# Stochastic processes in random environments and their limiting processes characterized by the semi-selfsimilarity

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# Chapter 1

# Introduction

Stochastic processes are often considered as mathematical models of time evolution of random phenomena. Brownian motion is the canonical example of such stochastic processes because it is in the intersection of many fundamental and important classes of processes. Namely, it is a Gaussian process, a Markov process, a Lévy process, and a selfsimilar process, and each of them has been developed in the probability theory. Here a selfsimilar process is the process whose finite dimensional distributions are invariant under a suitable scaling of time and space. For example, if $\{X(t), t \geq 0\}$ is a Brownian motion on $\mathbf{R}^d$, then for any $a > 0$, the processes $\{X(at), t \geq 0\}$ and $\{a^{1/2}X(t), t \geq 0\}$ have the same finite dimensional distributions. There are many selfsimilar processes other than Brownian motion, namely, $\alpha$-stable Lévy processes, fractional Brownian motions and so on. These processes are considered suitable to describe models of random phenomena and many applications of selfsimilar processes have been studied in the fields of probability theory, statistical physics and mathematical finance. See [EM02] and its reference for more details.

As for an extension of selfsimilarity, a notion of semi-selfsimilarity was introduced. Because of its weaker scaling property, semi-selfsimilar processes are expected to offer higher flexibility in stochastic modeling than selfsimilar ones. However, since semi-selfsimilarity is a new notion developed in [MS99], little is known of semi-selfsimilar processes in spite of their importance. In this thesis, we consider some problems of stochastic processes in random environments whose limiting processes can be characterized by the semi-selfsimilarity.

For studying properties of a stochastic process $\{X(t), t \geq 0\}$, it is important to characterize the distributions of $X(t)$ for fixed $t$ (we call such distributions marginal distributions of $\{X(t)\}$). If selfsimilar and semi-selfsimilar processes are Lévy processes, then their distributions are completely determined. However, it seems to exist no simple characterization of marginal distributions of selfsimilar and semi-selfsimilar processes with only stationary increments. On the other hand, for selfsimilar and semi-selfsimilar processes with independent increments the situation is better. For this reason, we also discuss some examples of selfsimilar and semi-selfsimilar processes with independent increments in this thesis.

In the next chapter, we give a survey about selfsimilar and semi-selfsimilar processes and their marginal distributions.

In Chapter 3, we construct a diffusion process on each of disconnected fractal sets on $\mathbf{R}$ (typical example is a Cantor set) as a limit of a suitably scaled random walk and show that the limiting process is semi-selfsimilar. Such diffusion processes were introduced by Fujita [Fu87] and these diffusion processes have been regarded as Brownian motions on disconnected fractal sets. He obtained a growth order of the eigenvalues of generators of Brownian motions. Analysis on fractal sets has shown that there exist some dimensions and each of them has an important role. From the growth order, the spectral dimensions of disconnected fractal sets are determined and we find a suitable scaling which implies the "random walk dimension" of each of disconnected fractal sets. This dimension shows how far a diffusion process spreads for a long time. Finding random walk dimension, we have a relationship among three dimension; random walk dimension, spectral dimension and Hausdorff dimension (that is a geometrical characteristic). This relation is satisfied in the cases of nested fractal sets (which are connected and finitely ramified ones, see [Ba98]).

In Chapter 4, we consider homogenization problems on the disconnected fractal sets discussed in Chapter 3. In the case of nested fractal sets, Kumagai and Kusuoka [KK96] studied these problems. On a nested fractal set they gave a sequence of independent and identically distributed positive random variables with a finite mean (such a sequence is called *an environment*) and showed that the influence of the environment implies that a stochastic process in a random environment converges weakly to a constant time-changed Brownian motion on each of nested fractal sets. In the case of disconnected fractal sets, we can treat an environment whose mean is infinite, which belongs to wider classes of environments than the cases of nested ones. We show that in the case of an environment whose mean is finite, the limiting process is a constant time-changed diffusion process obtained in Chapter 3, and whose mean is infinite, the limiting process belongs to a new class of semi-selfsimilar processes which are determined by environments.

In Chapter 5, we consider another type of stochastic processes in random environments. We study limiting behaviors of diffusion processes in semi-selfsimilar random environments $\{X(t)\}$ described by a formal stochastic differential equation

$$dX(t) = dB(t) - \frac{1}{2}W'(X(t))dt, \quad X(0) = 0,$$

where $\{B(t)\}$ is a one-dimensional Brownian motion and $\{W(x)\}$ is a semi-selfsimilar process which is independent of $\{B(t)\}$. It is known that the suitable scaling for convergence of $\{X(t)\}$ is not the same as that of homogenization problems and $W$ has an influence for the scaling, and thus in this chapter $W$ is regarded as an environment. In the case where $\{W(x)\}$ is a Brownian motion, Brox [Br86] had studied this problem. This is a continuous time analogue to the problem that Sinai [Si82] considered for the case of a one-dimensional random walk. His results was extended to the case of selfsimilar processes by Kawazu, Tamura and Tanaka [KTT88]. The selfsimilarity of $W$ has a very important role in studying the limiting behavior of $\{X(t)\}$. We try to relax the selfsimilarity of the environments to a more weaker scaling property: the semi-selfsimilarity, and see the difference between them. We show that the semi-selfsimilarity of environments implies that the limit distribution of a suitably scaled process $\{X(t)\}$ converges along a subsequence. It should be noted that environments do not have selfsimilarity in general, so that we cannot expect to have the limit distribution along a full sequence. But considering some characteristics of the limit distribution, we can show that the difference between the scaled diffusion process and the distribution (which is varying because of the semi-selfsimilarity of environments) converges to 0 along a full sequence.

In Chapter 6, we characterize the limit distributions obtained in Chapter 5 for some environments. To see them, firstly we show that all finite dimensional distributions of the suitably scaled process $\{X(t)\}$ considered in Chapter 5 converge to those of a semi-selfsimilar process which is determined by an environment. Secondly, we consider a case of a reflecting Lévy environment and show that the limiting process has independent increments. Sato [S91] studied some properties of marginal distributions of selfsimilar processes with independent increments. His results were extended to the case of semi-selfsimilar processes with independent increments by Maejima and Sato [MS99], and some examples were given (see Example 2.2.6). Our process obtained here is a new type of semi-selfsimilar processes with independent increments.

In Chapter 7, we study the limiting behaviors of diffusion processes in multi-dimensional random environments. It is well-known that $d$-dimensional Brownian motion $\{\mathbb{B}(t), t \geq 0\}$ (whose components are $d$ independent copies of one-dimensional Brownian motion $\{B(t)\}$) are recurrent for $d = 1$ or 2, namely, for any open subset $U \in \mathbf{R}^d$

$$P\{\mathbb{B}(t) \in U \text{ for some } t > 0\} = 1,$$

and this is no longer true otherwise, in which case Brownian motion is said to be transient. We consider the same problem for $\{X(t)\}$ considered in Chapter 5, instead of $\{B(t)\}$. However, whether the diffusion processes in semi-selfsimilar environments are recurrent or transient seems to be similar to that in the case of selfsimilar ones except few singular semi-selfsimilar ones, and thus we treat some selfsimilar environments to obtain the essence of the problems about recurrence and transience.

# Chapter 2

# Selfsimilar and semi-selfsimilar processes

In this chapter, we give a brief survey about selfsimilar and semi-selfsimilar processes. Most of the stochastic processes discussed in this chapter are $\mathbf{R}^d$-valued processes defined on a common probability space $(\Omega, \mathcal{F}, P)$. Hereafter in this thesis, for stochastic processes $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ we denote by $\{X(t)\} \stackrel{d}{=} \{Y(t)\}$ equality of all finite dimensional distributions with respect to $P$ and say that processes $\{X(t)\}$ and $\{Y(t)\}$ are *identical in law.*

## 2.1 Selfsimilar and semi-selfsimilar processes

We give definitions of selfsimilarity and semi-selfsimilarity of processes.

**Definition 2.1.1**

(i) A stochastic process $\{X(t), t \geq 0\}$ is called "selfsimilar" if for any $a > 0$, there exists $b = b(a) > 0$ such that

$$\{X(at), t \geq 0\} \stackrel{d}{=} \{bX(t), t \geq 0\}. \tag{2.1.1}$$

(ii) A stochastic process $\{X(t), t \geq 0\}$ is called "semi-selfsimilar" if there exist $a \in (0,1) \cup (1, \infty)$ and $b > 0$ satisfying (2.1.1). $a > 1$ and corresponding $b$ satisfying (2.1.1) are called an epoch and a span of $\{X(t)\}$, respectively.

In this thesis, for each of processes satisfying (i) and (ii) of Definition 2.1.1 we write that $\{X, P\}$ is *selfsimilar* and *semi-selfsimilar*, respectively. We say that a distribution $\mu$ on $\mathbf{R}^d$ is trivial if it is a $\delta$-distribution and nontrivial otherwise. We also say that $\{X(t), t \geq 0\}$ is trivial if the distribution of $\{X(t)\}$ is trivial for each $t > 0$ and nontrivial otherwise.

We give two examples of semi-selfsimilar processes.

**Example 2.1.2 ([MS99], [M01])** A probability distribution $\mu$ is called strictly semi-stable, if for some $a \in (0,1) \cup (1,\infty)$ there exists $b > 0$ such that $\hat{\mu}(\theta)^a = \hat{\mu}(b\theta)$, where $\hat{\mu}$ is the characteristic function of $\mu$ and $\theta \in \mathbf{R}^d$. It is known that a strictly semi-stable distribution can also be characterized as certain subsequential limits of normalized partial sums of independent and identically distributed random variables. Moreover, there exists a unique $\alpha \in (0,2]$ such that $b = a^{1/\alpha}$, so that we call such a distribution $\alpha$-semi-stable. 2-semi-stable is nothing but Gaussian. Let $\{X(t), t \geq 0\}$ be a nontrivial Lévy process (see Definition 2.2.1 below) such that the distribution of $X(1)$ is strictly $\alpha$-semi-stable. Then $\{X, P\}$ is semi-selfsimilar with $b = a^{1/\alpha}$. We call such a process a strictly $\alpha$-semi-stable Lévy process. This is an extension of strictly $\alpha$-stable Lévy processes defined below.

