Boundary value problems and crack propagation in elastic or viscoelastic media with cracks

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Contents

1 Introduction **2**

2 A boundary value problem for an infinite elastic strip with a semi-infinite crack **8**

2.1 Preliminaries 8
2.2 Integral equations on the boundary 11
2.3 Uniqueness and existence of solution 16

3 Propagation of cracks in an infinite elastic strip with a semi-infinite crack **24**

3.1 The model of crack propagation 24
3.2 The direction of crack extension 29

4 Existence of a weak solution in an infinite viscoelastic strip with a semi-infinite crack **32**

4.1 The model equations of viscoelasticity with crack 32
4.2 Solvability of the transformed problem 36
4.3 Weak solvability of the time-dependent problem 38

Chapter 1

Introduction

Earthquakes occur frequently in many parts of the world. The essence of the earthquake has been accurately identified only recently during the 1960's. According to the modern theory of plate tectonics [34] this essence is the fault movement that releases the strain energy accumulated in the bedrock (rock mass) underground. Indeed, first a strong force is exerted on the rock mass underground. As a result, this rock mass gradually becomes deformed. At the same time, energy accumulates in the form of strain in the rock mass. When the rock mass can no longer withstand the continually mounting pressure a rupture occurs. The stored strain energy is violently released in the form of a seismic wave. Earthquakes are caused by the occurrence of this phenomenon underground. The rupture that occurs underground and causes an earthquake consists of a rapid slipping movement along weak planes (fault planes or cracks) between giant slabs of rock- mass.

Various results of researches reveal that in the Japanese Archipelago and the surrounding areas several plates are converging each other, for example the Pacific Plate, the Philippine Sea Plate and a plate on land. At the boundary of these covering plates, both plates are pushing against each other or one plate subducts the other. It is easy to imagine that many large earthquakes (fault movements) are generated there. Therefore, the practical goal of seismology is to prevent or reduce damages due to earthquakes by estimating its hazard at a given site or by forecasting the occurrence of the next strong event. At present the prevailing approach to these problems is to extrapolate data from the record of past events and apply their information to the future. However, this purely phenomenological approach is unreliable, due mainly to the lack of representative data. From these reasons the motivation arises to face this thesis and we study fracture phenomena that are considered to occur in the case of earthquakes from the mathematical viewpoint.

An underground structure of the Earth constitutes three parts, crust, mantle and core, by means of a difference of chemical composition. On the other hand a dynamical outermost layer of the Earth is commonly called lithosphere or plate composed of crust and the upper part of mantle. It is well known that the lithosphere behaves elastic in regard to the movement in a time scale of shorter than several hundred thousands years. Moreover, under the lithosphere there exists a viscoelastic layer called athenosphere which is very fluid. It is necessary to take athenosphere into account in order to explain the movement of plate and the accumulated process of the strain energy. According to the above reason, we consider the elastic or viscoelastic media with cracks. Particularly, in this thesis we deal with the case that the material response is linearly elastic because of assuming that strain is very small. This is based on the fact that in stones or rocks such as a plate of the earth is led to the brittle fracture by a small strain.

Theory of elasticity has been thoroughly developed (see for example, [29], [30], [31]). Mathematical existence theorems in a linear elastic theory were established by Fichera [10] in 1972. Recently, Constanda studied the boundary value problems for the system of equilibrium equations of plane elasticity in [6]–[9]. By means of elastic single and double layer potentials he reduced the boundary value problems mentioned above to the integral equations. Then applying the theory of integral equations lead to the solvability of the interior and exterior Dirichlet and Neumann problems. However, the problems considered in [6]–[9] were those in a compact domain without any cracks.

On the other hand, for boundary value problems in a planar domain, Airy's stress function is, in general, used so that the system of partial differential equations is transformed into a biharmonic equation (see, for example [16]). Although the stress tensor is uniquely determined by this transformation, the boundary conditions seem to be inequivalent. Recently, Chudinovich and Constanda [4] in 1999 investigated plate problems for both an infinite and a finite plates with a finite crack and proved a unique solvability in Sobolev spaces. Krutitskiĭ [26]–[28] studied the Dirichlet and Neumann problems for Laplace and Helmholtz equations in a connected plane region with cuts. The problems were reduced to Fredholm integral equations of second and first kind, which were uniquely solvable with the help of a nonclassical angular potential.

Propagation of cracks is a phenomenon which leads to the brittle failure of materials. Analysis of the crack growth has been a major subject of fracture mechanics since Griffith's celebrated work [19] in 1920. Griffith wanted to know the difference between theoretical and practical strength of the glass. In the consideration of a brittle elastic body containing a crack, Griffith recognized that the macroscopic potential energy of the system, consisting of the internal stored elastic energy and the external potential energy of the applied loads, varied with the size of the crack. This expression is mistaken in equation (8) of [19]. To clarify this Sih [39] gave the correct version of Griffith's energy treatment. And Griffith supposed that a certain amount of work per unit area of crack surface must be expended at a microscopic level to create that area. In this context, the term "microscopic" implies that this work is not included in a continuum description of the process. He simply included this work as an additional potential energy of the system. Then, using the equilibrium principle of minimum potential energy for conservative systems, he considered the system to be in equilibrium when in an infinite plate with a particular crack length the uniform stress is loaded normal to the crack plate at the infinity. As a result, he postulated that the crack is at a critical state of incipient growth if the reduction in macroscopic potential energy associated with a small virtual crack advance from that state is equal to the microscopic work of creation of new crack surface by the virtual crack advance. A particular attraction of Griffith's energy fracture condition is that it obviates the need to examine the actual fracture process at the crack tip.

The formulation of fracture mechanics began with Irwin [20], [21] and his associates around 1950's. The impetus for the development of this discipline originally came from the increasing demand for more reliable safety criteria in engineering design. And the continuum field approach to fracture of solids was launched with the introduction of the elastic stress intensity factor as a crack tip field characterizing parameter by Irwin (1957)[20]. This idea provided a framework in discussing the strength of cracked solids of elastic material. He proposed that a crack begins to grow in a cracked body with limited plastic deformation when the elastic stress intensity factor reaches at a value called the fracture toughness of the material. The equivalence of the Irwin's stress intensity factor criterion and the Griffith's energy criterion for onset of growth of a tensile crack in a two-dimensional body of elastic material under plane stress conditions was demonstrated in [20], [21]. Irwin also introduced the energy release rate which means a rate of the energy, per unit length along the crack edge, that is supplied by the elastic energy in the body and by the loading system in creating the new fracture surface. He showed that the energy release rate is described by the elastic stress intensity factor.

Another idea that has been important in the evolution of fracture mechanics concerns the size scale over which different phenomena dominate. The idea is implicit in Irwin's stress intensity factor concept and it is a central feature of the crack tip cohesive zone model introduced by Barenblatt (1959)([1], [2], [3]). He considered a planar crack in a body subjected to tensile loading normal to the plane of the crack and supposed that the material response is linearly elastic, except for a region near the crack edge where the response departs from linearity. The source of nonlinearity can be plastic deformation, diffuse microcracking, nonlinear interatomic forces, or some other physical mechanism. The crack tip region is said to be autonomous at fracture initiation or during crack growth if the following two conditions are met:

1. the extent of the region of nonlinearity from the crack edge is very small compared to all other length dimensions of the body and loading system,

2. the mechanical state within this end region at incipient growth or during growth is independent of loading and geometrical configuration.

For the particular case of an elastic-plastic material, the property of autonomy implies that the crack tip plastic zone is completely surrounded by an elastic stress intensity factor field and that the state within the plastic zone is determined by the level of stress intensity of the surrounding field. This situation was termed small-scale yielding by Rice (1968)[38]. In this situation the stress intensity factor is a useful fracture characterizing parameter. Rice has made out standing contributions to virtually all fields in crack and fracture mechanics including the introduction of the J-integral concept for crack analysis [38], which laid the foundation of the nonlinear fracture mechanics and three-dimensional dynamic crack propagation. His impact on the whole field has been singular and enormous. Moreover, he [38] dealt with the application of linear elasticity to fracture and discussed dynamic running crack problems, the energy rate computations and the stress concentrations at smooth-ended notches.

In the dynamic field, the significant and pioneering contributions by Freund (see, [15]) deserve particular mention. Among his numerous contributions may be mentioned a series of four papers (1970's)([11], [12], [13], [14]) on crack propagation with nonconstant velocity and other dynamic problems, such as stress wave interaction with cracks. Friedman and Liu (1996)[16] described the energy release rate at the crack tip following [38] and [15]. Friedman, Hu and Velazquez (1998)[17] analyzed an asymptotic solution of fields near the moving crack tip. The coefficients of leading terms in this solution coincide with stress intensity factors. When a crack propagates in an elastic medium, the stress intensity factors evolve with the crack tip. Then, they (2000)[18] derived formulae which describe the evolution of these stress intensity factors for a homogeneous isotropic medium under plane strain conditions. At present, it is well known that there are some criteria which determine the crack extension. Ohtsuka [36] introduced the three famous criteria in homogeneous isotropic elastic plates and showed the crack extension is described by the stress intensity factor.

Various boundary value problems in smooth domains have already been investigated by many authors until now. However, in order to study more real physical phenomena it is necessary to treat such problems in less smooth domains, whose boundaries have corners or cuts or cracks. Constanda [4], [5] investigated Reissner-Mindlin-type model of bending of plates, which is an improvement of Kirchhoff's classical model in the sense of taking into account the effects of transverse shear deformation. Chudinovich and Constanda (2000)[5] studied the initial-boundary value problems for plates with transverse shear deformation in interior and exterior smooth domains without any cracks under Dirichlet and Neumann boundary conditions. And they proved the existence of a unique weak solution in Sobolev-type spaces. Before this they (1999)[4] investigated the existence and uniqueness of a weak solution represented by single and double layer potentials with distributional densities in the cases of both an infinite and a finite plate with a crack and its continuous dependence on the data. And Popelar and Atkinson (1980)[37] treated the dynamic propagation of a semi-infinite crack in an infinite linear viscoelastic strip subjected to mode 1 loading.

On the other hand, Kawashima and Shibata studied the initial-boundary value problem for nonlinear hyperbolic system of second order with third order viscosity. They showed in [24] the global existence of smooth solutions to the above problem for small and smooth initial data in a bounded domain, without any cracks, of n-dimensional Euclidean space with a smooth boundary. Nonlinear viscoelasticity was treated as an application of it.

In this thesis, as the first step to understand fracture phenomena that are supposed to occur in the case of earthquakes from the mathematical viewpoint, we deal with boundary value problems and crack propagation in elastic or viscoelastic media with cracks. This thesis is organized as follows.

In Chapter 2, we study a problem in a two-dimensional infinite elastic strip with a semi-infinite crack. On the boundaries of the strip asymmetric conditions are imposed for the purpose of determining the direction of crack propagation as the next problem. This problem leads to a singular integral equation by the potential theory. By proving the compactness of singular integral operator and using the results in [25], [33], [40], the existence of a unique solution is shown by the Fredholm alternative.

In Chapter 3, following Griffith's theory we apply the maximum energy release rate criterion of three famous criteria in [36], (see for example [41]). For virtual crack extension, by using the results of [35], [36], an energy release rate due to non-smooth crack growth can be represented by calculating the potential energy function. And in our situation we show that the direction of kinked crack extension can be determined only by the surface force without using the stress intensity factor.

In Chapter 4, we study an initial-boundary value problem in an infinite linear viscoelastic strip with a semi-infinite fixed crack. Taking into account an effect of dissipative forces, we consider a linear viscoelastic model constructed in [30] in a two-dimensional strip with a semi-infinite fixed crack, similar to the situation in [22], and define a weak solution of this problem. We prove the existence of a unique weak solution to the Laplace transformed problem by virtue of the Riesz theorem. Then by the Parseval's equality and the method in [5] we show that the solution of the transformed problem is a unique solution of the original viscoelastic problem and the latter solution belongs to the inverse Laplace transformed space corresponding to the Sobolev-type space in [5].

Chapter 2

A boundary value problem for an infinite elastic strip with a semi-infinite crack

In Chapter 2, we study a boundary value problem for an infinite elastic strip with a semi-infinite crack. By using the single and double layer potentials this problem is reduced to a singular integral equation, which is uniquely solved in the Hölder spaces by the Fredholm alternative.

