that $\omega^{i\theta}U'$ belongs to $\mathcal{P}(U)$ for each $i = 0, 1, \ldots, q^d - 2$. In fact, U is an (m/d)-flat of $\mathrm{AG}^*_{m/d}(n/d, q^d)$. We define $\overline{U}$ as an $(n/d - m/d)$-flat of $\mathrm{AG}^*_{n/d-m/d}(n/d, q^d)$ passing through the origin $\mathbf{0}$ such that $\overline{U} \cap U = \{\mathbf{0}\}$ holds. Then, if x belongs to $\overline{U}$, $\omega^{i\theta}x$, which is a scalar multiple of x over $\mathrm{GF}(q^d)$, also belongs to $\overline{U}$. And for each $U' \in \mathcal{P}(U)$, there exists a point $x \in \overline{U}$ such that $U' = U + x$. Therefore, $\omega^{i\theta}U' = \omega^{i\theta}(U + x) = U + \omega^{i\theta}x$ belongs to $\mathcal{P}(U)$ for each i.

Then $(V, \mathcal{B})$ is a $\mathrm{B}(q^n, q^m, (q^m - 1)/(q^d - 1))$. In fact, to count the number of blocks containing two points x and y, we have only to check the number of blocks containing the origin $\mathbf{0}$ and $z = x - y$ in $\mathcal{O}(U)$. U is given by

$\{\mathbf{0}\} \cup (S \circ T)$, where $S = \{\omega^0, \omega^\theta, \ldots, \omega^{(q^d-2)\theta}\}$ and T is an $(m/d-1)$-flat of $\mathrm{PG}_{m/d-1}(n/d-1, q^d)$. Therefore, the number of m-flats containing the origin $\mathbf{0}$ and $z = \omega^{l\theta+m} (\neq \mathbf{0})$ is $|T| = ((q^d)^{m/d} - 1)/(q^d - 1) = (q^m - 1)/(q^d - 1)$ since the point set of $\mathcal{O}(tS)$ is identified with the point set $\mathrm{AG}_0(n, q) \setminus \{\mathbf{0}\}$ for each $t \in T$. Thus the lemma is proved. $\square$

The following proposition is given by Jimbo and Kuriki [54].

Proposition 3.1.2 *Let q be a prime power. Then for any m_1 and m_2 such that $n = m_1 + m_2$ and $m_1 > m_2$, there exists a nested* $\mathrm{B}(q^n; q^{m_1}, \lambda_1; q^{m_2}, \lambda_2)$, *where $\lambda_1 = \phi(n-1, m_1-1, q)$ and $\lambda_2 = \phi(n-1, m_2-1, q)$.*

The nested BIB design constructed by Proposition 3.1.2 is generated by all m_1-flats and by all m_2-flats of $\mathrm{AG}(n, q)$. Here we will give another construction of a nested BIB design which has smaller λ's than that of Proposition 3.1.2.

Theorem 3.1.3 *Let q be a prime power and let d be a common divisor of integers n, m_1 and m_2 such that $n > m_1 > m_2 > 0$. Then there exists a nested* $\mathrm{B}(q^n; q^{m_1}, (q^{m_1}-1)/(q^d-1); q^{m_2}, (q^{m_2}-1)/(q^d-1))$.

Proof. For $V = \mathrm{AG}_0(n, q)$ and $\theta = (q^n - 1)/(q^d - 1)$, let U_1 and U_2 be an (m_1/d)-flat and an (m_2/d)-flat of $\mathrm{AG}^*(n/d, q^d)$, respectively, such that U_1 includes U_2 and that their cycle lengths are θ or its divisors. Then, U_1 and U_2 are also an m_1-flat and an m_2-flat of $\mathrm{AG}(n, q)$, respectively. Note that their cycle lengths are θ or its divisors also in $\mathrm{AG}(n, q)$.

Let

$$\mathcal{B}_1 = \bigcup_{U_1' \in \mathcal{P}(U_1)} \mathcal{O}(U_1') = \{\omega^i U_1' : i = 0, 1, \ldots, \theta - 1, U_1' \in \mathcal{P}(U_1)\},$$

and

$$\mathcal{B}_2 = \bigcup_{U_1' \in \mathcal{P}(U_2)} \mathcal{O}(U_2') = \{\omega^i U_2' : i = 0, 1, \ldots, \theta - 1, U_2' \in \mathcal{P}(U_2)\}.$$

Then, it is obvious that the blocks in $\mathcal{B}_2$ are nested within the blocks in $\mathcal{B}_1$. By Lemma 3.1.1, $(V, \mathcal{B}_1)$ and $(V, \mathcal{B}_2)$ are a $\mathrm{B}(q^n, q^{m_1}, (q^{m_1}-1)/(q^d-1))$ and a $\mathrm{B}(q^n, q^{m_2}, (q^{m_2}-1)/(q^d-1))$, respectively. Therefore, $(V, \mathcal{B}_1, \mathcal{B}_2)$ is a nested $\mathrm{B}(q^n; q^{m_1}, (q^{m_1}-1)/(q^d-1); q^{m_2}, (q^{m_2}-1)/(q^d-1))$. $\square$

Now, we compare a nested BIBD constructed by Theorem 3.1.3 with one of Proposition 3.1.2. A nested B(32; 8, 35; 4, 15) is generated by 3-flats and 2-flats of $\mathrm{AG}_3(5, 2)$ and $\mathrm{AG}_2(5, 2)$ by Proposition 3.1.2. On the other hand, we obtain a nested B(32; 8, 7; 4, 3) by Theorem 3.1.3. These two nested BIBDs have the same parameters v, k_1, k_2, but the design obtained by Theorem 3.1.3 has smaller λ than that of Proposition 3.1.2. In general, our method can generate a nested BIBD with smaller λ's than the ones constructed via Proposition 3.1.2, since our method uses a subclass of the m_1-flats and the m_2-flats of $\mathrm{AG}_{m_1}(n, q)$ and $\mathrm{AG}_{m_2}(n, q)$.

As a consequence of Theorem 3.1.3, we obtain the following corollary.

Corollary 3.1.4 *For any prime power q and for any integers n, m_1 and m_2 such that m_2 divides n and m_1, then there exists a nested* BIBD(q^n; q^{m_1}, $(q^{m_1}-1)/(q^{m_1}-1)$; q^{m_2}, 1).

By Theorem 3.1.3, we can construct nested BIBDs with parameters

$$(v;\, k_1,\, \lambda_1;\, k_2,\, \lambda_2) = (32;\, 4,\, 3;\, 2,\, 1),\ (32;\, 8,\, 7;\, 2,\, 1),\ (32;\, 8,\, 7;\, 4,\, 3).$$

These designs are not found in Morgan [72], Morgan, Preece and Rees [73], nor in their related website of tables of nested BIBDs for $v \leq 36$ and $r \geq v-1$ (see Rees [86]). Similarly, we can get more examples of nested BIBDs with parameters

$$(v;\, k_1,\, \lambda_1;\, k_2,\, \lambda_2) = (64;\, 4,\, 3;\, 2,\, 1),\ (64;\, 8,\, 7;\, 2,\, 1),\ (64;\, 8,\, 7;\, 4,\, 3) \text{ etc.}$$

which may be also new. The construction of Corollary 3.1.4 gives a nested BIBD with the smallest number of blocks for given $v = q^n$, $k_1 = q^{m_1}$ and $k_2 = q^{m_2}$ since $\lambda = 1$.

Example 3.1.1 Let $q = 2$, $n = 8$, $m_1 = 4$ and $m_2 = 2$ in Corollary 3.1.4. For a primitive element ω of GF(2^8), we represent each element ω^i of AG(8, 2) simply by the power i. For convenience, let ∞ be the origin $\mathbf{0}$. And let U_1 and U_2 be a 4-flat and 2-flat as following:

$$U_1 = \{(\infty, 0, 85, 170), (1, 25, 41, 157), (86, 110, 126, 242), (171, 195, 211, 72)\},$$
$$U_2 = \{\infty, 0, 85, 170\}.$$

These flats generate a nested B(256; 16, 5; 4, 1).

3.2 A construction of BIB designs with nested rows and columns

In this section, we give constructions of BIBRCs and completely balanced BIBRCs by using the previous method. The following proposition is given by Jimbo and Kuriki [54].

Proposition 3.2.1 *There exists a completely balanced* BIBRC(q^3, q, q, $q-1$) *for any prime power* q.

The design which is constructed by Proposition 3.2.1 is generated by the 1-flats and 2-flats of AG(3, q). The following theorem includes Proposition 3.2.1 as a special case.

Theorem 3.2.2 *Let q be a prime power and let d be a common divisor of positive integers n, m_1 and m_2 satisfying $m_1 + m_2 \leq n$. Then there exists a completely balanced* BIBRC(q^n, q^{m_1}, q^{m_2}, $(q^{m_1}-1)(q^{m_2}-1)/(q^d-1)$).

Proof. For $V = \mathrm{AG}_0(n, q)$ and $\theta = (q^n-1)/(q^d-1)$, let U, U_1 and U_2 be an $((m_1+m_2)/d)$-flat, an (m_1/d)-flat and an (m_2/d)-flat of $\mathrm{AG}^*(n/d, q^d)$, respectively, such that U is spanned by U_1 and U_2 and that their cycle lengths are θ or its divisors. Then, U, U_1 and U_2 are also an (m_1+m_2)-flat, an m_1-flat and m_2-flat of $\mathrm{AG}^*(n, q)$, respectively. And their cycle lengths are θ or its divisors.

Then, the element in U is arranged in a $q^{m_1} \times q^{m_2}$ array A. In fact, arrange the elements of U_1 in the first column of the array and those of U_2 in the first row such that the (1, 1)-element is the origin $\mathbf{0}$. And define the (i, j)-element by adding the i-th element of U_1 and the j-th element of U_2. Here we identify the array A with $U = U_1 \oplus U_2$. We define $\overline{U}$ as an $(n-(m_1+m_2))$-flat of $\mathrm{AG}^*_{n-(m_1+m_2)}(n, q)$ such that $\overline{U} \cap U = \{\mathbf{0}\}$ holds. Moreover, we define $\mathcal{P}(A) = \{A + x : x \in \overline{U}\}$ as a parallel class of A. Now, let

$$\mathcal{A} = \bigcup_{A' \in \mathcal{P}(A)} \mathcal{O}(A') = \{\omega^i A' : i = 0, 1, \ldots, \theta-1, A' \in \mathcal{P}(A)\},$$

$$\mathcal{B}_1 = \bigcup_{U_1' \in \mathcal{P}(U_1)} \mathcal{O}(U_1') = \{\omega^i U_1' : i = 0, 1, \ldots, \theta-1, U_1' \in \mathcal{P}(U_1)\} \text{ and}$$

$$\mathcal{B}_2 = \bigcup_{U_2' \in \mathcal{P}(U_2)} \mathcal{O}(U_2') = \{\omega^i U_2' : i = 0, 1, \ldots, \theta-1, U_2' \in \mathcal{P}(U_2)\}.$$

Note that the columns in $\mathcal{P}(A)$ is identified with m_1-flats in $\mathcal{P}(U_1)$. Let $\overline{U_1} = U_2 \oplus \overline{U}$. Then $\overline{U_1}$ is an $(n-m_1)$-flat of AG(n, q) passing through the origin $\mathbf{0}$. The set of columns of A is $\{U_1 + x : x \in U_2\}$ and the set of columns of the arrays in $\mathcal{P}(A)$ is $\{U_1 + x : x \in \overline{U_1}\}$, which is $\mathcal{P}(U_1)$. Thus, by Lemma 3.1.1, the family $\mathcal{B}_1$ of columns of arrays in $\mathcal{A}$ forms a B(q^n, q^{m_1}, $(q^{m_1}-1)/(q^d-1)$).

Similarly, $\mathcal{B}_2$ is the set of rows of the arrays in $\mathcal{A}$, which forms a B(q^n, q^{m_2}, $(q^{m_1}-2)/(q^d-1)$). And $(V, \mathcal{A})$ is a B(q^n, $q^{m_1+m_2}$, $(q^{m_1+m_2}-1)/(q^d-1)$) by identifying $A' \in \mathcal{A}$ with (m_1+m_2)-flats. Thus, $(V, \mathcal{A})$ is the desired completely balanced BIBRC(q^n, q^{m_1}, q^{m_2}, $(q^{m_1}-1)(q^{m_2}-1)/(q^d-1)$). □

Also, we have the following corollary.

Corollary 3.2.3 *For any prime power q and for any positive integers n, m_1 and m_2 such that m_2 divides n and m_1 and $m_1 + m_2 \leq n$, then there exists a completely balanced* BIBRC$(q^n,\, q^{m_1},\, q^{m_2},\, q^{m_1} - 1)$.

The BIBRC constructed by Theorem 3.2.2 gives a completely balanced BIBRC with minimum possible value of λ in the case of $\lambda_R = 1$ or $\lambda_C = 1$. If $k_1 = k_2 = q$ and n is even, the BIBRC which is constructed by Theorem 3.2.2 has the same parameters with those of Singh and Dey [90]. Corollary 3.2.3 includes the result given by Jimbo and Kuriki [54] as a special case by letting $k_1 = k_2 = q$ and $n = 3$. By Theorem 3.2.2, we can construct a BIBRC(16, 2, 4, 3) and a BIBRC(32, 2, 4, 3). The BIBRC(16, 2, 4, 3) can be also constructed by applying the method of Mukerjee and Gupta [75] and Cheng [26].

Moreover, when $k_1 = k_2$ and q is an odd prime power, we give another construction of a BIBRC whose λ is smaller than that of Theorem 3.2.2.

Theorem 3.2.4 *If $2m$ divides n and q is an odd prime power, then there exists a* BIBRC$(q^n,\, q^m,\, q^m,\, (q^m - 1)/2)$.

Proof. Let $V = \mathrm{AG}_0(n,\, q)$, $\theta = (q^n-1)/(q^m-1)$ and $\theta' = (q^n-1)/(q^{2m}-1)$. For $S = \{\omega^0,\, \omega^\theta,\, \ldots,\, \omega^{(q^m-2)\theta}\}$ and $S' = \{\omega^0,\, \omega^{\theta'},\, \ldots,\, \omega^{(q^{2m}-2)\theta'}\}$, let $U = \{\mathbf{0}\} \cup S$ and $U' = \{\mathbf{0}\} \cup S'$. U and U' are an m-flat and a $(2m)$-flat passing through the origin whose minimum cycle length are θ and θ', respectively. Obviously, U' includes U. By Lemma 3.1.1, the orbit of U and their parallel classes forms a $\mathrm{B}(q^n,\, q^m,\, 1)$ and the orbit of U' and their parallel classes also forms a $\mathrm{B}(q^n,\, q^{2m},\, 1)$.

Assuming that $2m$ divides n and q is an odd prime power, $\theta/2$ is an integer. Let $U_1 = U$ and $U_2 = \omega^{\theta/2}U_1 = \{\mathbf{0},\, \omega^{\theta/2},\, \omega^{3\theta/2},\, \ldots,\, \omega^{(q^m-1)\theta/2}\}$. U_2 is the m-flat with minimum cycle length θ. By arranging the elements U_1 and U_2 in the first row and in the first column of a $q^m \times q^m$ array A, similarly to the proof of Theorem 3.2.2, every element of U' which is spanned by U_1 and U_2 occurs as an entry of A. Let

$$\begin{aligned}
\mathcal{A} &= \{\omega^i A' : i = 0,\, 1,\, \ldots,\, \theta/2 - 1,\, A' \in \mathcal{P}(A)\} \text{ and} \\
\mathcal{B} &= \{\omega^i B : i = 0,\, 1,\, \ldots,\, \theta/2 - 1,\, B \in \mathcal{P}(U_1)\} \\
&\quad \cup \{\omega^i B : i = 0,\, 1,\, \ldots,\, \theta/2 - 1,\, B \in \mathcal{P}(U_2)\} \\
&= \{\omega^i B : i = 0,\, 1,\, \ldots,\, \theta - 1,\, B \in \mathcal{P}(U)\}.
\end{aligned}$$

Since the family $\mathcal{B}$ of rows and columns of the arrays in $\mathcal{A}$ forms a $\mathrm{B}(q^n,\, q^m,\, 1)$, $(V,\, \mathcal{A})$ has index $\lambda_R\{x,\, y\} + \lambda_C\{x,\, y\} = 1$. And by considering A as a block of size q^{2m}, $(V,\, \mathcal{A})$ is recognized as $\theta/(2\theta')$ copies of the $\mathrm{B}(q^n,\, q^{2m},\, 1)$.

In fact, for $i = 1, 2, \ldots, \theta/(2\theta')$, $(2m)$-flat A and $\omega^{i\theta'}A$ are the same by considering $(2m)$-flats as blocks of size q^{2m}. However, note that by considering them as arrays, $\omega^{i\theta'}$ is different from each element of A for $i = 1, 2, \ldots, \theta/(2\theta')$ and the array A and the transpose of $\omega^{\theta/2}A$ are the same. For $A' \in \mathcal{P}(A)$ not passing through the origin $\mathbf{0}$, $\omega^{i\theta}A'$ also belongs to $\mathcal{P}(A)$ for $i = 0, 1, \ldots, q^m - 1$. And the transpose of $\omega^{i\theta/2}A'$ belongs to $\mathcal{P}(A)$ for $i = 1, 3, \ldots, 2q^m - 1$. Moreover, by considering A' as a block of size q^{2m}, $\omega^{i\theta'}A'$ belongs to $\mathcal{P}(U')$ for $i = 0, 1, \ldots, q^{2m} - 1$.

Hence, $\lambda = q^m - \theta/(2\theta') = (q^m - 1)/2$ holds for each two distinct pairs $\{x, y\}$. Therefore, $(V, \mathcal{A})$ generates a BIBRC$(q^n, q^m, q^m, (q^m - 1)/2)$. $\square$

Example 3.2.1 For $V = \mathrm{AG}_0(4, 3)$, let A be blocks and $\mathcal{P}(A)$ be a parallel class of A as follows:

$$A = \begin{pmatrix} \infty & 0 & 40 \\ 20 & 30 & 50 \\ 60 & 10 & 70 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 4 & 53 \\ 55 & 62 & 38 \\ 49 & 37 & 76 \end{pmatrix}, \quad \omega^{40}A_1 = \begin{pmatrix} 41 & 44 & 13 \\ 15 & 22 & 78 \\ 9 & 77 & 36 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 11 & 72 & 6 \\ 47 & 63 & 65 \\ 48 & 59 & 14 \end{pmatrix}, \quad \omega^{40}A_2 = \begin{pmatrix} 51 & 32 & 46 \\ 7 & 23 & 25 \\ 8 & 19 & 54 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 21 & 75 & 69 \\ 24 & 2 & 57 \\ 73 & 58 & 16 \end{pmatrix}, \quad \omega^{40}A_3 = \begin{pmatrix} 61 & 35 & 29 \\ 64 & 42 & 17 \\ 33 & 18 & 56 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 31 & 67 & 68 \\ 12 & 3 & 79 \\ 26 & 5 & 34 \end{pmatrix}, \quad \omega^{40}A_4 = \begin{pmatrix} 71 & 27 & 28 \\ 52 & 43 & 39 \\ 66 & 45 & 74 \end{pmatrix},$$

where the elements ω^i in $\mathrm{AG}_4(3, q)$ is represented by its power i and the origin $\mathbf{0}$ is represented by ∞. The blocks are 2-flats and their rows and columns are 1-flats in $\mathrm{AG}_4(3, q)$. Then, $\mathcal{P}(A) = \{A, A_1, \ldots, A_4, \omega^{40}A_1, \ldots, \omega^{40}A_4\}$ holds. We define $\mathcal{A} = \{\omega^i A' : i = 0, 1, \ldots, 19, A' \in \mathcal{P}(A)\}$. Then, $(V, \mathcal{A})$ is a BIBRC(81, 3, 3, 1).

Here, define θ and θ' as in the proof of Theorem 3.2.4. Then $\theta = (3^4 - 1)/(3 - 1) = 40$ and $\theta' = (3^4 - 1)/(3^2 - 1) = 10$ holds. And

$$\omega^{\theta'}A = \omega^{10}A = \begin{pmatrix} \infty & 10 & 50 \\ 30 & 40 & 60 \\ 70 & 20 & 0 \end{pmatrix}$$

holds. By considering A and $\omega^{10}A$ as blocks with size 9, they are the same. However, note that the array $\omega^{10}A$ is different from A. Similarly, $\omega^{10}A_1$ and

A_2 are different arrays though they are the same block by recognizing them as sets. On the other hand, A_3 and the transpose of $\omega^{20}A_1$ are the same by considering them as arrays.

For given $v \le 512$, $3 \le \max\{k_1, k_2\} \le 5$ and $k_1k_2 < v$, Table 3.2.1 lists the smallest feasible new BIBRCs generated by Theorem 3.2.2, that is, these BIBRCs have the minimum possible value of λ. Note that the case of $k_1 = k_2 = 2$ is completely solved by Srivastav and Morgan [91].

Table 3.2.1: Examples of new BIBRCs constructed by our method

v	k_1	k_2	λ	our method
32	2	4	3	Th.3.2.2 AG(5, 2)
32	4	4	9	Th.3.2.2 AG(5, 2)
64	2	4	3	Th.3.2.2 AG(6, 2)
128	2	4	3	Th.3.2.2 AG(7, 2)
128	4	4	9	Th.3.2.2 AG(7, 2)
256	2	4	3	Th.3.2.2 AG(8, 2)
512	2	4	3	Th.3.2.2 AG(9, 2)
512	4	4	9	Th.3.2.2 AG(9, 2)

3.3 An asymptotic existence of BIB designs with nested rows and columns over GF(q)

In this section we will give a direct construction of a BIBRC which is based on the well known technique given by Wilson [99] for constructing a BIB design. Firstly, we define some symbols. For a prime power $q = mf + 1$, let $A = (a_{ij})$ be a $k_1 \times k_2$ array with elements in GF(q). We define $\Delta_d^R(A)$ as the number of ordered pairs $(a_{ij}, a_{ij'})$ lying in the same row of A such that the difference $a_{ij'} - a_{ij'}$ is d.

$$\Delta_d^R(A) = |\{(a_{ij}, a_{ij'}) : a_{ij'} - a_{ij} = d, 1 \le i \le k_1, 1 \le j \ne j' \le k_2\}|.$$

Similarly, we define $\Delta_d^C(A)$ and $\Delta_d^E(A)$ as the number of ordered pairs $(a_{ij}, a_{i'j'})$ such that $a_{i'j'} - a_{ij}$ is d and they occur in the same column and elsewhere, respectively. That is,

$$\Delta_d^C(A) = |\{(a_{ij}, a_{i'j}) : a_{i'j} - a_{ij} = d, 1 \le i \ne i' \le k_1, 1 \le j \le k_2\}| \text{ and}$$
$$\Delta_d^E(A) = |\{(a_{ij}, a_{i'j'}) : a_{i'j'} - a_{ij} = d, 1 \le i \ne i' \le k_1, 1 \le j \ne j' \le k_2\}|.$$

Let $\boldsymbol{A}$ be a family of f blocks A. Then, we define $\Delta_d^R(\boldsymbol{A}) = \sum_{A \in \boldsymbol{A}} \Delta_d^R(A)$, $\Delta_d^C(\boldsymbol{A}) = \sum_{A \in \boldsymbol{A}} \Delta_d^C(A)$ and $\Delta_d^E(\boldsymbol{A}) = \sum_{A \in \boldsymbol{A}} \Delta_d^E(A)$.