**Example 2.1.3 ([Ku87])** Let $\boldsymbol{a}_1 = (0,0), \boldsymbol{a}_2 = (1,0)$ and $\boldsymbol{a}_3 = (1/2, \sqrt{3}/2)$. Using these points, we define maps

$$\begin{aligned} \varphi_1(\boldsymbol{x}) &= \frac{\boldsymbol{x}}{2}, \\ \varphi_2(\boldsymbol{x}) &= \frac{\boldsymbol{x} + \boldsymbol{a}_2}{2}, \\ \varphi_3(\boldsymbol{x}) &= \frac{\boldsymbol{x} + \boldsymbol{a}_3}{2}. \end{aligned}$$

Then there exists a compact set $E$ such that

$$E = \bigcup_{i=1}^{3} \varphi_i(E),$$

uniquely. Such a set $E$ is called a two-dimensional Siérpinski Gasket. Let $G_0$ be a pre-Siérpinski Gasket (which is a graphical Siérpinski Gasket whose distance to the nearest point is one). Define a simple random walk $\{R_n : n = 0, 1, 2, \ldots\}$ on $G_0$ with the transition probabilities

$$P\{R_{n+1} = y | R_n = x\} = \begin{cases} \dfrac{1}{N(x)} & \text{if} \quad |y - x| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $N(x) = \sharp\{y \in G_0 : |y - x| = 1\}$. We set

$$X^{(n)}(t) = 2^{-n} R_{[5^n t]}, \quad t \geq 0, \; n = 0, 1, 2, \ldots, \tag{2.1.2}$$

where $[x]$ means the largest integer not larger than $x$. Then the scaled process $\{X^{(n)}(t)\}$ converges weakly to the process $\{X(t)\}$ such that $\{X, P\}$ has the following semi-selfsimilarity,

$$\{X(5^n t), t \geq 0\} \stackrel{d}{=} \{2^n X(t), t \geq 0\}.$$

Two examples above give a motivation to study semi-selfsimilar processes. These scaling properties are much weaker than selfsimilarity and expected to offer higher flexibility in stochastic modelings. Furthermore, the semi-selfsimilarity might also be important notion in analysis on fractal sets.

We say that $\{X(t), t \geq 0\}$ is stochastically continuous at $t$ if

$$\lim_{h \to 0} P\{|X(t + h) - X(t)| > \varepsilon\} = 0 \tag{2.1.3}$$

for any $\varepsilon > 0$. The following relations between $a$ and $b$ in Definition 2.1.1 are known.

**Theorem 2.1.4 ([La62], [MS99])**

(i) *If a selfsimilar process $\{X(t), t \geq 0\}$ is non-trivial and stochastically continuous at $t = 0$, then there exists a unique exponent $H \geq 0$ such that $b$ in (2.1.1) can be expressed as $b = a^H$.*

(ii) *If a semi-selfsimilar process $\{X(t), t \geq 0\}$ is non-trivial and stochastically continuous at each $t \geq 0$, then there exists a unique exponent $H \geq 0$ such that $b$ in (2.1.1) can be expressed as $b = a^H$.*

(iii) *In both cases $H > 0$ if and only if $X(0) = 0$ almost surely.*

Considering the selfsimilar and semi-selfsimilar processes that each of which has a unique exponent $H > 0$, we use notation $H$-selfsimilar and $H$-semi-selfsimilar processes, respectively. The condition that a semi-selfsimilar process is selfsimilar is the following.

**Theorem 2.1.5 ([MS99], [MSW99])**

(i) *We put*

$$\lambda_0 = \inf\{a > 1 : a \textit{ satisfies } (2.1.1) \textit{ for some } b > 0\}. \tag{2.1.4}$$

*If $\lambda_0 = 1$, then $\{X, P\}$ is selfsimilar.*

(ii) *If $\{X(t), t \geq 0\}$ is stochastically continuous at each $t$ and satisfies* (2.1.1) *for some $a_1$ and $a_2$ such that $\log a_1 / \log a_2$ is irrational, then $\{X, P\}$ is selfsimilar.*

For stochastic processes, we denote by $\stackrel{f.d.}{\Longrightarrow}$ the convergence of all finite-dimensional distributions. Semi-selfsimilar processes are characterized as limiting processes of geometric subsequences of the normalized processes as follows.

**Theorem 2.1.6 ([MS99])**

(i) *We assume the following conditions:*

(1) *$\{X(t), t \geq 0\}$ is stochastically continuous at $t = 0$.*

(2) *There exist another process $\{Y(t)\}$ and sequences $\{a_n\}$ and $\{b_n\}$ with $0 < a_n \nearrow \infty$ and $0 < b_n \nearrow \infty$ such that, for some $a > 1$,*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a,$$
$$\frac{1}{b_n}\{Y(a_{n+1}t) - Y(a \cdot a_n t)\} \to 0 \quad \textit{in probability},$$
$$\left\{\frac{1}{b_n} Y(a_n t), t \geq 0\right\} \stackrel{f.d.}{\Longrightarrow} \{X(t), t \geq 0\}.$$

(3) *There exists $t_0 > 0$ such that $X(t_0)$ and $X(at_0)$ are non-degenerated.*

*Then $\{X, P\}$ is $H$-semi-selfsimilar with some $H > 0$.*

(ii) *Conversely, if $\{X, P\}$ is non-trivial, $H$-semi-selfsimilar with $H > 0$ and stochastically continuous at $t = 0$, then $\{X(t)\}$ is such a limiting process.*

**Remark 2.1.7** Maejima and Sato [MS99] showed that the theorem above still holds for a wide-sense $H$-semi-selfsimilar process, namely,

$$\{X(at), t \geq 0\} \stackrel{d}{=} \{bX(t) + c(t), t \geq 0\}$$

for a non-random function $c : [0, \infty) \to \mathbf{R}^d$. In this thesis, we consider the case where (2.1.1) is satisfied.

## 2.2 Selfdecomposable and semi-selfdecomposable distributions

In this section, we consider the relationship between (semi-)selfsimilar processes and their marginal distributions. We give definitions of a Lévy process and an additive process, which are very important classes of stochastic processes.

**Definition 2.2.1** A stochastic process $\{X(t), t \geq 0\}$ is called a Lévy process if the following conditions are satisfied:

(1) For any choice of $n \geq 1$ and $0 \geq t_0 < t_1 < t_2 < \cdots < t_n$, random variables $X(t_0), X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent (called independent increment property).

(2) $X(0) = 0$ almost surely.

(3) The distribution of $X(s + t) - X(s)$ does not depend on $s$ (called stationary increment property).

(4) It is stochastically continuous (see (2.1.3) for the definition).

(5) Its sample paths are right-continuous and have left limits almost surely.

A stochastic process satisfying conditions (1), (2), (4) and (5) is called an additive process.

For example, the Brownian motion (which is modeled for continuous random motion) and the Poisson process (which is modeled for jumping random motion) are Lévy processes. Selfsimilar Lévy processes other than Brownian motion constitute an important class called strictly stable Lévy processes.

**Definition 2.2.2** A probability measure $\mu$ is called strictly stable, if for any $a > 0$, there exists $b > 0$ such that $\hat{\mu}(\theta)^a = \hat{\mu}(b\theta)$ for any $\theta \in \mathbf{R}^d$. Let $\{X(t), t \geq 0\}$ be a Lévy process. It is called a strictly stable Lévy process if the distribution of $X(1)$ is strictly stable.

Strictly stable distributions can be characterized as certain limits of normalized sums of independent and identically distributed random variables. The following properties are known:

(i) If $\mu$ is strictly stable, there exists a unique $\alpha \in (0, 2]$. We call such a distribution $\mu$ $\alpha$-stable (2-stable is nothing but Gaussian).

(ii) We assume $\{X(t)\}$ is a Lévy process. Then the distribution of $X(1)$ is $\alpha$-stable or $\alpha$-semi-stable if and only if $\{X(t)\}$ is $1/\alpha$-selfsimilar or $1/\alpha$-semi-selfsimilar, respectively.

This implies that marginal distributions of selfsimilar and semi-selfsimilar Lévy processes are completely determined by the distribution at time 1. We next consider selfsimilar and semi-selfsimilar additive processes, namely, they do not necessarily have stationary increments but have independent increments, due to Sato [S91] and Maejima and Sato [MS99], respectively. We start with a notion of *selfdecomposability* and *semi-selfdecomposability* of distributions.

**Definition 2.2.3** A probability measure $\mu$ is called selfdecomposable, if for any $b > 1$, there exists a probability measure $\rho_b$ on $\mathbf{R}^d$ such that

$$\hat{\mu}(\theta) = \hat{\mu}(b\theta)\hat{\rho}_b(\theta), \quad \theta \in \mathbf{R}^d, \tag{2.2.5}$$

and called semi-selfdecomposable if there exist a $b > 1$ and an infinitely divisible probability measure $\rho_b$ satisfying (2.2.5). Such a $b$ is called a span of $\mu$.

The selfdecomposable or semi-selfdecomposable distributions are obtained as a limit distribution of normalized sum or normalized partial sum, respectively. Refer to [S80] and [MN98] for more details. The following results link selfsimilar and semi-selfsimilar processes to selfdecomposablity and semi-selfdecomposablity of distributions, respectively.

**Theorem 2.2.4 ([S91], [MS99])**

(i) *If $\{X(t), t \geq 0\}$ is a selfsimilar additive process, then the distribution of $X(t)$ is selfdecomposable for each $t$.*

(ii) *If $\mu$ is a non-trivial selfdecomposable probability measure, then for any $H > 0$ there exists, identically in law, a non-trivial $H$-selfsimilar additive process $\{X(t), t \geq 0\}$ such that the distribution of $X(1)$ equals to $\mu$.*

(iii) *If $\{X(t), t \geq 0\}$ is a semi-selfsimilar additive process having $b$ as a span, then the distribution of $X(t)$ is semi-selfdecomposable having $b$ as a span for each $t$.*

(iv) *If $\mu$ is a non-trivial semi-selfdecomposable probability measure on $\mathbf{R}^d$ having $b$ as a span, then for any $H > 0$ there exists a non-trivial $H$-semi-selfsimilar additive process $\{X(t), t \geq 0\}$ having $b$ as a span such that the distribution of $X(1)$ equals to $\mu$.*

**Remark 2.2.5** Maejima and Sato [MS99] showed the theorem above still holds for a wide-sense $H$-semi-selfsimilar process.