2.1 Preliminaries

By $u = (u_i)_{i=1,2,3}$, $\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}$ and $\sigma = (\sigma_{ij})_{i,j=1,2,3}$ we denote the displacement vector, the strain tensor and the stress tensor, respectively. The linear elasticity equations for a homogeneous isotropic material consist of the constitutive law (Hooke's law)

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}, \quad i, j = 1, 2, 3 \tag{2.1}$$

and the equilibrium conditions without any body forces

$$\frac{\partial}{\partial x_j}\sigma_{ij} = 0, \quad i, j = 1, 2, 3. \tag{2.2}$$

Here and in what follows we use the summation convention. λ and μ are *Lamé* constants, δ_{ij} is the Kronecker's delta and the strain-displacement relation is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad u_{i,j} = \partial_j u_i, \quad i,j = 1,2,3. \tag{2.3}$$

In the state of a plane strain, the 3rd component u_3 of the displacement u is zero, while the components u_1 and u_2 are functions of x_1 and x_2 only, hence $\varepsilon_{i3} = 0$, $\sigma_{13} = \sigma_{23} = 0$. Let $\Omega = \{(x_1, x_2) \mid x_1 \in \mathbf{R}, -a < x_2 < a\}$ $(a > 0)$ be a strip in $\mathbf{R}^2$, representing a homogeneous elastic plate. Then (2.2) gives the system of equations

$$A\left(\partial_x\right) u = 0 \tag{2.4}$$

for $u = (u_1, u_2)^{\mathrm{T}}$, where $A(\partial_x) = A\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$,

$$A(\xi_1, \xi_2) = \begin{pmatrix} \mu\xi^2 + (\lambda+\mu)\xi_1^2 & (\lambda+\mu)\xi_1\xi_2 \\ (\lambda+\mu)\xi_1\xi_2 & \mu\xi^2 + (\lambda+\mu)\xi_2^2 \end{pmatrix}, \quad \xi^2 = \xi_1^2 + \xi_2^2.$$

We assume that shearing strain $\mu > 0$, modulus of compression $3\lambda + 2\mu \geq 0$, in which case it is easy to see that the operator A is elliptic. Moreover we introduce the boundary stress operator $T(\partial_x) = T\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ defined by

$$T(\xi_1, \xi_2) = \begin{pmatrix} (\lambda+2\mu)\nu_1\xi_1 + \mu\nu_2\xi_2 & \mu\nu_2\xi_1 + \lambda\nu_1\xi_2 \\ \lambda\nu_2\xi_1 + \mu\nu_1\xi_2 & \mu\nu_1\xi_1 + (\lambda+2\mu)\nu_2\xi_2 \end{pmatrix},$$

where $\nu = (\nu_1, \nu_2)^{\mathrm{T}}$ is the unit outward normal to $\partial\Omega$. In the case of $\nu = (0,1)^{\mathrm{T}}$

$$T(\xi_1, \xi_2) = \begin{pmatrix} \mu\xi_2 & \mu\xi_1 \\ \lambda\xi_1 & (\lambda+2\mu)\xi_2 \end{pmatrix}.$$

We denote by $\Gamma = \{(x_1, 0) \mid -\infty < x_1 \leq 0\}$ the crack in Ω. On the crack we assume the free traction condition

$$\sigma_{ij}^{+}\nu_j = \sigma_{ij}^{-}\nu_j = 0 \quad \text{on} \quad \Gamma^{\pm}, \tag{2.5}$$

where $\Gamma^{\pm}$ means both sides of Γ. Here for every $x \in \Gamma$ $\sigma_{ij}^{\pm} = \sigma_{ij}^{\pm}(x)$ means the limit of $(\nu_x, \sigma_{ij}(\bar{x}))$ as $\bar{x} \in \Omega \setminus \Gamma$ tends to $x \in \Gamma$ along the normal ν_x, in this case $\nu_x = (0, \mp 1)$. The limit values σ_{ij}^+ and σ_{ij}^- may be different in general, therefore σ_{ij} may have a jump on Γ. At the end-point $(0,0)$ of Γ we assume

$$\lim_{x_1 \to -0} \sigma_{ij}^{\pm} \nu_j \Big|_{x \in \Gamma^{\pm} \setminus \{(0,0)\}} = 0.$$

On $\partial\Omega_+ = \{(x_1, a) \mid x_1 \in \mathbf{R}\}$, $\partial\Omega_- = \{(x_1, -a) \mid x_1 \in \mathbf{R}\}$ $(a > 0)$ the boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega_-, \tag{2.6}$$

$$\sigma_{ij}\nu_j = p_i \quad \text{on} \quad \partial\Omega_+ \tag{2.7}$$

are imposed, where p_i are given continuous functions on $\partial\Omega_+$.

We introduce the class $\mathcal{K}$ of functions $u(x)$ with the properties (cf. [28]):

1) $u \in C^0(\overline{\Omega \setminus \Gamma}) \cap C^2(\Omega \setminus \Gamma)$,

2) $\nabla u \in C^0(\overline{\Omega \setminus \Gamma} \setminus \{(0,0)\})$,

3) in the neighborhood of $(0,0)$ there exist positive constant C and $\epsilon > -1$ such that

$$\mid \nabla u(x) \mid \leq C \mid x \mid^{\epsilon} \quad \text{as} \quad x \to 0, \tag{2.8}$$

4) for every $x \in \partial\Omega_\pm$ there exists a uniform limit of $(\nu_x, \nabla_{\bar{x}} u(\bar{x}))$ as $\bar{x} \in \Omega \setminus \Gamma$ tends to $x \in \partial\Omega_\pm$ along the normal $-\nu_x$.

We define the internal energy density by

$$E(u,u) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left\{\lambda(u_{1,1} + u_{2,2})^2 + 2\mu(u_{1,1}^2 + u_{2,2}^2) + \mu(u_{1,2} + u_{2,1})^2\right\}.$$

Then it is easy to see that $E(u,u)$ is a nonnegative quadratic form and that $E(u,u) = 0$ if and only if u is a rigid displacement

$$u = (c_1 + c_0 x_2, c_2 - c_0 x_1)^{\mathrm{T}} \tag{2.9}$$

with arbitrary constants c_0, c_1 and c_2. It is easily seen that

$$F_1 = (1,0)^{\mathrm{T}}, \quad F_2 = (0,1)^{\mathrm{T}}, \quad F_3 = (x_2, -x_1)^{\mathrm{T}}$$

consist of a basis of the space of such rigid displacements. For the matrix

$$F = \begin{pmatrix} F_1, & F_2, & F_3 \end{pmatrix}$$

it is clear that $AF = 0$ in $\mathbf{R}^2$, $TF = 0$ on $\partial\Omega_\pm \cup \Gamma$, and a generic vector of the form (2.9) can be written as Fk with an arbitrary constant vector k.

Furthermore, we introduce the class $\wp = \{u \mid u \to 0 \text{ as } \mid x \mid \to \infty\}$. One can easily verify for $u \in C^2(\Omega \setminus \Gamma) \cap C^1(\overline{\Omega \setminus \Gamma}) \cap \wp$

$$\int_{\Omega\setminus\Gamma} F^{\mathrm{T}} Au \,\mathrm{d}a = \int_{\partial\Omega_\pm} F^{\mathrm{T}} Tu \,\mathrm{d}s + 2\int_\Gamma F^{\mathrm{T}} Tu \,\mathrm{d}s.$$

Also, if $u \in C^2(\Omega \setminus \Gamma) \cap C^1(\overline{\Omega \setminus \Gamma}) \cap \wp$ is a solution of (2.4) in $\Omega \setminus \Gamma$, then

$$2\int_{\Omega\setminus\Gamma} E(u,u) \,\mathrm{d}a = \int_{\partial\Omega_\pm} u^{\mathrm{T}} Tu \,\mathrm{d}s + 2\int_\Gamma u^{\mathrm{T}} Tu \,\mathrm{d}s. \tag{2.10}$$

Indeed, Divergence Theorem and (2.4) yield that for any $u \in C^2(\Omega \setminus \Gamma) \cap C^1(\overline{\Omega \setminus \Gamma}) \cap \wp$

$$0 = \int_{\Omega\setminus\Gamma} u^{\mathrm{T}} Au \,\mathrm{d}a = -2\int_{\Omega\setminus\Gamma} E(u,u) \,\mathrm{d}a + \int_{\partial\Omega_\pm} u^{\mathrm{T}} Tu \,\mathrm{d}s + 2\int_\Gamma u^{\mathrm{T}} Tu \,\mathrm{d}s.$$

2.2 Integral equations on the boundary

It is well known that the fundamental matrix of $A(\partial_x)$ is given by

$$D(x,y) = A^*(\partial_x) t(x,y),$$

where A^* is the adjoint operator of A and $t(x,y)$ is a fundamental solution of $\mu(\lambda + 2\mu)\Delta^2$,

$$t(x,y) = -\{8\pi\mu(\lambda + 2\mu)\}^{-1} \mid x - y \mid^2 \ln \mid x - y \mid.$$

Hence, $D(x,y)$ is given explicitly by

$$D(x,y) = -\frac{1}{4\pi\mu(\tilde{\mu}+1)} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{2.11}$$

$$
\begin{aligned}
D_{11} &= 2\tilde{\mu}\ln \mid x-y \mid +2\tilde{\mu}-1+2\frac{(x_2-y_2)^2}{\mid x-y\mid^2},\\
D_{12} &= D_{21} = -2\frac{(x_1-y_1)(x_2-y_2)}{\mid x-y\mid^2},\\
D_{22} &= 2\tilde{\mu}\ln \mid x-y \mid +2\tilde{\mu}-1+2\frac{(x_1-y_1)^2}{\mid x-y\mid^2},\\
\tilde{\mu} &= \frac{\lambda+3\mu}{\lambda+\mu}.
\end{aligned}
$$

In view of (2.11), $D(x,y)=D(y,x)=D(y,x)^{\mathrm{T}}$.
Along with $D(x,y)$ we consider the matrix of singular solutions

$$
P(x,y)=(T(\partial_y)D(y,x))^{\mathrm{T}},
$$

which is written explicitly as

$$
\begin{aligned}
P(x,y) \quad = \quad & -\frac{1}{2\pi}\left(\frac{\partial}{\partial\nu_y}\ln\mid x-y\mid I+\frac{\tilde{\mu}-1}{\tilde{\mu}+1}\frac{\partial}{\partial\tau_y}\ln\mid x-y\mid\tilde{I}\right.\\
& \left.+\frac{2}{\tilde{\mu}+1}\tilde{I}\frac{\partial}{\partial\tau_y}\frac{(x-y)^{\mathrm{T}}(x-y)}{\mid x-y\mid^2}\right)
\end{aligned}
\tag{2.12}
$$

with $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\tilde{I}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ and $\tau=(\tau_1,\tau_2)^{\mathrm{T}}$ a unit tangential vector to $\partial\Omega_\pm\cup\Gamma$.

It is easily verified that the columns of $D(x,y)$ and $P(x,y)$ are solutions of equation (2.4) for any $x\in\mathbf{R}^2$, $y\in\partial\Omega_\pm\cup\Gamma$, $x\neq y$, and that

$$
D(x,y)=O(\ln\mid x\mid),\ P(x,y)=O(\mid x\mid^{-1})\ \text{as}\ \mid x\mid\to\infty. \tag{2.13}
$$

Now we denote by $\tilde{D}$ and $\tilde{P}$ the reflection of $D(x,y)$ and $P(x,y)$ with respect to $\partial\Omega_-=\{(x_1,-a)\mid x_1\in\mathbf{R}\}$

$$
\tilde{D}\,(x,y)=D\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}y_1\\y_2\end{pmatrix}\right)-D\left(\begin{pmatrix}x_1\\-2a-x_2\end{pmatrix},\begin{pmatrix}y_1\\y_2\end{pmatrix}\right), \tag{2.14}
$$

$$
\tilde{P}\,(x,y)=P\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}y_1\\y_2\end{pmatrix}\right)-P\left(\begin{pmatrix}x_1\\-2a-x_2\end{pmatrix},\begin{pmatrix}y_1\\y_2\end{pmatrix}\right). \tag{2.15}
$$

Then it is obvious that the columns of $\tilde{D}(x,y)$ and $\tilde{P}(x,y)$ vanish on $\partial\Omega_-$.

Using a potential theory, we will find a solution of problem (2.4)–(2.7) in the form

$$u(x_1,x_2)=\tilde{V}_{\partial\Omega_+}(g)+\tilde{V}_\Gamma(f)+\tilde{W}_\Gamma(g), \tag{2.16}$$

where

$$\begin{aligned}
\tilde{V}_{\partial\Omega_+}(g) &= \int_{\partial\Omega_+}\tilde{D}(x,y)g(y)\,\mathrm{d}y_1,\\
\tilde{V}_\Gamma(f) &= \int_\Gamma \tilde{D}(x,y)f(y)\,\mathrm{d}y_1,\\
\tilde{W}_\Gamma(g) &= \int_\Gamma \tilde{P}(x,y)g(y)\,\mathrm{d}y_1.
\end{aligned}$$

Now let us introduce function spaces. By $C^{0,\alpha}(G)$ we denote a Hölder space with exponent $\alpha\in(0,1)$ of functions defined on a domain G and by $C^{1,\beta}(G)$ the subspace of functions of C^1-class whose first order derivatives belong to $C^{0,\beta}(G)$, $\beta\in(0,1)$. If $(f,g)\in C^{0,\alpha}(\Gamma)\times(C^{0,\alpha}(\partial\Omega_+)\times C^{1,\beta}(\Gamma))$, then it is easily seen that u defined by (2.16) is continuous on $\partial\Omega_+\cup\Gamma^\pm$ and satisfies (2.4) and (2.6). In order to see that u satisfies boundary conditions (2.5) and (2.7) we substitute (2.16) into (2.5) and (2.7) so that we deduce the integral equations for g (cf. [7], [40]). From (2.7) it follows

$$\begin{aligned}
&\frac{1}{2}g\begin{pmatrix} x_1\\ a\end{pmatrix}+\mathrm{v.p.}\int_{\partial\Omega_+}T\tilde{D}\left(\begin{pmatrix} x_1\\ a\end{pmatrix},\begin{pmatrix} y_1\\ a\end{pmatrix}\right)g\begin{pmatrix} y_1\\ a\end{pmatrix}\,\mathrm{d}y_1\\
&\quad+\int_\Gamma T\tilde{D}\left(\begin{pmatrix} x_1\\ a\end{pmatrix},\begin{pmatrix} y_1\\ 0\end{pmatrix}\right)f\begin{pmatrix} y_1\\ 0\end{pmatrix}\,\mathrm{d}y_1\\
&\quad+\int_\Gamma T\tilde{P}\left(\begin{pmatrix} x_1\\ a\end{pmatrix},\begin{pmatrix} y_1\\ 0\end{pmatrix}\right)g\begin{pmatrix} y_1\\ 0\end{pmatrix}\,\mathrm{d}y_1=\begin{pmatrix} p_1\\ p_2\end{pmatrix},
\end{aligned}\tag{2.17}$$

where the integral on $\partial\Omega_+$ means a principal value. Let

$$\begin{aligned}
Q(x,y) &= -\frac{2\mu}{\pi(\tilde{\mu}+1)}\left(\ln\mid x-y\mid I-I+\frac{(x-y)^{\mathrm{T}}(x-y)}{\mid x-y\mid^2}\right),\\
\tilde{Q}(x,y) &= Q\left(\begin{pmatrix} x_1\\ x_2\end{pmatrix},\begin{pmatrix} y_1\\ y_2\end{pmatrix}\right)-Q\left(\begin{pmatrix} x_1\\ -2a-x_2\end{pmatrix},\begin{pmatrix} y_1\\ y_2\end{pmatrix}\right).
\end{aligned}$$