Now, let $\delta_l^R(A)$ be the sum of $\Delta_d^R(A)$'s over all d belonging to H_l^m. That is

$$\delta_l^R(A) = \sum_{d \in H_l^m} \Delta_d^R(A).$$

Similarly, we define $\delta_l^C(A)$ and $\delta_l^E(A)$ for the sums of $\Delta_d^C(A)$'s and $\Delta_d^E(A)$'s such that d belongs to H_l^m, respectively. That is,

$$\delta_l^C(A) = \sum_{d \in H_l^m} \Delta_d^C(A) \quad \text{and} \quad \delta_l^E(A) = \sum_{d \in H_l^m} \Delta_d^E(A).$$

The following theorem is obtained by generalizing the idea of Wilson [99]. By utilizing the following theorem, we can obtain many new designs having the smallest r and λ among the known constructions. In fact, such designs are listed in Table A.1 by name of "Th.3.3.1." Among them, there are non-completely balanced designs. For example, the following designs in Table 3.3.1 are non-completely balanced. Note that in the case of completely balanced, r, b and λ must be larger than these designs.

Table 3.3.1: Some examples of non-completely balanced BIBRCs

v	k_1	k_2	r	b	λ	v	k_1	k_2	r	b	λ
13	3	3	9	13	3	17	3	5	30	34	15
17	3	3	36	68	9	19	3	4	12	19	4
17	3	4	24	34	9	25	3	3	18	150	3

Theorem 3.3.1 *Let $q = mf + 1$ be a prime power and let A be a $k_1 \times k_2$ array with elements in* GF(q) *such that the following condition hold:*

(C1) *There is some constant λ such that*

$$(k_1 - 1)\delta_l^R(A) + (k_2 - 1)\delta_l^C(A) - \delta_l^E(A) = \lambda$$

for each $0 \le l < m$.

Then there exists a BIBRC(q, k_1, k_2, λ). *Moreover, if q is an odd prime power and $2m$ divides $q - 1$, then there exists a* BIBRC$(q, k_1, k_2, \lambda/2)$.

Proof. Let $A_u = \omega^{um}A$ for $u = 0, 1, \ldots, f-1$ and let $\boldsymbol{A} = \{A_0 = A, A_1, \ldots, A_{f-1}\}$, where ω is a primitive element of GF(q). Fix an element $d \in H_l^m$. Then,

$$\Delta_d^R(\boldsymbol{A}) = \sum_{u=0}^{f-1} \Delta_d^R(A_u) = \sum_{d' \in H_l^m} \Delta_d^R(A) = \delta_l^R(A)$$

holds. Similarly, $\Delta_d^C(\boldsymbol{A}) = \delta_l^C(A)$ and $\Delta_d^E(\boldsymbol{A}) = \delta_l^E(A)$ hold. Assuming the condition (C1),

$$(k_1-1)\Delta_d^R(\boldsymbol{A}) + (k_2-1)\Delta_d^C(\boldsymbol{A}) - \Delta_d^E(\boldsymbol{A}) = \lambda \tag{3.3.1}$$

holds for each $d \in \mathrm{GF}(q) \setminus \{0\}$. Hence, by defining $V = \mathrm{GF}(q)$ and $\mathcal{A} = \{A_u + x : A_u \in \boldsymbol{A}, x \in \mathrm{GF}(q)\}$, a pair $(V, \mathcal{A})$ is a BIBRC(q, k_1, k_2, λ).

Now, let q be an odd prime power and $2m|(q-1)$ such that $1 \neq -1$ and $-1 \in H_0^m$. Then $\pm d$ belong to the same coset H_l^m for any $d \in \mathrm{GF}(q) \setminus \{0\}$. Therefore, $\delta_l^R(A)$, $\delta_l^C(A)$, $\delta_l^E(A)$ and λ are even and

$$\sum_{d \in H_l^m/\{\omega^m, -\omega^m\}} (k_1-1)\Delta_d^R(A) + (k_2-1)\Delta_d^C(A) - \Delta_d^E(A) = \frac{\lambda}{2} \tag{3.3.2}$$

holds for each m. Now let $\{h_0 = 1, h_1, \ldots, h_{f/2-1}\} = H_0^m/\{1, -1\}$, $A'_u = h_u A$ for $u = 0, 1, \ldots, f/2-1$ and let $\boldsymbol{A'} = \{A'_0, A'_1, \ldots, A'_{f/2-1}\}$. By the equation (3.3.2),

$$(k_1-1)\Delta_d^R(\boldsymbol{A'}) + (k_2-1)\Delta_d^C(\boldsymbol{A'}) - \Delta_d^E(\boldsymbol{A'}) = \frac{\lambda}{2}$$

holds for each $d \in \mathrm{GF}(q) \setminus \{0\}$. Hence, by constructing blocks $\mathcal{A}' = \{A'_u + x : A'_u \in \boldsymbol{A'}, x \in \mathrm{GF}(q)\}$, we obtain a BIBRC$(q, k_1, k_2, \lambda/2)$. □

Example 3.3.1 Let $V = \mathrm{GF}(19)$, then $\omega = 2$ is a primitive element of GF(19). Furthermore, let $m = 3$ and

$$A = \begin{pmatrix} 0 & 1 & 4 \\ 16 & 6 & 17 \end{pmatrix}.$$

Then $\delta_0^R(A) = 6$, because there are (i) a pair $\{0, 1\}$ whose differences are $1 = \omega^0$ and $-1 = \omega^9$ in H_0^3 in the first row and (ii) two pairs $\{16, 17\}$ and $\{6, 17\}$ whose differences are ± 1, $8 = \omega^3$ and $11 = -8 = \omega^{12}$ in the second row. Similarly, $\delta_0^C(A) = 0$ and $\delta_0^E(A) = 2$ holds since the differences of a pair $\{4, 6\}$ are $7 = \omega^6$ and $-7 = 12 = \omega^{15}$. That is,

$$(k_1-1)\delta_0^R(A) + (k_2-1)\delta_0^C(A) - \delta_0^E(A) = 1 \cdot 6 + 2 \cdot 0 - 2 = 4.$$

Similarly,

$$(k_1-1)\delta_1^R(A)+(k_2-1)\delta_1^C(A)-\delta_1^E(A)=1\cdot 2+2\cdot 4-6=4 \text{ and}$$
$$(k_1-1)\delta_2^R(A)+(k_2-1)\delta_2^C(A)-\delta_2^E(A)=1\cdot 4+2\cdot 2-4=4$$

hold. Let $\mathcal{A}=\{hA+x : h\in H_0^3,\ x\in \mathrm{GF}(19)\}$. Then $(V,\ \mathcal{A})$ is a BIBRC(19, 2, 3, 4). Moreover, since $-1=18=\omega^9\in H_0^3$, if we utilize $\mathcal{A}'=\{hA+x : h\in H_0^3/\{1,\ -1\},\ x\in \mathrm{GF}(19)\}$ instead of $\mathcal{A}$, then $(V,\ \mathcal{A}')$ is a BIBRC(19, 2, 3, 2).

Moreover, if there exists a $k_1\times k_2$ array A with elements in GF(q) satisfying condition (C1) holds, there exists a BIBRC(q^n, k_1, k_2, λ).

Corollary 3.3.2 *Under the same assumptions of Theorem 3.3.1, there exists a* BIBRC(q^n, k_1, k_2, λ) *for* $n\geq 1$*. Moreover, if* q *is an odd prime power and* $2m$ *divides* $q-1$*, then there exists a* BIBRC(q^n, k_1, k_2, $\lambda/2$).

Proof. Let $\boldsymbol{A}=\{A_0,\ A_1,\ \ldots,\ A_{f-1}\}$, where $A_u=\omega^{um}A$ for $u=0,\ 1,\ \ldots,\ f-1$ and ω is a primitive element of GF(q). By Theorem 3.3.1, the equation (3.3.1) holds for each $d\in \mathrm{GF}(q)\setminus\{0\}$.

If we consider GF(q) as a subfield of GF(q^n), then $\mathrm{GF}(q)\setminus\{0\}$ is the multiplicative group H_0^g of g-th powers in GF(q^n) where $g=(q^n-1)/(q-1)$. Let S be any system of representatives for the cosets $\mathcal{H}^g$ modulo H_0^g in GF(q^n), i.e., S is a set of g field elements and $S\circ H_0^g=\mathrm{GF}(q^n)\setminus\{0\}$. We define $\boldsymbol{A}^*$ as $\{sA_u : A_u\in\boldsymbol{A},\ s\in S\}$. By the equation (3.3.1),

$$(k_1-1)\Delta_d^R(s\boldsymbol{A})+(k_2-1)\Delta_d^C(s\boldsymbol{A})-\Delta_d^E(s\boldsymbol{A})=\begin{cases}\lambda & \text{if } d\in sH_0^g\\ 0 & \text{if } d\notin sH_0^g\end{cases}$$

holds for each $s\in S$. Here, we add more blocks $\mathcal{A}^*=\{A'+x : A'\in\boldsymbol{A}^*,\ x\in \mathrm{GF}(q^n)\}$. Then, $(V^*,\ \mathcal{A}^*)$ is a BIBRC(q^n, k_1, k_2, λ), where $V^*=\mathrm{GF}(q^n)$.

When q is an odd prime power and $2m$ divides $q-1$, we utilize $\boldsymbol{A}'=\{A_0,\ A_1,\ \ldots,\ A_{f/2-1}\}$ instead of $\boldsymbol{A}$, where $A_u=h_uA$ and $H_0^m/\{1,\ -1\}=\{h_0=1,\ h_1,\ \ldots,\ h_{f/2-1}\}$. From this, we get a BIBRC(q^n, k_1, k_2, $\lambda/2$). □

If $m=1$ and q is a prime power, or $m=2$ and q is an odd prime power, then there always exists a $k_1\times k_2$ array satisfying (C1). That is, there exits a BIBRC(q, k_1, k_2, $k_1k_2(k_1-1)(k_2-1)$) for a prime power $q(>k_1k_2)$ and there exists a BIBRC(q, k_1, k_2, $k_1k_2(k_1-1)(k_2-1)/2$) for an odd prime power. If $m\geq 3$, we show an asymptotic existence of a BIBRC by utilizing Proposition 1.8.1.

Theorem 3.3.3 *For any positive integers* k_1*,* k_2 *and* λ*, let* $\lambda_0=\gcd(\lambda,\ k_1k_2(k_1-1)(k_2-1))$*. Assume that one of the following conditions holds:*

(i) *in the case when* $k_1k_2(k_1-1)(k_2-1)/\lambda_0$ *is even,*

$$(k_1-1)\left\lfloor\frac{\lambda_0}{k_1-1}\right\rfloor+(k_2-1)\left\lfloor\frac{\lambda_0}{k_2-1}\right\rfloor \geq \lambda_0 \ \textit{or} \tag{3.3.3}$$

$$\left(\frac{\lambda_0}{k_1-1}-\left\lfloor\frac{\lambda_0}{k_1-1}\right\rfloor\right)+\left(\frac{\lambda_0}{k_2-1}-\left\lfloor\frac{\lambda_0}{k_2-1}\right\rfloor\right) \geq 1, \tag{3.3.4}$$

(ii) *in the case when* $k_1k_2(k_1-1)(k_2-1)/\lambda_0$ *is odd,*

$$(k_1-1)\left\lfloor\frac{\lambda_0}{2(k_1-1)}\right\rfloor+(k_2-1)\left\lfloor\frac{\lambda_0}{2(k_2-1)}\right\rfloor \geq \frac{\lambda_0}{2} \ \textit{or} \tag{3.3.5}$$

$$\left(\frac{\lambda_0}{2(k_1-1)}-\left\lfloor\frac{\lambda_0}{2(k_1-1)}\right\rfloor\right)+\left(\frac{\lambda_0}{2(k_2-1)}-\left\lfloor\frac{\lambda_0}{2(k_2-1)}\right\rfloor\right) \geq 1. \tag{3.3.6}$$

Then there exists a constant $q_0 = q_0(k_1, k_2, \lambda)$ *such that a* BIBRC(q, k_1, k_2, λ) *exists for all prime powers* $q \geq q_0$ *satisfying*

$$\lambda(q-1) \equiv 0 \pmod{k_1k_2(k_1-1)(k_2-1)}. \tag{3.3.7}$$

As an application of Theorem 3.3.3, we can obtain the following corollaries. In the case of $k_1 = k_2 = k$, we can state the following corollary.

Corollary 3.3.4 *For any positive integer* k*, let* $\lambda = (k-1)/2$ *if* k *is odd or let* $\lambda = k/2$ *if* k *is even. Then there exists a* BIBRC(q, k, k, λ) *for sufficiently large prime powers* q *satisfying the congruence* (3.3.7)*.*

Proof. In Theorem 3.3.3, $\lambda_0 = \mathrm{lcm}(\lambda, k^2(k-1)^2) = \lambda$, which clearly satisfies the inequality (3.3.4). Thus the corollary is shown. □

Similarly, when k is even, the following corollary is obtained.

Corollary 3.3.5 *For an even integer* k*, let* $\lambda = k/2$*. Then there exists a* BIBRC(q, $k-1$, k, λ) *for sufficiently large prime powers* q *satisfying the congruence* (3.3.7)*.*

Moreover, we consider the case of a completely balanced BIBRC. In this case, λ is a multiple of $\mathrm{lcm}(k_1-1, k_2-1)$ by the equation (1.5.1). Then, the conditions (3.3.3) and (3.3.5) hold. Therefore, we obtain the following corollary.

Corollary 3.3.6 *For any positive integers* k_1 *and* k_2*, let* λ *be a multiple of* $\mathrm{lcm}(k_1-1, k_2-1)$*. If* q *is a sufficiently large prime power satisfying the congruence* (3.3.7)*, then there exists a completely balanced* BIBRC(q, k_1, k_2, λ)*.*

For the proof of Theorem 3.3.3, we use the same notations (2.5.1) in the proof of Theorem 2.5.7. And let $P_{k_1\times k_2} = P^R_{k_1\times k_2} \cup P^C_{k_1\times k_2} \cup P^E_{k_1\times k_2}$. We define ϵ^R_l by the number of ordered pairs $((i,\, j),\, (i,\, j'))$ such that $F((i,\, j),\, (i,\, j')) = H^m_l$ for all $((i,\, j),\, (i,\, j')) \in P^R_{k_1\times k_2}$. Similarly, we define ϵ^C_l and ϵ^E_l by the numbers of $((i,\, j),\, (i',\, j))$ and $((i,\, j),\, (i',\, j'))$ such that $F((i,\, j),\, (i',\, j)) = H^m_l$ and $F((i,\, j),\, (i',\, j')) = H^m_l$ for $((i,\, j),\, (i',\, j)) \in P^C_{k_1\times k_2}$ and $((i,\, j),\, (i',\, j')) \in P^E_{k_1\times k_2}$, respectively. For the numbers of the pairs $P^R_{k_1\times k_2}$, $P^C_{k_1\times k_2}$ and $P^E_{k_1\times k_2}$, the following equations obtained:

$$\sum_{l=0}^{m-1} \epsilon^R_l = k_1\binom{k_2}{2} = \frac{k_1k_2(k_2-1)}{2}, \tag{3.3.8}$$

$$\sum_{l=0}^{m-1} \epsilon^C_l = k_2\binom{k_1}{2} = \frac{k_1k_2(k_1-1)}{2} \text{ and} \tag{3.3.9}$$

$$\sum_{l=0}^{m-1} \epsilon^E_l = k_2(k_2-1)\binom{k_1}{2} = \frac{k_1k_2(k_1-1)(k_2-1)}{2}. \tag{3.3.10}$$

Conversely, for any nonnegative integers ϵ^R_l, ϵ^C_l and ϵ^E_l, $0 \le l < m$, satisfying the equations (3.3.8), (3.3.9) and (3.3.10), there is a corresponding choice F defined as above with these numbers. Moreover, if $-1 (\neq 1) \in H^m_0$ holds and if there exists a $k_1 \times k_2$ array A over GF(q) which is consistent with the choice F, then

$$\begin{aligned} &2\{(k_1-1)\epsilon^R_l + (k_2-1)\epsilon^C_l - \epsilon^E_l\} \\ &= (k_1-1)\delta^R_l(A) + (k_2-1)\delta^C_l(A) - \delta^E_l(A) \end{aligned}$$

holds for each l. Here we consider the following condition:

(C2) $(k_1-1)\epsilon^R_l + (k_2-1)\epsilon^C_l - \epsilon^E_l = \lambda$ holds for each $0 \le l < m$.

If the condition (C2) is satisfied, then we can construct a BIBRC$(q,\, k_1,\, k_2,\, \lambda)$ by Theorem 3.3.1.

Example 3.3.2 Let V = GF(19), then $\omega = 2$ is a primitive element of GF(19). Furthermore, let $m = 3$. We define a choice $F : P_{2\times 3} \to \mathcal{H}^3$ as follows:

$$\begin{array}{lll} F((1,\,1),\,(1,\,2)) = H^3_0 & F((1,\,1),\,(1,\,3)) = H^3_2 & F((1,\,2),\,(1,\,3)) = H^3_1 \\ F((2,\,1),\,(2,\,2)) = H^3_2 & F((2,\,1),\,(2,\,3)) = H^3_0 & F((2,\,2),\,(2,\,3)) = H^3_0 \\ \\ F((1,\,1),\,(2,\,1)) = H^3_1 & F((1,\,2),\,(2,\,2)) = H^3_1 & F((1,\,3),\,(2,\,3)) = H^3_2 \\ \\ F((1,\,1),\,(2,\,2)) = H^3_2 & F((1,\,1),\,(2,\,3)) = H^3_1 & F((1,\,2),\,(2,\,1)) = H^3_2 \\ F((1,\,2),\,(2,\,3)) = H^3_1 & F((1,\,3),\,(2,\,1)) = H^3_0 & F((1,\,3),\,(2,\,2)) = H^3_1 \end{array}.$$

The choice F satisfies $\epsilon_l^R + 2\epsilon_l^C - \epsilon_l^E = 2$ for each $l = 0, 1, 2$. Let A be the same 2×3 array over GF(19) in Example 3.3.1. Then, the array A is consistent with the choice F. Therefore a BIBRC(19, 2, 3, 2) exists by Theorem 3.3.1.

Proof of Theorem 3.3.3. Obviously, it is sufficient to show the existence of a BIBRC(q, k_1, k_2, λ_0), since a BIBRC(q, k_1, k_2, λ) can be obtained by making λ/λ_0 copies of the BIBRC.

Case (i). In the case when $k_1k_2(k_1-1)(k_2-1)/\lambda_0$ is even, q is always an odd prime power by the congruence (3.3.7). Let $m = k_1k_2(k_1-1)(k_2-1)/2\lambda_0$, then $2m$ divides $q-1$ by the congruence (3.3.7), that is, $-1 (\neq 1) \in H_0^m$. We define $s = k_1k_2(k_2-1)/2m = \lambda_0/(k_1-1)$, $t = k_1k_2(k_1-1)/2m = \lambda_0/(k_2-1)$. We set $\epsilon_l^R = s$, $\epsilon_l^C = t$ and $\epsilon_l^E = \lambda_0$ for all $0 \leq l < m$, then the equations (3.3.8), (3.3.9) and (3.3.10) and the condition (C2) are satisfied. But, s and t may be rational.

If we find nonnegative integers ϵ_l^R, ϵ_l^C and ϵ_l^E satisfying the equations (3.3.8), (3.3.9) and (3.3.10) and the condition (C2). Then we can fix a choice F. By Proposition 1.8.1, there exists a $k_1 \times k_2$ array which is consistent with the choice F for sufficiently large prime powers satisfying the congruence (3.3.7). Thus, we can construct a BIBRC(q, k_1, k_2, λ_0) by Theorem 3.3.1. In the following, we will find such nonnegative integers ϵ_l^R, ϵ_l^C and ϵ_l^E.

Case (i-a). In the case when the inequality (3.3.3) hold, that is, $(k_1 - 1)\lfloor s \rfloor + (k_2 - 1)\lfloor t \rfloor - \lambda_0 \geq 0$ holds. Let $\lceil a \rceil$ be the smallest integers which is not less than a. We arbitrarily define either $\epsilon_l^R = \lfloor s \rfloor$ or $\epsilon_l^R = \lceil s \rceil$ for $l = 0, 1, \ldots, m-1$ so that the numbers of $\lfloor s \rfloor$ and $\lceil s \rceil$ are $(\lceil s \rceil - s)m$ and $(s - \lfloor s \rfloor)m$, respectively. Similarly, we define either $\epsilon_l^C = \lfloor t \rfloor$ or $\epsilon_l^C = \lceil t \rceil$ so that the numbers of $\lfloor t \rfloor$ and $\lceil t \rceil$ are $(\lceil t \rceil - t)m$ and $(t - \lfloor t \rfloor)m$, respectively. Note that the numbers $(\lceil s \rceil - s)m$, $(s - \lfloor s \rfloor)m$, $(\lceil t \rceil - t)m$ and $(t - \lfloor t \rfloor)m$ are nonnegative integers. And let $\epsilon_l^E = (k_1 - 1)\epsilon_l^R + (k_2 - 1)\epsilon_l^C - \lambda_0$. Then ϵ_l^E are nonnegative integers for any l. It is easy to see that ϵ_l^R's, ϵ_l^C's and ϵ_l^E's satisfy the equations (3.3.8), (3.3.9) and (3.3.10) and the condition (C2) holds.

Case (i-b). In the case when the inequality (3.3.4) holds, that is, $(\lceil s \rceil - s)m \leq (t - \lfloor t \rfloor)m$ and $(s - \lfloor s \rfloor)m \geq (\lceil t \rceil - t)m$ hold. We define ϵ_l^R, ϵ_l^C and ϵ_l^E similarly to the Case (i-a). However, we define ϵ_l^R and ϵ_l^C so that $\epsilon_l^R = \lfloor s \rfloor$ and $\epsilon_l^C = \lfloor t \rfloor$ do not occur simultaneously. Then, ϵ_l^E is a nonnegative integer for each l. It is easy to see that ϵ_l^R's, ϵ_l^C's and ϵ_l^E's satisfy the equations (3.3.8), (3.3.9) and (3.3.10) and the condition (C2) holds.

Case (ii). We consider the case when $k_1k_2(k_1-1)(k_2-1)/\lambda_0$ is odd. Necessarily, λ_0 is a multiple of 4. We take $m = k_1k_2(k_1-1)(k_2-1)/\lambda_0$. If q is an odd prime power, $2m$ divides $(q-1)$ since the equation (3.3.7) holds. Otherwise q is a power of 2, and $-1 = 1$ holds. In any case, we have $-1 \in H_0^m$.