**Example 2.2.6 ([G79],[MS99])**

(i) Let $\{B(t), t \geq 0\}$ be a Brownian motion on $\mathbf{R}^d$ with $d = 3, 4, 5, \ldots$. Define the last exit time from the ball $\{x : |x| \leq r\}$ by

$$L(r) = \sup\{t \geq 0 : |B(t)| \leq r\}.$$

This process is a selfsimilar additive process such that

$$\{L(ar), r \geq 0\} \stackrel{d}{=} \{a^2 L(r), r \geq 0\}.$$

Therefore, the distribution of $L(r)$ is selfdecomposable for each $r > 0$.

(ii) Let $a > 1$ and $H > 0$. For $t \in [1, a)$, let $\nu_t$ be zero on $(-\infty, 0)$ and $\nu_t(B) = \int_B k_t(x) g_t(\log x) x^{-1} dx$ on $(0, \infty)$, where $g_t(x)$ and $k_t(x)$ satisfy the following conditions:

(1) For each $t \in [1, a)$, $g_t(x)$ is non-negative, bounded, Borel measurable, periodic with period $H \log a$ and not identically zero.

(2) For each $t \in [1, a)$, $k_t(x)$ is non-negative, decreasing with $\int_0^\infty x(1+x^2)^{-1} k_t(x) dx < \infty$ and not identically zero.

(3) For each $x > 0$, the function $k_t(x) g_t(\log x)$ is continuous and increasing in $t \in [1, a)$ and tends to $k_1(a^{-H} x) g_1(\log x)$ as $t \nearrow a$.

Then there exist stochastically continuous $H$-semi-selfsimilar processes $\{X(t), t \geq 0\}$ having $a$ as an epoch with independent increments. Therefore, the distribution of $X(t)$ is semi-selfdecomposable for each $t > 0$.

**Remark 2.2.7**

(i) Getoor [G79] showed that for each $r > 0$ the distributions of $L(r)$ is given by

$$P\{L(r) \in B\} = 2^{-(d-2)/2} (\Gamma(\tfrac{d-2}{2}))^{-1} r^{d-2} \int_B s^{-d/2} e^{-r^2/(2s)} ds$$

for any Borel set $B \in [0, \infty)$. In the case where $d = 3$, Pitman [Pi75] showed that $\{L(r)\}$ is a strictly $1/2$-stable increasing Lévy process, and otherwise not a Lévy process.

(ii) Semi-selfsimilar processes $\{X(t), t \geq 0\}$ are not determined identically in law by $\mu, b$ and $H$ because they are not determined by the distribution of $X(1)$ but of that of $\{X(t), 1 \leq t < \lambda_0\}$.

In Chapter 6, we give an example of the semi-selfsimilar process whose marginal distributions are semi-selfdecomposable. That is constructed in a different way from that of (ii) in Example 2.2.6.

# Chapter 3

# Diffusion processes with semi-selfsimilarity on disconnected fractal sets

In this and the next chapter, we consider semi-selfsimilar processes which are similar to that in Example 2.1.3, namely, the case where geometrical structure induces the semi-selfsimilarity of processes. In this chapter, we construct diffusion processes on disconnected fractal sets in **R**. The diffusion processes on such fractal sets were introduced by Fujita [Fu87]. Fujita obtained the growth order of the eigenvalues of the generators of diffusion processes. In [Fu90], he also studied the estimates of transition probability densities for diffusion processes with the generators. We construct such diffusion processes as scaling limits of these random walks by using a general limit theorem for one-dimensional generalized diffusion processes developed by Itô and Mckean [IM65] and Stone [St63], and regard the diffusion process as Brownian motion on each of disconnected fractal sets.

In Section 3.1, we give a setting for disconnected fractal sets on which we are going to construct diffusion processes. In Section 3.2, we construct diffusion processes as limits of scale-changed and time-changed random walks. This implies "*random walk dimension*" of each of disconnected fractal sets. In Section 3.3, we study the relationship among three dimensions, *random walk dimension* $d_w$, *spectral dimension* $d_s$ and *Hausdorff dimension* $d_f$. Analysis on connected fractals has shown that there exist some dimensions and they are important to study the property of fractal sets. In connected ones' cases, it is known that the relationship among the three dimensions $d_s = 2d_f/d_w$. We show that this relation is also valid in disconnected ones' cases.

## 3.1 Selfsimilar disconnected fractal sets on R.

Let $r > 1$ and let $\varphi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$ be a family of $r$-similitudes on $[0,1]$, namely, for $i = 1, 2, \dots, N$,

$$\varphi_i(x) = r^{-1}x + b_i, \quad 0 \le b_i \quad \text{and} \quad r^{-1} + b_i \le 1.$$

We set

$$\varphi_1(x) = r^{-1}x,$$
$$\varphi_N(x) = r^{-1}x + (1 - r^{-1}).$$

Then it is well-known that there exists a unique compact set $\widetilde{C} \subset [0,1]$ such that

$$\widetilde{C} = \bigcup_{i=1}^{N} \varphi_i(\widetilde{C}).$$

We assume

**Assumption 3.1.1** (**Strong separating condition**)

$$\varphi_i([0,1]) \cap \varphi_j([0,1]) = \emptyset \quad \text{for} \quad i \neq j.$$

Without loss of generality, we can assume that for $i = 1, 2, \dots, N-1$,

$$r^{-1} + b_i < b_{i+1}.$$

Then $\widetilde{C}$ is a disconnected fractal set on $[0,1]$ associated with $\varphi$. Assumption 3.1.1 implies that $\widetilde{C}$ satisfies the open set condition and that $N < r$. Let

$$\widetilde{m}(A) = \sum_{i=1}^{N} N^{-1}\widetilde{m}(\varphi_i^{-1}(A))$$

for any Borel set $A \subset [0,1]$. This $\widetilde{m}$ with support in $[0,1]$ is determined uniquely and let $\widetilde{m}(\widetilde{C}) = 2$. Set

$$\widetilde{F}_0 = \{0, 1\},$$

and define

$$\widetilde{F}_{n+1} = \bigcup_{i=1}^{N} \varphi_i(\widetilde{F}_n),$$

inductively. Let

$$\widetilde{F}_\infty = \bigcup_{n=0}^{\infty} \widetilde{F}_n.$$

Then $\widetilde{C}$ coincides with $Cl(\widetilde{F}_\infty)$.

Next we define an unbounded disconnected fractal set on $[0, \infty)$. Let

$$F_0 = \bigcup_{n=0}^{\infty} \varphi_1^{-n}(\widetilde{F}_n),\ F_n = \varphi_1^{(n)}(F_0),\ F_\infty = \bigcup_{n=0}^{\infty} F_n,\ C = Cl(F_\infty).$$

Then $C$ is an unbounded disconnected fractal set on $[0, \infty)$ associated with $\varphi$ and $F_0$ is called a pre-fractal set. We denote points of any $F_0$ and $F_n$ by $\{0 = a_0, a_1, a_2, \cdots\}$ and $\{0 = a_0^{(n)}, a_1^{(n)}, a_2^{(n)}, \cdots\}$ in the order from left, respectively. We define a measure $m$ on $C$ by

$$m(A) = N^n \widetilde{m}(\varphi_1^{(n)}(A)) \tag{3.1.1}$$

for any Borel set $A \subset [0, r^n]$ and $m(x) = 0$ for any $x \leq 0$.

## 3.2 Construction of diffusion processes

Using the infinitely extended measure $m$ defined by (3.1.1), we consider a generalized one-dimensional diffusion process $\{X(t), t \geq 0\}$ with a generator

$$\frac{d}{dm(x)} \frac{d}{dx}. \tag{3.2.2}$$

This $m$ is often called a speed measure. See [KW82] for more details. Fujita studied the estimates of transition probability densities for generalized one-dimensional diffusion processes with the generator (3.2.2). The process $\{X(t)\}$ can be regarded as a Brownian motion on $C$. In order to see this, we construct $\{X(t)\}$ from a limit of a suitably scaled random walk on a pre-fractal set $F_0$ by a similar manner developed in [St63]. Let

$$2h = \min_{i=1,2,\dots,N-1} (a_i - a_{i-1}). \tag{3.2.3}$$

We define a random walk $\{R_{hj}, j = 0, 1, 2, \ldots\}$ starting from 0 on $F_0$ whose jumps occur at integral multiples of $h$ as follows:
For $i \in \mathbf{N}$,

$$\begin{aligned}
P(R_{h(j+1)} = a_{i-1} | R_{hj} = a_i) &= \frac{h}{a_i - a_{i-1}}, \\
P(R_{h(j+1)} = a_i | R_{hj} = a_i) &= 1 - \left\{ \frac{h}{a_i - a_{i-1}} + \frac{h}{a_{i+1} - a_i} \right\}, \\
P(R_{h(j+1)} = a_{i+1} | R_{hj} = a_i) &= \frac{h}{a_{i+1} - a_i},
\end{aligned}$$

and at the origin,

$$\begin{aligned}
P(R_{h(j+1)} = a_1 | R_{hj} = a_0) &= \frac{h}{a_1}, \\
P(R_{h(j+1)} = a_0 | R_{hj} = a_0) &= 1 - \frac{h}{a_1}.
\end{aligned}$$

Let $\sigma_i$ be the waiting time of the random walk at the state $a_i$. From the definitions above, we see that $h^{-1}\sigma_i$ has a geometrical distribution with the mean

$$\left( \frac{h}{a_i - a_{i-1}} + \frac{h}{a_{i+1} - a_i} \right)^{-1} \quad \text{and} \quad \frac{a_1}{h}$$

in the case where $i \neq 0$ and $i = 0$, respectively.