Then

$$T\tilde{P} = -\frac{\partial^2}{\partial\tau_x\partial\tau_y}\tilde{Q}.$$

Substituting (2.16) with $\tilde{P}$ replaced by $\tilde{Q}$ into (2.5) yields

$$\begin{aligned}
&\pm\frac{1}{2}f\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \,\mathrm{d}y_1 \\
&\quad +\text{v.p.}\int_{\Gamma^\pm} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) f\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \,\mathrm{d}y_1 \\
&\quad -\frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\Bigg|_{y_1=-\infty}^{0} \\
&\quad +\text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \,\mathrm{d}y_1 \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned} \tag{2.18}$$

where the integrals on Γ are taken as principal values. The upper and lower signs correspond to the integrals on Γ^+ and Γ^-, respectively. One can easily check that the solution u of the form (2.16) satisfies condition (2.8) (cf. [27]). Subtracting two equations in (2.18) implies

$$f\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \Gamma. \tag{2.19}$$

Therefore the integral equation (2.17) on $\partial\Omega_+$ becomes

$$\left(Z + \frac{1}{2}I\right) g = p \quad \text{on} \quad \partial\Omega_+ \tag{2.20}$$

with $Z = T(\tilde{V}_{\partial\Omega_+} + \tilde{W}_\Gamma)$. And adding two equations in (2.18), we obtain

$$\begin{aligned}
&\int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \, \mathrm{d}y_1 \\
&\qquad - \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\Bigg|_{y_1=-\infty}^{0} \\
&\qquad + \text{v.p.} \int_{\Gamma^\pm} \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \, \mathrm{d}y_1 \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned} \tag{2.21}$$

hence

$$\begin{aligned}
&\text{v.p.} \int_\Gamma \frac{\partial}{\partial y_1} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \frac{1}{x_1 - y_1} \, \mathrm{d}y_1 \\
&\qquad + \text{v.p.} \int_\Gamma \left\{ \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) - \frac{1}{x_1 - y_1} \right\} \frac{\partial}{\partial y_1} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \, \mathrm{d}y_1 \\
&= -\int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \, \mathrm{d}y_1 \\
&\qquad + \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\Bigg|_{y_1=-\infty}^{0}.
\end{aligned} \tag{2.22}$$

Now we introduce the new space $C_\gamma^{0,\alpha}(G)$ defined by

$$C_\gamma^{0,\alpha}(G) = \{ f(x) \in C^{0,\alpha}(G) \mid f(x) = O(\mid x \mid^{-\gamma}) \text{ as } \mid x \mid \to \infty \} \quad (1 < \gamma)$$

equipped with the norm

$$\parallel g \parallel_{\gamma,\alpha} = \parallel g \parallel_{\gamma,\infty} + \mid g \mid_\alpha, \tag{2.23}$$

$$\parallel g \parallel_{\gamma,\infty} = \sup_{x \in G} \mid (1 + \mid x \mid^\gamma) g(x) \mid, \quad \mid g \mid_\alpha = \sup_{x, \tilde{x} \in G, x \neq \tilde{x}} \frac{\mid g(x) - g(\tilde{x}) \mid}{\mid x - \tilde{x} \mid^\alpha}.$$

Let $g \in C_\gamma^{0,\beta}(\Gamma)$ and vanish at the end of crack. Inverting the singular integral operator (2.22), we arrive at the integral equation of the second kind (cf.[33])

$$(I - Y_1)\frac{\partial}{\partial x_1} g(x) = \frac{1}{\pi^2 R(x)} \int_{-R}^{0} \frac{R(y)\,\mathrm{d}y_1}{y - x} \int_{\partial\Omega_+} T\tilde{D}(y,z) g(z)\,\mathrm{d}z_1,$$

$$\text{as} \quad R \to \infty, \quad x \in \Gamma, \quad (2.24)$$

where the integral on Γ is in the sense of principal value and

$$Y_1(f(x)) = \frac{1}{\pi^2 R(x)} \int_{-R}^{0} \frac{R(y)\,\mathrm{d}y_1}{y - x} \int_{\Gamma} \left(\frac{\partial}{\partial \tau_z} \tilde{Q}(z,y) - \frac{1}{z - y} \right) f(z)\,\mathrm{d}z_1,$$

$$R(x) = \sqrt{(x+R)x}.$$

2.3 Uniqueness and existence of solution

In this section we prove that problem (2.4)–(2.7) has a unique solution.

THEOREM 1. *Problem* (2.4)–(2.7) *has at most one solution of class* $\mathcal{K} \cap \wp$.

Proof. Let $\hat{u}$ be the difference of two solutions of class $\mathcal{K} \cap \wp$ to problem (2.4)–(2.7). Then, $\hat{u}$ satisfies (2.4)–(2.7) with $p = 0$. Therefore, (2.10) implies

$$E(\hat{u}, \hat{u}) = 0 \quad \text{in} \quad \Omega \setminus \Gamma.$$

Hence, $\hat{u}$ is of the form (2.9) in $\overline{\Omega \setminus \Gamma}$. Since $\hat{u} \in \wp$, we conclude that $\hat{u}(x) = 0$, $x \in \overline{\Omega \setminus \Gamma}$. $\square$

From (2.11), (2.12), (2.14), (2.15) and straightforward calculation one can easily obtain the following lemma. Similar result is proved in [8] in the case of a compact boundary.

LEMMA 1. *If* $f \in C_\gamma^{0,\alpha}(\partial\Omega_\pm \cup \Gamma)$, *then*
(i) $\tilde{W} f \in \wp$,

(ii) $\tilde{V} f \in \wp$.

Next we will prove the existence of the solution. As shown in the previous section, problem (2.4)–(2.7) is reduced to integral equation (2.20) for g on $\partial\Omega_+$. Since the kernels of Z are $1-$ *singular kernels* on $\partial\Omega_+$ defined below, it is not so easy to solve it.
Here upon, following [7], we call a matrix function $k(x,y)$ defined for all $x \in \partial\Omega_+$ and $y \in \partial\Omega_+$, $x \neq y$, and continuous there an $\omega -$ *singular kernel* on $\partial\Omega_+$, $\omega \in [0,1]$ if there exists a positive constant m such that

$$\mid k(x,y) \mid \leq m \mid x-y \mid^{-\omega} \quad \text{for all} \quad x,\ y \in \partial\Omega_+,\ x \neq y.$$

If an $\omega -$ *singular kernel* $k(x,y)$ on $\partial\Omega_+$ satisfies

$$\mid k(x,y) - k(\tilde{x},y) \mid \leq m \mid x-\tilde{x} \mid\mid x-y \mid^{-\omega-1}$$

for all $x,\ \tilde{x} \in \partial\Omega_+$ and $y \in \partial\Omega_+$, $0 <\mid x-\tilde{x} \mid< \frac{1}{2} \mid x-y \mid$, then $k(x,y)$ is called a *proper* $\omega -$ *singular kernel* on $\partial\Omega_+$.

THEOREM 2. *If $k(x,y)$ is a proper $\omega-$singular kernel on $\partial\Omega_+$, $\omega \in [0,1)$, $k(x,y) = k(y,x)$ and $k(x,y) = O(\mid x \mid^{-1})$ as $\mid x \mid \to \infty$ for any $y \in \partial\Omega_+$, then operator K defined on $C^{0,\alpha}_{\gamma}$ by*

$$(Kg)(x) = \int_{\partial\Omega_+} k(x,y) g(y)\, \mathrm{d}y, \quad x \in \partial\Omega_+$$

is compact.

Proof. This theorem was proved in [7] in the case of a compact domain. In the case where $\partial\Omega_+$ is unbounded, however, the compactness of K is not a direct consequence of that in the compact domain. We prove here that K as a mapping from $C^{0,\alpha}_{\gamma}(\partial\Omega_+)$ to $C^{0,\alpha}_{\tilde{\gamma}}(\partial\Omega_+)$, $\gamma > \tilde{\gamma} > 1$, with $\alpha = 1-\omega$ for $\omega \in (0,1)$ and any $\alpha \in (0,1)$ for $\omega = 0$ is compact.

Let M_1 be a bounded set in $C^{0,\alpha}_{\gamma}(\partial\Omega_+)$, that is, there exists a positive constant c such that

$$\parallel g \parallel_{\gamma,\alpha} \leq c \quad \text{for all} \quad g \in M_1, \tag{2.25}$$

and let $\{\theta_n\}_{n=1}^{\infty} \subset M_2 = K(M_1)$. Then there exists a sequence $\{g_n\}_{n=1}^{\infty}$ in M_1 such that $\theta_n = Kg_n$, $n = 1, 2, 3, ...$ It is obvious that $\theta_n \in C^{0,\alpha}(\partial\Omega_+)$.
(2.23), (2.25) imply that $\{g_n\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $C(\partial\Omega_+)$. Thus by applying *Ascoli* $-$ *Arzelà*'s theorem there exists a

uniformly convergent subsequence of $\{g_n\}_{n=1}^{\infty}$, which is denoted by $\{g_n\}_{n=1}^{\infty}$ for simplicity, and a $g \in C(\partial\Omega_+)$ such that

$$\| g_n - g \|_{\gamma,\infty} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.26}$$

Let $\theta = Kg$. Then, $\theta \in C^{0,\alpha}_{\tilde{\gamma}}(\partial\Omega_+)$ for some constant $\tilde{\gamma}$, $1 < \tilde{\gamma} < \gamma$. Really, we have

$$\begin{aligned}
| \theta_n(x) - \theta(x) | &\leq \int_{\partial\Omega_+} | k(x,y) || g_n(y) - g(y) | \ \mathrm{d}y_1 \\
&\leq c_1 \frac{1}{| x |^{\tilde{\gamma}}} \sup_{y\in\partial\Omega_+} \frac{| g_n(y) - g(y) |}{| 1 - \frac{y}{x} |^{\tilde{\gamma}} | x - y |^{-\gamma}} \int_{\partial\Omega_+} \frac{| k(x,y) |}{| x - y |^{\gamma-\tilde{\gamma}}} \, \mathrm{d}y_1,
\end{aligned}$$

consequently,

$$| \theta_n - \theta | (x) \leq c_2 | x |^{-\tilde{\gamma}} \| g_n - g \|_{\gamma-\tilde{\gamma},\infty}, \quad n = 1, 2, 3, ... \tag{2.27}$$

with some positive constants c_1, c_2. Since $k(x,y)$ is a *proper* $\omega -$ *singular kernel*,

$$\begin{aligned}
&| K(g_n - g)(x) - K(g_n - g)(\tilde{x}) | \\
&\quad = \left| \int_{\partial\Omega_+} [k(x,y) - k(\tilde{x},y)](g_n - g)(y) \, \mathrm{d}y_1 \right| \\
&\quad \leq c_3 | x - \tilde{x} |^{\alpha} \sup_{y\in\partial\Omega_+} | (1 + | y |^{\gamma})(g_n - g)(y) | .
\end{aligned}$$

Hence,

$$| \theta_n - \theta |_{\alpha} \leq c_3 \| g_n - g \|_{\gamma,\infty}, \quad n = 1, 2, 3, ... \tag{2.28}$$

The assertion now follows from the fact that the constants c_1, c_2, c_3 are independent of x and $\tilde{x}$. (2.27), (2.28), (2.23) and (2.26) yield

$$\| \theta_n - \theta \|_{\tilde{\gamma},\alpha} \to 0 \quad \text{as} \quad n \to \infty,$$

which proves that $K : C^{0,\alpha}_{\gamma}(\partial\Omega_+) \to C^{0,\alpha}_{\tilde{\gamma}}(\partial\Omega_+)$ is compact. $\square$

THEOREM 3. *Problem* (2.4)–(2.7) *has a unique solution* $u \in \mathcal{K} \cap \wp$ *for any* $p \in C^{0,\alpha}_{\gamma}(\partial\Omega_+)$ *with any* $\alpha \in (0,1)$ *and any* $\gamma > 1$.

Proof. In (2.20) Z is represented as $Zg = Z_1 g + Z_2 g$, where

$$\begin{aligned} Z_1 g &= \text{v.p.} \int_{\partial\Omega_+} \frac{1}{x_1 - y_1} g \begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1, \\ Z_2 g &= (Z - Z_1) g. \end{aligned} \tag{2.29}$$

Then Z_1 has a 1-singular kernel and Z_2 is a non-singular operator. Applying the operator $(Z_1 - \frac{1}{2}I)$ to both sides of (2.20) yields

$$\left((Z_1)^2 + Z_1 Z_2 - \frac{1}{2} Z_2 - \frac{1}{4} I \right) g = \left(Z_1 - \frac{1}{2} I \right) p. \tag{2.30}$$

Here we claim that

$$\begin{aligned} &((Z_1)^2 g)(x) \\ &= \int_{\partial\Omega_+} \frac{1}{x-y} \left[\int_{\partial\Omega_+} \frac{g(z)}{y-z} \mathrm{d}z_1 \right] \mathrm{d}y_1 \\ &= -\pi^2 g(x) + \int_{\partial\Omega_+} \left[\int_{\partial\Omega_+} \frac{g(z)}{(x-y)(y-z)} \mathrm{d}y_1 \right] \mathrm{d}z_1. \end{aligned} \tag{2.31}$$

LEMMA 2. *If $g \in C^{0,\alpha}_\gamma(\partial\Omega_+)$, then* (2.31) *holds.*

Proof. In the case of a compact boundary (2.31) is well-known as a Poincaré-Bertrand formula ([33], §23). For convenience we consider the functions of a real variable $x = (x_1, x_2)$ as the functions of a complex variable $x = x_1 + ix_2$. Let $x = x_1 + ix_2$, $y = y_1 + iy_2$ and $z = z_1 + iz_2$. In the present case where $\partial\Omega_+$ is unbounded first we prove the formula

$$\begin{aligned} &\int_{\partial\Omega_+} \frac{1}{x-y} \left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{y-z} \mathrm{d}z_1 \right] \mathrm{d}y_1 \\ &\qquad = -\pi^2 \phi(x,x) + \int_{\partial\Omega_+} \left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{(x-y)(y-z)} \mathrm{d}y_1 \right] \mathrm{d}z_1 \end{aligned}$$

for $\phi \in C^{0,\alpha}_\gamma(\partial\Omega_+ \times \partial\Omega_+)$. Let

$$\begin{aligned} \Phi(t) &= \int_{\partial\Omega_+} \frac{1}{t-y} \left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{y-z} \mathrm{d}z_1 \right] \mathrm{d}y_1, \\ \Psi(t) &= \int_{\partial\Omega_+} \left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{(t-y)(y-z)} \mathrm{d}y_1 \right] \mathrm{d}z_1, \end{aligned}$$

where $t = t_1 + it_2$ is a point on the plane, not on $\partial\Omega_+$. Then,

$$\Phi(t) = \Psi(t) \tag{2.32}$$

holds. Indeed, it is sufficient to prove

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} \frac{1}{t-y} \left[\int_{y_1-\varepsilon}^{y_1+\varepsilon} \frac{\phi(y,z)}{y-z} \, \mathrm{d}z_1 \right] \mathrm{d}y_1 \;\rightarrow\; 0, \\ I_2 &= \int_{-\infty}^{\infty} \left[\int_{z_1-\varepsilon}^{z_1+\varepsilon} \frac{\phi(y,z)}{(t-y)(y-z)} \, \mathrm{d}y_1 \right] \mathrm{d}z_1 \;\rightarrow\; 0 \\ \text{as} \quad & \varepsilon \;\rightarrow\; 0^+. \end{aligned}$$