Similarly, put $\epsilon_l^R = \lfloor \lambda_0/2(k_1-1) \rfloor$ or $\lceil \lambda_0/2(k_1-1) \rceil$, $\epsilon_l^C = \lfloor \lambda_0/2(k_2-1) \rfloor$ or $\lceil \lambda_0/2(k_2-1) \rceil$ and $\epsilon_l^E = (k_1-1)\epsilon_l^R + (k_2-1)\epsilon_l^C - \lambda_0/2$, so that the numbers of $\lfloor \lambda_0/2(k_1-1) \rfloor$, $\lceil \lambda_0/2(k_1-1) \rceil$, $\lfloor \lambda_0/2(k_2-1) \rfloor$ and $\lceil \lambda_0/2(k_2-1) \rceil$ are $(\lceil \lambda_0/2(k_1-1) \rceil - \lambda_0/\ 2(k_1-1))m$, $(\lambda_0/2(k_1-1) - \lfloor \lambda_0/2(k_1-1) \rfloor)m$, $(\lceil \lambda_0/2(k_2-1) \rceil - \lambda_0/2(k_2-1))m$ and $(\lambda_0/2(k_2-1) - \lfloor \lambda_0/2(k_2-1) \rfloor)m$, respectively.

Similarly, we can find nonnegative integers ϵ_l^R's, ϵ_l^C's and ϵ_l^E's satisfying the equations (3.3.8), (3.3.9) and (3.3.10) and the condition (C2) holds. That is, in any case, we can fix a choice F such that the condition (C2) holds. Thus, the theorem is shown. $\square$

Chapter 4

Multiple edge-colored graph decompositions

In this chapter, the asymptotic existence of colorwise simple edge-colored graph decompositions of complete graphs is discussed. In Section 4.1, we introduce a simple property "tree-ordered" to show the asymptotic existence of colorwise simple edge-colored graph decompositions of complete graphs. In Section 4.2, outline of a proof of the main theorem is given. In Sections 4.3, 4.4 and 4.5, some theorems are prepared to show the main theorem. In Section 4.6, we introduce a notion of "balanced" decomposition of graphs. And we show the asymptotic existence of balanced graph decompositions of complete graphs. In Section 4.7, the asymptotic existence of graph decompositions is shown for any edge-colored graph $K_v^{\boldsymbol{\lambda}}$. Also, balanced case is treated.

4.1 Tree-ordered structure of edge-colored graphs

First, we define the property "tree-ordered." Let $G = (X(G), E(G), \theta_G, \psi_G)$ be an edge-c-colored graph. For each distinct vertices x and y of $X(G)$, let $\langle x, y\rangle$ be the *edge set* $\psi_G^{-1}(\{x, y\})$ between x and y in G and let $\mathcal{E}\langle G\rangle$ be the family of all edge sets in G. We define $C(\langle x, y\rangle)$ as a *color multiset* of an edge set $\langle x, y\rangle$, that is, $C(\langle x, y\rangle) = (\theta_G(e) : e \in \langle x, y\rangle)$ and $\mathcal{C}(G)$ as the family of all color multisets over all edge sets of G. If G is colorwise simple, a color multiset is simply a set. And we denote an edge with color i between vertices x and y by $\{x, y\}_i$.

Let $\mathcal{G}$ be a family of edge-c-colored graphs. We define $\mathcal{C}(\mathcal{G}) = \bigcup_{G\in\mathcal{G}} \mathcal{C}(G)$. Then $\mathcal{G}$ is said to be *tree-ordered* if (i) $C_1 \subset C_2$, $C_1 \supset C_2$, or $C_1 \cap C_2 = \emptyset$

holds for any distinct color multisets C_1 and C_2 in $\mathcal{C}(\mathcal{G})$ and (ii) the color multisets (i) belongs to $\mathcal{C}(\mathcal{G})$ for any i. Especially, G is called *tree-ordered edge-c-colored graph* if $\mathcal{C}(\{G\})$ is tree-ordered. If $\mathcal{G}$ consists only of colorwise simple edge-c-colored graphs, then we use the term "color sets" instead of color multisets.

Then we obtain the following theorem.

Theorem 4.1.1 *Let $\mathcal{G}$ be a tree-ordered $\boldsymbol{\lambda}$-admissible family of colorwise simple edge-c-colored graphs, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_c)$ is a vector of positive integers. Then there exists a constant $v_0 = v_0(\mathcal{G}, \boldsymbol{\lambda})$ such that $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ exist for all integers $v \geq v_0$ satisfying the congruences* (1.6.2).

To prove Theorem 4.1.1, we firstly show the following theorem which is a simple version of Theorem 4.1.1.

Theorem 4.1.2 *Let $\mathcal{G}$ be a tree-ordered admissible family of colorwise simple edge-c-colored graphs. Then there exits a constant $v_0 = v_0(\mathcal{G})$ such that $\mathcal{G}$-decompositions of $K_v^{[c]}$ exist for all integers $v \geq v_0$ satisfying the congruences* (1.6.3).

A proof of Theorem 4.1.2 will be stated in Sections 4.2, 4.3, 4.4 and 4.5. And a proof of Theorem 4.1.1 will be stated in Section 4.7. Moreover, we obtain a similar theorem to Theorem 4.1.2 in the case of "balanced" graph decompositions of complete graphs in Section 4.6.

4.2 Outline of the proof of an asymptotic theorem for graph decompositions

For a set K of positive integers, let $B(K)$ be the set of integers v such that there exists a pairwise balanced design $\mathrm{B}(v, K, 1)$. K is called a *PBD-closed set* if $B(K) = K$ holds.

For given c, we fix a tree-ordered admissible family $\mathcal{G}$ of colorwise simple graphs with c colors. Let $(V, \mathcal{B})$ be a $\mathrm{B}(v, K, 1)$. It is readily seen that there exists a $\mathrm{D}(K_v^{[c]}, \mathcal{G})$ if there exists a $\mathrm{D}(K_{|B|}^{[c]}, \mathcal{G})$ for every $B \in \mathcal{B}$ by combining all such decompositions. That is, in the terminology of Wilson [101], the set of integers

$$D(\mathcal{G}) = \{v : \mathrm{D}(K_v^{[c]}, \mathcal{G}) \text{ exists}\}$$

is PBD-closed. The main result of Wilson [101] asserts the following proposition.

Proposition 4.2.1 *If a PBD-closed set D contains integers greater than* 1*, then D is eventually periodic with some positive period $\beta(D)$, that is,*

$$v \in D \Rightarrow v + t\beta(D) \in D \textit{ for all sufficiently large } t.$$

Now the assumption that $\mathcal{G}$ is admissible implies that there exists a positive integer m such that the constant vector $(m, m, \ldots, m)$ of length c is a nonnegative integral linear combination of the $\mu(G)$'s for $G \in \mathcal{G}$. This in turn means that we can obtain a colorwise simple graph G_0 with c colors which consists of the disjoint union of graphs isomorphic to members of $\mathcal{G}$ and such that G_0 has exactly m edges of each color. Then, the following theorem is obtained, which is proved in Section 4.3.

Theorem 4.2.2 *Let G_0 be a tree-ordered colorwise simple graph with c colors and m edges of each of c colors. Then there exists a constant $q_0 = q_0(m, k)$ such that $K_q^{[c]}$ admits a G_0-decompositon for every prime power $q \equiv 2m + 1 \pmod{4m}$ with $q \geq q_0$, where k is the number of vertices of G_0.*

By Theorem 4.2.2, there are (infinitely many) values of v for which there exist $\mathrm{D}(K_q^{[c]}, G_0)$'s, and hence a $\mathcal{G}$-decomposition of $K_q^{[c]}$. Thus we have the existence of an eventual period $\beta_0 \neq 0$ for $D(\mathcal{G})$ by Wilson [101]. A multiple of an eventual period is also an eventual period of $D(\mathcal{G})$, so we may assume β_0 is divisible by $\beta(\mathcal{G})$. To complete the proof of Theorem 4.1.2, it will suffice to show the following theorem which is proved in Section 4.5.

Theorem 4.2.3 *Let $\mathcal{G}$ be a tree-ordered admissible family of colorwise simple graphs with c colors. Let n be a positive integer satisfying the congruences* (1.6.3)*. Then there exists an integer v_0 such that $v_0 \equiv n \pmod{\beta_0}$ and that $K_{v_0}^{[c]}$ admits a $\mathcal{G}$-decomposition, where β_0 is an eventual period of $D(\mathcal{G})$.*

In order to prove Theorem 4.2.3, we first show the following theorem. The proof is given in Section 4.4.

Theorem 4.2.4 *Let $\mathcal{G}$ be a tree-ordered admissible family of colorwise simple graphs with c colors. Let v be a positive integer satisfying the congruences* (1.6.3) *and $v \geq 2 + |V(G)|$ for all G in $\mathcal{G}$. Then, for an eventual period β_0 of $D(\mathcal{G})$, there exists a prime power $q \equiv 1 \pmod{\beta_0}$ such that $qK_v^{[c]}$ admits $\mathcal{G}$-decomposition.*

In summary, the proof of Theorem 4.1.2 will be completed by the material in the next three sections.

4.3 A construction from cyclotomy in finite fields

In this section, we prove Theorem 4.2.2 by utilizing Proposition 1.8.1.

Proof of Theorem 4.2.2. Let Γ denote the group of $q(q-1)/2m$ permutations

$$\{x \mapsto ax + b : a \in H_0^{2m},\, b \in \mathrm{GF}(q)\}$$

of $\mathrm{GF}(q)$. Then by letting Γ act naturally on the set

$$\{(x,\, y) : x,\, y \in \mathrm{GF}(q),\, x \neq y\},$$

we obtain $2m$ orbits

$$\{(x,\, y) : y - x \in H_i^{2m},\, x,\, y \in \mathrm{GF}(q)\},$$

on which Γ is sharply transitive. Here we consider the following condition:

(C3) There is an injective mapping $\phi : V(G_0) \to \mathrm{GF}(q)$ such that for each color i, the $2m$ field elements

$$\{\pm(\phi(x) - \phi(y)) : \{x,\, y\} \in E_i(G_0)\}$$

form a system of representatives for the cyclotomic classes H_0^{2m}, H_1^{2m}, $\ldots$, H_{2m-1}^{2m} of index $2m$.

By virtue of the condition (C3), we claim that $K_q^{[c]}$ can be decomposed into G_0's. When we apply the permutations in Γ to the vertices of the image of G_0, we obtain a decomposition of $K_q^{[c]}$ into $q(q-1)/2m$ subgraphs isomorphic to G_0 by the condition (C3).

Proposition 1.8.1 asserts that, provided q is sufficiently large, we can map vertices of G_0 to field elements so that the difference $\phi(x)-\phi(y)$ ($x,\, y \in V(G_0)$) in one direction is in some cyclotomic class H_i^{2m} we may wish, but then the difference $\phi(y) - \phi(x)$ in the other direction will belong to the cyclotomic class $H_{i+\ell}^{2m}$ where ℓ is an integer satisfying $-1 \in H_\ell^{2m}$. If q is a prime power with $q \equiv 2m + 1 \pmod{4m}$, $-1 \in H_m^{2m}$, since $-1 = \omega^{(q-1)/2}$ and $(q-1)/2 \equiv m \pmod{2m}$. Thus if $a \in H_i^{2m}$, then $-a \in H_{i+m}^{2m}$.

It is clear, from Proposition 1.8.1, that there exists an injection ϕ satisfying the condition (C3) if G_0 is a simple edge-c-colored graph. To handle a colorwise simple graph with c colors, we want to find an injection ϕ which is a choice for any color i and is well-defined for each edge set.

Let $\mathcal{E}\langle G_0\rangle$ be a family of all edge sets in G_0. A subfamily $\mathcal{E}_1 \subset \mathcal{E}\langle G_0\rangle$ is called a *resolution class of color set* C if (i) $\cup_{\langle x,\, y\rangle \in \mathcal{E}_1} C(\langle x,\, y\rangle) = C$ and

(ii) $C(\langle x, y\rangle) \cap C(\langle x', y'\rangle) = \emptyset$ for any distinct edge sets $\langle x, y\rangle$ and $\langle x', y'\rangle$ in $\mathcal{E}_1$. $\mathcal{E}\langle G_0\rangle$ is said to be *resolvable with respect to color set* C if $\mathcal{E}\langle G_0\rangle$ is partitioned into resolution classes of color set C.

By the assumption, $\mathcal{C}(G_0)$ is tree-ordered. We can choose a resolution class $\mathcal{E}_1$ from $\mathcal{E}\langle G_0\rangle$ since G_0 has m edges of each color i and $C_1 \cap C_2 = \emptyset$, $C_1 \subset C_2$, or $C_1 \supset C_2$ holds for any distinct color sets C_1 and C_2 in $\mathcal{C}(G_0)$. In fact, we fix a color $i_1 \in C$ and chose the maximal color set C_1 containing i_1. Secondly, we fix a color $i_2 \in C \setminus C_1$ and choose the maximal color set C_2 containing i_2. Then $C_1 \cap C_2 = \emptyset$ since $\mathcal{C}(G_0)$ is tree-ordered. By continuing this process, we obtain a resolution class $\mathcal{E}_1$ of color set.

We define G_1 as the graph having the same vertex set with G_0 and the edge set $\mathcal{E}\langle G_0\rangle \setminus \mathcal{E}_1$. Then, we can choose a resolution class $\mathcal{E}_2$ from $\mathcal{E}\langle G_1\rangle$ since $\mathcal{C}(G_1)$ is also tree-ordered. By continuing this process, we obtain m resolution classes $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\}$ of $\mathcal{E}\langle G_0\rangle$. Each $\mathcal{E}_l$ has exactly one edge of color i for each $i \in C$. We can choose an injection ϕ such that

$$\begin{aligned}&\phi(x) - \phi(y) \in H_l^{2m} \text{ and}\\ &\phi(y) - \phi(x) \in H_{l+m}^{2m}\end{aligned}$$

hold for any edge $\{x, y\}$ in $\mathcal{E}_l$, $l = 0, 1, \ldots, m-1$. With such a choice ϕ satisfying the condition (C3), by applying Proposition 1.8.1, we obtain the required G_0-decomposition of $K_q^{[c]}$ for sufficiently large q. □

4.4 Integral solutions for a certain linear system

In this section, we will show Theorem 4.2.4, which will be utilized to show Theorem 4.2.3 in the next section. To show Theorem 4.2.4, we give a lemma which says that the congruences (1.6.3) are sufficient for the existence of an integral solution of a certain system of linear equations. Here, we use the following well known proposition to show the lemma (see, for example, [88]).

Proposition 4.4.1 *Let M be a rational $s \times t$ matrix and $\boldsymbol{c}$ be a rational column vector of length s. The equation $M\boldsymbol{x} = \boldsymbol{c}$ has an integral solution $\boldsymbol{x}$, a column vector of length t, if and only if $\boldsymbol{y}M$ integral implies $\boldsymbol{yc}$ is an integer for all rational row vectors $\boldsymbol{y}$ of length s.*

For a family of colorwise simple graphs $\mathcal{G}$ with c colors, let $\mathcal{F}$ denote the set of all subgraphs F of $K_v^{[c]}$ each of which is isomorphic to some member of $\mathcal{G}$. And let M be the matrix whose rows are indexed by the $cv(v-1)/2$

edges of $K_v^{[c]}$ and whose columns are indexed by the members in $\mathcal{F}$, where the entry in row e and column F of M is 1 if $e \in E(F)$ and 0 otherwise. Let $\mathbf{1}$ be all-one vector of length $cv(v-1)/2$.

Lemma 4.4.2 *Let $\mathcal{G}$ be a tree-ordered admissible family of colorwise simple graphs with c colors and let $\mathcal{F}$ denote the set of all subgraphs F of $K_v^{[c]}$ each of which is isomorphic to some member of $\mathcal{G}$. In addition, assume $v \geq 2+|V(G)|$ for all G in $\mathcal{G}$. The equation $M\boldsymbol{x} = \mathbf{1}$ has an integral solution $\{s_F : F \in \mathcal{F}\}$ if and only if v satisfies the congruences* (1.6.3).

Proof. The proof is similar to that of Lamken and Wilson (see [63, Theorem 5.4]) though the first part of the proof is different from them because of the existence of multiple edges. So, we show only different parts of the proof.

We assume that rationals $b(e)$ for $e \in E(K_v^{[c]})$ are given such that $b(F) = \sum_{e\in E(F)} b(e)$ is integral for each $F \in \mathcal{F}$. For an edge $e = \{x, y\}_i$, $b_i\{x, y\} = b(e)$. For rational numbers a and b, $a \equiv b$ means that the difference $b - a$ is an integer.

For each color $i \in C$, let G_i be a graph in a tree-ordered admissible family $\mathcal{G}$ having an edge of color set $\{i\}$. Note that G_i and $G_{i'}$ may be the same graph. Let x, y, u and v be any four vertices of $K_v^{[c]}$ and let $F_{i,1}$ be an isomorphic copy of G_i in $K_v^{[c]}$ such that $F_{i,1}$ contains the edge $\{x, y\}_i$ of color i and that $u, v \notin V(F_{i,1})$. Let $F_{i,2}$, $F_{i,3}$ and $F_{i,4}$ be the isomorphic graphs to $F_{i,1}$ obtained by applying the permutations (xu), (yv) and $(xu)(yv)$, respectively. Now since $b(F_{i,l})$ is integral for $l = 1, 2, 3, 4$, we have

$$b(F_{i,1}) + b(F_{i,4}) \equiv b(F_{i,2}) + b(F_{i,3}). \tag{4.4.1}$$

Each side of this congruence consists of sums of $b(e)$'s. Since $F_{i,l}$ ($l = 1, 2, 3, 4$) have common edges, by deleting $b(e)$'s corresponding to these edges from both side of the congruence (4.4.1), the congruence (4.4.1) is reduced to

$$b_i\{x, y\} + b_i\{u, v\} \equiv b_i\{x, v\} + b_i\{u, y\}. \tag{4.4.2}$$

Since the congruence (4.4.2) holds for any x, y, u and v in $V(K_v^{[c]})$, there exist rationals $\gamma_i(x)$ of each $x \in V(K_v^{[c]})$ such that

$$b_i\{x, y\} \equiv \gamma_i(x) + \gamma_i(y). \tag{4.4.3}$$

To prove the congruence (4.4.3), choose distinct vertices p, q, r and solve the equations $b_i\{p, q\} = \gamma_i(p) + \gamma_i(q)$, $b_i\{q, r\} = \gamma_i(q) + \gamma_i(r)$, $b_i\{r, p\} = \gamma_i(r) + \gamma_i(p)$ and define $\gamma_i(x) = b_i\{x, p\} - \gamma_i(p)$ for any $x \neq p, q, r$. Then the congruence (4.4.3) holds for any two vertices x and y. Note that γ's may be rationals.

The reminder of the proof of this lemma is the same as that of Lamken and Wilson (see [63, Theorem 5.4]). $\square$

Now we are ready to prove Theorem 4.2.4.

Proof of Theorem 4.2.4. The congruences (1.6.1) holds for some positive rationals a_G, since $\mathcal{G}$ is admissible. Given $G \in \mathcal{G}$, the number of $F \in \mathcal{F}$ with $F \cong G$ that contain an edge e of $K_v^{[c]}$ depends only on its color. More precisely, there is a constant M_G such that the number of $F \in \mathcal{F}$ with $F \cong G$ containing an edge e of color i is $m_i M_G$, where $\mu(G) = (m_1, m_2, \ldots, m_c)$. Let $d_F = a_G/M_G$ for $F \cong G$. Then

$$\sum_{F:e\in E(F)} d_F = 1 \quad \text{for every edge } e \text{ of } K_v^{[c]}.$$

Define $z_F = Md_F$, where M is a positive integer chosen so that all z_F are (positive) integers. Then

$$\sum_{F:e\in E(F)} z_F = M \quad \text{for every edge } e \text{ of } K_v^{[c]}.$$

Let v be a positive integer satisfying the congruences (1.6.3). We assume that $v \geq 2 + |V(G)|$ for all G in $\mathcal{G}$. We define $\{s_F : F \in \mathcal{F}\}$ as in Lemma 4.4.2. Let ${s'}_F = s_F + tz_F$ for each $F \in \mathcal{F}$ and for any integer t, then

$$\sum_{F:\, e\in E(F)} s'_F = 1 + tM \quad \text{for every edge } e \text{ of } K_v^{[c]}.$$

We fix t so that

(i) $s'_F = s_F + tz_F \geq 0$ for each $F \in \mathcal{F}$ and

(ii) $q = 1 + tM$ is a prime or a power of prime congruent to 1 modulo β_0.

The existence of t satisfying (ii) is due to the well-known theorem by Dirichlet (see, for example, [36]). Thus, we obtain a $\mathcal{G}$-decomposition of $qK_v^{[c]}$. $\square$

4.5 A linear algebraic construction

To prove Theorem 4.2.3, we use Theorem 4.2.4 together with the following techniques utilized in Wilson [103] to show that there is at least one $\mathcal{G}$-decomposition of $K_v^{[c]}$ for each feasible congruence class modulo β_0. The following proposition is necessary to show Theorem 4.2.3 (see, for example, [103]).

Proposition 4.5.1 *Let W be a d-dimensional vector space over* $\mathrm{GF}(q)$*, and let $\ell : W \to \mathrm{GF}(q)$ be any nonzero linear functional. If $d \geq n^2$, there exist linear transformations $T_1, T_2, \ldots, T_n$ of W to itself with the following properties: $S_{ij} = (T_j - T_i)^{-1}$ exists whenever $i \neq j$ and for any $n(n-1)/2$ scalars ρ_{ij}, $1 \leq i < j \leq n$, there exist vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \in W$ such that*

$$\ell(S_{ij}(\boldsymbol{x}_j - \boldsymbol{x}_i)) = \rho_{ij}$$

holds for $1 \leq i < j \leq n$.

Fix an integer n satisfying the congruences (1.6.3). By Theorem 4.2.4, there exists a $\mathcal{G}$-decomposition $\mathcal{F}$ of $qK_n^{[c]}$, where $q \equiv 1 \pmod{\beta_0}$ is some prime power. Take each subgraph $F \in \mathcal{F}$ with multiplicity s'_F to get a multiset $F_1, F_2, \ldots, F_N$ of subgraphs in $\mathcal{F}$ such that each edge $\{x, y\}$ of color i in $K_n^{[c]}$ appears in exactly q of these subgraphs, $i = 1, 2, \ldots, c$.

For a positive integer d, let $v_0 = nq^d$. Then $v_0 \equiv n \pmod{\beta_0}$ holds. Let $\{1, 2, \ldots, n\}$ be the vertex set of $K_v^{[c]}$, and $V = W \times \{1, 2, \ldots, n\}$ be the vertex set of $K_{v_0}^{[c]}$, where W is a d-dimensional vector space over $\mathrm{GF}(q)$. We note that the following lemma is proved by Proposition 4.2.1 together with Theorem 4.2.2.