We next consider the suitable scaling. For simplicity we consider a triadic Cantor set, and thus $h = 1/2$. For any $m, n, k \in \mathbf{N}$, there exist $k_0, k_1, k_2 \in \mathbf{N}$ such that

$$a_{k-1}^{(n)} = a_{k_0}^{(n)}, \; a_k^{(n)} = a_{k_1}^{(n+m)}, \; a_{k+1}^{(n)} = a_{k_2}^{(n+m)}.$$

In the same manner as that of $\{R_{j/2}\}$, we define a random walk $\{R_{j/2}^{(n)}\}$ and $\{R_{j/2}^{(n+m)}\}$ on $F_n$ and $F_{n+m}$, respectively. Then we have the following relation:

$$\begin{aligned}
&6^m E \left[ \min \left\{ j : R_{(i+j)/2}^{(n)} \text{ hits } a_{k-1}^{(n)} \text{ or } a_{k+1}^{(n)} \right\} \mid R_{i/2}^{(n)} = a_k^{(n)} \right] \qquad (3.2.4) \\
= \;& E \left[ \min \left\{ j : R_{(i+j)/2}^{(n+m)} \text{ hits } a_{k-1}^{(n+m)} \text{ or } a_{k+1}^{(n+m)} \right\} \mid R_{i/2}^{(n+m)} = a_k^{(n+m)} \right].
\end{aligned}$$

For each disconnected fractal set, we put

$$R(t) = R_{hj}, \qquad hj \le t < h(j+1).$$

From the relation (3.2.4), we should set $3^{-n}R(6^n t)$ and then expect to have a non-trivial limiting process as $n \to \infty$. Let $\mathrm{D} = \mathrm{D}([0,\infty);\mathbf{R})$ be the space of $\mathbf{R}$-valued right continuous functions on $(0,\infty)$ with left limits with the Skorohod topology. We have the following convergence of a scale-changed and time-changed random walk to the generalized diffusion process $\{X(t), t \ge 0\}$ with the generator (3.2.2) in D.

**Theorem 3.2.1** *A scale-changed and time-changed random walk*

$$\left\{r^{-n}R\left((rN)^n t\right), t \ge 0\right\}$$

*on* $F_0$ *converges weakly in* D *to the process* $\{X(t), t \ge 0\}$ *on* $C$ *starting from* 0 *with the generator* (3.2.2) *as* $n \to \infty$.

**Remark 3.2.2** In the case of a triadic Cantor set, $r = 3, N = 2$ and $rN = 6$.

This theorem can be proved in a similar way to that of Theorem 4.1.1 (i) below.

## 3.3 Relationship among dimensions of disconnected fractal sets

Theorem 2.1.6 and Theorem 3.2.1 imply that $\{X, P\}$ above has the following semi-selfsimilar property,

$$\{rX(t), t \ge 0\} \stackrel{d}{=} \{X((rN)t), t \ge 0\}. \qquad (3.3.5)$$

Therefore, the random walk dimension $d_w$ of $C$ is given by

$$d_w = \log(rN)/\log r.$$

Fujita [Fu87] obtained the growth order of the eigenvalues of (3.2.2). That implied the *spectral dimension* of a disconnected fractal set as follows. Let

$\{\lambda_n\}$ be eigenvalues such that $\lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots$. He showed that there exist positive constants $C_1, C_2$ and $n_0$ such that

$$C_1 n^{\log(Nr)/\log N} < -\lambda_n < C_2 n^{\log(Nr)/\log N}$$

for any $n \geq n_0$. Let

$$\rho(x) = \sharp\{-\lambda \geq x : \lambda'\text{s are eigenvalues of (3.2.2)}\}.$$

This implies the existence of $d_s = 2\log N/\log(rN)$ such that

$$0 < \liminf_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \limsup_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \infty,$$

and this $d_s$ is called a spectral dimension. By Assumption 3.1.1, the Hausdorff dimension $d_f = \log N/\log r$, therefore the following relationship is also satisfied in the case of disconnected fractal sets,

$$d_s d_w = 2d_f. \tag{3.3.6}$$

It is known that this relation is valid in the case of connected fractal sets (see [Ba98]).

# Chapter 4

# Homogenization problems on disconnected fractal sets

In this chapter, we consider the homogenization problems of diffusion processes constructed in Chapter 3, and thus we use the same notation as those of Chapter 3. The problem is as follows. Consider a sequence of independent and identically distributed random variables $\alpha = \{\alpha_i\}_{i=1}^{\infty}$. We regard this $\alpha$ as an environment. For a given realization of the environment $\alpha$, we define a Markov process on a pre-fractal set $F_0$ which jumps to neighbors with a rate determined by $\alpha_i$'s. Such a process is considered as a model for hopping conduction in a disordered medium, see the book of Hughes [H96].

In Section 4.1, we construct a birth and death process $\{Y(t), t \geq 0\}$ on a pre-fractal set whose jumping rate and waiting time are determined by $\alpha_i$'s. Under some assumptions on $\alpha_i$, some limit theorems are proved for suitably normalized and scaled processes $\{Y(t)\}$. In the case of nested fractal sets (which are connected and finite ramified) the same problems were studied by Kumagai and Kusuoka [KK96]. They dealt with an environment whose mean is finite. In the present case, we can treat environments whose mean is infinite and show that the limiting process is a semi-selfsimilar process whose semi-selfsimilarity is not (3.3.5) but determined by the environment. In Section 4.2, we consider the random walks' case, namely, their jumps occur at integral multiples of a unit time.

## 4.1 Homogenization problems

Let $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ be a sequence of independent and identically distributed random variables with values in $(0, \infty)$. For $x \in [0, \infty)$ and $k = 0, 1, 2, \ldots$, we set

$$I(0) = 0 \quad \text{and} \quad I(x) = a_l + k, \quad a_l + k < x \leq \{a_l + (k+1)\} \wedge a_{l+1},$$

$$J_0(0) = 0 \quad \text{and} \quad J_0(l) = \sum_{i=1}^{l} \lfloor a_i - a_{i-1} \rfloor,$$

$$J_1(x) = J_0(l) + k, \quad a_l + k < x \leq \{a_l + (k+1)\} \wedge a_{l+1}, \tag{4.1.1}$$

where $\lfloor x \rfloor$ denotes the function of rounding up to the integer and $a_i \in F_0$ (that is a pre-fractal set, see Section 3.1). For a given $\alpha$, we set

$$S(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ S(I(x)) + \alpha_{J_1(x)+1}(x - I(x)) & \text{for } x > 0. \end{cases} \tag{4.1.2}$$

Denote by $\mathcal{A}$ the $\sigma$-field generated by $\alpha$. For a given $\mathcal{A}$, we define a birth and death process $\{Y(t), t \geq 0\}$ starting from 0 on a pre-fractal set $F_0$ in the random environment $\alpha$ by using this $S(x)$ as follows:
For $i \in \mathbf{N}$,

$$P\{Y(t+h) = a_{i-1} | Y(t) = a_i, \mathcal{A}\} = \frac{h}{S(a_i) - S(a_{i-1})} + o(h),$$

$$P\{Y(t+h) = a_i | Y(t) = a_i, \mathcal{A}\} = 1 - \left\{ \frac{h}{S(a_i) - S(a_{i-1})} + \frac{h}{S(a_{i+1}) - S(a_i)} \right\} + o(h),$$

$$P\{Y(t+h) = a_{i+1} | Y(t) = a_i, \mathcal{A}\} = \frac{h}{S(a_{i+1}) - S(a_i)} + o(h),$$

$$P\{Y(t+h) = a_j | Y(t) = a_i, \mathcal{A}\} = o(h) \quad \text{for } j \notin \{i-1, i, i+1\},$$

and at the origin,

$$P\{Y(t+h) = a_1 | Y(t) = 0, \mathcal{A}\} = \frac{h}{S(a_1)} + o(h),$$

$$P\{Y(t+h) = 0 | Y(t) = 0, \mathcal{A}\} = 1 - \frac{h}{S(a_1)} + o(h),$$

$$P\{Y(t+h) = a_j | Y(t) = 0, \mathcal{A}\} = o(h) \quad \text{for } j \in \{2, 3, 4, \ldots\},$$

as $h \to 0$. Let $\mathrm{D} = \mathrm{D}([0,\infty);\mathbf{R})$. We have the following theorem.

**Theorem 4.1.1** *Let $\{X(t), t \geq 0\}$ be the diffusion process on $C$ with the generator* (3.2.2)*.*

(i) *If $E[\alpha_1] = a < \infty$, then the process*

$$\{r^{-n}Y(a(rN)^n t), t \geq 0\}$$

*converges weakly in* D *to the process $\{X(t)\}$ as $n \to \infty$.*

(ii) *If there exists a slowly varying function $L_1(\cdot)$ such that*

$$\frac{1}{nL_1(n)}\sum_{i=1}^{n}\alpha_i \to 1$$

*in probability, then the process*

$$\{r^{-n}Y((rN)^n L_1(r^n)t), t \geq 0\}$$

*converges weakly in* D *to the process $\{X(t)\}$ as $n \to \infty$.*

(iii) *If $\alpha_i$'s belong to the domain of attraction of a one-sided positive strictly stable distribution of index $\alpha \in (0,1)$, namely, there exists a slowly varying function $L_2(\cdot)$ such that*

$$\frac{1}{nL_2(n)}\sum_{i=1}^{n}\alpha_i$$

*converges in law to a one-sided positive strictly stable distribution of index $\alpha$. Then there exists a semi-selfsimilar process $\{\widetilde{X}(t), t \geq 0\}$ on $C$ such that the process*

$$\{r^{-n}Y(\{r^{1/\alpha}N\}^n L_2(r^n)t), t \geq 0\}$$

*converges weakly in* D *to the process $\{\widetilde{X}(t)\}$ as $n \to \infty$. This $\{\widetilde{X}, P\}$ has the following semi-selfsimilar property,*

$$\{\widetilde{X}(t), t \geq 0\} \stackrel{d}{=} \{r^{-1}\widetilde{X}(r^{1/\alpha}t)N, t \geq 0\}. \tag{4.1.3}$$

**Proof of Theorem 4.1.1**

We can construct the birth and death process $\{Y(t), t \geq 0\}$ on $F_0$ as a generalized one-dimensional diffusion process with a generator

$$\frac{d}{dm(x)}\frac{d}{dS(x)},$$

where $m(x) = 0$ for $x < 0$ and $m(x) = 2i$ for $a_{i-1} \leq x < a_i$. This $S(x)$ is often called a scale function. $\{Y(t)\}$ is realized by a scale-changed and a time-changed one-dimensional Brownian motion in the following manner. Let

$$\Omega_0 = \{\omega \in \mathrm{C}([0,\infty); \mathbf{R}) : \omega(0) = 0\} \tag{4.1.4}$$

and let $P_0$ be the Wiener measure on $\Omega_0$. For an element $\omega \in \Omega_0$ we write $B(t) = B(t,\omega) = \omega(t) =$ the value of $\omega$ at time $t$. Then $\{B(t), t \geq 0, P_0\}$ is a Brownian motion starting from 0. We define $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ on another probability space $(\Omega_1, \mathcal{F}_1, P_1)$ and consider the homogenization problem on $(\Omega_0 \times \Omega_1, P_0 \times P_1)$. For a fixed $\alpha$ we set

$$\begin{aligned}
M_0(x) &= m_0 \circ S^{-1}(x), \\
l(t,x) &= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{[x-\varepsilon, x+\varepsilon]}(B_s)\,ds, \\
A_0(t) &= \int_{\mathbf{R}} l(t,x) M_0(dx).
\end{aligned}$$

Then two processes $\{Y(t)\}$ and $\{S^{-1}(B(A_0^{-1}(t)))\}$ are identical in law.