For I_1, we divide the integral over $(-\infty, \infty)$ three

$$\int_{-\infty}^{\infty} = \int_{R}^{\infty} + \int_{-\infty}^{-R} + \int_{-R}^{R}.$$

The above assertion for the third integral was proved in [33], so that it is sufficient to consider them for the first and second integrals. Since $\phi(y,z) \in C_{\gamma}^{0,\alpha}(\partial\Omega_+ \times \partial\Omega_+)$, the first integral can be estimated as follows when R is sufficiently large.

$$\begin{aligned} &\left| \int_{R}^{\infty} \frac{1}{t-y} \left[\int_{y_1-\varepsilon}^{y_1+\varepsilon} \frac{\phi(y,z)}{y-z} \, \mathrm{d}z_1 \right] \mathrm{d}y_1 \right| \\ &= \left| \int_{R}^{\infty} \frac{1}{t-y} \left[\int_{y_1-\varepsilon}^{y_1+\varepsilon} \left(\frac{\phi(y,z)-\phi(y,y)}{y-z} + \frac{\phi(y,y)}{y-z} \right) \mathrm{d}z_1 \right] \mathrm{d}y_1 \right| \\ &\leq \left| \int_{R}^{\infty} \frac{1}{t-y} \left[\int_{y_1-\varepsilon}^{y_1+\varepsilon} \left(\frac{C(\mid \phi(y,z) \mid + \mid \phi(y,y) \mid)^{\tilde{\alpha}}}{\mid y-z \mid^{1-(1-\tilde{\alpha})\alpha}} + \frac{\phi(y,y)}{y-z} \right) \mathrm{d}z_1 \right] \mathrm{d}y_1 \right| \\ &\leq \varepsilon^{(1-\tilde{\alpha})\alpha} \sup_{R-\varepsilon<y_1<\infty} \| y_1 \mid \phi(y,y)|^{\tilde{\alpha}} \\ &\rightarrow \;\; 0 \quad \text{as} \quad \varepsilon \;\rightarrow\; 0^+, \end{aligned}$$

where C is a constant and $1 > \alpha > \tilde{\alpha} > 0$. In the same way the second integral tends to 0 as $\varepsilon \rightarrow 0^+$. Similarly one can show that $I_2 \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

We denote by $\Phi^+(x)$ and $\Phi^-(x)$ the limits of $\Phi(t)$ as $t \to x$ from the upper and from the lower of $\partial\Omega_+$, respectively. By the Plemelj's formula, the relation

$$\Phi^+(x) + \Phi^-(x) = 2\int_{\partial\Omega_+} \frac{1}{x-y} \left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{y-z}\, \mathrm{d}z_1\right] \mathrm{d}y_1 \tag{2.33}$$

holds. Furthermore, $\Psi(t)$ is represented as

$$\Psi(t) = \int_{\partial\Omega_+} \frac{\psi(z;t)}{z-t}\, \mathrm{d}z_1, \tag{2.34}$$

$$\psi(z;t) = \int_{\partial\Omega_+} \left(\frac{1}{y-t} - \frac{1}{y-z}\right) \phi(y,z)\, \mathrm{d}y_1.$$

Denoting by $\psi^+(z;x)$ and $\psi^-(z;x)$ the limits of $\psi(z;t)$ as $t \to x$ from the upper and from the lower of $\partial\Omega_+$, respectively. Again by the Plemelj's formula we obtain

$$\begin{aligned} \psi^+(z;x) - \psi^-(z;x) &= 2\pi i\phi(x,z), \qquad (2.35) \\ \psi^+(z;x) + \psi^-(z;x) &= 2\int_{\partial\Omega_+} \left(\frac{1}{y-x} - \frac{1}{y-z}\right) \phi(y,z)\, \mathrm{d}y_1 \\ &= 2(z-x)\int_{\partial\Omega_+} \frac{\phi(y,z)}{(x-y)(y-z)}\, \mathrm{d}y_1. \end{aligned}$$

Put

$$\begin{aligned} \psi(z;t) &= \psi^+(z;x) + \varepsilon^+ \quad (\text{if } t \text{ is in the upper of } \partial\Omega_+), \\ \psi(z;t) &= \psi^-(z;x) + \varepsilon^- \quad (\text{if } t \text{ is in the lower of } \partial\Omega_+). \end{aligned} \tag{2.36}$$

Then it is obvious that $\varepsilon^+ \to 0$, $\varepsilon^- \to 0$ as $t \to x$. Moreover, one can prove

$$\int_{\partial\Omega_+} \frac{\varepsilon^+}{z-t}\, \mathrm{d}z_1 \to 0, \quad \int_{\partial\Omega_+} \frac{\varepsilon^-}{z-t}\, \mathrm{d}z_1 \to 0 \tag{2.37}$$

as $t \to x$ along $\pm\nu_x$. In fact,

$$\mid \varepsilon^+ \mid = \mid \psi(z;t) - \psi^+(z;x) \mid \leq C\delta^{\alpha(1-\tilde{\alpha})} \mid \psi(z;t) - \psi^+(z;x) \mid^{\tilde{\alpha}},$$

where C is a constant, $\delta = \mid t - x \mid$, and α, $\tilde{\alpha}$ are the same as above. Therefore

$$\left| \int_{\partial\Omega_+} \frac{\varepsilon^+}{z-t} \, \mathrm{d}z_1 \right| \leq C\delta^{\alpha(1-\tilde{\alpha})} \int_{\partial\Omega_+} \frac{\mid \psi(z;t) - \psi^+(z;x) \mid^{\tilde{\alpha}}}{\mid z-t \mid} \, \mathrm{d}z_1 \ \to 0 \quad \text{as} \quad \delta \to 0.$$

The case of ε^- can be treated in exactly the same manner. Replacing $\psi(z;t)$ in (2.34) by expression (2.36) and using (2.37), we obtain

$$\begin{aligned} \Psi^+(x) &= \pi i \psi^+(x;x) + \int_{\partial\Omega_+} \frac{\psi^+(z;x)}{z-x} \, \mathrm{d}z_1, \\ \Psi^-(x) &= -\pi i \psi^-(x;x) + \int_{\partial\Omega_+} \frac{\psi^-(z;x)}{z-x} \, \mathrm{d}z_1, \end{aligned}$$

hence by (2.35)

$$\begin{aligned} &\Psi^+(x) + \Psi^-(x) \\ &\quad = -2\pi^2\phi(x,x) + 2\int_{\partial\Omega_+} \left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{(x-y)(y-z)} \, \mathrm{d}y_1 \right] \mathrm{d}z_1. \end{aligned} \tag{2.38}$$

Since from (2.32) the left sides of (2.33) and (2.38) are equal, the formula is proved. Hence, for any $g \in C^{0,\alpha}_{\gamma}(\partial\Omega_+)$ and $x \in \partial\Omega_+$ (2.31) holds. $\square$

Now we return to the proof of THEOREM 3. Using Cauchy's integral theorem to the integral in the right-hand side of (2.31) yields

$$\begin{aligned} ((Z_1)^2 g)(x) &= -\pi^2 g(x) \\ &\quad + \int_{\partial\Omega_+} \left[\frac{1}{x-z} \left(\int_{\partial\Omega_+} \frac{\mathrm{d}y_1}{x-y} - \int_{\partial\Omega_+} \frac{\mathrm{d}y_1}{z-y} \right) g(z) \right] \mathrm{d}z_1 \\ &= -\pi^2 g(x). \end{aligned}$$

Hence, equation (2.30) can be written as

$$\left(Z_1 Z_2 - \frac{1}{2} Z_2 - (\frac{1}{4} + \pi^2) I \right) g = \left(Z_1 - \frac{1}{2} I \right) p. \tag{2.39}$$

It is easily seen that $Z_2 g$ satisfies the Lipschitz condition if $g \in C^{0,\alpha}_{\gamma}(\partial\Omega_+)$ and the right-hand side of (2.39) also belongs to $C^{0,\alpha}_{\tilde{\gamma}}(\partial\Omega_+)$ if $p \in C^{0,\alpha}_{\gamma}(\partial\Omega_+)$.

Since Z_1Z_2 and Z_2 have *proper* $0-$ *singular kernels*, by THEOREM 2, we can apply Fredholm's theorem to problem (2.39) in the dual system

$$\langle \bigcup_{\gamma<\gamma_0} C_\gamma^{0,\alpha}(\partial\Omega_+), \bigcup_{\tilde{\gamma}<\gamma<\gamma_0} C_{\tilde{\gamma}}^{0,\alpha}(\partial\Omega_+)\rangle$$

with a fixed $\gamma_0 > 1$ (cf. [9]).

We can apply the same argument to (2.24). The operator Y_1 can be decomposed into

$$Y_1 = Y_{11} + Y_{10},$$

where Y_{11} has a 1-singular kernel and Y_{10} is a non-singular operator. Similarly, if $\frac{\partial}{\partial x_1} g \in C_\gamma^{0,\beta}(\Gamma)$, then we can apply Fredholm's theorem in the dual system

$$\langle \bigcup_{\gamma<\gamma_0} C_\gamma^{0,\beta}(\Gamma), \bigcup_{\tilde{\gamma}<\gamma<\gamma_0} C_{\tilde{\gamma}}^{0,\beta}(\Gamma)\rangle.$$

It is not difficult to prove that u defined by (2.16) with g given above is a desired solution to problem (2.4)–(2.7). $\square$

Moreover, we require stronger regularity of g.

THEOREM 4. *If $p \in C_\gamma^{1,\alpha}(\partial\Omega_+)$, then $g \in C_\gamma^{1,\alpha}(\partial\Omega_+) \times C_\gamma^{2,\beta}(\Gamma)$ whose first order derivative vanishes at the crack tip.*

This THEOREM 4 can be proved in a similar way as in the proof of THEOREM 2.

Chapter 3

Propagation of cracks in an infinite elastic strip with a semi-infinite crack

In Chapter 3, we study a quasi-stationary model of crack propagation in an infinite elastic strip with a semi-infinite crack and how to determine the real crack propagation from virtual crack extension by applying maximum energy release rate criterion at the crack tip. Then we prove that the crack propagates the direction only given by a surface force.

3.1 The model of crack propagation

In this section we consider a quasi-stationary model of crack propagation. To obtain an explicit formula we adopt the energy criterion given by Griffith [19]. According to his theory, when a crack is extended, there is a flow of energy from the stress field in the body to the crack tip. This energy is stored on both faces of the newly enlarged crack. In the case of linear elasticity, we call the released potential energy G as the crack increases a unit area the energy release rate. Following [36], we represent G in the form

$$G = -\lim_{\varepsilon \to 0} \frac{\Pi(u_\varepsilon) - \Pi(u)}{\varepsilon}, \tag{3.1}$$

where Π is the potential energy functional defined by

$$\Pi(u) = \int_{\Omega\setminus\Gamma} E(u,u)\,\mathrm{d}x - \int_{\partial\Omega_\pm} s\cdot u\,\mathrm{d}x_1 \tag{3.2}$$

and $s = (s_i) = (\sigma_{ij}\nu_j) = Tu$.

Now let us consider the virtual kinked crack extension

$$\Gamma_\varepsilon = \{x_\varepsilon \mid x_\varepsilon = x_0 + \varepsilon X, x_0 \in \Gamma\} \tag{3.3}$$

with $X = (\cos\theta_0,\ \sin\theta_0)$ and $\varepsilon > 0$. This means that the virtual crack extension Γ_ε shifts with an angle θ_0 from Γ. Then we deduce the boundary value problem with respect to the displacement u_ε

$$(*)\begin{cases} Au_\varepsilon = 0 \quad \text{in} \quad \Omega\setminus\Gamma_\varepsilon, \\ Tu_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon^\pm, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_-, \\ Tu_\varepsilon = p \quad \text{on} \quad \partial\Omega_+, \end{cases}$$

where $\Gamma_\varepsilon^\pm$ mean both sides of Γ_ε. We seek a solution u_ε of problem $(*)$ in the form

$$u_\varepsilon = u + \varepsilon\hat{u}, \tag{3.4}$$

where u is a solution of problem (2.4)–(2.7). Differentiation of Tu_ε on $\Gamma_\varepsilon^\pm$ with respect to ε yields

$$0 = T\left(\frac{\partial u_\varepsilon}{\partial\varepsilon} + \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial}{\partial\varepsilon}\varepsilon\cos\theta_0 + \frac{\partial u_\varepsilon}{\partial x_2}\frac{\partial}{\partial\varepsilon}\varepsilon\sin\theta_0\right)\bigg|_{\Gamma_\varepsilon^\pm}.$$

Letting $\varepsilon \longrightarrow 0$, we get

$$T\left(\hat{u} + \frac{\partial u}{\partial x_1}\cos\theta_0 + \frac{\partial u}{\partial x_2}\sin\theta_0\right)\bigg|_{\Gamma^\pm} = 0.$$

In view of (2.4)–(2.7), (3.4) and $(*)$ we obtain the boundary value problem of $\hat{u}$:

$$(**)\begin{cases} A\hat{u} = 0 \quad \text{in} \quad \Omega\setminus\Gamma, \\ T\hat{u} = -T\left(\frac{\partial u}{\partial x_1}\cos\theta_0 + \frac{\partial u}{\partial x_2}\sin\theta_0\right) \quad \text{on} \quad \Gamma^\pm, \\ \hat{u} = 0 \quad \text{on} \quad \partial\Omega_-, \\ T\hat{u} = 0 \quad \text{on} \quad \partial\Omega_+. \end{cases}$$

Similarly for u we can apply the potential theory to problem $(**)$, so that the solution of $(**)$ is described in the form

$$\hat{u}(x_1, x_2) = \tilde{V}_{\partial\Omega_+}(h_1) + \tilde{V}_\Gamma(h_2) + \tilde{W}_\Gamma(h_1), \tag{3.5}$$

where $(h_2, h_1) \in C^{0,\alpha}_\gamma(\Gamma) \times (C^{0,\alpha}_\gamma(\partial\Omega_+) \times C^{1,\beta}_\gamma(\Gamma))$, $\gamma > 1$, have the similar properties as (f, g). In order for $\hat{u}$ in (3.5) to satisfy the boundary condition in $(**)$ we substitute (3.5) into $(**)$ and derive the integral equations on $\partial\Omega_+$ and Γ.