Lemma 4.5.2 *Let $\mathcal{G}$ be a tree-ordered admissible family of colorwise simple graphs with c colors. There exists a positive integer β_0 which is divisible by $2\beta(\mathcal{G})$ with the property: If $K_{v_0}^{[c]}$ admits a $\mathcal{G}$-decomposition for some positive integer v_0, then $K_v^{[c]}$ can be $\mathcal{G}$-decomposed for all sufficiently large integers $v \equiv v_0 \pmod{\beta_0}$.*

Again, we utilize Theorem 4.2.2 as follows. In Theorem 4.2.2, it is obvious that $\beta(\mathcal{G})$ divides m, where m is the number of edges of each color in G_0. Let G'_0 be a graph having β_0 components which are isomorphic to G_0 and let $m' = \beta_0 m$ be the number of edges of each color in G'_0. By applying Theorem 4.2.2 to G'_0, there exist G'_0-decompositions ($\mathcal{G}$-decompositions) of $K_p^{[c]}$ for sufficiently large prime power $p \equiv 2m' + 1 \pmod{4m'}$. It is obvious that β_0 divides $2m'$, thus $p \equiv 1 \pmod{\beta_0}$. By Lemma 4.5.2, there exist $\mathcal{G}$-decompositions of $K_v^{[c]}$ for sufficiently large integer $v \equiv 1 \pmod{\beta_0}$. Hence, there exist $\mathcal{G}$-decompositions of $K_{q^d}^{[c]}$ for $q \equiv 1 \pmod{\beta_0}$ and for sufficiently large integers d.

By choosing an integer $d \geq n^2$ which is large enough, $K_{q^d}^{[c]}$ defined on the vertex set $W \times \{x\}$ can be $\mathcal{G}$-decomposed for each x. Let $K_{n(q^d)}^{[c]}$ be a colorwise simple complete n-partite graph with c colors. Then we obtain the following lemma.

Lemma 4.5.3 *For a tree-ordered admissible family $\mathcal{G}$ of colorwise simple edge-c-colored graphs, if there exists a $\mathcal{G}$-decomposition of $qK_n^{[c]}$ for a prime power q and an integer n satisfying the congruences* (1.6.3)*, then there exists a $\mathcal{G}$-decomposition of $K_{n(q^d)}^{[c]}$ for any $d \geq n^2$.*

Proof. For each subgraph F_h, $h = 1, 2, \ldots, N$, by decomposing $qK_n^{[c]}$ as in the previous section, we want to assign scalars $\rho_h(x, y)$ in $\mathrm{GF}(q)$ to all ordered pairs (x, y) of vertices of F_h with $x < y$ and for which x and y are adjacent so that: for every pair (x, y) with $1 \leq x < y \leq n$ and every color i, $1 \leq i \leq c$, the following condition is satisfied:

(C4) For each $l \in \mathrm{GF}(q)$ and x, y in $qK_n^{[c]}$, there is a unique edge $\{x, y\}$ of color i in $\mathcal{F}$ to which scalar $l = \rho_h(x, y)$ is assigned.

Note that for each $\{x, y\}_i \in \langle x, y\rangle$ such that $x < y$, $\rho_h(x, y)$ is assigned to the same element of $\mathrm{GF}(q)$ not depending i.

We regard the subgraphs $F_1, F_2, \ldots, F_N$ as "formally disjoint" by distinguishing the q edges $\{x, y\}$ in $qK_n^{[c]}$ as distinct edges. Let $\mathcal{E}\langle x, y\rangle$ be a family of the edge sets $\langle x, y\rangle$ that appear in $F_1, F_2, \ldots, F_N$. Assume that $\mathcal{G}$ is tree-ordered, then $\mathcal{E}\langle x, y\rangle$ is partitioned into resolution classes $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_q$ in a similar manner to the proof of Theorem 4.2.2. For an edge set $\langle x, y\rangle \in F_h$, if $\langle x, y\rangle$ belongs to $\mathcal{E}_l$, we define $\rho_h(x, y) = l$, where $x < y$ and $l \in \mathrm{GF}(q)$.

The reminder of the proof of this theorem is the same as that of Lamken and Wilson (see [63, Theorem 6.2]) together with Proposition 4.5.1. That is, there exists a $\mathcal{G}$-decomposition of $K_{n(q^d)}^{[c]}$. $\square$

A $\mathcal{G}$-decomposition of $K_{nq^d}^{[c]}$ is obtained by applying the decompositions of $K_{q^d}^{[c]}$ in Theorem 4.2.2 and Lemma 4.5.2 and $K_{n(q^d)}^{[c]}$ in Lemma 4.5.3. Thus, Theorem 4.1.2 is proved.

4.6 Balanced graph decompositions

In this section, we introduce a property "balanced" to a $\mathcal{G}$-decomposition of $K_v^{[c]}$. For a family $\mathcal{G}$ of edge-c-colored graphs, a $\mathcal{G}$-decomposition $\mathcal{F}$ of $K_v^{[c]}$ is called *balanced* if each vertex of $K_v^{[c]}$ belongs to exactly the same number$(= r)$, which is called *replication number*, of subgraphs $F \in \mathcal{F}$. In the reminder of this section, we consider only a case when $\mathcal{G}$ consists of graphs which have the same number of vertices and edges for each color.

Let $\mathcal{G}$ be a family of colorwise simple edge-c-colored graphs with k vertices and m edges of each color and $\mathcal{F}$ be a balanced $\mathcal{G}$-decomposition of $K_v^{[c]}$ with

$b = |\mathcal{F}|$ members and the replication number r. Then,

$$vr = bk \text{ and } b = \frac{v(v-1)}{2m}$$

hold, hence

$$r = \frac{k(v-1)}{2m}. \tag{4.6.1}$$

Moreover, we need the following condition (C5).

(C5) There exist integers $u_G(x)$ for $x \in V(G)$ and $G \in \mathcal{G}$ such that

$$\begin{aligned} \sum_{G \in \mathcal{G}} \sum_{x \in V(G)} u_G(x) \tau_G(x) = (v-1,\, v-1,\, \ldots,\, v-1) \text{ and} \\ \sum_{G \in \mathcal{G}} \sum_{x \in V(G)} u_G(x) = \frac{k(v-1)}{2m}, \end{aligned} \tag{4.6.2}$$

where $\tau_G(x)$ is the degree vector of vertex x in G.

It can be shown that an integer $v \equiv 1 \pmod{2m}$ satisfies the formulas (1.6.3) and (4.6.1) and the condition (C5) by letting $u_G(x) = (v-1)/(2m)$ and $u_{G'}(x') = 0$ for all vertices x and x' in a graph G and the graphs $G' \in \mathcal{G} \backslash \{G\}$, respectively.

If there exists an integer n such that there are integers $u_G(x)$ satisfying the condition (C5). Then, all integers $v \equiv n \pmod{2m}$ satisfy the condition (C5). In fact, let $s = (v-n)/(2m)$ and let $u_{G,v}(x) = u_G(x) + s$ and $u_{G',v}(x') = u_{G'}(x')$ for each x and x' in a graph G and the graphs $G' \in \mathcal{G} \backslash \{G\}$, respectively. Then, it is easy to show that the equations (4.6.2) hold. Conversely, if there exists an integer n not satisfying the condition (C5) but the formulas (1.6.3) and (4.6.1), then every integer $v \equiv n \pmod{2m}$ does not satisfy the condition (C5). In fact, assume that there exists an integer $v \equiv n \pmod{2m}$ satisfying the condition (C5) and the formulas (1.6.3) and (4.6.1), then all integers $v_0 \equiv v \pmod{2m}$ satisfy the condition (C5) by the above discussion, which is contradiction. Thus, we define T as the subset of integers in $\mathbb{Z}_{2m}$ such that they satisfy the formulas (1.6.3) and (4.6.1) and the condition (C5). Then, we obtain the following lemma.

Lemma 4.6.1 *Let $\mathcal{G}$ be a family of colorwise simple graphs with k vertices, c colors and m edges for each of c colors. Then, necessary conditions for the existence of balanced $\mathcal{G}$-decompositions of $K_v^{[c]}$ are*

$$v \equiv t \pmod{2m} \quad \textit{for each } t \in T. \tag{4.6.3}$$

Then, the following theorem is obtained.

Theorem 4.6.2 *Let $\mathcal{G}$ be a family of tree-ordered colorwise simple edge-c-colored graphs with k vertices and m edges for each of c colors. Then there exists a constant $v_0 = v_0(\mathcal{G})$ such that balanced $\mathcal{G}$-decompositions of $K_v^{[c]}$ exist for all integers $v \geq v_0$ satisfying the congruence* (4.6.3).

A proof is similar to that of Theorem 4.1.2. Firstly, we have to show the following lemma.

Lemma 4.6.3 *Let $\mathcal{G}$ be a family of tree-ordered colorwise simple edge-c-colored graphs with m edges for each of c colors and $D(\mathcal{G})$ be the set of integers v such that there exists a balanced* $\mathrm{D}(K_v^{[c]}, \mathcal{G})$*, that is,*

$$D(\mathcal{G}) = \{v : \text{balanced } \mathrm{D}(K_v^{[c]}, \mathcal{G}) \text{ exists}\}.$$

Then, $D(\mathcal{G})$ is a PBD-closed set.

Proof. For any $v \in B(D(\mathcal{G}))$, we have only to show that $v \in D(\mathcal{G})$ since it is obvious that $D(\mathcal{G}) \subset B(D(\mathcal{G}))$. For any $v \in B(D(\mathcal{G}))$, there exists a $\mathrm{B}(v, K, 1)$ for $K = D(\mathcal{G})$. Let $(V, \mathcal{B})$ be a $\mathrm{B}(v, K, 1)$. Since $K = D(\mathcal{G})$, for any block size $k \in D(\mathcal{G})$, there exists a balanced $\mathrm{D}(K_{|B|}^{[c]}, \mathcal{G})$. For each block of a $(V, \mathcal{B})$, it is readily seen that if there exists a balanced $\mathrm{D}(K_{|B|}^{[c]}, \mathcal{G})$ for every $B \in \mathcal{B}$, then there exists a $\mathrm{D}(K_v^{[c]}, \mathcal{G})$. It is sufficient to show that the constructed $\mathrm{D}(K_v^{[c]}, \mathcal{G})$ is balanced.

For each $B \in \mathcal{B}$, a balanced $\mathrm{D}(K_{|B|}^{[c]}, \mathcal{G})$ has the replication number $k(|B|-1)/2m$ by the equation (4.6.1). For each $x \in V$, let $\mathcal{B}_x$ be the family of blocks B such that x belongs to B and r_x be the replication number of x in $\mathrm{D}(K_v^{[c]}, \mathcal{G})$. Then,

$$\sum_{B \in \mathcal{B}_x} (|B| - 1) = v - 1$$

and

$$r_x = \sum_{B \in \mathcal{B}_x} \frac{k(|B| - 1)}{2m} = \frac{k(v-1)}{2m}$$

hold. That is, the replication number r_x is a constant for each $x \in V$. That is, $\mathrm{D}(K_v^{[c]}, \mathcal{G})$ is balanced. Thus $v \in D(\mathcal{G})$, which proves the lemma. □

A $\mathcal{G}$-decomposition of $K_q^{[c]}$, which is obtained by Theorem 4.2.2, is always balanced. Thus the existence of an eventual period $\beta_0 \neq 0$ for $D(\mathcal{G})$ is shown, and β_0 is divisible by m. To complete the proof of Theorem 4.6.2, it is sufficient to show the following theorem.

Theorem 4.6.4 *Let $\mathcal{G}$ be a family of tree-ordered colorwise simple edge-c-colored graphs with k vertices and m edges for each of c colors. Let n be a positive integer satisfying the congruences* (4.6.3). *Then there exists an integer v_0 such that $v_0 \equiv n \pmod{\beta_0}$ and that $K_v^{[c]}$ admits a balanced $\mathcal{G}$-decomposition.*

In order to prove Theorem 4.6.4, we first show the following theorem.

Theorem 4.6.5 *Let $\mathcal{G}$ be a family of tree-ordered colorwise simple edge-c-colored graphs with k vertices and m edges for each of c colors. Let v be a positive integer satisfying the congruences* (4.6.3) *and $v \geq \max\{k+3, 7\}$. Then there exists a prime power $q \equiv 1 \pmod{\beta_0}$ such that $qK_v^{[c]}$ admits a balanced $\mathcal{G}$-decomposition.*

If we show Theorem 4.6.5, there exists a balanced $\mathrm{D}(qK_n^{[c]}, \mathcal{G})$ for all n satisfying the congruences (4.6.3). Thus, we can show that there exists a balanced $\mathcal{G}$-decomposition of $K_{nq^d}^{[c]}$ by the similar manner in the proof of Theorem 4.2.3, which shows Theorem 4.6.4.

In order to prove Theorem 4.6.5, it is sufficient to show the following lemma by the similar way in the proof of Theorem 4.2.4. Let $\mathcal{G}$ be a family of colorwise simple edge-c-colored graphs with k vertices and m edges for each c colors, and let $\mathcal{F}$ denote the set of all subgraphs F of $K_v^{[c]}$ each of which is isomorphic to some member of $\mathcal{G}$. Let M be the matrix whose rows are indexed by the edges and the vertices of $K_v^{[c]}$ and whose columns are indexed by the members in $\mathcal{F}$, where the entry in row e and column F of M is 1 if $e \in E(F)$ and 0 otherwise and the entry in row x and column F of M is 1 if $x \in V(F)$ and 0 otherwise. Let

$$\boldsymbol{c}^T = (\overbrace{1, 1, \ldots, 1}^{cv(v-1)/2}, \overbrace{r, r, \ldots, r}^{v})$$

be a vector of length $cv(v-1)/2 + v$ whose coordinates are indexed by the edges and the vertices of $K_v^{[c]}$, where $r = k(v-1)/2m$.

Lemma 4.6.6 *Let $\mathcal{G}$ be a family of tree-ordered colorwise simple edge-c-colored graphs with k vertices and m edges for each of c colors and let $\mathcal{F}$ denote the set of all subgraphs F of $K_v^{[c]}$ each of which is isomorphic to some member of $\mathcal{G}$. In addition, assume that $v \geq \max\{k+3, 7\}$ holds. The equation $M\boldsymbol{x} = \boldsymbol{c}$ has an integral solution $\{s_F : F \in \mathcal{F}\}$ if and only if v satisfies the congruences* (4.6.3).

Proof. The proof is similar to that of Lemma 4.4.2 but note that the "balanced" property must be considered. We define rationals $b(e)$ for $e \in E(K_v^{[c]})$ and $a(x)$ for $x \in V(K_v^{[c]})$ so that

$$s(F) = (b+a)(F) = \sum_{e \in E(F)} b(e) + \sum_{x \in V(F)} a(x)$$

is integral for each $F \in \mathcal{F}$. We use the same notations in the proof of Lemma 4.4.2.

For each color $i \in C$, let G_i be a graph having an edge of color set $\{i\}$. Let x, y, u and v be any four vertices of $K_v^{[c]}$ and let $F_{i,1}$ be an isomorphic copy of G in $K_v^{[c]}$ such that $F_{i,1}$ contains the edge $\{x, y\}$ in $K_v^{[c]}$ of color i and that $u, v \notin V(F_{i,1})$. Let $F_{i,2}$, $F_{i,3}$ and $F_{i,4}$ be the isomorphic graphs to $F_{i,1}$ obtained by applying the permutations (xu), (yv) and $(xu)(yv)$, respectively. Now since $s(F_{i,l})$ is integral for $l = 1, 2, 3, 4$, we have

$$s(F_{i,1}) + s(F_{i,4}) \equiv s(F_{i,2}) + s(F_{i,3}). \tag{4.6.4}$$

Each side of this congruence consists of sums of $b(e)$'s and $a(x)$'s. Since $F_{i,l}$ ($l = 1, 2, 3, 4$) have common edges and vertices, by deleting $b(e)$'s and $a(x)$'s corresponding to these edges and vertices from both side of the congruence (4.6.4), the congruence (4.6.4) is reduced to

$$b_i\{x, y\} + b_i\{u, v\} \equiv b_i\{x, v\} + b_i\{u, y\}. \tag{4.6.5}$$

By the same method in Lemma 4.4.2, there exist rationals $\gamma_i(x)$ for each $x \in V(K_v^{[c]})$ such that

$$b_i\{x, y\} \equiv \gamma_i(x) + \gamma_i(y) \tag{4.6.6}$$

holds.

Let z be a vertex of $G \in \mathcal{G}$. Given vertices x, y of $K_v^{[c]}$, choose isomorphic copy $F \in \mathcal{F}$ of G such that $x \in V(F)$, $y \notin V(F)$ and x corresponds to z under the isomorphism. Let F' be the image of F under the permutation (xy). We have $\tau_F(x) = \tau_{F'}(y) = \tau_G(z)$.

Of course, $s(F) \equiv s(F')$, as both have been assumed to be integers. After cancelling terms $b(e)$'s and $a(p)$'s that appear on both sides, we have

$$\begin{aligned} &\sum (b(e) : e \in E(F) \text{ incident with } x) + a(x) \\ &\quad \equiv \sum (b(e) : e \in E(F) \text{ incident with } y) + a(y) \end{aligned} \tag{4.6.7}$$

Let U_i denote the set of vertices u of F for which the edge $\{x, u\}$ in $K_v^{[c]}$ of color i is in F (or, equivalently, such that the edge $\{y, u\}$ of color i is in F'). Then we obtain

$$\sum_{i=1}^{c} \sum_{u \in U_i} b_i\{x, u\} + a(x) \equiv \sum_{i=1}^{c} \sum_{u \in U_i} b_i\{y, u\} + a(y) \tag{4.6.8}$$

from the congruence (4.6.7).

Choose and fix a vertex p of $K_v^{[c]}$ distinct from x and y. By the congruence (4.6.5), we have

$$b_i\{x, u\} - b_i\{x, p\} \equiv b_i\{y, u\} - b_i\{y, p\}$$

for $u \in U_i$. The point is that even if we replace $b_i\{x, u\}$ and $b_i\{y, u\}$ in the congruence (4.6.8) by $b_i\{x, p\}$ and $b_i\{y, p\}$, the congruence is preserved. Thus,

$$\sum_{i=1}^{c} |U_i| b_i\{x, p\} + a(x) \equiv \sum_{i=1}^{c} |U_i| b_i\{y, p\} + a(y)$$

holds modulo an integer. From the congruence (4.6.6), the expression of the congruence (4.6.7) is, modulo an integer,

$$\sum_{i=1}^{c} |U_i|(\gamma_i(x) + \gamma_i(p)) + a(x) \equiv \sum_{i=1}^{c} |U_i|(\gamma_i(y) + \gamma_i(p)) + a(y).$$

The congruence (4.6.7), after canceling terms involving p on both sides, reduces to

$$\sum_{i=1}^{c} \deg_i(z)\gamma_i(x) + a(x) \equiv \sum_{i=1}^{c} \deg_i(z)\gamma_i(y) + a(y), \tag{4.6.9}$$

where $\deg_i(z) = |U_i|$ since $\tau_F(x) = \tau_{F'}(y) = \tau_G(z)$ hold. The congruence (4.6.9) hold for all vertices x, y of $K_v^{[c]}$ and vertices z of any member of $\mathcal{G}$. Let $\boldsymbol{\gamma}(x)$ be the vector $(\gamma_1(x), \gamma_2(x), \ldots, \gamma_c(x))$. It can be written

$$\langle \tau_G(z), \boldsymbol{\gamma}(x) \rangle + a(x) \equiv \langle \tau_G(z), \boldsymbol{\gamma}(y) \rangle + a(y), \tag{4.6.10}$$

where the angle brackets denote the dot product of vectors. Fix the vertices x and y of $K_v^{[c]}$. By the assumption, there exist integers $u_G(z)$ such that the condition (C5) holds. That is, by applying equations (4.6.2) to the congruence (4.6.10), we have

$$\begin{aligned} &\sum_{G \in \mathcal{G}} \sum_{z \in V(G)} u_G(z)(\langle \tau_G(z), \boldsymbol{\gamma}(x) \rangle + a(x)) \\ &\quad \equiv \sum_{G \in \mathcal{G}} \sum_{z \in V(G)} u_G(z)(\langle \tau_G(z), \boldsymbol{\gamma}(y) \rangle + a(y)). \end{aligned}$$

Hence, we obtain

$$(v-1)\sum_{i=1}^{c}\gamma_i(x)+\frac{k(v-1)}{2m}a(x)\equiv(v-1)\sum_{i=1}^{c}\gamma_i(y)+\frac{k(v-1)}{2m}a(y). \quad (4.6.11)$$

This congruence holds for all vertices x, y in $K_v^{[c]}$.

By hypothesis, $s(F)$ is an integer, and we have

$$s(F)=\sum_{i=1}^{c}\Bigg(\sum_{\{x,\,y\}\in E(F)} b_i\{x,\,y\}\Bigg)+\sum_{x\in V(F)} a(x).$$

We apply the congruence (4.6.6) to the terms $b_i\{x,\,y\}$'s and use the congruence (4.6.9) for the second congruence, to find that

$$\begin{aligned}
s(F) &\equiv \sum_{u\in V(F)}\Bigg(\sum_{i=1}^{c}\deg_i(u)\gamma_i(u)+a(u)\Bigg)\\
&\equiv \sum_{u\in V(F)}\Bigg(\sum_{i=1}^{c}\deg_i(u)\gamma_i(p)+a(p)\Bigg)\\
&\equiv 2m\sum_{i=1}^{c}\gamma_i(p)+ka(p)\equiv 0. \qquad (4.6.12)
\end{aligned}$$

Finally, we will show that $s'(K_v^{[c]})=\sum_e b(e)+\frac{k(v-1)}{2m}\sum_x a(x)$ is an integer.

$$\begin{aligned}
s'(K_v^{[c]}) &= \sum_{i=1}^{c}\Bigg(\sum_{\{x,\,y\}} b_i\{x,\,y\}+\frac{k(v-1)}{2m}\sum_x a(x)\Bigg)\\
&\equiv \sum_{i=1}^{c}\Bigg(\sum_{\{x,\,y\}}(\gamma_i(x)+\gamma_i(y))+\frac{k(v-1)}{2m}\sum_x a(x)\Bigg)\\
&\equiv \sum_{i=1}^{c}\Bigg((v-1)\sum_x\gamma_i(x)+\frac{k(v-1)}{2m}\sum_x a(x)\Bigg)\\
&\equiv \sum_x\Bigg((v-1)\sum_{i=1}^{c}\gamma_i(x)+\frac{k(v-1)}{2m}a(x)\Bigg).
\end{aligned}$$

By the congruence (4.6.11),

$$\begin{aligned}
s'(K_v^{[c]}) &\equiv v(v-1)\sum_{i=1}^{c}\gamma_i(x)+\frac{kv(v-1)}{2m}a(x)\\
&\equiv \frac{v(v-1)}{2m}\Bigg(2m\sum_{i=1}^{c}\gamma_i(x)+ka(x)\Bigg).
\end{aligned}$$

By the assumption, $v(v-1) \equiv 0 \pmod{2m}$ and the congruence (4.6.12) hold. Thus, $s(K_v^{[c]})$ is an integer and the lemma is proved. $\square$

Thus, Theorems 4.6.5 and 4.6.2 are shown.

4.7 Generalization to decompositions of multiple edge graphs

In this section, we show Theorem 4.1.1. The proof of Theorem 4.1.1 is similar to that of Theorem 4.1.2. To show Theorem 4.1.1, we prepare three theorems which are generalized versions of Theorems 4.2.2, 4.2.4 and 4.2.3.