**Proof of (i) of Theorem 4.1.1**

From the definition of $J_1(x)$ (see (4.1.1) for the definition) and $N < r$ (that is induced by Assumption 3.1.1), we have

$$|J_1(r^n) - r^n| \leq (N-1)\sum_{k=0}^{n-1} r^k \leq N^n.$$

Hence, for any $x \in [0, r^m]$ we have

$$|J_1(r^n x) - r^n x| \leq N^{n+m}. \tag{4.1.5}$$

The definition of $S(x)$ implies

$$\left| S(I(x)) - \sum_{i=1}^{J_1(x)} \alpha_i \right| \le \sum_{\{i: a_i - I(a_i) < 1\}} \alpha_i. \tag{4.1.6}$$

From (4.1.5), (4.1.6) and $E[\alpha_1] = a < \infty$, we see

$$\frac{S(r^n x)}{ar^n} \to x \tag{4.1.7}$$

uniformly on any compact set almost surely with respect to $P_1$. Set

$$Y^{(n)}(t) = r^{-n} Y(a(rN)^n t), \quad t \ge 0, \ n = 1, 2, 3, \ldots. \tag{4.1.8}$$

Then the generator of $\{Y^{(n)}(t)\}$ is given by

$$\frac{d}{dm_n(x)} \frac{d}{dS_n(x)},$$

where $m_n(x) = N^{-n} m_0(r^n x)$ and $S_n(x) = (ar^n)^{-1} S(r^n x)$. Put

$$M_n(x) = m_n \circ S_n^{-1}(x),$$
$$Z^{(n)}(t) = S_n(Y^{(n)}(t)).$$

Then the generator of $\{Z^{(n)}(t)\}$ is given by

$$\frac{d}{dM_n(x)} \frac{d}{dx}.$$

Letting

$$A_n(t) = \int_{\mathbf{R}} l(t, x) M_n(dx),$$
$$A(t) = \int_{\mathbf{R}} l(t, x) m(dx),$$

we have

**Lemma 4.1.2** $\{B(A_n^{-1}(t)), t \ge 0\}$ *converges to* $\{B(A^{-1}(t)), t \ge 0\}$ *in the* $J_1$*-topology in* D *almost surely with respect to* $P_1$ *as* $n \to \infty$.

**Proof**

It is enough to check that for $P_1$-almost every $\omega_1 \in \Omega_1$, the measure $M_n$ and $m$ satisfy conditions (i)-(viii) of Theorem 1 of [St63]. In our case, $k_n(x) \equiv 0$, $a_0 = 0$ (reflecting boundary) and $b_0 = \infty$, and we see that (4.1.7) implies conditions (i)-(iii) and (v)-(viii). Hence, we have only to check condition (iv). Assume that $y_n$ is the increasing point of $M_n (n = 1, 2, 3, \ldots)$ and that $y_n$ converges to $y_0$. Then there exists $a_j^{(n)} \in F_n$ such that $y_n = S_n(a_j^{(n)})$ and $a_j^{(n)}$ converges to $c \in C$, therefore $S_n(a_j^{(n)})$ converges to $c$ which is the increasing point of $m$. □

Lemma 4.1.2 and (4.1.7) imply that the scaled process $\{S_n^{-1}(B(A_n^{-1}(t)))\}$ converges to $\{B(A^{-1}(t))\}$ in the $J_1$-topology in D as $n \to \infty$ almost surely, which completes the proof of (i) of Theorem 4.1.1. □

**Proof of (ii) of Theorem 4.1.1**

Set $S_n(x) = \{r^n L_1(r^n)\}^{-1} S_0(r^n x)$. Then $S_n(x)$ converges to $x$ uniformly on each of compact sets in probability with respect to $P_1$. Therefore, we can prove (ii) of Theorem 4.1.1 by a repetition of the argument of proof of (i). □

**Proof of (iii) of Theorem 4.1.1**

Set $S_n(x) = \{r^{n/\alpha} L_2(r^n)\}^{-1} S_0(r^n x)$ with $0 < \alpha < 1$. Then noting $\alpha_i > 0$ for any $i \in \mathbf{N}$ and $0 < \alpha < 1$, we can show that $\{S_n(x)\}$ converges in law to an increasing strictly $\alpha$-stable Lévy process $\{\xi(x)\}$ with $0 < \alpha < 1$ (we call such a process *a subordinator process*) by a similar argument to that of proof of (i). By Skorohod's realization theorem of almost sure convergence, there exist a probability space $(\widehat{\Omega}_1, \widehat{\mathcal{F}}_1, \widehat{P}_1)$ and D-valued random variables $\widehat{S}_n$, $\widehat{\xi}$ such that

(i) $\widehat{S}_n$ converges to $\widehat{\xi}$ in D almost surely with respect to $\widehat{P}_1$ as $n \to \infty$,

(ii) the distributions of $\widehat{S}_n$ and $\widehat{\xi}$ are equal to those of $S_n$ and $\xi$, respectively.

We set

$$\begin{aligned}
\widehat{M}_n(x) &= m_n \circ \widehat{S}_n^{-1}(x), \\
\widehat{M}(x) &= m \circ \widehat{\xi}^{-1}(x), \\
\widehat{A}_n(t) &= \int_{\mathbf{R}} l(t,x)\widehat{M}_n(dx), \\
\widehat{A}(t) &= \int_{\mathbf{R}} l(t,x)\widehat{M}(dx).
\end{aligned}$$

Then we have

**Lemma 4.1.3** $\{B(\widehat{A}_n^{-1}(t)), t \geq 0\}$ *converges to* $\{B(\widehat{A}^{-1}(t)), t \geq 0\}$ *in the* $J_1$*-topology in* D *almost surely with respect to* $\widehat{P}_1$ *as* $n \to \infty$.

From Lemma 4.1.3 and Skorohod's realization theorem, we see that the scaled process $\{\widehat{S}_n^{-1}(B(\widehat{A}_n^{-1}(t)))\}$ converges to $\{\widehat{\xi}^{-1}(B(\widehat{A}^{-1}(t)))\}$ in the $J_1$-topology in D almost surely with respect to $\widehat{P}_1$. Set

$$\widehat{X}^{\xi}(t) = \widehat{\xi}^{-1}(B(\widehat{A}^{-1}(t))),\ t \geq 0. \tag{4.1.9}$$

Then we have the following scaling property:

**Lemma 4.1.4** $\{\widehat{X}^{\xi}, t \geq 0\}$ *has the semi-selfsimilar property such that*

$$\{\widehat{X}^{\xi}(t), t \geq 0\} \stackrel{L}{=} \{r^{-n}\widehat{X}^{\xi}((r^{1/\alpha}N)^n t), t \geq 0\}$$

*for any* $n \in \mathbf{Z}$, *where* $\stackrel{L}{=}$ *denotes identity in law with respect to* $P_0 \times P_1$.

**Proof**
Since $\widehat{\xi}$ is a subordinator process, we have that $\{\widehat{\xi}(x)\} \stackrel{D}{=} \{r^{-n/\alpha}\widehat{\xi}(r^n x)\}$ and $\{\widehat{M}(x)\} \stackrel{D}{=} \{N^{-n}\widehat{M}(r^{n/\alpha}x)\}$ for any $n \in \mathbf{Z}$, where $\stackrel{D}{=}$ denotes identity in law with respect to $P_1$. Hence, in the same way as that of Lemma 3 of [KK84] we have the assertion. $\square$

Therefore, Lemma 4.1.3 and Lemma 4.1.4 complete the proof of (iii) of Theorem 4.1.1. $\square$

## 4.2 Random walks' case

Next, we consider the homogenization problems for random walks. For any $\theta > 0$ we set $\alpha_{i,\theta} = \alpha_i \vee \theta$. From $\alpha_{i,\theta}$'s, we define $S_\theta(x)$ as

$$S_\theta(x) = \begin{cases} 0 & \text{for } x \le 0, \\ S_\theta(I(x)) + \alpha_{J_1(x)+1,\theta}(x - I(x)) & \text{for } x > 0. \end{cases} \tag{4.2.1}$$

Then we have $S_\theta(x) \to S(x)$ as $\theta \to 0$. Let $\theta_n > 0, n \in \mathbf{N}$ satisfy $\theta_n \searrow 0$ as $n \to \infty$ and let

$$h_n = h\theta_n \tag{4.2.2}$$

(see (3.2.3) for the definition of $h$). For $n \in \mathbf{N}$ we define a random walk $\{W_n(t), t \ge 0\}$ starting from 0 on a pre-fractal set $F_0$ in the random environment $\alpha$ whose jumps occur at integral multiples of $h_n$ as follows:
For $i \in \mathbf{N}$ and $j = 0, 1, 2, \dots,$

$$P\{W_n(h_n(j+1)) = a_{i-1} | W_n(h_n j) = a_i, \mathcal{A}\} = \frac{h_n}{S_{\theta_n}(a_i) - S_{\theta_n}(a_{i-1})},$$

$$\begin{aligned} &P\{W_n(h_n(j+1)) = a_i | W_n(h_n j) = a_i, \mathcal{A}\} \\ &\qquad = 1 - \left\{ \frac{h_n}{S_{\theta_n}(a_i) - S_{\theta_n}(a_{i-1})} + \frac{h_n}{S_{\theta_n}(a_{i+1}) - S_{\theta_n}(a_i)} \right\}, \end{aligned}$$

$$P\{W_n(h_n(j+1)) = a_{i+1} | W_n(h_n j) = a_i, \mathcal{A}\} = \frac{h_n}{S_{\theta_n}(a_{i+1}) - S_{\theta_n}(a_i)},$$

and at the origin,

$$P\{W_n(h_n(j+1)) = a_1 | W_n(h_n j) = 0, \mathcal{A}\} = \frac{h_n}{S_{\theta_n}(a_1)},$$

$$P\{W_n(h_n(j+1)) = 0 | W_n(h_n j) = 0, \mathcal{A}\} = 1 - \frac{h_n}{S_{\theta_n}(a_1)}.$$

We set

$$W_n(t) = W_n(h_n j), \quad h_n j \le t < h_n(j+1). \tag{4.2.3}$$

Then we have the following.