It is easily obtained

$$\begin{aligned}
&\frac{1}{2}h_1\begin{pmatrix} x_1 \\ a \end{pmatrix} + \text{v.p.}\int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ a \end{pmatrix}\,\mathrm{d}y_1 \\
&\quad + \int_\Gamma T\tilde{D}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\,\mathrm{d}y_1 \\
&\quad + \int_\Gamma T\tilde{P}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\,\mathrm{d}y_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\quad \text{on} \quad \partial\Omega_+.
\end{aligned} \tag{3.6}$$

It yields

$$\begin{aligned}
&\pm\frac{1}{2}h_2\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ a \end{pmatrix}\,\mathrm{d}y_1 \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\,\mathrm{d}y_1 \\
&\quad - \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\Bigg|_{y_1=-\infty}^{0} \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1}h_1\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\,\mathrm{d}y_1 \\
&= \; -T\left(\frac{\partial u}{\partial x_1}\cos\theta_0 + \frac{\partial u}{\partial x_2}\sin\theta_0\right) \quad \text{on} \quad \Gamma^\pm,
\end{aligned} \tag{3.7}$$

since h_1 vanishes at the crack tip. Note that

$$\begin{aligned}\frac{\partial^2}{\partial x_2 \partial \tau_x} \ln \mid x-y \mid &= \frac{\partial^2}{\partial x_1 \partial \nu_x} \ln \mid x-y \mid, \\ \frac{\partial^2}{\partial x_2 \partial \nu_x} \ln \mid x-y \mid &= -\frac{\partial^2}{\partial x_1 \partial \tau_x} \ln \mid x-y \mid .\end{aligned} \tag{3.8}$$

Then using integration by parts and THEOREM 4, we can rewrite (3.7) to

$$\begin{aligned}&\pm\frac{1}{2}h_2\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ a \end{pmatrix} \,\mathrm{d}y_1 \\ &\quad +\text{v.p.}\int_{\Gamma^\pm} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \,\mathrm{d}y_1 \\ &\quad +\text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial \tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1} h_1\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \,\mathrm{d}y_1 \\ &= -\Bigg\{\Bigg(\int_{\partial\Omega_+} \frac{\partial}{\partial x_1} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \,\mathrm{d}y_1 \\ &\quad +\text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial \tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial^2}{\partial y_1^2} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \,\mathrm{d}y_1\Bigg)\cos\theta_0 \\ &\quad +\Bigg(\int_{\partial\Omega_+} \frac{\partial}{\partial x_2} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \,\mathrm{d}y_1 \pm \frac{1}{2}\frac{\partial^2}{\partial x_1^2} g(x) \\ &\quad +\text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial \nu_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial^2}{\partial y_1^2} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \,\mathrm{d}y_1\Bigg)\sin\theta_0\Bigg\}.\end{aligned} \tag{3.9}$$

Subtracting and adding two equations in (3.9) yield

$$h_2(x) = -\frac{\partial^2}{\partial x_1^2} g(x), \tag{3.10}$$

$$
\begin{aligned}
&\int_{\partial\Omega_+} T\tilde{D}\left(\left(\begin{array}{c} x_1 \\ 0 \end{array}\right),\left(\begin{array}{c} y_1 \\ a \end{array}\right)\right) h_1\left(\begin{array}{c} y_1 \\ a \end{array}\right)\, \mathrm{d}y_1 \\
&\quad +\text{v.p.}\int_\Gamma T\tilde{D}\left(\left(\begin{array}{c} x_1 \\ 0 \end{array}\right),\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\right) h_2\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\, \mathrm{d}y_1 \\
&\quad +\text{v.p.}\int_\Gamma \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\left(\begin{array}{c} x_1 \\ 0 \end{array}\right),\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\right) \frac{\partial}{\partial y_1} h_1\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\, \mathrm{d}y_1 \\
= &\quad -\Bigg\{\Bigg(\int_{\partial\Omega_+} \frac{\partial}{\partial x_1} T\tilde{D}\left(\left(\begin{array}{c} x_1 \\ 0 \end{array}\right),\left(\begin{array}{c} y_1 \\ a \end{array}\right)\right) g\left(\begin{array}{c} y_1 \\ a \end{array}\right)\, \mathrm{d}y_1 \\
&\quad +\text{v.p.}\int_\Gamma \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\left(\begin{array}{c} x_1 \\ 0 \end{array}\right),\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\right) \frac{\partial^2}{\partial y_1^2} g\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\, \mathrm{d}y_1\Bigg) \cos\theta_0 \\
&\quad +\Bigg(\int_{\partial\Omega_+} \frac{\partial}{\partial x_2} T\tilde{D}\left(\left(\begin{array}{c} x_1 \\ 0 \end{array}\right),\left(\begin{array}{c} y_1 \\ a \end{array}\right)\right) g\left(\begin{array}{c} y_1 \\ a \end{array}\right)\, \mathrm{d}y_1 \\
&\quad +\text{v.p.}\int_\Gamma \frac{\partial}{\partial\nu_x}\tilde{Q}\left(\left(\begin{array}{c} x_1 \\ 0 \end{array}\right),\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\right) \frac{\partial^2}{\partial y_1^2} g\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\, \mathrm{d}y_1\Bigg) \sin\theta_0\Bigg\}.
\end{aligned}
\tag{3.11}
$$

Substituting (3.10) into (3.11) leads to the similar formula as (2.24)

$$
\begin{aligned}
(I-Y_1)\frac{\partial}{\partial x_1}h_1(x) \quad = \quad & \frac{1}{\pi^2 R(x)}\int_{-R}^0 \frac{R(y)}{y-x}\Bigg\{T\tilde{V}_{\partial\Omega_+}h_1 - T\tilde{V}_\Gamma\frac{\partial^2}{\partial x_1^2}g \\
& + \cos\theta_0\left(\frac{\partial}{\partial x_1}T\tilde{V}_{\partial\Omega_+}g + \frac{\partial}{\partial\tau_x}Y_2 g\right) \\
& + \sin\theta_0\left(\frac{\partial}{\partial x_2}T\tilde{V}_{\partial\Omega_+}g + \frac{\partial}{\partial\nu_x}Y_2 g\right)\Bigg\}\ \mathrm{d}y_1 \\
& \text{as}\quad R\to\infty,\quad x\in\Gamma,
\end{aligned}
\tag{3.12}
$$

where

$$
Y_2(f) = \text{v.p.}\int_\Gamma \tilde{Q}\left(\left(\begin{array}{c} x_1 \\ 0 \end{array}\right),\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\right)\frac{\partial^2}{\partial y_1^2} f\left(\begin{array}{c} y_1 \\ 0 \end{array}\right)\, \mathrm{d}y_1.
$$

Applying THEOREMs 3 and 4 to problem (∗∗), we can get a unique solution $\hat{u}$.

3.2 The direction of crack extension

In this section we calculate G defined by (3.1). Taking into account (2.10), if u is a solution of problem $(*)$, then $\Pi(u)$ vanishes except on $\partial\Omega_+$. Then from (3.2), (3.4) $\Pi(u_\varepsilon)$ is written by

$$\Pi(u_\varepsilon) = -\frac{1}{2}\int_{\partial\Omega_+} p^{\mathrm{T}} \cdot u_\varepsilon \, \mathrm{d}x_1 = \Pi(u) + \varepsilon\Pi(\hat{u}). \tag{3.13}$$

In order to determine the crack direction θ_0 we apply maximum energy release rate criterion in 2-dimensional plane (cf. Wu [41]). Thus by virtue of (3.1), (3.13) we seek the angle θ_0 such that

$$\max_{-\pi<\theta_0<\pi} G = \max_{-\pi<\theta_0<\pi} (-\Pi(\hat{u})). \tag{3.14}$$

From (2.24) it implies that

$$\frac{\partial}{\partial x_1} g(x) = Y_3(T\tilde{V}_{\partial\Omega_+} g) \quad \text{on} \quad \Gamma, \tag{3.15}$$

where

$$\begin{aligned} Y_3 g \quad &= \quad \lim_{R\to\infty} \left((1+\pi^2)I - Y_{11}Y_{10} - Y_{10}\right)^{-1} \\ &\qquad \left\{(I+Y_{11})\left(\lim_{R\to\infty} \frac{1}{\pi^2 R(z)} \int_{-R}^{0} \frac{R(y)g}{y-z} \, \mathrm{d}y_1\right)\right\}. \end{aligned}$$

Substituting (3.15) into (2.39) yields that

$$g(x) = \left(Z_1 Z_2 - \frac{1}{2}Z_2 - (\frac{1}{4}+\pi^2)I\right)^{-1} \left\{\left(Z_1 - \frac{1}{2}I\right)p\right\} \text{ on } \partial\Omega_+. \tag{3.16}$$

Similarly, h_1 is described by g and θ_0. Indeed, from (3.12) it follows that

$$\frac{\partial}{\partial x_1} h_1(x) = Y_3\left(T\tilde{V}_{\partial\Omega_+} h_1 - T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g\right) + A_1 \cos\theta_0 + B_1 \sin\theta_0 \text{ on } \Gamma, \tag{3.17}$$

where A_1, B_1 are functions defined by

$$\begin{aligned} A_1 \quad &= \quad Y_3\left(\frac{\partial}{\partial x_1} T\tilde{V}_{\partial\Omega_+} g + \frac{\partial}{\partial \tau_x} Y_2 g\right), \\ B_1 \quad &= \quad Y_3\left(\frac{\partial}{\partial x_2} T\tilde{V}_{\partial\Omega_+} g + \frac{\partial}{\partial \nu_x} Y_2 g\right). \end{aligned}$$

Substituting (3.10), (3.17) into (3.6), we have

$$h_1(x) = C + A_2 \cos\theta_0 + B_2 \sin\theta_0 \quad \text{on} \quad \partial\Omega_+, \tag{3.18}$$

where

$$\begin{aligned}
C &= \left(Z_1 Z_2 - \frac{1}{2} Z_2 - (\frac{1}{4} + \pi^2) I \right)^{-1} \\
&\qquad \left\{ \left(Z_1 - \frac{1}{2} I \right) \left(I + \int_\Gamma \frac{\partial}{\partial \tau_x} \tilde{Q}(x,y) Y_3 \right) \left(T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g \right) \right\}, \\
A_2 &= \left(Z_1 Z_2 - \frac{1}{2} Z_2 - (\frac{1}{4} + \pi^2) I \right)^{-1} \\
&\qquad \left\{ \left(Z_1 - \frac{1}{2} I \right) \left(\int_\Gamma \frac{\partial}{\partial \tau_x} \tilde{Q}(x,y) \right) (-A_1) \right\}, \\
B_2 &= \left(Z_1 Z_2 - \frac{1}{2} Z_2 - (\frac{1}{4} + \pi^2) I \right)^{-1} \\
&\qquad \left\{ \left(Z_1 - \frac{1}{2} I \right) \left(\int_\Gamma \frac{\partial}{\partial \tau_x} \tilde{Q}(x,y) \right) (-B_1) \right\}.
\end{aligned}$$

Since A_i, B_i and C are functions depending on g, h_i depends only on a surface force p for $i = 1, 2$. Hence, substituting (3.10), (3.17), (3.18) into (3.5), we have

$$\begin{aligned}
\hat{u} &= \tilde{V}_{\partial\Omega_+} (C + A_2 \cos\theta_0 + B_2 \sin\theta_0) + \tilde{V}_\Gamma \left(-\frac{\partial^2}{\partial x_1^2} g \right) \\
&\quad + \tilde{V}_\Gamma' \left(Y_3 \left(T\tilde{V}_{\partial\Omega_+} (C + A_2 \cos\theta_0 + B_2 \sin\theta_0) - T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g \right) \right. \\
&\qquad \left. + A_1 \cos\theta_0 + B_1 \sin\theta_0 \right),
\end{aligned}$$

since (3.8) leads to

$$\tilde{W}_\Gamma = \frac{\partial}{\partial x_1} \tilde{V}_\Gamma'.$$

Thus, from (3.13) $\Pi(\hat{u})$ is written as

$$-2\Pi(\hat{u}) = D + A_3 \cos\theta_0 + B_3 \sin\theta_0, \tag{3.19}$$

where

$$
\begin{aligned}
D &= \int_{\partial\Omega_+} p^{\mathrm{T}} \cdot \left(\tilde{V}_{\partial\Omega_+} C + \tilde{V}_\Gamma \left(-\frac{\partial^2}{\partial x_1^2} g \right) \right. \\
&\qquad \left. + \tilde{V}_\Gamma' \left(Y_3 \left(T\tilde{V}_{\partial\Omega_+} C - T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g \right) \right) \right) \, \mathrm{d}x_1, \\
A_3 &= \int_{\partial\Omega_+} p^{\mathrm{T}} \cdot \left(\tilde{V}_{\partial\Omega_+} A_2 + \tilde{V}_\Gamma' \left(Y_3 \left(T\tilde{V}_{\partial\Omega_+} A_2 \right) + A_1 \right) \right) \, \mathrm{d}x_1, \\
B_3 &= \int_{\partial\Omega_+} p^{\mathrm{T}} \cdot \left(\tilde{V}_{\partial\Omega_+} B_2 + \tilde{V}_\Gamma' \left(Y_3 \left(T\tilde{V}_{\partial\Omega_+} B_2 \right) + B_1 \right) \right) \, \mathrm{d}x_1.
\end{aligned}
$$

Formula (3.1) is equivalent to

$$
G = \frac{1}{2}(D + A_3 \cos\theta_0 + B_3 \sin\theta_0).
$$

From this it is easy to see that G attains the maximum value in $(-\pi, \pi)$ at

$$
\theta_0 = \mathrm{Tan}^{-1} \left(\frac{B_3}{A_3} \right). \tag{3.20}
$$

Summing up the above, we have

THEOREM 5. *Suppose a homogeneous elastic body Ω with a crack Γ is loaded a surface force p. Then according to maximum energy release rate criterion Γ propagates along the direction θ_0 given by* (3.20) *dependent only on a surface force p.*

Chapter 4

Existence of a weak solution in an infinite viscoelastic strip with a semi-infinite crack

In Chapter 4, we study an initial-boundary value problem in an infinite viscoelastic strip with a semi-infinite fixed crack. For this problem we prove the existence and uniqueness of a weak solution which is prescribed on each side of the extended crack in Sobolev-type spaces.