Theorem 4.7.1 *Let G_0 be a tree-ordered colorwise simple graph with c colors and $m\lambda_i$ edges of color i. Then there exists a constant $q_0 = q_0(m, k)$ such that $K_q^{\boldsymbol{\lambda}}$ admits a G_0-decomposition for every prime power $q \equiv 2m+1 \pmod{4m}$ with $q \geq q_0$, where k is the number of vertices of G_0.*

Proof. It is sufficient to show the following generalized condition of (C3):

(C3)' There is an injective mapping $\phi : V(G_0) \to \mathrm{GF}(q)$ such that for each color i, $m\lambda_i$ field elements $\pm(\phi(x)-\phi(y))$ belong to the cyclotomic class H_l^m λ_i times for each $l = 0, 1, \ldots, 2m-1$ when $e = \{x, y\}$ ranges over the edges of color i in G_0.

Let $\boldsymbol{C}$ be a multiset of color set C which contains color i λ_i times. Let $\mathcal{E}\langle G_0\rangle$ be a family of all edge sets in G_0. A subfamily $\mathcal{E}_1 \subset \mathcal{E}\langle G_0\rangle$ is called a $\boldsymbol{\lambda}$-*resolution class* if (i) $\cup_{\langle x,y\rangle \in \mathcal{E}_1} C(\langle x, y\rangle) = \boldsymbol{C}$ and (ii) $C(\langle x, y\rangle) \subset C(\langle x', y'\rangle)$, $C(\langle x, y\rangle) \supset C(\langle x', y'\rangle)$, or $C(\langle x, y\rangle) \cap C(\langle x', y'\rangle) = \emptyset$ for any distinct edge sets $\langle x, y\rangle$ and $\langle x', y'\rangle$. $\mathcal{E}\langle G_0\rangle$ is said to be $\boldsymbol{\lambda}$-*resolvable* if $\mathcal{E}\langle x, y\rangle$ is partitioned into $\boldsymbol{\lambda}$-resolution classes.

Assume that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$. We can choose a class $\mathcal{E}_1$ from $\mathcal{E}\langle G_0\rangle$ in a similar manner to Theorem 4.2.2 such that $\mathcal{E}_1$ has λ_1 edges of each color i because of the assumption of tree-ordered property. We define G_0' as the graph having the same vertex set with G_0 and edge sets $\mathcal{E}\langle G_0\rangle \setminus \mathcal{E}_1$. G_0' has $m\lambda_i - \lambda_1$ edges of each color i, $i = 1, 2, \ldots, c$. For color i, $i = 2, 3, \ldots, c$, $\lambda_i - \lambda_1$ edges are not included in any edge set with color 1. Then, a class $\mathcal{E}_1'$ can be chosen from $\mathcal{E}\langle G_0'\rangle$ such that $\mathcal{E}_1'$ has $\lambda_2 - \lambda_1$ edges of each color i, $i = 2, 3, \ldots, c$ since $\mathcal{C}(G_0')$ is tree-ordered. We add the class $\mathcal{E}_1'$ to $\mathcal{E}_1$. We continue this for each $i = 1, 2, \ldots, c$. Then, we can get a $\boldsymbol{\lambda}$-resolution class $\mathcal{E}_1$ from $\mathcal{E}\langle G_0\rangle$.

Similarly, we define G_1 as the graph with the edge set $\mathcal{E}\langle G_0\rangle \backslash \mathcal{E}_1$. Then, we can choose a $\boldsymbol{\lambda}$-resolution class $\mathcal{E}_2$ from $\mathcal{E}\langle G_1\rangle$. We continue this process for each $i = 1, 2, \ldots, m$, then $\mathcal{E}\langle G_0\rangle$ has m $\boldsymbol{\lambda}$-resolution classes $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\}$. Each $\mathcal{E}_l$ has exactly λ_i edges of color i for each $i \in C$. Thus, we can show that there exists such injective mapping ϕ by Proposition 1.8.1. $\square$

Theorem 4.7.2 *Let $\mathcal{G}$ be a tree-ordered $\boldsymbol{\lambda}$-admissible family of colorwise simple graphs with c colors. Let v be a positive integer satisfying the congruences* (1.6.2) *and $v \geq 2 + |V(G)|$ for all G in $\mathcal{G}$. Then there exists a prime power $q \equiv 1 \pmod{\beta_0}$ such that $qK_v^{\boldsymbol{\lambda}}$ admits a $\mathcal{G}$-decomposition.*

The proof of Theorem 4.7.2 is similar to that of Theorem 4.2.4.

Theorem 4.7.3 *Let $\mathcal{G}$ be a tree-ordered $\boldsymbol{\lambda}$-admissible family of colorwise simple graphs with c colors. Let n be a positive integer satisfying the congruences* (1.6.2)*. Then there exists an integer v_0 such that $v_0 \equiv n \pmod{\beta_0}$ and that $K_{v_0}^{\boldsymbol{\lambda}}$ admits a $\mathcal{G}$-decomposition.*

Proof. By Theorem 4.7.2, there exists a family $\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$ which is a $\mathcal{G}$-decomposition of $qK_n^{\boldsymbol{\lambda}}$. For any vertex x and y in $qK_n^{\boldsymbol{\lambda}}$, there are $\lambda_i q$ F_h's including an edge $\{x, y\}$ of color i. Now, we consider an assignment $\rho_h(x, y)$ of GF(q) to each edge set $\langle x, y\rangle$ in F_h so that the following condition holds:

(C4)' For each $l \in \mathrm{GF}(q)$ and x, y in $K_n^{\boldsymbol{\lambda}}$, there are exactly λ_i edges $\{x, y\}$ of color i in $\mathcal{F}$ to which the same value $l = \rho_h(x, y)$ is assigned.

It suffices to show that there exists an assignment $\{\rho_h(x, y)\}$ satisfying the condition (C4)'. Let $\mathcal{E}\langle x, y\rangle$ be a collection of the edge sets $\langle x, y\rangle$ that appear in $F_1, F_2, \ldots, F_N$. Assume that $\mathcal{G}$ is tree-ordered, then $\mathcal{E}\langle x, y\rangle$ is partitioned into $\boldsymbol{\lambda}$-resolution classes $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_q$ in the similar manner to the proof of Theorems 4.2.3 and 4.7.1. For an edge set $\langle x, y\rangle \in F_h$, if $\langle x, y\rangle$ belongs to $\mathcal{E}_l$, we define $\rho_h(x, y) = l$, where $x < y$ and $l \in \mathrm{GF}(q)$. Thus, the theorem is proved. $\square$

By utilizing Theorems 4.7.1, 4.7.2 and 4.7.3, we can show Theorem 4.1.1 similarly to the proof of Theorem 4.1.2 in Sections 4.2, 4.3, 4.4 and 4.5.

Now, we will give a "balanced" version of Theorem 4.1.1. For $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_c)$, let $\mathcal{G}$ be a family of colorwise simple edge-c-colored graphs with k vertices and $m\lambda_i$ edges of each color i. Then $\mu(G) = m\boldsymbol{\lambda}$ holds for each $G \in \mathcal{G}$. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_b\}$ be a balanced $\mathcal{G}$-decomposition of $K_v^{\boldsymbol{\lambda}}$ with the replication number r. Then,

$$vr = bk \text{ and } b = \frac{v(v-1)}{2m}$$

hold, hence we obtain

$$r = \frac{k(v-1)}{2m}. \tag{4.7.1}$$

Moreover, we need the following condition:

(C5)' There exist integers $u_G(x)$ for $x \in V(G)$ and $G \in \mathcal{G}$ such that

$$\sum_{G\in\mathcal{G}} \sum_{x\in V(G)} u_G(x)\tau_G(x) = (v-1)\boldsymbol{\lambda} \text{ and}$$

$$\sum_{G\in\mathcal{G}} \sum_{x\in V(G)} u_G(x) = \frac{k(v-1)}{2m}$$

hold, where $\tau_G(x)$ is the degree vector of vertex x in G.

We define T as the subset of integers in $\mathbb{Z}_{2m}$ satisfying the formulas (1.6.2) and (4.7.1) and the condition (C5)'. Then, we obtain the following lemma.

Lemma 4.7.4 *For $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_c)$, let $\mathcal{G}$ be a family of colorwise simple graphs with k vertices and $m\lambda_i$ edges of each color i. Then necessary conditions for the existence of balanced $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ are*

$$v \equiv t \pmod{2m} \quad \textit{for each } t \in T. \tag{4.7.2}$$

Then, the following theorem is obtained.

Theorem 4.7.5 *For $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_c)$, let $\mathcal{G}$ be a family of tree-ordered colorwise simple edge-c-colored graphs with k vertices and $m\lambda_i$ edges of each color i. Then there exists a constant $v_0 = v_0(\mathcal{G}, \boldsymbol{\lambda})$ such that balanced $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ exist for all integers $v \geq v_0$ satisfying the congruence* (4.7.2).

A proof of Theorem 4.7.5 is similar to that of Theorems 4.6.2 and 4.1.1. Finally, we obtain the following corollary.

Corollary 4.7.6 *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_c)$ be a vector whose entries λ_i are positive integers such that the greatest common divisor of λ_i's is* 1. *And let $\mathcal{G}$ be a family of tree-ordered colorwise simple edge-c-colored graphs with k vertices and $m\lambda_i$ edges of each color i such that the congruences* (1.6.2) *are equivalent to the congruence* (4.7.2). *For $\lambda \geq 1$, there exists a constant $v_0 = v_0(\mathcal{G}, \lambda\boldsymbol{\lambda})$ such that balanced $\mathcal{G}$-decompositions of $K_v^{\lambda\boldsymbol{\lambda}}$ exist for all integers $v \geq v_0$ satisfying the congruences*

$$\begin{aligned} \lambda(v-1) &\equiv 0 \pmod{\alpha(\mathcal{G}; \boldsymbol{\lambda})} \text{ and} \\ \lambda v(v-1) &\equiv 0 \pmod{2m}. \end{aligned} \tag{4.7.3}$$

Proof. By the assumption, $\beta(\mathcal{G};\, \boldsymbol{\lambda}) = m$ holds. the congruence $\lambda v(v-1) \equiv 0 \pmod{2m}$ means that $\lambda v(v-1)\boldsymbol{\lambda}/2$ is an integral linear combination of vectors $\mu(G)$, $G \in \mathcal{G}$. While, the congruence $v(v-1) \equiv 0 \pmod{2\beta(\mathcal{G};\, \lambda\boldsymbol{\lambda})}$ means that $v(v-1)\lambda\boldsymbol{\lambda}/2$ is an integral linear combination of $\mu(G)$, $G \in \mathcal{G}$. These are obviously equivalent. Similarly, $\lambda(v-1) \equiv 0 \pmod{\alpha(\mathcal{G};\, \boldsymbol{\lambda})}$ is equivalent to $v - 1 \equiv 0 \pmod{\alpha(\mathcal{G};\, \lambda\boldsymbol{\lambda})}$. Hence by Theorem 4.7.5, the corollary is proved. $\square$

Chapter 5

Asymptotic existence of BIB designs with nested rows and columns

In this chapter, the asymptotic existence of BIBRCs with some λ's is discussed. Theorem 4.7.5 is applied to show the asymptotic existence of BIBRCs. In Section 5.1, a relationship between BIBRCs and some balanced edge-colored graph decompositions of complete graphs is discussed. We consider the balanced edge-colored graph decompositions of complete graphs instead of BIBRCs. In Section 5.2, the asymptotic existence of completely balanced BIBRCs is shown, which is derived from the result of Lamken and Wilson [63]. In Section 5.3, 5.4, it is also shown that BIBRCs with some λ's exist for sufficiently large v by utilizing Theorem 4.7.5. These results can not be obtained by the result of Lamken and Wilson [63]. In Section 5.5, the asymptotic existence of BIBRCs in the case of $\lambda \geq k_1k_2(k_1-1)(k_2-1)$ is shown by combining the results in Sections 5.3, 5.4.

5.1 A relationship between BIBRCs and edge-colored graph decompositions

In this section, we define edge-colored graphs such that balanced decompositions by those graphs are equivalent to BIBRCs.

Example 5.1.1 Let $V = \mathbb{Z}_{13}$ and $\mathcal{A} = \{A_i + x : i = 1, 2, 3, x \in \mathbb{Z}_{13}\}$ be a family of 2×3 arrays, where

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 10 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 3 & 6 \\ 9 & 12 & 4 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 9 & 5 \\ 1 & 10 & 12 \end{pmatrix}.$$

Then, the pair $(V, \mathcal{A})$ is a BIBRC(13, 2, 3, 3). Half of unordered pairs $\{x, y\}$ of V have

$$(\lambda_R\{x, y\}, \lambda_C\{x, y\}, \lambda_E\{x, y\}) = (3, 1, 2) \tag{5.1.1}$$

and the rest of the pairs have

$$(\lambda_R\{x, y\}, \lambda_C\{x, y\}, \lambda_E\{x, y\}) = (3, 2, 4), \tag{5.1.2}$$

both give a constant $\lambda = 3$.

Let G be an edge-3-colored graph shown in Figure 5.1.1, where the solid edges represent color 1, the dashed edges color 2 and the dotted edges color 3. It is not a colorwise simple graph with 3 colors. Let $\mathcal{F} = \{G_i + x : i =$

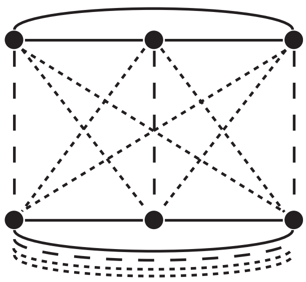


Figure 5.1.1: G

$1, 2, 3, x \in \mathbb{Z}_{13}\}$ be a family of subgraphs of $K_{13}^{(3,2,4)}$, where G_1, G_2 and G_3 are shown in Figure 5.1.2.

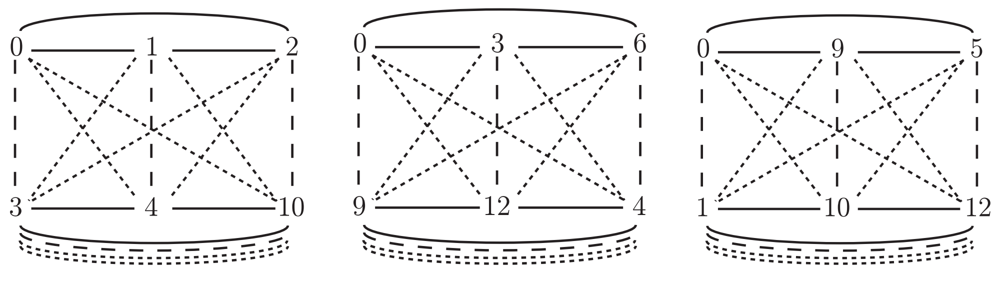


Figure 5.1.2: G_1, G_2 and G_3

Then, we claim that a BIBRC(13, 2, 3, 3) is equivalent to a balanced G-decomposition of $K_{13}^{(3,2,4)}$

Assume that there exists a balanced G-decomposition of $K_{13}^{(3,2,4)}$. For unordered pair $\{x, y\}$ of distinct vertices of $K_{13}^{(3,2,4)}$, when x and y occur together exactly once in an edge set with the color multiset (1, 2, 3, 3), they do not occur in any other edges of the color sets (1), (2) and (3). On the other hand, when x and y do not occur in an edge set with the color multiset (1, 2, 3, 3), they occur exactly once in an edge of the color sets (1), (2) and twice in edges of the color set (3). We identify the vertex set of $K_{13}^{(3,\,2,\,4)}$ with V and subgraphs with 2×3 arrays. And let $\mathcal{A}$ be a family of such arrays.

Then, for any two distinct pair $\{x,\, y\}$ of V, the equation (5.1.1) holds if two vertices corresponding to x and y occur in an edge set of a graph in G with the color multiset (1, 2, 3, 3). The equation (5.1.2) holds if two vertices corresponding to x and y do not occur in an edge set with the same color multiset. Thus, $\lambda_R\{x,\, y\} + 2\lambda_C\{x,\, y\} - \lambda_E\{x,\, y\} = 3$ holds for any distinct pair x and y in V. That is, $(V,\, \mathcal{A})$ is a BIBRC(13, 2, 3, 3).

Conversely, assume that there exists a BIBRC(13, 2, 3, 3) $(V,\, \mathcal{A})$. For any two distinct pair $\{x,\, y\}$ in V, either the equation (5.1.1) or (5.1.2) holds. We identify V with a vertex set of $K_{13}^{(3,\,2,\,4)}$. For any $A = (a_{ij})$ in $\mathcal{A}$, let $\boldsymbol{\lambda}\{a_{ij},\, a_{i'j'}\} = (\lambda_R\{a_{ij},\, a_{i'j'}\},\, \lambda_C\{a_{ij},\, a_{i'j'}\},\, \lambda_E\{a_{ij},\, a_{i'j'}\})$ and we define a subgraph with vertices a_{ij}'s as follows:

(i) For a_{ij} and $a_{ij'}$, $j \neq j'$, if $\boldsymbol{\lambda}\{a_{ij},\, a_{ij'}\}$ is of type (5.1.1), then we put four edges between a_{ij} and $a_{ij'}$ and colored by color multiset (1, 2, 3, 3), otherwise, we put an edge of color 1.

(ii) For a_{ij} and $a_{i'j}$, $i \neq i'$, we put an edge of color 2.

(iii) For a_{ij} and $a_{i'j'}$, $i \neq i'$ and $j \neq j'$, we put an edge of color 3.

By permuting rows and columns in 2×3 arrays, all such subgraphs are equivalent to G. Then, it is easy to show that the family of subgraphs is a balanced G-decomposition of $K_{13}^{(3,\,2,\,4)}$.

Thus, to show that there exist BIBRC(v, 2, 3, 3)'s for sufficiently large integers $v \equiv 1 \pmod 2$, it is sufficient to show that there exist balanced G-decompositions of $K_v^{(3,\,2,\,4)}$ for sufficiently large integers $v \equiv 1 \pmod 2$. Unfortunately, since G is not a colorwise simple graph with 3 colors. We can not apply Theorem 4.7.5. However, by replacing some of the edge of color 3 by another color 4 which is represented by the dashed-dotted edges and define G' as a colorwise simple graph with 4 colors shown in Figure 5.1.3 instead of G. If there is a balanced G'-decomposition of $K_v^{(3,\,2,\,2,\,2)}$, then it can be considered as a G-decomposition of $K_v^{(3,\,2,\,4)}$. It is easy to check that $\alpha(G;\, \boldsymbol{\lambda}) = 2$, $\beta(G;\, \boldsymbol{\lambda}) = 2$, G' is tree-ordered and the condition (C5) holds for $v \equiv 1 \pmod 2$, where $\boldsymbol{\lambda} = (3,\, 2,\, 2,\, 2)$. By Theorem 4.7.5, there exist G-decompositions of $K_v^{(3,\,2,\,2,\,2)}$ for sufficiently large integers $v \equiv 1 \pmod 2$. That is, BIBRC(v, 2, 3, 3)'s exist for all sufficiently large $v \equiv 1 \pmod 2$.

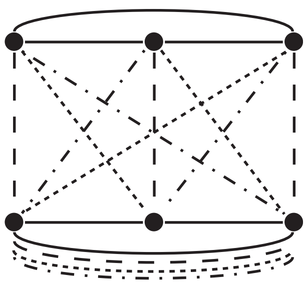


Figure 5.1.3: G'

In the sequel of this chapter, assume that $2 \leq k_1 \leq k_2$. For a positive integer λ, fix positive integers λ_R, λ_C and λ_E such that $\lambda_R \geq \lceil \frac{\lambda}{k_1-1} \rceil$, $\lambda_C \geq \lceil \frac{\lambda}{k_2-1} \rceil$ and $\lambda = (k_1-1)\lambda_R + (k_2-1)\lambda_C - \lambda_E$ hold. We use three colors $\{R, C, E\}$. Let $G(0, 0)$ be a simple edge-3-colored graph with k_1k_2 vertices $V = \{v_{ij} \,|\, 1 \leq i \leq k_1, 1 \leq j \leq k_2\}$, which is colored as follows and is shown in Figure 5.1.4, where the solid edges, the dashed edges and the dotted edges represent color R, C and E, respectively:

(i) Each edge $\{v_{ij}, v_{ij'}\}$ is colored by R for $1 \leq i \leq k_1$ and $1 \leq j < j' \leq k_2$.

(ii) Each edge $\{v_{ij}, v_{i'j}\}$ is colored by C for $1 \leq i < i' \leq k_1$ and $1 \leq j \leq k_2$.

(iii) Each edge $\{v_{ij}, v_{i'j'}\}$ is colored by E for $1 \leq i < i' \leq k_1$ and $1 \leq j \neq j' \leq k_2$.

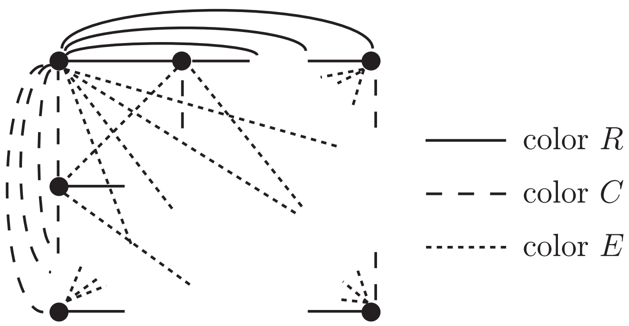


Figure 5.1.4: $G(0, 0)$

For given integers $0 \leq a_R \leq k_1k_2(k_2-1)/2$ and $0 \leq a_C \leq k_1k_2(k_1-1)/2$, let $G(a_R, a_C)$ be an edge-3-colored graph such that (i) an edge of color C and $k_2 - 1$ edges of color E are added to each of a_R edges of color R in $G(0, 0)$

and (ii) an edge of color C and k_1-1 edges of color E are added to a_C edges of color C in $G(0, 0)$. That is, a_R edges of color R in $G(0, 0)$ are replaced by a_R edge sets with color multiset $(R, C, E, \ldots, E)$ where the number of E's is k_2-1 and a_C edges of color C in $G(0, 0)$ are replaced by a_C edge sets with color multiset $(R, C, E, \ldots, E)$ where the number of E's is k_1-1. And let $\mathcal{G}(a_R, a_C)$ be the family of all $G(a_R, a_C)$'s.