**Theorem 4.2.1** *Let $\{X(t), t \ge 0\}$ be the diffusion process on $C$ with the generator* (3.2.2).

(i) *If* $E[\alpha_1] = a < \infty$, *then the process* $\{r^{-n}W_n(a(rN)^n t), t \geq 0\}$ *converges weakly in* D *to the process* $\{X(t)\}$ *as* $n \to \infty$.

(ii) *If there exists a slowly varying function* $L_1(\cdot)$ *such that*

$$\frac{1}{nL_1(n)} \sum_{i=1}^{n} \alpha_i \to 1$$

*in probability, then the process* $\{r^{-n}W_n((rN)^n L_1(r^n)t), t \geq 0\}$ *converges weakly in* D *to the process* $\{X(t)\}$ *as* $n \to \infty$.

(iii) *If* $\alpha_i$*'s belong to the domain of attraction of a one-side positive strictly stable distribution of index* $\alpha \in (0,1)$, *then the process*

$$\{r^{-n}W_n((r^{1/\alpha}N)^n L_2(r^n)t), t \geq 0\}$$

*converges weakly in* D *to the semi-selfsimilar process* $\{\widetilde{X}(t)\}$ *as* $n \to \infty$, *where* $L_2(x)$ *and* $\widetilde{X}(t)$ *were appeared in* (iii) *of Theorem* 4.1.1.

**Remark 4.2.2** If there exists $\theta_1 > 0$ such that $\alpha_i > \theta_1$ for any $i \in \mathbf{N}$, then we can take a constant time unit $h_1 = h\theta_1$ instead of $h_n$ in the same manner as that in Theorem 3.2.1.

**Proof of Theorem 4.2.1**

First we construct a random walk $\{W_n(t), t \geq 0\}$ on $F_0$ from a birth and death process in the following manner. Let $\{Y_{\theta_n}(t), t \geq 0\}$ be a generalized one-dimensional diffusion process with a generator

$$\frac{d}{dm(x)} \frac{d}{dS_{\theta_n}(x)}.$$

From $\{Y_{\theta_n}(t)\}$, we define $\tau_{n,j}$ and $T_{n,j}$ inductively as

$$\begin{aligned} T_{n,0} &= 0, \\ T_{n,j+1} &= T_{n,j} + \tau_{n,j+1}, \qquad (4.2.4) \\ \tau_{n,j+1} &= \inf\{t > 0 : Y_{\theta_n}(T_{n,j} + t) \neq Y_{\theta_n}(T_{n,j})\}. \end{aligned}$$

For $a_i \in F_0$ we set

$$w_{n,i} = \left( \frac{1}{S_{\theta_n}(a_{i+1}) - S_{\theta_n}(a_i)} + \frac{1}{S_{\theta_n}(a_i) - S_{\theta_n}(a_{i-1})} \right) h_n.$$

Then we have $w_{n,i} \in (0,1]$. Let $\theta_{n,i}$ be the unique solution of

$$1 - w_{n,i} = \exp\left\{ -\frac{w_{n,i}\theta_{n,i}}{h_n} \right\}.$$

In the case where $Y_{\theta_n}(T_j) = a_i$, using this $\theta_{n,i}$, we set

$$\sigma_{n,j+1} = \begin{cases} h_n & \text{if} \quad w_{n,i} = 1, \\ mh_n & \text{if} \quad 0 < w_{n,i} < 1 \text{ and } (m-1)\theta_{n,i} \leq T_j < m\theta_{n,i}, \end{cases} \tag{4.2.5}$$

and

$$\begin{aligned} U_{n,0} &= 0, \\ U_{n,j+1} &= U_{n,j} + \sigma_{n,j+1}. \end{aligned} \tag{4.2.6}$$

If $w_{n,i} \in (0,1)$, then the value of $h_n^{-1}\sigma_{n,j}$ is geometrically distributed with the mean $w_{n,i}^{-1}$. By using notation above, we can express the random walk $\{W_n(t)\}$ by

$$W_n(t) = Y_{\theta_n}(T_{n,j}), \quad U_{n,j} \leq t < U_{n,j+1}. \tag{4.2.7}$$

In the case where the state space is $F_n$, we take $\theta_n \searrow 0$ and set

$$Y_{\theta_n}^{(n)}(t) = \begin{cases} r^{-n} Y_{\theta_n}(a(rN)^n t) & \text{in case (i)}, \\ r^{-n} Y_{\theta_n}((rN)^n L_1(r^n) t) & \text{in case (ii)}, \\ r^{-n} Y_{\theta_n}((r^{1/\alpha} N)^n L_2(r^n) t) & \text{in case (iii)}. \end{cases}$$

for $n \in \mathbf{N}$. Let

$$S_{\theta_n}^{(n)}(x) = \begin{cases} (ar^n)^{-1} S_{\theta_n}(r^n x) & \text{in case (i)}, \\ \{r^n L_1(r^n)\}^{-1} S_{\theta_n}(r^n x) & \text{in case (ii)}, \\ \{r^{n/\alpha} L_2(r^n)\}^{-1} S_{\theta_n}(r^n x) & \text{in case (iii)}. \end{cases}$$

Then the generator of $\{Y_{\theta_n}^{(n)}(t)\}$ is given by

$$\frac{d}{dm_n(x)} \frac{d}{S_{\theta_n}^{(n)}(x)}.$$

We see

$$\begin{aligned}
S^{(n)}_{\theta_n} &\to x \quad \text{almost surely} && \text{in case (i)},\\
S^{(n)}_{\theta_n} &\to x \quad \text{in probability} && \text{in case (ii)},\\
S^{(n)}_{\theta_n} &\to \xi(x) \quad \text{in law} && \text{in case (iii)},
\end{aligned}$$

as $n \to \infty$. Hence, the birth and death process $\{Y^{(n)}_{\theta_n}(t)\}$ also converges weakly to each of the limiting processes in Theorem 4.1.1 respectively as $n \to \infty$.

From the scaled birth and death process $\{Y^{(n)}_{\theta_n}(t)\}$, we define $T^{(n)}_{n,j}$ and $\tau^{(n)}_{n,j}$ in the same way as (4.2.4). We set

$$h^{(n)}_n = \begin{cases} \{a(rN)^n\}^{-1} h_n & \text{in case (i)},\\ \{(rN)^n L_1(r^n)\}^{-1} h_n & \text{in case (ii)},\\ \{(r^{1/\alpha}N)^n L_2(r^n)\}^{-1} h_n & \text{in case (iii)}. \end{cases}$$

Using $h_n$, we define $\sigma^{(n)}_{n,j}$ and $U^{(n)}_{n,j}$ in the same way as (4.2.5) and (4.2.6), respectively. By a similar argument to that in Section 4 of [St63], we have

**Lemma 4.2.3** *for each $t > 0$ and any $\varepsilon > 0$,*

$$P\left\{ \sup_{\substack{1\le l\le k\\ T^{(n)}_{n,k}\le t}} |T^{(n)}_{n,l} - U^{(n)}_{n,l}| > \varepsilon \right\} \le \frac{14}{\varepsilon^2}(h^{(n)}_n)^2. \tag{4.2.8}$$

Since $\sum_{n=1}^{\infty}((rN)^{-n}h_n)^2 < \infty$, we have

**Proposition 4.2.4** *for each $t > 0$,*

$$\max_{\substack{1\le l\le k\\ T^{(n)}_{n,k}\le t}} \left| T^{(n)}_{n,l} - U^{(n)}_{n,l} \right| \to 0$$

*uniformly almost surely as $n \to \infty$.*

This and Theorem 4.1.1 imply Theorem 4.2.1. □

# Chapter 5

# One-dimensional diffusion processes in semi-selfsimilar random environments

In this and the next chapter, we consider semi-selfsimilar processes which are similar to that in Example 2.1.2, namely, the case where a distribution converges along a subsequence and the distribution is wandering along a full sequence.

In this chapter, we consider a different type of stochastic processes in random environments from those of Chapter 4, and thus we use different notation. Let $\{X(t), t \geq 0\}$ be a one-dimensional diffusion process described by a formal stochastic differential equation

$$dX(t) = dB(t) - \frac{1}{2}W^{'}(X(t))dt,\ t > 0, \quad X(0) = 0, \tag{5.0.1}$$

where $\{B(t), t \geq 0\}$ is a one-dimensional Brownian motion starting at 0, $\{W(x), x \in \mathbf{R}\}$ is a stochastic process which is independent of $\{B(t)\}$ and $W^{'}(x)$ denotes the formal derivative of $W(x)$.