4.1 The model equations of viscoelasticity with crack

We consider that dissipative forces occur through processes of viscosity. The expression for the dissipative forces can be described by the time derivatives of the strain tensor ($\dot{\varepsilon}_{ij}$). In an isotropic body the dissipative stress tensor ($\hat{\sigma}_{ij}$) can be written by the analogous of (2.1)

$$\hat{\sigma}_{ij} = 2\eta\dot{\varepsilon}_{ij} + \left(\zeta - \frac{2}{3}\eta\right)\dot{\varepsilon}_{kk}\delta_{ij}, \tag{4.1}$$

where ζ and η are two viscosities assumed to be positive constants. From the balance law of momentum the equations of motion in the absence of body forces become

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i, j = 1, 2, 3. \tag{4.2}$$

Here t is a time variable, ρ is the density of the medium. The viscosity effect can be taken into account in (4.2) by replacing σ_{ij} in (4.2) by the sum $\sigma_{ij} + \hat{\sigma}_{ij}$. Hence in the state of a plane strain we obtain

$$\rho \partial_t^2 u - \tilde{A} \partial_t u - Au = 0, \tag{4.3}$$

where A is defined in section **2.1** and

$$\tilde{A}(\xi_1, \xi_2) = \begin{pmatrix} \eta\xi^2 + (\zeta + \frac{1}{3}\eta)\xi_1^2 & (\zeta + \frac{1}{3}\eta)\xi_1\xi_2 \\ (\zeta + \frac{1}{3}\eta)\xi_1\xi_2 & \eta\xi^2 + (\zeta + \frac{1}{3}\eta)\xi_2^2 \end{pmatrix}.$$

Moreover, we introduce the boundary stress operator $\tilde{T}(\partial_x)$ defined by

$$\tilde{T}(\xi_1, \xi_2) = \begin{pmatrix} (\zeta + \frac{4}{3}\eta)\nu_1\xi_1 + \eta\nu_2\xi_2 & \eta\nu_2\xi_1 + (\zeta - \frac{2}{3}\eta)\nu_1\xi_2 \\ (\zeta - \frac{2}{3}\eta)\nu_2\xi_1 + \eta\nu_1\xi_2 & \eta\nu_1\xi_1 + (\zeta + \frac{4}{3}\eta)\nu_2\xi_2 \end{pmatrix}.$$

Particularly, in the case of $\nu = (0, 1)^{\mathrm{T}}$

$$\tilde{T}(\xi_1, \xi_2) = \begin{pmatrix} \eta\xi_2 & \eta\xi_1 \\ (\zeta - \frac{2}{3}\eta)\xi_1 & (\zeta + \frac{4}{3}\eta)\xi_2 \end{pmatrix}.$$

Let $\Gamma_1 = \{(x_1, 0) \mid 0 < x_1 < \infty\}$. On the crack we assume the free traction condition

$$(\sigma_{ij}^+ + \hat{\sigma}_{ij}^+)\nu_j = (\sigma_{ij}^- + \hat{\sigma}_{ij}^-)\nu_j = 0 \quad \text{on} \quad \Gamma^\pm, \tag{4.4}$$

where $\Gamma^\pm$ mean both sides of Γ. Here for every $x \in \Gamma$ $\sigma_{ij}^\pm + \hat{\sigma}_{ij}^\pm = \sigma_{ij}^\pm(x) + \hat{\sigma}_{ij}^\pm(x)$ mean the limits of $(\nu_x, (\sigma_{ij} + \hat{\sigma}_{ij})(\bar{x}))$ as $\bar{x} \in \Omega \setminus \Gamma$ tends to $x \in \Gamma$ along the normals $\nu_x = (0, \mp 1)$. At the end-point $(0, 0)$ of Γ we assume

$$\lim_{x_1 \to -0} (\sigma_{ij}^\pm + \hat{\sigma}_{ij}^\pm)\nu_j \big|_{x \in \Gamma^\pm \setminus \{(0,0)\}} = 0.$$

On $\partial\Omega_+$, $\partial\Omega_-$ the boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega_-, \tag{4.5}$$

$$(\sigma_{ij} + \hat{\sigma}_{ij})\nu_j = f_i \quad \text{on} \quad \partial\Omega_+ \tag{4.6}$$

are imposed, where f_i are given continuous functions on $\partial\Omega_+$. Summing up, our problem is to find u such that

$$(***)\begin{cases} \rho\partial_t^2 u(x,t) - \tilde{A}\partial_t u(x,t) - Au(x,t) = 0, \quad (x,t) \in \Upsilon, \\ Tu(x,t) + \tilde{T}\partial_t u(x,t) = 0, \quad x \in \Gamma^{\pm}, \\ u(x,t) = 0, \quad x \in \partial\Omega_-, \\ Tu(x,t) + \tilde{T}\partial_t u(x,t) = f(x,t), \quad x \in \partial\Omega_+, \\ u(x,0) = \partial_t u(x,0) = 0, \quad x \in \Omega\setminus\Gamma, \end{cases}$$

where $\Upsilon \equiv \Omega\setminus\Gamma \times (0,\infty)$ and T is defined in section **2.1**.

For any $m \in \mathbf{R}$ and $s \in \mathbf{C}$, let $H_{m,s}(\mathbf{R}^2)$ be the Sobolev space equipped with the norm

$$\| u \|_{m,s} = \left\{ \int_{\mathbf{R}^2} (1+|s|+|\chi|)^{2m} |\tilde{u}(\chi)|^2 \, \mathrm{d}\chi \right\}^{\frac{1}{2}},$$

where $\tilde{u}$ is the Fourier transform of u. Let $H_{m,s}(\Omega_\pm)$ be the space of the restrictions to $\Omega_\pm$ of all elements of $H_{m,s}(\mathbf{R}^2)$ equipped with norm

$$\| u \|_{m,s;\Omega_\pm} = \inf_{v\in H_{m,s}(\mathbf{R}^2), v|_{\Omega_\pm}=u} \| v \|_{m,s} .$$

Clearly, for any fixed $s \in \mathbf{C}$, the norms in $H_{m,s}(\Omega_\pm)$ and in the standard Sobolev space $H_m(\Omega_\pm)$ are equivalent.

Let π_0 and π_1 be the operators of restriction from $\{x_2 = 0\}$ to Γ and Γ_1, and let γ^+ and γ^- be the continuous trace operators from $H_1(\Omega_+)$ and $H_1(\Omega_-)$ to $H_{1/2}(\{x_2 = 0\})$ along the normal. Also, let $\gamma_i^\pm = \pi_i\gamma^\pm, i = 0, 1$.

Let $H_{m,s}(\Omega\setminus\Gamma)$ be the space of all $u = u^+ + u^-$ such that $u^+ \in H_{m,s}(\Omega_+)$, $u^- \in H_{m,s}(\Omega_-)$ and $\gamma_1^+ u^+ = \gamma_1^- u^-$. The norm in $H_{m,s}(\Omega\setminus\Gamma)$ is defined by

$$\| u \|^2_{m,s;\Omega\setminus\Gamma} = \| u^+ \|^2_{m,s;\Omega_+} + \| u^- \|^2_{m,s;\Omega_-} .$$

Note that the traces of $u \in H_{m,s}(\Omega\setminus\Gamma)$ on opposite sides of Γ may be distinct: $\gamma_0^+ u^+ \neq \gamma_0^- u^-$.

By $H_{-m,s}(\Omega\setminus\Gamma)$ we denote the dual space of $\dot{H}_{m,s}(\Omega\setminus\Gamma) = \{u \in H_{m,s}(\Omega\setminus\Gamma) \mid u = 0 \text{ on } \partial\Omega_-,\ \mathrm{supp} u \subset \overline{\Omega\setminus\Gamma}\}$, with respect to the duality generated by $(\cdot,\cdot)_{0,\Omega\setminus\Gamma}$ in L^2. Its norm is denoted by $\| \cdot \|_{-m,s;\Omega\setminus\Gamma}$; the dual space of

$H_{m,s}(\Omega \setminus \Gamma)$ is $\dot{H}_{-m,s}(\Omega \setminus \Gamma)$, which can be identified with the subspace of $H_{-m,s}(\mathbf{R}^2)$.

We denote by $\bar{f}$ the Laplace transform of the function $f(x,t)$ with respect to t,

$$\bar{f}(x,s) = \mathcal{L}[f(x,t)] = \int_0^\infty e^{-st} f(x,t)\, \mathrm{d}t,$$

where $s \in \mathbf{C}_\tau = \{s = s_1 + is_2 \mid s_1 > \tau\}$. We now fix $\tau > 0$. For any $m, k \in \mathbf{R}$, we define the space $H^{\mathcal{L}}_{m,k,\tau}(\Omega \setminus \Gamma)$ of all $\bar{u}(x,s)$ regarded as functions of s with values in $H_m(\Omega \setminus \Gamma)$, whose norm in $H^{\mathcal{L}}_{m,k,\tau}(\Omega \setminus \Gamma)$ is defined by

$$\| \bar{u} \|^2_{m,k,\tau;\Omega\setminus\Gamma} = \sup_{s_1 > \tau} \int_{-\infty}^{\infty} (1 + |s|)^{2k} \| \bar{u} \|^2_{m,s;\Omega\setminus\Gamma} \ \mathrm{d}s_2.$$

Finally, for any $m, k \in \mathbf{R}$, let $H^{\mathcal{L}^{-1}}_{m,k,\tau}(\Upsilon)$ be the space of the inverse Laplace transforms u of $\bar{u} \in H^{\mathcal{L}}_{m,k,\tau}(\Omega \setminus \Gamma)$ equipped with the norm

$$\| u \|_{m,k,\tau;\Upsilon} = \| \bar{u} \|_{m,k,\tau;\Omega\setminus\Gamma} .$$

Let us introduce the sesquilinear forms

$$B[u^\pm, v] = 2 \int_{\Omega_\pm} E(u^\pm, v)\, \mathrm{d}x, \quad \tilde{B}[u^\pm, v] = 2 \int_{\Omega_\pm} \tilde{E}(u^\pm, v)\, \mathrm{d}x,$$

where

$$\begin{aligned} E(u,v) &= \frac{1}{2} \{ (\lambda + 2\mu)\, (u_{1,1} v^*_{1,1} + u_{2,2} v^*_{2,2}) + \lambda (u_{1,1} v^*_{2,2} + u_{2,2} v^*_{1,1}) \\ &\qquad + \mu (u_{1,2} + u_{2,1})(v_{1,2} + v_{2,1})^* \}, \end{aligned}$$

$$\begin{aligned} \tilde{E}(u,v) &= \frac{1}{2} \left\{ \left(\zeta + \frac{4}{3}\eta \right) (u_{1,1} v^*_{1,1} + u_{2,2} v^*_{2,2}) + \left(\zeta - \frac{2}{3}\eta \right) (u_{1,1} v^*_{2,2} + u_{2,2} v^*_{1,1}) \right. \\ &\qquad \left. + \eta (u_{1,2} + u_{2,1})(v_{1,2} + v_{2,1})^* \right\} \end{aligned}$$

and u^* is a complex conjugate of u. Let $u \in H^{\mathcal{L}^{-1}}_{m,k,\tau}(\Upsilon)$ satisfy $(***)$. Then taking the L^2-inner product of both sides in $(***)$ with any v in the class $\mathcal{K}_0(\bar{\Upsilon})$ defined by

$$\mathcal{K}_0(\bar{\Upsilon}) \equiv \{ v \in C^\infty_0(\bar{\Upsilon}) \mid \gamma_1^+ v = \gamma_1^- v,\ v|_{\partial\Omega_-} = 0 \}$$

and integrating by parts since

$$
\begin{aligned}
& (Au^+ + \tilde{A}\partial_t u^+, v)_{0;\Omega_+} \\
= \quad & -B[u^+, v] - \tilde{B}[\partial_t u^+, v] + (Tu^+ + \tilde{T}\partial_t u^+, \gamma_1^+ v)_{0;\Gamma_1} + (f, v)_{0;\partial\Omega_+},
\end{aligned}
$$

$$
(Au^- + \tilde{A}\partial_t u^-, v)_{0;\Omega_-} = -B[u^-, v] - \tilde{B}[\partial_t u^-, v] - (Tu^- + \tilde{T}\partial_t u^-, \gamma_1^- v)_{0;\Gamma_1},
$$

we arrive at

$$
\int_0^\infty \left\{ \tilde{B}[\partial_t u, v] + B[u, v] - (\rho^{\frac{1}{2}}\partial_t u, \rho^{\frac{1}{2}}\partial_t v)_{0;\Omega\setminus\Gamma} \right\} \, \mathrm{d}t = \int_0^\infty (f, v)_{0;\partial\Omega_+} \, \mathrm{d}t. \quad (4.7)
$$

Conversely, integration by parts in (4.7) implies that if $u \in C^2(\Upsilon) \cap C^1(\bar{\Upsilon})$ satisfying the initial and boundary conditions in $(***)$, then u is also a solution of $(***)$. Therefore, we call $u \in H_{1,0,\tau}^{\mathcal{L}^{-1}}(\Upsilon)$ satisfying (4.7) a weak solution of $(***)$.