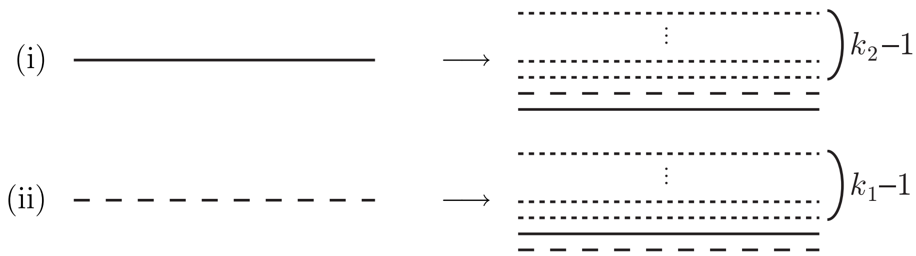


Figure 5.1.5:

By identifying $G \in \mathcal{G}(a_R, a_C)$ with $k_1 \times k_2$ array, a balanced $\mathcal{G}$-decomposition $\mathcal{F}$ of $K_v^{(\lambda_R, \lambda_C, \lambda_E)}$ is equivalent to a BIBRC(v, k_1, k_2, λ). In fact, for any two distinct vertices x and y of a balanced $\mathcal{G}$-decomposition of $K_v^{(\lambda_R, \lambda_C, \lambda_E)}$, if they occur in s_R $(\leq \lambda_R)$ edge sets of color multiset $(R, C, E, \ldots, E)$ with k_2-1 E's and in s_C $(\leq \lambda_C)$ edge sets of color multiset $(R, C, E, \ldots, E)$ with k_1-1 E's, then they occur in $\lambda_R - s_R$ edges of color set (R), in $\lambda_C - s_C$ edges of color set (C) and in $\lambda_E - (k_2-1)s_R - (k_1-1)s_C$ edges of color set (E), where $\lambda_E = (k_1-1)\lambda_R + (k_2-1)\lambda_C - \lambda$. By identifying the vertices of each graph with entries of $k_1 \times k_2$ array, $\lambda_R\{x, y\} = \lambda_R - s_C$, $\lambda_C\{x, y\} = \lambda_C - s_R$ and $\lambda_E\{x, y\} = \lambda_E - (k_2-1)s_R - (k_1-1)s_C$ hold. That is,

$$(k_1-1)\lambda_R\{x, y\} + (k_2-1)\lambda_C\{x, y\} - \lambda_E\{x, y\} = \lambda$$

holds. Hence, hereafter we consider a balanced $\mathcal{G}$-decomposition of $K_v^{(\lambda_R, \lambda_C, \lambda_E)}$ instead of a BIBRC(v, k_1, k_2, λ). In the following sections we will show asymptotic existence of BIBRCs for four cases of λ.

5.2 The case of completely balanced

In this section we consider the case when λ is a multiple of $\mathrm{lcm}(k_1-1, k_2-1)$. In this case we have only to consider a completely balanced BIBRCs to show the asymptotic existence since the equations (1.5.1) are satisfied.

Theorem 5.2.1 *For positive integers $k_1 \le k_2$, let λ be a multiple of* $\mathrm{lcm}(k_1-1,\, k_2-1)$*. Then there exists a constant $v_0 = v_0(k_1,\, k_2,\, \lambda)$ such that completely balanced* $\mathrm{BIBRC}(v,\, k_1,\, k_2,\, \lambda)$*'s exist for all $v \ge v_0$ satisfying the congruences* (1.5.2).

Proof. By Corollary 4.7.6 ([63, Corollary 13.3]), it is sufficient to show an asymptotic existence of BIBRCs with $\lambda_0 = \mathrm{lcm}(k_1-1,\, k_2-1)$. Let $\lambda_R = \lambda_0/(k_1-1)$, $\lambda_C = \lambda_0/(k_2-1)$, $\lambda_E = \lambda_0$ and $\boldsymbol{\lambda} = (\lambda_R,\, \lambda_C,\, \lambda_E)$, and let $G = G(0,\, 0)$.

Then, G is a simple edge-3-colored graph. It is obvious that G is $\boldsymbol{\lambda}$-admissible, that is,

$$\tau_G(x) = (k_2-1,\, k_1-1,\, (k_1-1)(k_2-1)) = \frac{(k_1-1)(k_2-1)}{\lambda_0}\boldsymbol{\lambda}$$

hold for any $x \in V(G)$. Hence, G-decompositions of $K_v^{\boldsymbol{\lambda}}$ are obviously balanced and $\alpha(G;\, \boldsymbol{\lambda}) = (k_1-1)(k_2-1)/\lambda_0$. And

$$\begin{aligned}\mu(G) &= (k_1k_2(k_2-1),\, k_1k_2(k_1-1),\, k_1k_2(k_1-1)(k_2-1)) \\ &= \frac{k_1k_2(k_1-1)(k_2-1)}{\lambda_0}\boldsymbol{\lambda}\end{aligned}$$

hold. Thus, $\beta(G;\, \boldsymbol{\lambda}) = k_1k_2(k_1-1)(k_2-1)/\lambda_0$ holds. That is, the necessary conditions (1.5.2) are equivalent to the congruences $v-1 \equiv 0 \pmod{\alpha(G;\, \boldsymbol{\lambda})}$ and $v(v-1) \equiv 0 \pmod{2\beta(G;\, \boldsymbol{\lambda})}$. Hence by Corollary 4.7.6 ([63, Corollary 13.3]), there exists a G-decomposition for sufficiently large v satisfying the necessary conditions. Thus, the theorem is proved. □

Moreover, if k_1 equals to k_2 and they are odds, then we obtain the following corollary by identifying the colors R and C as the same color.

Corollary 5.2.2 *For an odd integer k, let λ be a multiple of* $(k-1)/2$*. Then there exists a constant $v_0 = v_0(k,\, \lambda)$ such that* $\mathrm{BIBRC}(v,\, k,\, k,\, \lambda)$*'s exist for all $v \ge v_0$ satisfying the congruences* (1.5.2).

5.3 The case when λ is a multiple of $k_1 - 1$ or $k_2 - 1$

Theorem 5.3.1 *For positive integers $k_1 \le k_2$, let λ be a multiple of $k_1 - 1$ or $k_2 - 1$. Then there exists a constant $v_0 = v_0(k_1,\, k_2,\, \lambda)$ such that* $\mathrm{BIBRC}(v,\, k_1,\, k_2,\, \lambda)$*'s exist for all $v \ge v_0$ satisfying the congruences* (1.5.2).

Proof. In the case when $k_1 = k_2$ holds, we obtain the theorem by Theorem 5.2.1. Thus, we assume that $k_1 < k_2$ and consider the case when λ is a multiple of $k_1 - 1$. Note that the proof for the case when λ is a multiple of $k_2 - 1$ is similar to the present case. Now, let $\lambda_0 = k_1 - 1$.

We use $k_2 + 1$ colors $\{R, C, E_1, \ldots, E_{k_2-1}\}$. We define integers $\lambda_R = \lambda_0/(k_1 - 1) = 1$, $\lambda_C = \lceil \lambda_0/(k_2 - 1) \rceil$, $\lambda_E = (k_1 - 1)\lambda_R + (k_2 - 1)\lambda_C - \lambda_0 = (k_2 - 1)\lambda_C$ and a vector $\boldsymbol{\lambda} = (1, \lambda_C, \lambda_C, \ldots, \lambda_C)$ of length $k_2 + 1$. (Note that λ_C is 1 in the case of $k_1 < k_2$ but in case of $k_1 > k_2$, $\lambda_C > 1$.) Let $\mathcal{G} = \mathcal{G}'(m_0, 0)$ be the family of all edge-(k_2+1)-colored graphs $G'(m_0, 0)$ which is defined as follows, where $m_0 = m(\lambda_E - \lambda_0)/(k_2 - 1)$ and $m = k_1k_2(k_2 - 1)/2$. Note that $0 < m_0 < k_1k_2(k_2 - 1)/2$ holds.

(i) The edges of color E of $G(m_0, 0)$ are replaced by λ_C colors E_i for $i = 1, 2, \ldots, k_2 - 1$ such that each color does not occur twice in each edge of the color multiset $(R, C, E, \ldots, E)$.

Then, a family of color sets in $\mathcal{G}$ is

$$\mathcal{C}(\mathcal{G}) = \{\{R\}, \{C\}, \{E_1\}, \ldots, \{E_{k_2-1}\}, \{R, C, E_1, \ldots, E_{k_2-1}\}\}.$$

And

$$\mu(\mathcal{G}) = (m, m\lambda_C, m\lambda_C, \ldots, m\lambda_C) = \frac{k_1k_2(k_2 - 1)}{2} \cdot \boldsymbol{\lambda} \tag{5.3.1}$$

holds since the number of edges with color C is $k_1k_2(k_1 - 1)/2 + m_0 = m\lambda_C$. That is, G is tree-ordered and $\boldsymbol{\lambda}$-admissible.

Next, we claim that $(v - 1) \equiv 0 \pmod{(k_2 - 1)}$ and $v(v - 1) \equiv 0 \pmod{k_1k_2(k_2 - 1)}$ together imply $v - 1 \equiv 0 \pmod{\alpha(\mathcal{G}; \boldsymbol{\lambda})}$, $v(v - 1) \equiv 0 \pmod{2\beta(\mathcal{G}; \boldsymbol{\lambda})}$ and the condition (C5). The second congruence can be derived from the equation (5.3.1). To show the first congruence, we have only to show that $(v-1)\cdot\boldsymbol{\lambda}$ is an integral linear combination of the vector $\tau_G(x)$ for $x \in V(G)$ in $G \in \mathcal{G}$. Since $\mathcal{G}$ is the family of all $G'(m_0, 0)$'s, there exist G_1, $G_2 \in \mathcal{G}$ and vertices $x_1 \in V(G_1)$, $x_2 \in V(G_2)$ such that the degree vectors are

$$\begin{aligned}
&\tau_{G_1}(x_1) = (k_2 - 1, k_1 - 1, k_1 - 1, \ldots, k_1 - 1) \text{ and} \\
&\tau_{G_2}(x_2) = (k_2 - 1, k_1, k_1, \ldots, k_1),
\end{aligned}$$

respectively. Since the following equation

$$\boldsymbol{\lambda} = \Big(\frac{k_1 + k}{k_2 - 1} - \lambda_C\Big)\tau_{G_1}(x_1) + \Big(\lambda_C - \frac{k_1 + k - 1}{k_2 - 1}\Big)\tau_{G_2}(x_2)$$

holds, we have $v-1 \equiv 0 \pmod{\alpha(\mathcal{G};\, \boldsymbol{\lambda})}$. Also, this implies that the condition (C5) is satisfied since

$$\Big(\frac{k_1+k}{k_2-1} - \lambda_C\Big) + \Big(\lambda_C - \frac{k_1+k-1}{k_2-1}\Big) = \frac{1}{k_2-1} = \frac{k_1k_2}{2m}$$

holds. By Theorem 4.7.5, there exists a balanced $\mathcal{G}$-decomposition of $K_v^{\boldsymbol{\lambda}}$ for sufficiently large v satisfying the necessary conditions. By Corollary 4.7.6, it is shown that there exist BIBRCs for sufficiently large v in the case when λ is a multiple of k_1-1 or k_2-1. Thus, the theorem is shown. $\square$

5.4 The case when λ is a multiple of k_2 and $k_1 \le k_2$

Theorem 5.4.1 *For positive integers $k_1 \le k_2$, let λ be a multiple of k_2. Then there exists a constant $v_0 = v_0(k_1,\, k_2,\, \lambda)$ such that* BIBRC$(v,\, k_1,\, k_2,\, \lambda)$*'s exist for all $v \ge v_0$ satisfying the congruences* (1.5.2)*.*

Proof. When k_2 is a multiple of k_1-1, there exists BIBRC$(v,\, k_1,\, k_2,\, \lambda)$'s for sufficiently large integers satisfying the necessary conditions by Theorem 5.3.1. Assume that k_2 is not a multiple of k_1-1 and that k_1 is greater than 2. Let $\lambda_0 = k_2$ and let $\lambda_R = \lceil k_2/(k_1-1)\rceil \ge 2$, $\lambda_C = \lceil k_2/(k_2-1)\rceil = 2$, $\lambda_E = (k_1-1)\lambda_R + (k_2-1)\lambda_C - \lambda_0 = (k_1-1)\lambda_R + k_2 - 2$, $\lambda'_R = \lambda_R - 1$ and $\lambda'_E = \lambda_E - (k_1+k_2-2)$. We use k_1+k_2+3 colors $\{R',\, R_1,\, C',\, C_1,\, E',\, E_1,\, E_2,\, \dots,\, E_{k_1+k_2-2}\}$ and define a vector of length k_1+k_2+3 as

$$\boldsymbol{\lambda} = (\lambda'_R,\, 1,\, 1,\, 1,\, \lambda'_E,\, \overbrace{1,\, 1,\, \dots,\, 1}^{k_1-1},\, \overbrace{1,\, 1,\, \dots,\, 1}^{k_2-1}).$$

Let

$$\varepsilon_R = \frac{k_1k_2(k_2-1)}{2}, \quad \varepsilon_C = \frac{k_1k_2(k_1-1)}{2},$$
$$m = \frac{k_1k_2(k_1-1)(k_2-1)}{2\lambda_0} = \frac{k_1(k_1-1)(k_2-1)}{2}.$$

Let $\mathcal{G}'(0,\, 0)$ be the family of all edge-(k_1+k_2+3)-colored graphs which are obtained by the following replacement of colors:

(i) The ε_R edges of color R of $G(0,\, 0)$ are replaced by $m\lambda'_R$ edges of color R' and $\varepsilon_R - m\lambda'_R$ edges of color R_1.

(ii) The ε_C edges of color C are replaced by m edges of color C' and $\varepsilon_C - m$ edges of color C_1

(iii) The $m\lambda_0$ edges of color E are replaced by $m\lambda'_E$ edges of color E', $\varepsilon_R - m\lambda'_R$ edges of colors $E_1, E_2, \ldots, E_{k_1-1}$, and $\varepsilon_C - m$ edges of colors $E_{k_1}, E_{k_1+1}, \ldots, E_{k_1+k_2-2}$.

For nonnegative integers a_R and a_C, $0 \le a_R \le \varepsilon_R$ and $0 \le a_C \le \varepsilon_C$, let $\mathcal{G}'(a_R, a_C)$ be the family of all edge-(k_1+k_2+3)-colored graphs such that a_R edge sets with the color set $\{C_1, E_{k_1}, \ldots, E_{k_1+k_2-2}\}$ are added to a_R edges of color R' in G, respectively and such that a_C edge sets with the color set $\{R_1, E_1, \ldots, E_{k_1-1}\}$ are added to a_C edges of color C' in G, respectively, where G belongs to $\mathcal{G}'(0, 0)$. That is, for $G \in \mathcal{G}'(0, 0)$, a_R edges of color R in G are replaced with a_R edge sets of color set $\{R', C_1, E_{k_1}, \ldots, E_{k_1+k_2-1}\}$ and a_C edges of color C' in G are replaced with a_C edge sets of color set $\{R_1, C', E_1, \ldots, E_{k_1-1}\}$.

Since $3 \le k_1 \le k_2$ holds, we have $0 < m\lambda_R - \varepsilon_R < m$ and $0 < 2m - \varepsilon_C < m\lambda'_R$. Moreover,

$$m - (k_1 - 1) > \varepsilon_R - m\lambda_R \text{ and } m\lambda'_R - (k_2 - 1) > 2m - \varepsilon_C\lambda_R \tag{5.4.1}$$

hold. Let $\mathcal{G} = \mathcal{G}'(2m - \varepsilon_C, m\lambda_R - \varepsilon_R)$. Then, a family of color set $\mathcal{G}$ is

$$\begin{aligned}\mathcal{C}(\mathcal{G}) = \{&\{R'\}, \{R_1\}, \{C'\}, \{C_1\}, \{E'\}, \{E_1\}, \ldots, \{E_{k_2-1}\},\\ &\{R_1, C', E_1, \ldots, E_{k_1-1}\}, \{R', C_1, E_{k_1}, \ldots, E_{k_1+k_2-2}\}\}.\end{aligned}$$

And

$$\begin{aligned}\mu(\mathcal{G}) &= (m\lambda'_R, m, m, m, m\lambda'_E, \overbrace{m, m, \ldots, m}^{k_1-1}, \overbrace{m, m, \ldots, m}^{k_2-1})\\ &= \frac{k_1(k_1-1)(k_2-1)}{2} \cdot \boldsymbol{\lambda}\end{aligned} \tag{5.4.2}$$

holds. That is, $\mathcal{G}$ is tree-ordered and $\boldsymbol{\lambda}$-admissible.

Next, we claim that $k_2(v-1) \equiv 0 \pmod{(k_1-1)(k_2-1)}$ and $v(v-1) \equiv 0 \pmod{k_1(k_1-1)(k_2-1)}$ together imply $v-1 \equiv 0 \pmod{\alpha(\mathcal{G}; \boldsymbol{\lambda})}$, $v(v-1) \equiv 0 \pmod{2\beta(\mathcal{G}; \boldsymbol{\lambda})}$ and the condition (C5). Then, $v(v-1) \equiv 0 \pmod{2\beta(\mathcal{G}; \boldsymbol{\lambda})}$ holds by the equation (5.4.2) and the second congruences. Assuming the first congruence, we will show that $(v-1) \cdot \boldsymbol{\lambda}$ is an integral linear combination of the vectors $\tau_G(x)$ for $x \in V(G)$ in $G \in \mathcal{G}$.

By the inequalities (5.4.1) and $m\lambda'_E > (k_1-1)(k_2-1)$, there exist vertices

x_i in G_i, for $i = 1, 2, \ldots, 5$, with degree vectors

$$\begin{aligned}
\tau_{G_1}(x_1) &= (k_2-1,\, 0,\, k_1-1,\, 0,\, (k_1-1)(k_2-1),\, \overbrace{0, \ldots, 0}^{k_1-1},\, \overbrace{0, \ldots, 0}^{k_2-1}),\\
\tau_{G_2}(x_2) &= (k_2-1,\, 1,\, k_1-1,\, 0,\, (k_1-1)(k_2-1),\, 1, \ldots, 1,\, 0, \ldots, 0),\\
\tau_{G_3}(x_3) &= (k_2-1,\, 0,\, k_1-1,\, 1,\, (k_1-1)(k_2-1),\, 0, \ldots, 0,\, 1, \ldots, 1),\\
\tau_{G_4}(x_4) &= (k_2-2,\, 1,\, k_1-1,\, 0,\, (k_1-1)(k_2-2),\, 1, \ldots, 1,\, 0, \ldots, 0),\\
\tau_{G_5}(x_5) &= (k_2-1,\, 0,\, k_1-2,\, 1,\, (k_1-2)(k_2-1),\, 0, \ldots, 0,\, 1, \ldots, 1).
\end{aligned}$$

Since the equation

$$\begin{aligned}
\boldsymbol{\lambda} = &\Big(\frac{k_2}{(k_1-1)(k_2-1)} - 2\Big)\tau_{G_1}(x_1) + \Big(\lambda_R - \frac{k_2}{k_1-1}\Big)\tau_{G_2}(x_2)\\
&+ \frac{k_2-2}{k_2-1}\tau_{G_3}(x_3) + \Big(\frac{k_2}{k_1-1} - (\lambda_R - 1)\Big)\tau_{G_4}(x_4) + \frac{1}{k_2-1}\tau_{G_5}(x_5)
\end{aligned}$$

and $k_2(v-1) \equiv 0 \pmod{(k_1-1)(k_2-1)}$ hold, we obtain $v - 1 \equiv 0 \pmod{\alpha(\mathcal{G};\, \boldsymbol{\lambda})}$. Also, this implies that the condition (C5) is satisfied since

$$\begin{aligned}
&\Big(\frac{k_2}{(k_1-1)(k_2-1)} - 2\Big) + \Big(\lambda_R - \frac{k_2}{k_1-1}\Big)\\
&\quad + \frac{k_2-2}{k_2-1} + \Big(\frac{k_2}{k_1-1} - (\lambda_R - 1)\Big) + \frac{1}{k_2-1}\\
&= \frac{k_2}{(k_1-1)(k_2-1)} = \frac{k_1k_2}{2m}
\end{aligned}$$

holds. By Theorem 4.7.5, there exists a balanced $\mathcal{G}$-decomposition of $K_v^{\boldsymbol{\lambda}}$ for sufficiently large v satisfying the necessary conditions. By Corollary 4.7.6, it is shown that there exist BIBRCs for sufficiently large v in the case when λ is a multiple of k_2. Thus, the theorem is shown. □

5.5 The case of $\boldsymbol{\lambda \geq k_1k_2(k_1-1)(k_2-1)}$

By Theorems 5.3.1 and 5.4.1, we obtain the following theorem.

Theorem 5.5.1 *For positive integers $k_1 \leq k_2$, let λ be an integer which is greater than or equals to $k_1k_2(k_1-1)(k_2-1)$. Then there exists a constant $v_0 = v_0(k_1, k_2, \lambda)$ such that* BIBRC(v, k_1, k_2, λ)*'s exist for all $v \geq v_0$ satisfying the congruences* (1.5.2).

Proof. Note that $k_1 \leq k_2$ is assumed. Let $\lambda' = \lambda/(k_1(k_1-1), \lambda)$, $\lambda_1 = (k_2-1)(k_2, \lambda')(k_1(k_1-1), \lambda))$ and $\lambda_2 = k_2(k_2-1, \lambda')(k_1(k_1-1), \lambda)$. Then, $\lambda' \geq k_2(k_2-1)$ holds since $\lambda \geq k_1k_2(k_1-1)(k_2-1)$.

Now $k_2/(k_2, \lambda')$ and $(k_2-1)/(k_2-1, \lambda')$ are relatively prime integers, so there exist integers s_1 and s_2 such that $0 < s_1 < k_2/(k_2, \lambda')$ and

$$\frac{\lambda'}{(k_2(k_2-1), \lambda')} = s_1\frac{k_2-1}{(k_2-1, \lambda')} + s_2\frac{k_2}{(k_2, \lambda')}$$

hold. Multiplying the above equation by $(k_2(k_2-1), \lambda') = (k_2-1, \lambda')(k_2, \lambda')$, we obtain $\lambda' = s_1(k_2, \lambda')(k_2-1) + s_2(k_2-1, \lambda')k_2$. Thus s_2 is positive. Moreover, by multiplying $(k_1(k_1-1), \lambda)$, we find $\lambda = s_1\lambda_1 + s_2\lambda_2$ for some positive integers s_1 and s_2.

Let v be an integer satisfying the congruences (1.5.2). Then, it is obvious that

$$\begin{aligned} \lambda_i(v-1) &\equiv 0 \pmod{(k_1-1)(k_2-1)} \\ \lambda_i v(v-1) &\equiv 0 \pmod{k_1k_2(k_1-1)(k_2-1)} \end{aligned}$$

hold for $i = 1, 2$. By Theorems 5.3.1 and 5.4.1, there exist BIBRC(v, k_1, k_2, λ_1)'s and BIBRC(v, k_1, k_2, λ_2)'s for sufficiently large integers v satisfying the necessary conditions (1.5.2). That is, we can obtain a BIBRC(v, k_1, k_2, λ) by combining s_1 copies of BIBRC(v, k_1, k_2, λ_1) and s_2 copies of BIBRC(v, k_1, k_2, λ_2). $\square$

Further Research and Open Problems

In Chapter 2, we discussed the existence of grid-block designs. As was stated in Sections 1.3, 2.2 and 2.3, the existence problem for a GB(v, k_1, k_2) in the case of (i) $k_1 = k_2 = 2$, (ii) $k_1 = 2$ and $k_2 = 3$, (iii) $k_1 = 2$ and $k_2 = 4$ and (iv) $k_1 = k_2 = 3$ were completely solved. However, it remains open for (a) $k_1 = 2$ and $k_2 = 5$, (b) $k_1 = 2$ and $k_2 = 6$, (c) $k_1 = 3$ and $k_2 = 4$, (d) $k_1 = k_2 = 4$ and so on. We list such problems for the existence of a GB(v, k_1, k_2).