In the case where $\{W(x)\}$ is a Brownian motion, Brox [Br86] had showed that the distribution of $(\log t)^{-2}X(t)$ converges as $t \to \infty$. This is a continuous time analogue to the problem that Sinai [Si82] had considered for the case of a one-dimensional random walk. That is a Markov process on $\mathbf{Z}$ which jumps from $i$ to $i+1$ and from $i+1$ to $i$ with a different rate determined by an environment. Using Brox's idea, Kawazu, Tamura and Tanaka [KTT88] extended to the case of selfsimilar processes. In this chapter, we study the limiting behaviors of one-dimensional diffusion processes in the case of semi-selfsimilar processes, which do not belong to the class of selfsimilar ones in general. In Section 5.1, we explain the models and the results. In Section 5.2, we show that the distribution of a suitably scaled one-dimensional diffusion process in a semi-selfsimilar environment satisfying some conditions converges *along a subsequence*. It should be noted that environments do not have the selfsimilarity in general, hence we cannot expect to have the limit distribution along a full sequence. But concidering the characteristics of the limit distribution, we can show that the difference between the scaled diffusion process and some distribution (it is varying as $t \to \infty$ because of the semi-selfsimilarity of environments) converges to 0 *along a full sequence*.

## 5.1 The model and the result

Since $W(x)$ is not differentiable in general, the meaning of a solution (5.0.1) is not clear. Hence, we give another description of $\{X(t)\}$ in (5.0.1). Let

$$\mathcal{W} = \{W \in \mathrm{D}(\mathbf{R};\mathbf{R}) : W(0) = 0\}, \tag{5.1.2}$$

namely, $\mathcal{W}$ is the space of $\mathbf{R}$-valued right continuous functions on $\mathbf{R}$ with left limits and vanishing at 0. Let $Q$ be the probability measure on $\mathcal{W}$ with respect to which $\{W(x), x \geq 0\}$ and $\{W(x), x \leq 0\}$ are independent and $H$-semi-selfsimilar,

$$\{W(ax), x \in \mathbf{R}\} \stackrel{\mathcal{D}}{=} \{a^H W(x), x \in \mathbf{R}\}, \tag{5.1.3}$$

where $\stackrel{\mathcal{D}}{=}$ denotes identity in law with respect to $Q$. In this and the next chapter, for each process satisfying (5.1.3) we write that $\{W, Q\}$ is $H$-semi-selfsimilar and we regard $\{W, Q\}$ as an environment.

For a fixed $W \in \mathcal{W}$ we deal with a diffusion process with a generator

$$\mathcal{L}_W = \frac{1}{2} e^{W(x)} \frac{d}{dx}\left(e^{-W(x)} \frac{d}{dx}\right). \tag{5.1.4}$$

We construct the diffusion process above in the same way as that in [Br86]. Let

$$\Omega = \{\omega \in \mathrm{C}\left([0,\infty);\mathbf{R}\right) : \omega(0) = 0\}$$

and let $P$ be the Wiener measure on $\Omega$. For an element $\omega \in \Omega$ we write $B(t) = B(t,\omega) = \omega(t)$ =the value of $\omega$ at time $t$. Then $\{B(t), t \geq 0, P\}$ is a Brownian motion starting from 0. In the same manner as those in Chapter 4, we set

$$S(x) = \int_0^x \exp\{W(y)\}dy,$$

$$A(t) = \int_0^t \exp\left\{-2W\left(S^{-1}(B(s))\right)\right\} ds.$$

Then the process

$$X(t,W) = S^{-1}\left(B(A^{-1}(t))\right), \quad t \geq 0, \tag{5.1.5}$$

defined on the probability space $(\Omega, P)$ is a diffusion process with the generator (5.1.4). We define a probability measure $\mathcal{P}$ on $\mathcal{W} \times \Omega$ by $\mathcal{P} = Q \times P$ and denote by $\mathcal{E}$ the expectation with respect to $\mathcal{P}$. We assume that $\{W, Q\}$ and $\{B, P\}$ are independent. Then $\{X(t), t \geq 0\}$ is regarded as a process defined on the probability space $(\mathcal{W} \times \Omega, \mathcal{P})$ and we call it a diffusion process in a semi-selfsimilar random environment. If the environment is fixed, then $\{X(t,W), t \geq 0\}$ is governed by $P$.

We restrict the class of environments. Let $\mathcal{W}_0$ be the set of elements $W \in \mathcal{W}$ which satisfy the following conditions:

(i) $\displaystyle\limsup_{x\to\infty}\left\{W(x) - \inf_{[0,x]} W\right\} = \limsup_{x\to-\infty}\left\{W(x) - \inf_{[x,0]} W\right\} = \infty.$

(ii) $W$ does not take the same local maximum (minimum) value of $W$ at different points of local maximum (minimum). Here $W$ is said to take a local maximum at $x$ if $\sup\{W(y) : y \in I(x)\} = W(x) \vee W(x-)$ for some $\varepsilon$, where $I(x) = (x-\varepsilon, x+\varepsilon)$ and $a \vee b = \max\{a,b\}$.

(iii) $\inf\{x > 0 : W(x) > 0\} = \sup\{x < 0 : W(x) > 0\} = 0.$

(iv) If $W$ is discontinuous at $x$, then $\sup\{W(y) : y \in I_\pm\} > W(x\pm)$ and $\inf\{W(y) : y \in I_\pm\} < W(x\pm)$ for any $\varepsilon > 0$, where $I_+ = (x, x+\varepsilon)$ and $I_- = (x-\varepsilon, x)$.

We next explain the notion of a valley of $W$ introduced by [KTT88]. Let $W \in \mathcal{W}_0$. If the following are satisfied, then $V = (a,b,c)$ is called a valley of $W$;

(1) $a \le b \le c$,

(2) $W(a) > W(x) > W(b)$ for all $x \in (a, b)$,

(3) $W(b) < W(x) < W(c)$ for all $x \in (b, c)$.

For a valley $V = (a, b, c)$, $D = (W(c) - W(b)) \wedge (W(a) - W(b))$ is called the depth of $V$. A valley $V = (a, b, c)$ is said to be proper if

$$H_{a,b} < W(c) - W(b), \quad H_{c,b} < W(a) - W(b),$$

where

$$H_{x,y} = \begin{cases} \sup\limits_{x \le x' \le y' \le y} \{W(y') - W(x')\} & \text{for } x < y, \\ \sup\limits_{y \le y' \le x' \le x} \{W(y') - W(x')\} & \text{for } y < x. \end{cases}$$

$A = H_{a,b} \vee H_{c,b}$ is called the inner directed ascent of $V$. A valley $V = (a, b, c)$ is said to contain 0 if $a < 0 < c$. The following proposition is shown in [KTT88].

**Proposition 5.1.1** *Let $r > 0$ and $W \in \mathcal{W}_0$. Then there exists a proper valley $V$ of $W$ containing 0 such that $A < r \le D$.*

For a fixed $W \in \mathcal{W}_0$ the bottom of such a proper valley is uniquely determined by $r$, and thus it is denoted by $b_r(W)$. For all $\lambda > 0$ we define the scaled environment $W_\lambda$ by

$$W_\lambda(x) = \lambda^{-1} W(\lambda^{1/H} x), \tag{5.1.6}$$

where $H$ is the exponent of $\{W, Q\}$. If environments are selfsimilar, then the distribution of $b_1(W_\lambda)$ is the same as that of $b_1(W)$ for any $\lambda > 0$. However, in the case of semi-selfsimilar environments this is not true in general. Using the notation above, we state our main result.

**Theorem 5.1.2** *Let $\{W, Q\}$ be an $H$-semi-selfsimilar environment such that $Q(\mathcal{W}_0) = 1$. We assume that almost all sample functions of $\{W, Q\}$ have proper valleys of $W$ containing 0 with $A < 1 < D$. Then we have the following:*

(i) (1) *If $\lambda_0 = 1$ (see (2.1.4) for the definition of $\lambda_0$), then the distribution of $\lambda^{-1/H} X(e^\lambda)$ converges to that of $b_1(W)$ as $\lambda \to \infty$.*

(2) *If* $\lambda_0 > 1$, *then the distribution of* $\lambda^{-1/H}X(e^\lambda)$ *converges to that of* $b_1(W_\xi)$ *as* $\lambda \to \infty$ *along a subsequence* $\{\lambda_0^n \xi, n = 0, 1, 2, \ldots\}$, *for any* $\xi \in [1, \lambda_0)$.

(ii) *If* $\{W, Q\}$ *is stochastically continuous for any* $x \in \mathbf{R}$ (*see* (2.1.3) *for the definition of stochastically continuous*), *then for any* $\varepsilon > 0$,

$$P\left\{\left|\lambda^{-1/H}X\left(e^\lambda, W\right) - b_1(W_\lambda)\right| > \varepsilon\right\} \to 0 \tag{5.1.7}$$

*in probability with respect to* $Q$ *as* $\lambda \to \infty$.

**Remark 5.1.3** In the case of selfsimilar environments it was shown that if $Q(\mathcal{W}_0) = 1$, then for each $r > 0$ almost all sample functions of $\{W, Q\}$ have proper valleys containing 0 with $A < r < D$. On the other hand, as mentioned in Remark 2.2.7 (ii), in order to construct a semi-selfsimilar environment we need to determine the law of $\{W(x), 1 \le x < \lambda_0\}$, and we can construct a semi-selfsimilar environment $\{W, Q\}$ such that almost all sample functions of $\{W, Q\}$ have proper valleys containing 0 with $D = 1$ even though $Q(\mathcal{W}_0) = 1$. Hence, we need the assumption that almost all environments have 'good' valleys. We give some examples satisfying such an assumption in the next section.

Let $R$ be a random variable defined on $(\mathcal{W}\times\Omega, \mathcal{P})$ with density $(x \log \lambda_0)^{-1}$ in $[1, \lambda_0)$ and be independent of $\{X(t)\}$. Then the assertion (2) above is expressed as follows.

**Corollary 5.1.4** *The distribution of* $(\lambda R)^{-1/H}X(e^{\lambda R})$ *converges to that of* $b_1(W_R)$ *as* $\lambda \to \infty$.