4.2 Solvability of the transformed problem

In this section, we consider the transformed problem. Applying the Laplace transform to $(***)$, we obtain the transformed problem

$$
(****) \begin{cases} \rho s^2 \bar{u} - s\tilde{A}\bar{u} - A\bar{u} = 0 \quad \text{in} \quad \Omega \setminus \Gamma, \\ (T + s\tilde{T})\bar{u} = 0 \quad \text{on} \quad \Gamma^\pm, \\ \bar{u} = 0 \quad \text{on} \quad \partial\Omega_-, \\ (T + s\tilde{T})\bar{u} = \bar{f} \quad \text{on} \quad \partial\Omega_+. \end{cases}
$$

We take the L^2-inner product of both sides in $(****)$ with $v \in \mathcal{K}_0(\overline{\Omega \setminus \Gamma})$. Then, for any $v \in H_{1,s}(\Omega \setminus \Gamma)$ satisfying $\gamma_1^+ v = \gamma_1^- v$ we have

$$
s^2(\rho^{\frac{1}{2}}\bar{u}, \rho^{\frac{1}{2}}v)_{0;\Omega\setminus\Gamma} + s\tilde{B}[\bar{u}, v] + B[\bar{u}, v] = (\bar{f}, v)_{0;\partial\Omega_+}. \quad (4.8)
$$

Formally separating the real and imaginary parts in (4.8) yields

$$
(s_1^2 - s_2^2)(\rho^{\frac{1}{2}}\bar{u}, \rho^{\frac{1}{2}}v)_{0;\Omega\setminus\Gamma} + s_1\tilde{B}[\bar{u}, v] + B[\bar{u}, v] = \mathrm{Re}\{(\bar{f}, v)_{0;\partial\Omega_+}\}, \quad (4.9)
$$

$$
2s_1 s_2(\rho^{\frac{1}{2}}\bar{u}, \rho^{\frac{1}{2}}v)_{0;\Omega\setminus\Gamma} + s_2\tilde{B}[\bar{u}, v] = \mathrm{Im}\{(\bar{f}, \bar{u})_{0;\partial\Omega_+}\}. \quad (4.10)
$$

Whenever $s \in \mathbf{C}_\tau$, $s \neq 0$. Hence we can divide both sides of (4.8) by s and get

$$s(\rho^{\frac{1}{2}}\bar{u}, \rho^{\frac{1}{2}}v)_{0;\Omega\setminus\Gamma} + \tilde{B}[\bar{u}, v] + \frac{1}{s}B[\bar{u}, v] = \frac{1}{s}(\bar{f}, v)_{0;\partial\Omega_+}.$$

If $s \in \mathbf{C}_\tau$, then we define a positive definite Hermitian form on $H_1(\Omega \setminus \Gamma)$

$$a_\tau^{\mathrm{Re}}(\bar{u}, v) = (\rho^{\frac{1}{2}}\bar{u}, \rho^{\frac{1}{2}}v)_{0;\Omega\setminus\Gamma} + \frac{1}{s_1}\tilde{B}[\bar{u}, v] + \frac{1}{s_1^2 + s_2^2}B[\bar{u}, v].$$

It is easily seen that this form defines a norm equivalent to that of $H_1(\Omega \setminus \Gamma)$. Let $\mathcal{H}(\Omega \setminus \Gamma)$ be a Hilbert space whose inner product is defined by this Hermitian form $(\bar{u}, v)_{\mathcal{H}(\Omega\setminus\Gamma)} = a_\tau^{\mathrm{Re}}(\bar{u}, v)$.

Next, we introduce another Hermitian form on $\mathcal{H}(\Omega \setminus \Gamma)$ defined by

$$a_\tau^{\mathrm{Im}}(\bar{u}, v) = (\rho^{\frac{1}{2}}\bar{u}, \rho^{\frac{1}{2}}v)_{0;\Omega\setminus\Gamma} - \frac{1}{s_1^2 + s_2^2}B[\bar{u}, v].$$

Then, for any $s \in \mathbf{C}_\tau$

$$|a_\tau^{\mathrm{Im}}(\bar{u}, \bar{u})| \leq c \parallel \bar{u} \parallel^2_{\mathcal{H}(\Omega\setminus\Gamma)} .$$

Hence, by Riesz theorem one can see that there exists a bounded Hermitian operator H and $a_\tau^{\mathrm{Im}}(\bar{u}, v) = (H\bar{u}, v)_{\mathcal{H}(\Omega\setminus\Gamma)}$ holds for arbitrary $\bar{u}, v \in \mathcal{H}(\Omega \setminus \Gamma)$. Also there exists a continuous operator $\hat{H}$ from $H_{-\frac{1}{2}}(\partial\Omega_+)$ to $H_{\frac{1}{2}}(\partial\Omega_+) = \mathcal{H}(\partial\Omega_+)$ such that

$$(\bar{f}, v)_{0;\partial\Omega_+} = (\hat{H}\bar{f}, v)_{\mathcal{H}(\partial\Omega_+)}.$$

Therefore, we can rewrite (4.8) as follows:

$$s\left((s_1 I + s_2 i H)\bar{u}, v\right)_{\mathcal{H}(\Omega\setminus\Gamma)} = (\hat{H}\bar{f}, v)_{\mathcal{H}(\partial\Omega_+)},$$

where I is the identity matrix. From [32] we see:

PROPOSITION 1. *Let H be a bounded Hermitian operator defined on a Hilbert space $\mathcal{H}$, and let*

$$m = \inf_{\|g\|=1}(Hg, g), \quad M = \sup_{\|g\|=1}(Hg, g).$$

If $\tilde{\lambda} \notin [m, M]$, then the inverse operator $(\tilde{\lambda}I - H)^{-1}$ exists.

Now we only need to remark that $\frac{s_1}{s_2}i \notin [m, M]$ for any $s \in \mathbf{C}_\tau$. Indeed, as $|s_2| \to \infty$

$$(H\bar{u}, \bar{u})_{\mathcal{H}(\Omega\setminus\Gamma)} = \rho \parallel \bar{u} \parallel_0^2 > 0.$$

Therefore as in Theorem 2 in [5], (4.8) has a unique solution $\bar{u} \in H_{1,s}(\Omega \setminus \Gamma)$ for every $\bar{f} \in H_{-\frac{1}{2},s}(\partial\Omega_+)$. Then, we call this solution $\bar{u} \in H_{1,s}(\Omega \setminus \Gamma)$ of (4.8) a weak solution of $(* * **)$.

Moreover, adding (4.10) multiplied by $s_1^{-1}s_2$ to (4.9) with $v = \bar{u}$, we obtain

$$\begin{aligned} & |s|^2 \parallel \rho^{\frac{1}{2}}\bar{u} \parallel^2_{0;\Omega\setminus\Gamma} + \frac{|s|^2}{s_1}\tilde{B}[\bar{u}, \bar{u}] + B[\bar{u}, \bar{u}] \\ = \quad & \mathrm{Re}\{(\bar{f}, \bar{u})_{0;\partial\Omega_+}\} + \frac{s_2}{s_1}\mathrm{Im}\{(\bar{f}, \bar{u})_{0;\partial\Omega_+}\} = \frac{1}{s_1}\mathrm{Re}\{s^*(\bar{f}, \bar{u})_{0;\partial\Omega_+}\}. \end{aligned}$$

From this it follows that $\parallel \bar{u} \parallel^2_{1,s;\Omega\setminus\Gamma} \leq c|s||(\bar{f}, \bar{u})_{0;\partial\Omega_+}|$, hence

$$\parallel \bar{u} \parallel_{1,s;\Omega\setminus\Gamma} \leq c|s| \parallel \bar{f} \parallel_{-\frac{1}{2},s;\partial\Omega_+} \tag{4.11}$$

by the trace theorem.

4.3 Weak solvability of the time-dependent problem

In this section we will establish the existence of a unique solution to problem (4.7) with initial conditions in $(* * *)$.

THEOREM 6. *Let for any* $f \in H^{\mathcal{L}^{-1}}_{-\frac{1}{2},1,\tau}(\partial\Upsilon_+)$, $\partial\Upsilon_+ \equiv \partial\Omega_+ \times (0, \infty)$ *and* $\tau > 0$. *Problem* (4.7) *with initial conditions in* $(* * *)$ *has a unique weak solution* $u \in H^{\mathcal{L}^{-1}}_{1,0,\tau}(\Upsilon)$. *Moreover, if* $f \in H^{\mathcal{L}^{-1}}_{-\frac{1}{2},k,\tau}(\partial\Upsilon_+)$ *with any* $k \in \mathbf{R}$, *then* $u \in H^{\mathcal{L}^{-1}}_{1,k-1,\tau}(\Upsilon)$ *and*

$$\parallel u \parallel_{1,k-1,\tau;\Upsilon} \leq c \parallel f \parallel_{-\frac{1}{2},k,\tau;\partial\Upsilon_+} .$$

Proof. Let $\bar{u}(\cdot, s) \in H_{1,s}(\Omega \setminus \Gamma)$ be the weak solution of $(* * **)$. Now, we regard $U(s) = U^+(s) + U^-(s) = \bar{u}^+(\cdot, s) + \bar{u}^-(\cdot, s)$ and $F(s) = \bar{f}(\cdot, s)$ as functions with values in $H_1(\Omega\setminus\Gamma)$ and $H_{-\frac{1}{2}}(\partial\Omega_+)$, respectively. One can find the following result in [32]:

PROPOSITION 2. *If a holomorphic function $\bar{f}(s)$ is at most of the same increasing degree as polynomials of $\mid s \mid$, then there exists a unique distribution F which satisfies the condition: $\bar{f}(s) = \mathcal{L}[F]$.*

Firstly, we will prove that U is holomorphic. Let $s_0 \in \mathbf{C}_\tau$ and $K_R(s_0)$ be a circle with a centre s_0 and a radius R such that $\overline{K_R(s_0)} \subset \mathbf{C}_\tau$. And let $U(s_0)$ be the solution of the problem

$$\rho s_0^2 U(s_0) - s_0 \tilde{A} U(s_0) - AU(s_0) = 0,$$

$$(T + s_0 \tilde{T}) U(s_0)|_{\partial\Omega_+} = F(s_0),$$

which satisfies

$$\begin{aligned} \| U(s_0) \|_{1;\Omega\setminus\Gamma} &\leq c|s_0| \; \| F(s_0) \|_{-\frac{1}{2},s_0;\partial\Omega_+} \\ &\leq c \, \| F(s_0) \|_{-\frac{1}{2};\partial\Omega_+} . \end{aligned}$$

Rewriting $(\ast\ast\ast\ast)$ with $(T + s\tilde{T})U(s)|_{\partial\Omega_+} = F(s)$ in the form

$$\begin{aligned} & \rho s_0^2 U(s) - s_0 \tilde{A} U(s) - AU(s) \\ &= -\rho(s^2 - s_0^2)U(s) + (s - s_0)\tilde{A}U(s), \end{aligned} \tag{4.12}$$

we find from (4.11) and the result in [5] that

$$\begin{aligned} \| U(s) \|_{1;\Omega\setminus\Gamma} \leq \; & c(\| F(s) \|_{-\frac{1}{2};\partial\Omega_+} + |s^2 - s_0^2| \; \| U(s) \|_{-1;\Omega\setminus\Gamma} \\ & + \| (s\tilde{A} + A)U(s) - (s_0\tilde{A} + A)U(s) \|_{-1;\Omega\setminus\Gamma}). \end{aligned}$$

Since

$$\begin{aligned} & \| (s\tilde{A} + A)U(s) - (s_0\tilde{A} + A)U(s) \|_{-1;\Omega\setminus\Gamma} \\ &= \frac{|((s\tilde{A} + A)U(s) - (s_0\tilde{A} + A)U(s), v)_{0;\Omega\setminus\Gamma}|}{\| v \|_{1;\Omega\setminus\Gamma}} \\ &\leq c \, \frac{| - (s - s_0)\tilde{B}[U(s), v] - (1 - 1)B[U(s), v] + (F, v)_{0;\partial\Omega_+}|}{\| v \|_{1;\Omega\setminus\Gamma}} \\ &\leq c \, (\| F(s) \|_{-\frac{1}{2};\partial\Omega_+} + |s - s_0| \; \| U(s) \|_{1;\Omega\setminus\Gamma}), \quad \forall v \in \dot{H}_1(\Omega \setminus \Gamma), \end{aligned}$$

by choosing $R > 0$ such that $c|s^2 - s_0^2| < \frac{1}{4}$ and $c|s - s_0| < \frac{1}{4}$ for $s \in \overline{K_R(s_0)}$ one can get

$$\| U(s) \|_{1;\Omega\setminus\Gamma} \leq c \| F(s) \|_{-\frac{1}{2};\partial\Omega_+} . \tag{4.13}$$

Since F is holomorphic, $U(s)$ is bounded in $H_1(\Omega \setminus \Gamma)$ for $s \in \overline{K_R(s_0)}$. From (4.12) it follows that

$$\begin{aligned} \rho s_0^2[U(s) - U(s_0)] - s_0\tilde{A}[U(s) - U(s_0)] - A[U(s) - U(s_0)] \\ = -\rho(s^2 - s_0^2)U(s) + (s - s_0)\tilde{A}U(s), \end{aligned}$$

$$((T + s\tilde{T})U(s) - (T + s_0\tilde{T})U(s_0))|_{\partial\Omega_+} = F(s) - F(s_0).$$

Hence, for $s \in \overline{K_R(s_0)}$

$$\begin{aligned} &\| U(s) - U(s_0) \|_{1;\Omega\setminus\Gamma} \\ &\leq \; c\Big\{\| F(s) - F(s_0) \|_{-\frac{1}{2};\partial\Omega_+} + |s^2 - s_0^2| \, \| U(s) \|_{-1;\Omega\setminus\Gamma} \\ &\qquad + \| (s\tilde{A} + A)U(s) - (s_0\tilde{A} + A)U(s) \|_{-1;\Omega\setminus\Gamma}\Big\} . \end{aligned} \tag{4.14}$$

Since $U(s)$ is bounded in $H_{-1}(\Omega \setminus \Gamma)$, as s tends to s_0 the right-hand side of (4.14) tends to 0, which means U is continuous at s_0.