Problem 1 Is the necessary condition $v \equiv 1 \pmod{25}$ for the existence of a GB(v, 2, 5) sufficient?

Problem 2 Is the necessary condition $v \equiv 1 \pmod{72}$ for the existence of a GB(v, 2, 6) sufficient?

Problem 3 Is the necessary condition $v \equiv 1, 16, 21, 36 \pmod{60}$ for the existence of a GB(v, 3, 4) sufficient?

Problem 4 Is the necessary condition $v \equiv 1 \pmod{96}$ for the existence of a GB(v, 4, 4) sufficient?

In Problem 3, if there exists a GB($60m + 1$, 3, 4) for $m = 1, 2, \ldots, 11$, a D($K_{4(60)}$, $G_{3,4}$) and a D($K_{5(60)}$, $G_{3,4}$), where $G_{3,4} = K_3 \times K_4$, then we can show that there exists a GB(v, 3, 4) for any $v \equiv 1 \pmod{60}$ by utilizing some similar recursive constructions to those in Section 2.3. In Appendix B, we show examples of GB($60m + 1$, 3, 4)'s for $m = 1, 2, 3, 4, 6, 7, 9, 10, 11$ and a D($K_{4(60)}$, $G_{3,4}$). If there are a D($K_{5(60)}$, $G_{3,4}$) and a GB(481, 3, 4), then a GB($60m + 1$, 3, 4) will exist for any positive integer m. In other cases of $v \equiv 16, 21, 36 \pmod{60}$, it may not be easy to show the existence for GB(v, 3, 4).

Similarly, in Problem 4, if there exists a GB($96m + 1$, 4, 4) for $m = 1, 2, \ldots, 11$, a D($K_{4(96)}$, $G_{4,4}$) and a D($K_{5(96)}$, $G_{4,4}$), where $G_{4,4} = K_4 \times$

K_4, then we can show that the necessary condition $v \equiv 1 \pmod{96}$ for the existence of a GB(v, 4, 4) is sufficient by utilizing some recursive constructions. In Appendix B, we show examples of GB($96m+1$, 4, 4)'s for $m = 1, 2, 3, 6, 7, 8, 10$. If there are a D($K_{4(96)}$, $G_{4,4}$), a D($K_{5(96)}$, $G_{4,4}$), a GB(865, 4, 4) and a GB(1057, 4, 4), then a GB($96m+1$, 4, 4) exist for any positive integer m. Thus, Problem 4 will be solved by obtaining these designs.

Now we turn our attention to the case of resolvable. In this case, the another condition $k_1k_2|v$ is added to the congruences (1.3.3). The smallest possible size for the existence of a resolvable GB(v, k_1, k_2) is $k_1 = k_2 = 3$. In this case, we can construct some resolvable GB(v, 3, 3)'s by Theorems 2.1.9 and 2.5.5.

Problem 5 Is the necessary condition $v \equiv 9 \pmod{36}$ for the existence of a resolvable GB(v, 3, 3) sufficient?

The second smallest possible size is $k_1 = 3$ and $k_2 = 4$.

Problem 6 Is the necessary condition $v \equiv 36 \pmod{60}$ for the existence of a resolvable GB(v, 3, 4) sufficient?

In this case, as far as the author knows, no resolvable GB(v, 3, 4) has been found yet. Another problem is asymptotic existence of resolvable GB(v, k_1, k_2)'s.

Problem 7 For any positive integers k_1 and k_2, is there a constant $v_0 = v_0(k_1, k_2)$ such that resolvable GB(v, k_1, k_2)'s exist for all $v \geq v_0$ satisfying the congruences (1.3.3) and the condition $k_1k_2|v$?

In Chapters 3 and 5, we mentioned BIBRCs and showed the asymptotic existence for BIBRCs with some λ. As was stated in Section 1.5, the existence problem for a BIBRC(v, k_1, k_2, λ) was completely solved only in the case of $k_1 = k_2 = 2$.

Problem 8 Are the necessary conditions $\lambda(v-1) \equiv 0 \pmod 2$ and $\lambda v(v-1) \equiv 0 \pmod{12}$ for the existence of a BIBRC(v, 2, 3, λ) sufficient?

Problem 9 Are the necessary conditions $\lambda(v-1) \equiv 0 \pmod 3$ and $\lambda v(v-1) \equiv 0 \pmod{36}$ for the existence of a BIBRC(v, 2, 4, λ) sufficient?

Problem 10 Are the necessary conditions $\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v(v-1) \equiv 0 \pmod{36}$ for the existence of a BIBRC(v, 3, 3, λ) sufficient?

While, when $\lambda \geq k_1k_2(k_1-1)(k_2-1)$ holds, there exist BIBRC(v, k_1, k_2, λ)'s for sufficiently large integers satisfying the necessary conditions. In the case when $k_1 \leq k_2$ and $\lambda < k_1k_2(k_1-1)(k_2-1)$, if λ is a multiple of k_1-1, k_2-1, or k_2 there exist BIBRC(v, k_1, k_2, λ)'s for sufficiently large integers satisfying the necessary conditions. On the other hand, in the case when $\lambda = 1$, if $3 \leq k_1 \leq k_2$ holds except for $k_1 = k_2 = 3$, it is easy to show that there does not exist any BIBRC(v, k_1, k_2, λ). However, the author does not know a boundary of λ whether a BIBRC(v, k_1, k_2, λ) exists or not for fixed k_1 and k_2.

Problem 11 For a positive integer k_1 and k_2, find a condition of λ for which there exist BIBRC(v, k_1, k_2, λ)'s. Moreover if λ satisfies such condition, is there a constant $v_0 = v_0(k_1, k_2, \lambda)$ such that BIBRC(v, k_1, k_2, λ)'s exist for all $v \geq v_0$ satisfying the congruences (1.5.2)?

In Chapter 4, we assumed the "tree-ordered" property. But there are some cases a $\mathcal{G}$-decomposition exists even when $\mathcal{G}$ is not tree-ordered. As an example, let G_1 be a colorwise simple graph with 3 colors shown in Figure 1. Though G_1 is not tree-ordered, in this case, there exist many G_1-decompositions of $K_v^{[3]}$. (For example, we may construct some G_1-decompositions by utilizing Heffter's difference triples in [14, pp. 481–488] and [89].)

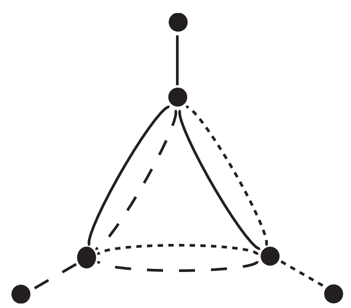


Figure 1: G_1

We find $\alpha(G_1) = 1$ and $\beta(G_1) = m = 3$. We can show that there exist G_1-decompositions of $K_q^{[3]}$ for sufficiently large prime powers $q \equiv 1 \pmod 6$ by utilizing Theorem 4.2.2. By utilizing a notion of PBD-closed set, there exist G_1-decompsotions of $K_v^{[3]}$ for sufficiently large integers $v \equiv 1 \pmod 6$. Similarly, in the case of $v \equiv 0, 3, 4 \pmod 6$, we could show that there exist G_1-decompositions for sufficiently large such integers if we construct a G_1-decomposition of $K_v^{[3]}$ for each of $v \equiv 0, 3, 4 \pmod 6$.

If we fix a $\boldsymbol{\lambda}$-admissible family $\mathcal{G}$ of colorwise simple graphs with c colors which are not necessarily tree-ordered, then we can show an asymptotic existence theorem for $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$. However, for any $\boldsymbol{\lambda}$-admissible family of colorwise simple graphs which are not tree-ordered, we can not show asymptotic existence theorem, as far as author knows. But we can obtain partial asymptotic existence by utilizing Theorem 4.2.2.

On the other hand, as an example, let G_2 be a colorwise simple graph with 3 colors shown in Figure 2. In this case, it is easy to show that there do not exist any G_2-decompositions of $K_v^{[3]}$.

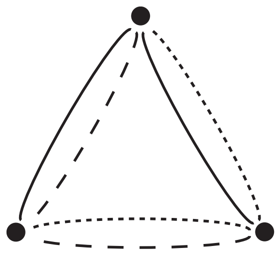


Figure 2: G_2

The first problem is as follows:

Problem 12 Find a more general condition for family $\mathcal{G}$ of colorwise simple edge-c-colored graphs such that there exist $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$.

Problem 13 If $\mathcal{G}$ satisfies the above condition, is there a constant $v_0 = v_0(\mathcal{G}, \boldsymbol{\lambda})$ such that $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ exist for all integers $v \geq v_0$ which satisfy the congruences (1.6.2)?

We know that there exist G_2-decompositoins of $K_v^{(2,\,2,\,2)}$. Thus, we consider the following problem.

Problem 14 Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_c)$ such that the greatest common divisor λ_i's in $\boldsymbol{\lambda}$ is 1. For any $\boldsymbol{\lambda}$-admissible family $\mathcal{G}$ of colorwise simple edge-c-colored graphs, find a condition of λ such that there exist $\mathcal{G}$-decompositions of $K_v^{\lambda\boldsymbol{\lambda}}$. Moreover if λ satisfies such condition, is there a constant $v_0 = v_0(\mathcal{G}, \lambda)$ such that $\mathcal{G}$-decompositions of $K_v^{\lambda\boldsymbol{\lambda}}$ exist for all $v \geq v_0$ satisfying the congruences (1.6.2)?

More generally, we can consider $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ even in the case where $\mathcal{G}$ is not necessarily a family of colorwise simple edge-c-colored graphs.

Problem 15 For any $\boldsymbol{\lambda}$-admissible family of edge-c-colored graphs, find a condition family $\mathcal{G}$ of edge-c-colored graphs such that there exist $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$. Moreover if $\mathcal{G}$ satisfies such condition, is there a constant $v_0 = v_0(\mathcal{G}, \boldsymbol{\lambda})$ such that $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ exist for all integers $v \geq v_0$ which satisfy the congruences (1.6.2)?

Also, we can extend edge-colored graphs to edge-colored directed graphs whose edges are ordered pairs of the vertex set instead of unordered pairs. Similarly, we can define $\alpha(\mathcal{G}; \boldsymbol{\lambda})$, $\beta(\mathcal{G}; \boldsymbol{\lambda})$ and $\boldsymbol{\lambda}$-admissible and show the necessary conditions.

Problem 16 For any $\boldsymbol{\lambda}$-admissible family of edge-c-colored directed graphs, find a condition family $\mathcal{G}$ of edge-c-colored directed graphs such that there exist $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$. Moreover if $\mathcal{G}$ satisfies such condition, is there a constant $v_0 = v_0(\mathcal{G}, \boldsymbol{\lambda})$ such that $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ exist for all integers $v \geq v_0$ which satisfy the necessary conditions?

Moreover, these problems can be extended to balanced cases. Lastly, the author believes that the graph decomposition problem can be applied to many types of combinatorial designs.

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Appendices

A. A table of BIB designs with nested rows and columns having small parameters

In Section 3.3, we considered the existence theorem for sufficiently large prime power $v = q$. In this section, we list the parameters of the designs for $q \leq 101$ and $3 \leq k_1 \leq k_2 \leq 11$ which are obtained by computer according to Theorem 3.3.1. In Table A.1, q is restricted to the case of prime power. For the actual base blocks of the designs see Mutoh [77]. The replication number of a BIBRC is defined by $r = \lambda(q-1)/(k_1-1)(k_2-1)$. Though Table A.1 does not include the value b to save the space, it can be computed by $b = \lambda q(q-1)/k_1k_2(k_1-1)(k_2-1)$. It is obvious that in the case when q is an odd prime power, there exists a BIBRC(q, k_1, k_2, $k_1k_2(k_1-1)(k_2-1)/2$) with $r = k_1k_2(q-1)/2$. And in the case when q is a power of 2, there exists a BIBRC(q, k_1, k_2, $k_1k_2(k_1-1)(k_2-1)$) with $r = k_1k_2(q-1)$. Thus we list the designs whose replication number r is smaller than $k_1k_2(q-1)/2$ or $k_1k_2(q-1)$. In Table A.1, r is the smallest integer found by a computer according to Theorem 3.3.1 or other literatures. The designs are constructed by the methods listed in Sources. Finally, we list the possible smallest λ_m satisfying the equation (1.5.2) in the case when there are no known constructions attaining λ_m. Here, "$-$" implies that the minimum λ_m is attained by the construction given in the table.

Table A.1: Constructed BIBRCs for $q \leq 101$

q	k_1	k_2	r	λ	Source	λ_m	q	k_1	k_2	r	λ	Source	λ_m
13	3	3	9	3	IJ, UM5, Th3.3.1	$-$	19	3	3	9	2	P, Th3.3.1	$-$
13	3	4	12	6	JK5, Th3.3.1	$-$	19	3	4	12	4	Th3.3.1	$-$
16	3	3	45	12	JK5, Th3.3.1	$-$	19	3	5	45	20	JK5, Th3.3.1	$-$
16	3	4	60	24	JK5, Th3.3.1	6	19	3	6	18	10	Th3.3.1	$-$
16	3	5	15	8	JK5, Th3.3.1	$-$	19	4	4	48	24	Th3.3.1	8
17	3	3	36	9	UM4, Th3.3.1	$-$	25	3	3	18	3	UM4, Th3.3.1	$-$
17	3	4	24	9	Th3.3.1	$-$	25	3	4	24	6	AP1, Th3.3.1	3
17	3	5	30	15	Th3.3.1	$-$	25	3	5	30	10	Th3.3.1	1
17	4	4	16	9	Th3.3.1	$-$	25	3	6	36	15	Th3.3.1	$-$

Table A.1: (cont.)

q	k_1	k_2	r	λ	Source	λ_m
25	3	7	84	42	UM3, JK5, Th3.3.1	21
25	3	8	24	14	JK5	–
25	4	4	32	12	UM5, Th3.3.1	6
25	4	5	12	6	C1	2
25	4	6	48	30	JK5	15
27	3	3	13	2	JK9, Th3.2.2	–
27	3	4	156	36	Th3.3.1	12
27	3	5	195	60	Th3.3.1	20
27	3	6	234	90	Th3.3.1	10
27	3	7	273	126	Th3.3.1	42
27	3	8	312	168	Th3.3.1	56
27	4	6	312	180	Th3.3.1	60
29	3	3	63	9	UM4, Th3.3.1	–
29	3	4	84	18	JK5, Th3.3.1	–
29	3	5	105	30	Th3.3.1	–
29	3	6	126	45	Th3.3.1	–
29	3	7	42	18	JK5	9
29	3	8	168	84	JK5, Th3.3.1	12
29	3	9	378	216	JK5, Th3.3.1	108
29	4	4	112	36	C1(AP1, 4 × 7), UM1, JK5, Th3.3.1	–
29	4	5	140	60	UM1, JK5, Th3.3.1	–
29	4	6	168	90	JK5, Th3.3.1	–
29	4	7	28	18	JK5	–
29	5	5	175	100	UM1, Th3.3.1	–
31	3	3	45	6	S6, UM2, JK5, Th3.3.1	–
31	3	4	60	12	S6, C1(P, 3 × 5), UM2, JK5, Th3.3.1	–
31	3	5	15	4	JK5, Th3.3.1	–
31	3	6	54	18	Th3.3.1	6
31	3	7	105	42	UM3, JK5, Th3.3.1	–
31	3	8	120	56	JK5, Th3.3.1	–
31	3	9	135	72	JK5, Th3.3.1	–
31	3	10	30	18	JK5	–
31	4	4	80	24	UM2, Th3.3.1	–
31	4	5	60	24	JK5, Th3.3.1	8
31	4	6	120	60	JK5, Th3.3.1	12
31	4	7	140	84	Th3.3.1	–
31	5	5	75	40	JK5, Th3.3.1	–
31	5	6	30	20	JK5	–
32	3	4	372	72	Th3.3.1	18
32	3	6	558	180	Th3.3.1	90
32	3	8	744	336	Th3.3.1	42
32	3	10	930	540	Th3.3.1	270
32	4	4	31	9	Th3.2.2	–
32	4	5	620	240	Th3.3.1	60
32	4	6	744	360	Th3.3.1	45
32	4	7	868	504	Th3.3.1	126
32	5	6	930	600	Th3.3.1	300
37	3	3	18	2	Th3.3.1	1
37	3	4	24	4	Th3.3.1	2
37	3	5	45	10	Th3.3.1	–
37	3	6	54	15	Th3.3.1	5
37	3	7	126	42	UM3, JK5, Th3.3.1	7
37	3	8	72	28	JK5, Th3.3.1	–
37	3	9	162	72	JK5, Th3.3.1	12
37	3	10	180	90	JK5, Th3.3.1	15
37	3	11	198	110	JK5, Th3.3.1	55
37	4	4	32	8	Th3.3.1	4
37	4	5	60	20	Th3.3.1	–
37	4	6	72	30	JK5	10
37	4	7	168	84	UM3, Th3.3.1	14
37	4	8	192	112	Th3.3.1	56
37	4	9	36	24	JK5	–
37	5	5	225	100	UM1, Th3.3.1	–
37	5	6	180	100	JK5, Th3.3.1	50
37	5	7	630	420	JK5, Th3.3.1	70
37	6	6	216	150	JK5, Th3.3.1	25
41	3	3	90	9	UM4, Th3.3.1	–
41	3	4	60	9	Th3.3.1	–
41	3	5	60	12	JK5, Th3.3.1	3
41	3	6	72	18	Th3.3.1	9
41	3	7	210	63	Th3.3.1	–
41	3	8	120	42	JK5, Th3.3.1	–
41	3	9	135	54	Th3.3.1	–
41	3	10	120	54	JK5	27
41	3	11	330	165	Th3.3.1	33
41	4	4	80	18	SD2, UM4, Th3.3.1	–
41	4	5	40	12	AP1, JK5	6
41	4	6	120	45	Th3.3.1	9
41	4	7	140	63	Th3.3.1	–
41	4	8	160	84	Th3.3.1	–
41	4	9	360	216	JK5, Th3.3.1	108
41	4	10	80	54	JK5	27
41	5	5	50	20	UM5	10
41	5	6	120	60	JK5	15
41	5	7	140	84	JK5	21
41	5	8	40	28	JK5	–
41	6	6	360	225	Th3.3.1	45
43	3	3	63	6	S6, C1(AP2, 3 × 7), UM2, JK5, Th3.3.1	–
43	3	4	84	12	S6, C1(AP2, 3 × 7), UM2, JK5, Th3.3.1	–
43	3	5	105	20	JK5, Th3.3.1	–
43	3	6	126	30	S6, C1(3 × 7), UM3, JK5, Th3.3.1	–
43	3	7	21	6	AP2, JK5	–
43	3	8	168	56	JK5, Th3.3.1	8
43	3	9	189	72	JK5, Th3.3.1	–
43	3	10	210	90	JK5, Th3.3.1	–
43	3	11	231	110	JK5, Th3.3.1	–
43	4	4	112	24	UM2, Th3.3.1	–
43	4	5	140	40	Th3.3.1	–
43	4	6	168	60	C1(AP1, 6 × 7), UM3, JK5, Th3.3.1	–
43	4	7	84	36	JK5	12
43	4	8	224	112	Th3.3.1	16
43	4	9	252	144	JK5, Th3.3.1	–
43	4	10	280	180	Th3.3.1	–
43	5	6	210	100	JK5, Th3.3.1	–
43	5	7	105	60	JK5	20
43	5	8	840	560	JK5, Th3.3.1	80
43	6	6	252	150	JK5, Th3.3.1	150
43	6	7	42	30	JK5	–
49	3	3	36	3	UM5, Th3.3.1	–
49	3	4	24	3	Th3.3.1	–
49	3	5	60	10	Th3.3.1	5
49	3	6	72	15	Th3.3.1	–
49	3	7	48	12	C1(4 × 7)	3
49	3	8	48	14	JK5	7
49	3	9	216	72	JK5, Th3.3.1	9
49	3	10	240	90	JK5, Th3.3.1	45
49	3	11	264	110	JK5, Th3.3.1	55
49	4	4	64	12	UM5, Th3.3.1	3
49	4	5	80	20	Th3.3.1	5
49	4	6	96	30	C1(4 × 7), JK5, Th3.3.1	15
49	4	7	16	6	C1	3
49	4	8	192	84	Th3.3.1	14
49	4	9	144	72	JK5	18
49	4	10	240	135	Th3.3.1	45
49	4	11	352	220	Th3.3.1	55
49	5	5	150	50	UM4, Th3.3.1	25
49	5	6	180	75	Th3.3.1	25
49	5	7	210	105	Th3.3.1	5
49	5	8	240	140	Th3.3.1	70
49	5	9	360	240	JK5, Th3.3.1	30
49	6	6	144	75	C1(6 × 7), UM4	–
49	6	7	24	15	C1	–
49	6	8	96	70	JK5	35
53	3	3	117	9	UM4, Th3.3.1	–
53	3	4	156	18	JK5, Th3.3.1	–
53	3	5	195	30	Th3.3.1	–
53	3	6	234	45	Th3.3.1	–
53	3	7	273	63	Th3.3.1	–
53	3	8	312	84	JK5, Th3.3.1	–
53	3	9	351	108	Th3.3.1	–
53	3	10	390	135	Th3.3.1	–
53	3	11	429	165	Th3.3.1	–
53	4	4	208	36	C1(AP1, 4 × 13), UM1, JK5, Th3.3.1	–
53	4	5	260	60	UM1, JK5, Th3.3.1	–

Table A.1: (cont.)