Finally, let $\mathcal{V} \in H_1(\Omega \setminus \Gamma)$ be the solution of the problem

$$\rho s_0^2\mathcal{V} - s_0\tilde{A}\mathcal{V} - A\mathcal{V} = -2\rho s_0 U(s_0) + \tilde{A}U(s_0),$$

$$\gamma_1^+\mathcal{V} = \gamma_1^-\mathcal{V}, \quad (T + s_0\tilde{T})\mathcal{V}|_{\partial\Omega_+} = F'(s_0) - \tilde{T}U(s_0)|_{\partial\Omega_+}.$$

Then

$$\mathcal{W}(s) \equiv \frac{U(s) - U(s_0)}{s - s_0} - \mathcal{V}(s) \in H_1(\Omega \setminus \Gamma)$$

satisfies $\gamma_1^+\mathcal{W} = \gamma_1^-\mathcal{W}$ and

$$\begin{aligned} &s_0^2\rho\mathcal{W}(s) - s_0\tilde{A}\mathcal{W}(s) - A\mathcal{W}(s) \\ &= \; -\rho[(s + s_0)U(s) - 2s_0U(s_0)] + \tilde{A}U(s) - \tilde{A}U(s_0), \end{aligned}$$

$$\left\{\frac{(T+s\tilde{T})U(s)-(T+s_0\tilde{T})U(s_0)}{s-s_0}-(T+s_0\tilde{T})\mathcal{V}\right\}\Bigg|_{\partial\Omega_+}$$

$$=\frac{F(s)-F(s_0)}{s-s_0}-F'(s_0)+\tilde{T}(U(s)-U(s_0))|_{\partial\Omega_+}.$$

In the same way as (4.13) we obtain

$$\begin{aligned}
&\| \mathcal{W}(s) \|_{1;\Omega\setminus\Gamma} \\
&\leq \; c\left(\left\|\frac{F(s)-F(s_0)}{s-s_0}-F'(s_0)\right\|_{-\frac{1}{2};\partial\Omega_+}\right. \\
&\qquad + \| \tilde{T}(U(s)-U(s_0)) \|_{-\frac{1}{2};\partial\Omega_+} \\
&\qquad + \| (s+s_0)U(s)-2s_0U(s_0) \|_{-1;\Omega\setminus\Gamma} \\
&\qquad \left. + \| (s-s_0)\tilde{A}\mathcal{W}+(s-s_0)\tilde{A}\mathcal{V} \|_{-1;\Omega\setminus\Gamma}\right) \\
&\leq \; c\left(\left\|\frac{F(s)-F(s_0)}{s-s_0}-F'(s_0)\right\|_{-\frac{1}{2};\partial\Omega_+}\right. \\
&\qquad + \| (s+s_0)U(s)-2s_0U(s_0) \|_{-1;\Omega\setminus\Gamma} \\
&\qquad +|s-s_0| \, \| \mathcal{W} \|_{1;\Omega\setminus\Gamma} +|s-s_0| \, \| \mathcal{V} \|_{1;\Omega\setminus\Gamma} \\
&\qquad \left. + \| U(s)-U(s_0) \|_{1;\Omega\setminus\Gamma}\right).
\end{aligned}$$

Since U is continuous at s_0, $\| \mathcal{W}(s) \|_{1;\Omega\setminus\Gamma}$ tends to 0 as s tends to s_0. This means that $U'(s_0)$ exists, $U'(s_0)=\mathcal{V}$ and the mapping U is holomorphic from $\mathbf{C}_\tau$ to $H_1(\Omega\setminus\Gamma)$.

Moreover, noting that

$$\| U(s) \|_{1,s;\Omega\setminus\Gamma}\leq c|s| \, \| F(s) \|_{-\frac{1}{2},s;\partial\Omega_+},$$

we have for $k \in \mathbf{R}$

$$\begin{aligned}
& \| \, u \, \|^2_{1,k-1,\tau;\Upsilon} \\
& = \sup_{s_1>\tau} \int_{-\infty}^{\infty} (1+|s|)^{2(k-1)} \| \, U(s) \, \|^2_{1,s;\Omega\setminus\Gamma} \ \mathrm{d}s_2 \\
& \leq c \sup_{s_1>\tau} \int_{-\infty}^{\infty} (1+|s|)^{2(k-1)} |s|^2 \| \, F(s) \, \|^2_{-\frac{1}{2},s;\partial\Omega_+} \ \mathrm{d}s_2 \\
& \leq c \| \, f \, \|^2_{-\frac{1}{2},k,\tau;\partial\Upsilon_+} .
\end{aligned}$$

Secondly, we shall prove that u satisfies (4.7). We recall that if φ_1 and φ_2 satisfy

$$\int_0^\infty e^{-2\tau_i t} |\varphi_i(t)|^2 \ \mathrm{d}t < \infty, \quad i = 1, 2,$$

then Parseval's equality holds :

$$\int_0^\infty e^{-(\tau_1+\tau_2)t} \varphi_1(t)(\varphi_2(t))^* \ \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\varphi_1}(\tau_1 + is_2)(\overline{\varphi_2}(\tau_2 + is_2))^* \ \mathrm{d}s_2.$$

For any $u \in H^{\mathcal{L}^{-1}}_{1,k,\tau}(\Upsilon)$, $k = 0, 1, 2, \cdots$, $u = 0$ for $t < 0$ and

$$\int \int_\Upsilon e^{-2\tau t} \sum_{|\alpha|\leq 1, \alpha_t \leq 1} |\partial_x^\alpha \partial_t^{\alpha_t + k} u(x,t)|^2 \ \mathrm{d}x \ \mathrm{d}t < \infty, \qquad (4.15)$$

where α is a two-component multi-index and α_t is a non-negative integer. Then, the norm in $H^{\mathcal{L}^{-1}}_{1,k,\tau}(\Upsilon)$ defined by (4.15) is equivalent to $\| \cdot \|_{1,k,\tau;\Upsilon}$.

Let $v \in \mathcal{K}_0(\bar{\Upsilon})$. We write $\overline{v(\cdot,t)} = \mathcal{V}(s)$ and $v(\cdot, 0) = v_0 = \mathcal{V}_0 \in \dot{H}_1(\Omega \setminus \Gamma)$, and set $s \in \mathbf{C}_\tau$. Using Parseval's equality with $\tau_1 = s_1$ and $\tau_2 = -s_1$ and the fact that the Laplace transform of $\partial_t v$ at $-s^*$ is $-s^*\mathcal{V}(-s^*) - \mathcal{V}_0$, we find that

$$\begin{aligned}
& \int_0^\infty \{ \tilde{B}[\partial_t u, v] + B[u,v] - (\rho^{\frac{1}{2}} \partial_t u, \rho^{\frac{1}{2}} \partial_t v)_{0;\Omega\setminus\Gamma} - (f,v)_{0;\partial\Omega_+} \} \ \mathrm{d}t \\
= & \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big\{ \tilde{B}[sU(s), \mathcal{V}(-s^*)] + B[U(s), \mathcal{V}(-s^*)] \\
& \qquad + (\rho^{\frac{1}{2}} sU(s), \rho^{\frac{1}{2}} (s^* \mathcal{V}(-s^*) + \mathcal{V}_0))_{0;\Omega\setminus\Gamma} \\
& \qquad - (F(s), \mathcal{V}(-s^*))_{0;\partial\Omega_+} \Big\} \ \mathrm{d}s_2. \qquad (4.16)
\end{aligned}$$

Since $U(s)$ is a weak solution of $(****)$, $\mathcal{W} \in \dot{H}_1(\Omega \setminus \Gamma)$ satisfies

$$s^2(\rho^{\frac{1}{2}}U(s), \rho^{\frac{1}{2}}\mathcal{W})_{0;\Omega\setminus\Gamma} + s\tilde{B}[U(s), \mathcal{W}] + B[U(s), \mathcal{W}] = (F(s), \mathcal{W})_{0;\partial\Omega_+}.$$

Taking $\mathcal{W} = \mathcal{V}(-s^*)$, we arrive at

$$\begin{aligned}(\rho^{\frac{1}{2}}sU(s), \rho^{\frac{1}{2}}(s^*\mathcal{V}(-s^*) + \mathcal{V}_0))_{0;\Omega\setminus\Gamma} + s\tilde{B}[U(s), \mathcal{V}(-s^*)]\\ +B[U(s), \mathcal{V}(-s^*)] - (F(s), \mathcal{V}(-s^*))_{0;\partial\Omega_+} = (\rho^{\frac{1}{2}}sU(s), \rho^{\frac{1}{2}}\mathcal{V}_0)_{0;\Omega\setminus\Gamma}.\end{aligned}$$

Therefore, (4.16) can be written as

$$\begin{aligned}\int_0^\infty \{\tilde{B}[\partial_t u, v] + B[u, v] \quad &- \quad (\rho^{\frac{1}{2}}\partial_t u, \rho^{\frac{1}{2}}\partial_t v)_{0;\Omega\setminus\Gamma} - (f, v)_{0;\partial\Omega_+}\}\,\mathrm{d}t\\ &= \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}(\rho^{\frac{1}{2}}sU(s), \rho^{\frac{1}{2}}\mathcal{V}_0)_{0;\Omega\setminus\Gamma}\,\mathrm{d}s_2. \qquad (4.17)\end{aligned}$$

Now let us show that the right-hand side of (4.17) vanishes. Since

$$\begin{aligned}\int_{-\infty}^{\infty} |(\rho^{\frac{1}{2}}U(s), \rho^{\frac{1}{2}}\mathcal{V}_0)_{0;\Omega\setminus\Gamma}|^2\,\mathrm{d}s_2 \quad &\leq \quad \rho \parallel \mathcal{V}_0 \parallel_1^2 \int_{-\infty}^{\infty} \parallel U(s) \parallel_{1,s;\Omega\setminus\Gamma}^2\ \mathrm{d}s_2\\ &\leq \quad c \parallel \mathcal{V}_0 \parallel_1^2 \parallel u \parallel_{1,0,\tau;\Upsilon}^2 < \infty,\end{aligned}$$

the function $(u, v_0)_{0;\Omega\setminus\Gamma} = \phi(t)$ satisfies $\int_0^\infty e^{-2\delta t}|\phi(t)|^2\ \mathrm{d}t < \infty$ for any δ. Let $\psi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t)$. Then the initial condition of $\partial_t u$ implies $\psi(0) = 0$. Hence, we obtain that $0 = \psi(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(sU(s), \mathcal{V}_0)_{0;\Omega\setminus\Gamma}\,\mathrm{d}s_2$. From (4.17) it follows that u satisfies (4.7).

For the uniqueness, suppose that $u \in H_{1,0,\tau}^{\mathcal{L}^{-1}}(\Upsilon)$ satisfies

$$\int_0^\infty \left\{\tilde{B}[\partial_t u, v] + B[u, v] - (\rho^{\frac{1}{2}}\partial_t u, \rho^{\frac{1}{2}}\partial_t v)_{0;\Omega\setminus\Gamma}\right\}\,\mathrm{d}t = 0 \qquad (4.18)$$

for all $v \in \mathcal{K}_0(\bar{\Upsilon})$. Fix an arbitrary $T > 0$. It is obvious that $u(\cdot, t) = U(t)$ can be regarded as a function in $H_1(0, T)$ with values in $H_1(\Omega \setminus \Gamma)$. Then we see that every u has a finite norm $\parallel u \parallel_{1,0;\Upsilon_T}^2$ which is defined by (4.15) with $k = 0$ and Υ replaced by $\Upsilon_T \equiv \Omega \setminus \Gamma \times (0, T)$.

We now define the function

$$\mathcal{Z}(t) = z(\cdot, t) = \begin{cases} -\displaystyle\int_t^T u(\cdot, \omega)\,\mathrm{d}\omega & \text{if } t \leq T,\\ 0 & \text{if } t > T.\end{cases} \qquad (4.19)$$

and

$$\mathcal{Z}'(t) = (\partial_t z)(\cdot, t) = \begin{cases} u(\cdot, t) & \text{if } t < T, \\ 0 & \text{if } t > T. \end{cases} \tag{4.20}$$

Clearly, the restriction of the function $\mathcal{Z}(t)$ to $(0, T)$ belongs to $H_1(0, T; H_1(\Omega \setminus \Gamma))$ and $z(x, t)$ can be approximated in the norm $\| \cdot \|_{1,0;\Upsilon_T}$ by elements $v \in \mathcal{K}(\bar{\Upsilon})$. We can set $v = z$ in (4.18), so that

$$\int_0^T \left\{ \tilde{B}[\partial_t u, z] + B[u, z] - (\rho^{\frac{1}{2}} \partial_t u, \rho^{\frac{1}{2}} \partial_t z)_{0;\Omega\setminus\Gamma} \right\} \, \mathrm{d}t = 0.$$

From this, (4.19) and (4.20) we can derive

$$\int_0^T \left\{ \tilde{B}[\partial_t u, z] + B[\partial_t z, z] - (\rho^{\frac{1}{2}} \partial_t u, \rho^{\frac{1}{2}} u)_{0;\Omega\setminus\Gamma} \right\} \, \mathrm{d}t = 0,$$

or

$$\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \tilde{B}[u, z] + B[z, z] - \| \rho^{\frac{1}{2}} u \|^2_{0;\Omega\setminus\Gamma} \right\} - \tilde{B}[u, u] \right) \, \mathrm{d}t = 0. \tag{4.21}$$

Since $U \in H_1(0, T; H_1(\Omega \setminus \Gamma))$,

$$\begin{aligned} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \| \rho^{\frac{1}{2}} u \|^2_{0;\Omega\setminus\Gamma} \, \mathrm{d}t &= \| \rho^{\frac{1}{2}} U(T) \|^2_{0;\Omega\setminus\Gamma} - \| \rho^{\frac{1}{2}} U(0) \|^2_{0;\Omega\setminus\Gamma} \\ &= \| \rho^{\frac{1}{2}} U(T) \|^2_{0;\Omega\setminus\Gamma} . \end{aligned} \tag{4.22}$$

From (4.19) and (4.20) it follows that $\mathcal{Z} \in H_1(0, T; H_1(\Omega \setminus \Gamma))$. Hence,

$$\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} B[z, z] \, \mathrm{d}t = B[\mathcal{Z}(T), \mathcal{Z}(T)] - B[\mathcal{Z}(0), \mathcal{Z}(0)] = -B[\mathcal{Z}(0), \mathcal{Z}(0)] \tag{4.23}$$

and

$$\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \tilde{B}[u, z] \, \mathrm{d}t = \tilde{B}[U(T), \mathcal{Z}(T)] - \tilde{B}[U(0), \mathcal{Z}(0)] = 0. \tag{4.24}$$

Equalities (4.21)–(4.24) imply that

$$B[\mathcal{Z}(0), \mathcal{Z}(0)] + \| \rho^{\frac{1}{2}} U(T) \|^2_{0;\Omega\setminus\Gamma} + \int_0^T \tilde{B}[u, u] \, \mathrm{d}t = 0.$$

Since B and $\tilde{B}$ are non-negative, $U(T) = u(\cdot, T) = 0$ for any $T > 0$, which completes the proof of the theorem. $\square$

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