q	k_1	k_2	r	λ	Source	λ_m
53	4	6	312	90	JK5, Th3.3.1	–
53	4	7	364	126	JK5, Th3.3.1	–
53	4	8	416	168	JK5, Th3.3.1	–
53	4	9	468	216	JK5, Th3.3.1	–
53	4	10	520	270	JK5, Th3.3.1	–
53	4	11	572	330	JK5, Th3.3.1	–
53	5	5	325	100	UM1, Th3.3.1	–
53	5	6	390	150	Th3.3.1	–
53	5	7	455	210	Th3.3.1	–
53	5	8	520	280	JK5, Th3.3.1	–
53	5	9	585	360	Th3.3.1	–
53	5	10	1300	900	JK5, Th3.3.1	450
53	6	6	468	225	Th3.3.1	–
53	6	7	546	315	Th3.3.1	–
53	6	8	624	420	Th3.3.1	–
53	7	7	637	441	Th3.3.1	–
61	3	3	45	3	UM5, Th3.3.1	–
61	3	4	60	6	AP1, JK5, Th3.3.1	–
61	3	5	30	4	JK5	2
61	3	6	90	15	Th3.3.1	3
61	3	7	105	21	Th3.3.1	–
61	3	8	120	28	Th3.3.1	–
61	3	9	270	72	JK5, Th3.3.1	36
61	3	10	60	18	JK5	9
61	3	11	330	110	JK5, Th3.3.1	11
61	4	4	80	12	SD2, UM5, Th3.3.1	–
61	4	5	60	12	AP1, JK5	4
61	4	6	120	30	JK5, Th3.3.1	6
61	4	7	280	84	UM3, Th3.3.1	42
61	4	8	320	112	Th3.3.1	56
61	4	9	180	72	JK5	–
61	4	10	120	54	JK5	18
61	4	11	440	220	Th3.3.1	22
61	5	5	75	20	UM5	–
61	5	6	60	20	AP1, JK5	10
61	5	7	175	70	Th3.3.1	14
61	5	8	120	56	JK5	–
61	5	9	90	48	JK5	24
61	5	10	500	300	Th3.3.1	30
61	5	11	330	220	JK5	110
61	6	6	180	75	UM4	15
61	6	7	420	210	UM1, JK5, Th3.3.1	21
61	6	8	240	140	JK5	28
61	6	9	540	360	JK5, Th3.3.1	36
61	6	10	130	90	JK5	45
61	7	7	245	147	UM4	–
61	7	8	560	392	Th3.3.1	196
64	3	3	63	4	Th3.3.1	–
64	3	4	252	24	JK5, Th3.3.1	2
64	3	5	315	40	JK5, Th3.3.1	–
64	3	6	378	60	JK5, Th3.3.1	10
64	3	7	63	12	JK5	4
64	3	8	504	112	JK5, Th3.3.1	2
64	3	9	189	48	JK5	–
64	3	10	630	180	JK5, Th3.3.1	30
64	3	11	693	220	JK5, Th3.3.1	–
64	4	4	21	3	JK9, Th3.2.2	1
64	4	5	1260	240	Th3.3.1	20
64	4	6	504	120	JK5, Th3.3.1	5
64	4	7	252	72	JK5	2
64	4	8	63	21	Th3.2.2	1
64	4	9	252	96	JK5	24
64	4	10	840	360	Th3.3.1	15
64	4	11	2772	1320	Th3.3.1	110
64	5	6	630	200	JK5, Th3.3.1	100
64	5	7	315	120	JK5	40
64	5	8	2520	1120	Th3.3.1	20
64	5	9	315	160	JK5	–
64	5	10	3150	1800	Th3.3.1	100
64	6	6	756	300	JK5, Th3.3.1	25
64	6	7	126	60	JK5	10
64	6	8	1008	560	JK5, Th3.3.1	5
64	6	9	378	240	JK5	120
64	6	10	1260	900	JK5	75
64	7	7	441	252	JK5	28
64	7	8	504	336	JK5	14

q	k_1	k_2	r	λ	Source	λ_m
64	7	9	63	48	JK5	–
67	3	3	99	6	S6, UM2, JK5, Th3.3.1	–
67	3	4	132	12	S6, UM2, JK5, Th3.3.1	–
67	3	5	165	20	JK5, Th3.3.1	–
67	3	6	198	30	S6, C1(AP2, 3 × 11), UM3, JK5, Th3.3.1	–
67	3	7	231	42	S6, UM3, JK5, Th3.3.1	–
67	3	8	264	56	JK5, Th3.3.1	–
67	3	9	297	72	JK5, Th3.3.1	–
67	3	10	330	90	JK5, Th3.3.1	–
67	3	11	33	10	JK5	–
67	4	4	176	24	UM2, Th3.3.1	–
67	4	5	220	40	Th3.3.1	–
67	4	6	264	60	UM3, JK5, Th3.3.1	–
67	4	7	308	84	UM3, Th3.3.1	–
67	4	8	352	112	Th3.3.1	–
67	4	9	396	144	JK5, Th3.3.1	–
67	4	10	440	180	Th3.3.1	–
67	4	11	132	60	JK5	20
67	5	6	330	100	JK5, Th3.3.1	–
67	5	7	385	140	Th3.3.1	–
67	5	9	495	240	JK5, Th3.3.1	–
67	5	10	550	300	Th3.3.1	–
67	5	11	165	100	JK5	–
67	6	6	396	150	C1(AP1, 6 × 11), UM1, JK5, Th3.3.1	–
67	6	7	462	210	UM1, JK5, Th3.3.1	–
67	6	8	528	280	JK5, Th3.3.1	–
67	6	9	594	360	JK5, Th3.3.1	–
67	6	10	660	450	JK5, Th3.3.1	–
67	6	11	66	50	JK5	–
67	7	7	539	294	UM1, Th3.3.1	–
67	7	8	616	392	Th3.3.1	–
67	7	9	693	504	JK5, Th3.3.1	–
71	3	5	105	12	JK5, Th3.3.1	–
71	3	6	126	18	Th3.3.1	–
71	3	7	105	18	JK5, Th3.3.1	–
71	3	8	840	168	JK5, Th3.3.1	24
71	3	10	210	54	JK5, Th3.3.1	–
71	3	11	231	66	Th3.3.1	–
71	4	5	140	24	JK5, Th3.3.1	–
71	4	6	168	36	Th3.3.1	–
71	4	7	140	36	JK5	–
71	4	8	1120	336	JK5, Th3.3.1	48
71	4	10	280	108	JK5	–
71	4	11	1540	660	JK5, Th3.3.1	132
71	5	5	175	40	S6, UM2, JK5, Th3.3.1	–
71	5	6	210	60	S6, C1(AP2, 5 × 7), UM2, JK5, Th3.3.1	–
71	5	7	35	12	JK5	–
71	5	8	280	112	JK5, Th3.3.1	16
71	5	9	315	144	JK5, Th3.3.1	–
71	5	10	350	180	JK5	–
71	5	11	385	220	JK5	–
71	6	6	252	90	UM2, Th3.3.1	–
71	6	7	210	90	JK5	18
71	6	8	1680	840	JK5, Th3.3.1	24
71	6	9	1890	1080	JK5, Th3.3.1	216
71	6	10	420	270	JK5	–
71	6	11	2310	1650	JK5, Th3.3.1	330
71	7	7	245	126	JK5	–
71	7	8	280	168	JK5	–
71	7	9	315	216	JK5	–
71	7	10	70	54	JK5	–
71	8	8	2240	1568	JK5	224
73	3	3	36	2	SD2, Th3.3.1	1
73	3	4	36	3	Th3.3.1	1
73	3	5	90	10	Th3.3.1	5
73	3	6	108	15	Th3.3.1	5
73	3	7	126	21	Th3.3.1	7
73	3	8	72	14	JK5	–
73	3	9	108	24	JK5	6
73	3	10	360	90	JK5, Th3.3.1	15
73	3	11	396	110	JK5, Th3.3.1	55
73	4	4	96	12	UM5, Th3.3.1	2
73	4	5	120	20	Th3.3.1	10

Table A.1: (cont.)

q	k_1	k_2	r	λ	Source	λ_m
73	4	6	144	30	JK5	5
73	4	7	252	63	Th3.3.1	7
73	4	8	288	84	JK5, Th3.3.1	28
73	4	9	72	24	JK5	12
73	4	10	360	135	Th3.3.1	15
73	4	11	396	165	Th3.3.1	55
73	5	5	225	50	UM4, Th3.3.1	–
73	5	6	270	75	Th3.3.1	25
73	5	7	315	105	Th3.3.1	35
73	5	8	360	140	JK5, Th3.3.1	–
73	5	9	180	80	JK5	20
73	5	10	450	225	Th3.3.1	25
73	5	11	495	275	Th3.3.1	–
73	6	6	216	75	UM4, Th3.3.1	25
73	6	7	504	210	UM1, Th3.3.1	35
73	6	8	144	70	JK5	–
73	6	9	216	120	JK5	30
73	6	10	720	450	JK5, Th3.3.1	75
73	6	11	792	550	JK5, Th3.3.1	275
73	7	7	294	147	UM4	49
73	7	8	504	294	JK5, Th3.3.1	98
73	7	9	252	168	JK5	42
73	7	10	840	630	Th3.3.1	105
73	8	8	576	392	JK5, Th3.3.1	–
73	8	9	72	56	JK5	–
79	3	3	117	6	S6, UM2, JK5, Th3.3.1	–
79	3	4	156	12	S6, C1(AP2, 3×13), UM2, JK5, Th3.3.1	–
79	3	5	195	20	JK5, Th3.3.1	–
79	3	6	234	30	S6, UM3, JK5, Th3.3.1	–
79	3	7	273	42	S6, UM3, JK5, Th3.3.1	–
79	3	8	312	56	JK5, Th3.3.1	–
79	3	9	351	72	JK5, Th3.3.1	–
79	3	10	390	90	JK5, Th3.3.1	–
79	3	11	429	110	JK5, Th3.3.1	–
79	4	4	208	24	UM2, Th3.3.1	–
79	4	5	260	40	Th3.3.1	–
79	4	6	312	60	UM3, JK5, Th3.3.1	–
79	4	7	364	84	UM3, Th3.3.1	–
79	4	8	416	112	Th3.3.1	–
79	4	9	468	144	JK5, Th3.3.1	–
79	4	10	520	180	Th3.3.1	–
79	4	11	572	220	Th3.3.1	–
79	5	6	390	100	JK5, Th3.3.1	–
79	5	7	455	140	Th3.3.1	–
79	5	9	585	240	JK5, Th3.3.1	–
79	5	10	650	300	Th3.3.1	–
79	6	6	468	150	UM1, JK5, Th3.3.1	–
79	6	7	546	210	UM1, JK5, Th3.3.1	–
79	6	8	624	280	JK5, Th3.3.1	–
79	6	9	702	360	JK5, Th3.3.1	–
79	6	10	780	450	JK5, Th3.3.1	–
79	6	11	858	550	JK5, Th3.3.1	–
79	7	7	637	294	UM1, Th3.3.1	–
79	7	8	728	392	Th3.3.1	–
79	7	9	819	504	JK5, Th3.3.1	–
79	7	10	910	630	Th3.3.1	–
79	7	11	3003	2310	JK5, Th3.3.1	770
79	8	9	936	672	JK5, Th3.3.1	–
81	3	3	20	1	Th3.2.4	–
81	3	4	120	9	Th3.3.1	3
81	3	5	120	12	JK5	1
81	3	6	360	45	Th3.3.1	1
81	3	7	420	63	Th3.3.1	21
81	3	8	240	42	JK5, Th3.3.1	7
81	3	9	40	8	Th3.2.2	1
81	3	10	240	54	JK5	9
81	3	11	660	165	Th3.3.1	11
81	4	4	80	9	SD2, Th3.3.1	–
81	4	5	80	12	AP1, JK5	3
81	4	6	240	45	Th3.3.1	3
81	4	7	280	63	Th3.3.1	–
81	4	8	320	84	JK5, Th3.3.1	42
81	4	9	360	108	Th3.3.1	6
81	4	10	160	54	JK5	27
81	4	11	440	165	Th3.3.1	33
81	5	5	100	20	UM5	5
81	5	6	240	60	S6, UM2, JK5	5
81	5	7	280	84	JK5	21
81	5	8	80	28	JK5	14
81	5	9	360	144	JK5	2
81	5	10	400	180	UM3, JK5	45
81	5	11	440	220	UM3, JK5	55
81	6	6	144	45	UM5	5
81	6	7	840	315	Th3.3.1	21
81	6	8	480	210	Th3.3.1	7
81	6	9	540	270	Th3.3.1	1
81	6	10	480	270	UM3, JK5	45
81	6	11	528	330	UM3	55
81	7	7	980	441	Th3.3.1	–
81	7	8	560	294	JK5	147
81	7	9	1260	756	Th3.3.1	21
81	7	10	560	378	JK5	189
81	7	11	1540	1155	Th3.3.1	231
81	8	8	320	196	C1(8), UM4	–
81	8	9	40	28	C1	–
81	8	10	160	126	JK5	63
89	3	3	198	9	UM4, Th3.3.1	–
89	3	4	132	9	Th3.3.1	–
89	3	5	165	15	Th3.3.1	–
89	3	6	396	45	Th3.3.1	–
89	3	7	462	63	Th3.3.1	–
89	3	8	264	42	JK5, Th3.3.1	–
89	3	9	297	54	Th3.3.1	–
89	3	10	660	135	Th3.3.1	–
89	3	11	132	30	JK5	15
89	4	4	176	18	UM4, Th3.3.1	–
89	4	5	220	30	Th3.3.1	–
89	4	6	264	45	Th3.3.1	–
89	4	7	308	63	Th3.3.1	–
89	4	8	352	84	Th3.3.1	–
89	4	9	396	108	Th3.3.1	–
89	4	10	440	135	Th3.3.1	–
89	4	11	88	30	JK5	15
89	5	5	275	50	UM4, Th3.3.1	–
89	5	6	330	75	Th3.3.1	–
89	5	7	385	105	Th3.3.1	–
89	5	8	440	140	JK5, Th3.3.1	–
89	5	9	495	180	Th3.3.1	–
89	5	10	550	225	Th3.3.1	–
89	5	11	220	100	JK5	25
89	6	6	792	225	Th3.3.1	–
89	6	7	924	315	Th3.3.1	–
89	6	8	528	210	JK5, Th3.3.1	–
89	6	9	594	270	Th3.3.1	–
89	6	10	1320	675	Th3.3.1	–
89	6	11	264	150	JK5	75
89	7	7	1078	441	Th3.3.1	–
89	7	8	616	294	JK5, Th3.3.1	–
89	7	9	693	378	Th3.3.1	–
89	7	10	1540	945	Th3.3.1	–
89	7	11	308	210	JK5	105
89	8	8	704	392	JK5, Th3.3.1	–
89	8	9	792	504	JK5, Th3.3.1	–
89	8	10	880	630	JK5, Th3.3.1	–
89	8	11	88	70	JK5	–
89	9	9	891	648	Th3.3.1	–
97	3	3	72	3	UM5, Th3.3.1	–
97	3	4	96	6	AP1, JK5, Th3.3.1	3
97	3	5	120	10	Th3.3.1	5
97	3	6	288	30	S6, UM3, JK5, Th3.3.1	15
97	3	7	336	42	S6, UM3, JK5, Th3.3.1	21
97	3	8	96	14	JK5	7
97	3	9	324	54	Th3.3.1	9
97	3	10	480	90	JK5, Th3.3.1	45
97	3	11	528	110	JK5, Th3.3.1	55
97	4	4	128	12	UM5, Th3.3.1	3
97	4	5	160	20	Th3.3.1	5
97	4	6	192	30	JK5	15
97	4	7	336	63	Th3.3.1	21
97	4	8	384	84	JK5, Th3.3.1	7
97	4	9	288	72	JK5	9

Table A.1: (cont.)

q	k_1	k_2	r	λ	Source	λ_m	q	k_1	k_2	r	λ	Source	λ_m
97	4	10	480	135	Th3.3.1	45	101	4	4	400	36	C1(4 × 5),	–
97	4	11	528	165	Th3.3.1	55						UM1, JK5, Th3.3.1	
97	5	5	300	50	UM4, Th3.3.1	25	101	4	5	100	12	AP1, JK5	–
97	5	6	360	75	Th3.3.1	25	101	4	6	600	90	JK5, Th3.3.1	18
97	5	7	420	105	Th3.3.1	35	101	4	7	700	126	JK5, Th3.3.1	–
97	5	8	480	140	JK5, Th3.3.1	35	101	4	8	800	168	JK5, Th3.3.1	–
97	5	9	540	180	Th3.3.1	15	101	4	9	900	216	JK5, Th3.3.1	–
97	5	10	600	225	Th3.3.1	75	101	4	10	200	54	JK5	–
97	5	11	660	275	Th3.3.1	–	101	4	11	1100	330	JK5, Th3.3.1	66
97	6	6	288	75	UM4, Th3.3.1	–	101	5	5	125	20	UM5	4
97	6	7	672	210	UM1, JK5, Th3.3.1	105	101	5	6	300	60	S6, UM2, JK5	6
97	6	8	192	70	JK5	35	101	5	7	350	84	JK5	42
97	6	9	648	270	Th3.3.1	45	101	5	8	200	56	JK5	–
97	6	10	960	450	JK5, Th3.3.1	225	101	5	9	450	144	JK5	72
97	6	11	1056	550	JK5, Th3.3.1	275	101	5	10	500	180	AP4, S6, UM3, JK5	18
97	7	7	392	147	UM4	–	101	5	11	550	220	UM3, JK5	22
97	7	8	672	294	JK5, Th3.3.1	49	101	6	6	180	45	UM5	9
97	7	9	756	378	Th3.3.1	63	101	6	7	1050	315	Th3.3.1	63
97	7	10	1120	630	Th3.3.1	315	101	6	8	1200	420	JK5, Th3.3.1	84
97	7	11	1232	770	Th3.3.1	385	101	6	9	1350	540	Th3.3.1	108
97	8	8	384	196	UM4	98	101	6	10	600	270	JK5	27
97	8	9	288	168	JK5	42	101	6	11	1650	825	Th3.3.1	33
97	8	10	960	630	JK5, Th3.3.1	105	101	7	7	1225	441	Th3.3.1	–
97	8	11	1056	770	JK5	385	101	7	8	1400	588	JK5, Th3.3.1	–
97	9	9	486	324	UM4	54	101	7	9	1575	756	Th3.3.1	–
97	9	10	1440	1080	JK5, Th3.3.1	135	101	7	10	700	378	JK5	189
101	3	3	225	9	UM4, Th3.3.1	–	101	7	11	1925	1155	Th3.3.1	231
101	3	4	300	18	JK5, Th3.3.1	–	101	8	8	1600	784	JK5, Th3.3.1	–
101	3	5	150	12	JK5	6	101	8	9	1800	1008	JK5, Th3.3.1	–
101	3	6	450	45	Th3.3.1	9	101	8	10	400	252	JK5	–
101	3	7	525	63	Th3.3.1	–	101	8	11	2200	1540	JK5, Th3.3.1	308
101	3	8	600	84	JK5, Th3.3.1	–	101	9	9	2025	1296	Th3.3.1	–
101	3	9	675	108	Th3.3.1	–	101	9	10	900	648	JK5	324
101	3	10	300	54	JK5	27	101	9	11	4950	3960	JK5, Th3.3.1	396
101	3	11	825	165	Th3.3.1	33	101	10	10	1000	810	JK5	81

P, Preece [80]; SD2, Singh & Dey [90, Th.2]; S6, Street [94, Th.6]; AP1, AP2 and AP4, Theorems 1, 2 and 4 of Agrawal & Prasad [7]; JK5 and JK9, Theorems 5 and 9 of Jimbo & Kuriki [54]; IJ, Ipinyomi & John [53]; C1, Cheng [26, Th.2.1]; UM1, UM2, UM3, UM4 and UM5, Theorems 1, 2, 3, 4 and 5 of Uddin & Morgan [98]; Cheng's result combines a BIBD with a BIBRC to give a new BIBRC, both row-column designs having the same v; when the referenced design does not explicitly appear in his paper, the dimensions $k_1 \times k_2$ of the required initial BIBRC are in parentheses. See http://jim.math.keio.ac.jp/˜yukiyasu/table.html.

B. Examples of grid-block designs with small parameters

In this appendix, grid-block designs mentioned in the open problems in page 102 are listed. Firstly, GB($60m+1$, 3, 4)'s for $m = 1, 2, 3, 6, 7, 9, 10, 11$ are listed in Table B.1. They are constructed by utilizing finite fields as in Lemma 2.5.6. We list m, primitive elements (or polynomials) and base grid-blocks A which satisfies the condition of Lemma 2.5.6.

Table B.1: Table of the base grid-blocks of 3×4 grid-block designs

m	primitive element or polynomial	A			
1	2	0	1	3	7
		5	25	56	43
		19	47	30	59
2	$\omega^2+\omega^1+7$	ω^∞	ω^0	ω^1	ω^2
		ω^3	ω^5	ω^{15}	ω^{98}
		ω^{24}	ω^{95}	ω^{97}	ω^{45}
3	2	0	1	3	7
		5	13	23	63
		99	90	142	39
6	$\omega^2+\omega^1+2$	ω^∞	ω^0	ω^1	ω^2
		ω^3	ω^4	ω^7	ω^8
		ω^{52}	ω^{210}	ω^{289}	ω^{93}
7	2	0	1	3	7
		5	13	23	37
		105	365	281	86
9	2	0	1	3	7
		5	14	22	46
		64	474	250	521
10	7	0	1	3	7
		5	13	28	56
		141	130	414	307
11	2	0	1	5	11
		7	15	53	100
		50	67	331	218

Let $V = \mathbb{Z}_{240}$ and $\boldsymbol{A}$ be a family of base grid-blocks as follows:

$$\boxed{\begin{array}{cccc} 0 & 1 & 6 & 15 \\ 13 & 30 & 3 & 48 \\ 2 & 23 & 60 & 101 \end{array}}, \boxed{\begin{array}{cccc} 0 & 25 & 74 & 143 \\ 34 & 195 & 140 & 97 \\ 123 & 84 & 233 & 210 \end{array}}, \boxed{\begin{array}{cccc} 0 & 73 & 26 & 135 \\ 85 & 230 & 127 & 20 \\ 179 & 148 & 77 & 58 \end{array}}$$

We define $\mathcal{A} = \{A + x : A \in \boldsymbol{A},\, x \in \mathbb{Z}_{240}\}$. Then a pair $(\mathbb{Z}_{240}, \mathcal{A})$ is a $\mathrm{D}(K_{4(60)}, G_{3,4})$ since $\partial\boldsymbol{A} = \mathbb{Z}_{240} \setminus \{0, 4, 8, \ldots, 236\}$ holds.

Secondly, GB$(96m + 1, 4, 4)$'s for $m = 1, 2, 3, 6, 7, 8, 10$ are listed in Table B.2. They are also obtained by utilizing Lemma 2.5.6.

Table B.2: Table of the base grid-blocks of 4×4 grid-block designs

m	primitive element or polynomial	A
1	5	$\begin{array}{cccc} 0 & 1 & 3 & 7 \\ 5 & 13 & 81 & 38 \\ 16 & 60 & 26 & 86 \\ 46 & 74 & 61 & 29 \end{array}$
2	5	$\begin{array}{cccc} 0 & 1 & 3 & 7 \\ 5 & 14 & 25 & 39 \\ 35 & 72 & 131 & 62 \\ 82 & 150 & 110 & 183 \end{array}$
3	$\omega^2 + \omega^1 + 3$	$\begin{array}{cccc} \omega^{\infty} & \omega^{0} & \omega^{1} & \omega^{2} \\ \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} \\ \omega^{20} & \omega^{17} & \omega^{155} & \omega^{83} \\ \omega^{46} & \omega^{70} & \omega^{221} & \omega^{7} \end{array}$
6	5	$\begin{array}{cccc} 0 & 1 & 3 & 7 \\ 5 & 13 & 26 & 41 \\ 14 & 67 & 258 & 418 \\ 229 & 490 & 357 & 279 \end{array}$
7	5	$\begin{array}{cccc} 0 & 1 & 3 & 7 \\ 5 & 13 & 22 & 38 \\ 15 & 37 & 172 & 338 \\ 515 & 581 & 481 & 186 \end{array}$

Table B.2: (cont.)

m	primitive element or polynomial	A			
8	11	0	1	3	7
		5	13	23	37
		16	52	168	473
		572	345	739	80
10	$\omega^2 + \omega^1 + 12$	ω^∞	ω^0	ω^1	ω^2
		ω^3	ω^4	ω^5	ω^6
		ω^{10}	ω^7	ω^{434}	ω^{569}
		ω^{318}	ω^{427}	ω^{211}	ω^{660}