**Proof**

For a fixed $\lambda > 1$ we have $\lambda = \lambda_0^n \xi$, where $n \in \{0, 1, 2, \ldots\}$ and $\xi = \xi(\lambda) \in [1, \lambda_0)$. We set $c = (\log \lambda_0)^{-1}$. Then for every bounded Borel function $f$ we have

$$\begin{aligned} & \mathcal{E}\left[f\left((\lambda R)^{-1/H}X(e^{\lambda R}, W)\right)\right] \\ = \quad & \int_1^{\lambda_0} \mathcal{E}\left[f\left((\lambda r)^{-1/H}X(e^{\lambda r}, W)\right)\right]\frac{cdr}{r} \end{aligned}$$

$$
\begin{aligned}
&= \int_{\xi}^{\lambda_0\xi} \mathcal{E}\left[f\left((\lambda_0^n\rho)^{-1/H}X(e^{\lambda_0^n\rho},W)\right)\right]\frac{cd\rho}{\rho} \\
&\qquad \text{(by changing the variables } \lambda=\lambda_0^n\xi \text{ and } \xi r=\rho) \\
&= \int_{\xi}^{\lambda_0} \mathcal{E}\left[f\left((\lambda_0^n\rho)^{-1/H}X(e^{\lambda_0^n\rho},W)\right)\right]\frac{cd\rho}{\rho} \\
&\qquad\qquad + \int_{\lambda_0}^{\lambda_0\xi} \mathcal{E}\left[f\left((\lambda_0^n\rho)^{-1/H}X(e^{\lambda_0^n\rho},W)\right)\right]\frac{cd\rho}{\rho} \\
&= \int_{\xi}^{\lambda_0} \mathcal{E}\left[f\left((\lambda_0^n\rho)^{-1/H}X(e^{\lambda_0^n\rho},W)\right)\right]\frac{cd\rho}{\rho} \\
&\qquad\qquad + \int_{1}^{\xi} \mathcal{E}\left[f\left((\lambda_0^{n+1}\widetilde{\rho})^{-1/H}X(e^{\lambda_0^{n+1}\widetilde{\rho}},W)\right)\right]\frac{cd\widetilde{\rho}}{\widetilde{\rho}} \\
&\qquad \text{(by changing the variable } \rho=\lambda_0\widetilde{\rho} \text{ in second term)} \\
&\longrightarrow \int_{1}^{\lambda_0} \mathcal{E}\left[f\left(b_1(W_\rho)\right)\right]\frac{cd\rho}{\rho} \\
&\qquad \text{(by the assertion (b) in } (\mathrm{I\!I}) \text{ of Theorem 2.2)} \\
&= \mathcal{E}\left[f(b_1(W_R))\right].
\end{aligned}
$$

□

## 5.2 Proof of Theorem 5.1.2

**Proof of** (i) **of Theorem 5.1.2**
If $\lambda_0 = 1$, then environments are selfsimilar and nothing is left to be proved. We consider the case where $\lambda_0 > 1$. For a fixed $\xi \in [1, \lambda_0)$ we set

$$\lambda_n = \lambda_n(\xi) = \lambda_0^n \xi, \quad n = 0, 1, 2, \ldots.$$

Then $\{W_{\lambda_n}, Q\}$ and $\{W_\xi, Q\}$ are identical in law. Denote by $\mu_\lambda$ the distribution of $b_1(W_\lambda)$, and the identity in law above implies that $\mu_{\lambda_n} = \mu_\xi$. Using Skorohod's realization theorem of almost sure convergence, we can find $\mathcal{W}$-valued random variables $\overline{W}_{\lambda_n}$ and $\overline{W}_\xi$ defined on a suitable probability space $(\overline{\mathcal{W}}, \overline{Q})$ with the following properties;

(a) $\{W_{\lambda_n}, Q\}$ and $\{\overline{W}_{\lambda_n}, \overline{Q}\}$ are identical in law, $\{W_\xi, Q\}$ and $\{\overline{W}_\xi, \overline{Q}\}$ identical in law,

(b) $\overline{W}_{\lambda_n} \to \overline{W}_\xi$ in the Skorohod topology almost surely with respect to $\overline{Q}$ as $n \to \infty$.

Using the same method as that in the proof of Theorem I-A-1 of [KTT88], for any $\varepsilon > 0$ and sequence $\{r_n\}$ satisfying $r_n \to 1$ as $n \to \infty$ we have

$$P\{|X(e^{\lambda_n r_n}, \lambda_n \overline{W}_{\lambda_n}) - b_1(\overline{W}_\xi)| > \varepsilon\} \to 0 \tag{5.2.8}$$

almost surely with respect to $\overline{Q}$. It is known that for a fixed $W \in \mathcal{W}_0$ and $r > 0$

$$B_r(W) = \{b \in \mathbf{R} \mid (\text{the depth of the valley } V = (a, b, c) \text{ containing } 0) \le r\}$$

is a locally finite set and $b_r, r > 0$ is a step function, which has only finitely many jumps in each bounded interval away from 0. Hence, (b) and the assumption that $A < 1 < D$ imply

$$b_1(\overline{W}_{\lambda_n}) \to b_1(\overline{W}_\xi)$$

almost surely with respect to $\overline{Q}$. Using this convergence, we have

$$P\{|X(e^{\lambda_n r_n}, \lambda_n \overline{W}_{\lambda_n}) - b_1(\overline{W}_{\lambda_n})| > \varepsilon\} \to 0$$

almost surely with respect to $\overline{Q}$ as $n \to \infty$, and (a) implies

$$P\{|X(e^{\lambda_n r_n}, \lambda_n W_{\lambda_n}) - b_1(W_{\lambda_n})| > \varepsilon\} \to 0$$

in probability with respect to $Q$.

For a fixed $W$ and each $\lambda > 0$ we have the following scaling property (c.f. Lemma 5.3 of [KTT89] for the proof);

$$\{X(t, \lambda W_\lambda), t \ge 0\} \stackrel{d}{=} \{\lambda^{-1/H} X(\lambda^{2/H} t, W), t \ge 0\}, \tag{5.2.9}$$

where $\stackrel{d}{=}$ denotes identity in law with respect to $P$. Using the above scaling with setting

$$r_n = 1 - (2/H) \log \lambda_n / \lambda_n, \tag{5.2.10}$$

we have

$$P\{|\lambda_n^{-1/H} X(e^{\lambda_n}, W) - b_1(W_{\lambda_n})| > \varepsilon\} \to 0 \tag{5.2.11}$$

for any $\varepsilon > 0$ in probability with respect to $Q$ as $n \to \infty$. Since $\mu_{\lambda_n} = \mu_\xi$, the proof is finished. $\square$

**Proof of** (ii) **of Theorem 5.1.2**
For each $\lambda > 1$, there exists $\theta(\lambda) \in [1, \lambda_0)$ such that

$$\{W_\lambda(x), x \in \mathbf{R}\} \stackrel{\mathcal{D}}{=} \{W_{\theta(\lambda)}(x), x \in \mathbf{R}\}, \tag{5.2.12}$$

which implies that $\mu_\lambda = \mu_{\theta(\lambda)}$. Using (5.2.9) and (5.2.12), we have

$$\begin{aligned}\{\lambda^{-1/H} X(\lambda^{2/H} t, W), t \geq 0\} &\stackrel{d}{=} \{X(t, \lambda W_\lambda), t \geq 0\} \\ &\stackrel{\mathcal{L}}{=} \{X(t, \lambda W_{\theta(\lambda)}), t \geq 0\}, \end{aligned}\tag{5.2.13}$$

where $\stackrel{\mathcal{L}}{=}$ denotes identity in law with respect to $\mathcal{P}$. Since $\theta(\lambda)$ is varying on $[1, \lambda_0)$ as $\lambda \to \infty$, we cannot use the same method as that in the proof of Theorem I-A-1 of [KTT88] for semi-selfsimilar environments in general. But we can show that $\{W_\lambda\}_{\lambda > 1}$ is tight as follows:
The stochastic continuity property implies that for each $M > 0$

$$W_\lambda(M) = W_\lambda(M-)$$

almost surely with respect to $Q$. Hence, we need to show that for each $M > 0$ and any $\varepsilon > 0$

$$\lim_{a \to \infty} \limsup_{\lambda \to \infty} Q\left\{\sup_{0 \leq x \leq M} |W_\lambda(x)| > a\right\} = 0, \tag{5.2.14}$$

$$\lim_{\delta \to 0} \limsup_{\lambda \to \infty} Q\left\{w^\delta(W_\lambda, M) > \varepsilon\right\} = 0, \tag{5.2.15}$$

where $w^\delta(W_\lambda, M) = \sup\{|W_\lambda(x) - W_\lambda(y)| : 0 \leq x \leq y \leq M,\ y - x \leq \delta\}$. (5.2.12) implies

$$\begin{aligned} &\lim_{a \to \infty} \limsup_{\lambda \to \infty} Q\left\{\sup_{0 \leq x \leq M} |W_\lambda(x)| > a\right\} \\ = &\lim_{a \to \infty} \sup_{\lambda \in [1, \lambda_0)} Q\left\{\sup_{0 \leq x \leq M} |W_\lambda(x)| > a\right\} \\ \leq &\lim_{a \to \infty} Q\left\{\sup_{0 \leq x \leq M} |W(\lambda_0^{1/H} x)| > a\right\} = 0, \end{aligned}$$

and the stochastic continuity property of environments implies

$$\begin{aligned}
&\lim_{\delta\to 0}\limsup_{\lambda\to\infty} Q\left\{w^{\delta}(W_\lambda, M) > \varepsilon\right\} \\
= &\lim_{\delta\to 0}\sup_{\lambda\in[1,\lambda_0)} Q\left\{w^{\lambda^{1/H}\delta}(W_1, \lambda^{1/H}M) > \lambda\varepsilon\right\} \\
\le &\lim_{\delta\to 0} Q\left\{w^{\lambda_0^{1/H}\delta}(W_1, \lambda_0^{1/H}M) > \varepsilon\right\} = 0.
\end{aligned}$$

Therefore, for any subsequence of $\{\lambda_n\}$ we can take a further subsequence $\{\lambda_n'\}$ such that $W_{\lambda_n'} - W_{\theta(\lambda_n')} \to 0$ in the Skorohod topology as $n \to \infty$. In the same way as that in the case of (i) we can show that for any $\varepsilon > 0$

$$P\{|\lambda_n'^{\,-1/H} X(e^{\lambda_n'}, W) - b_1(W_{\lambda_n'})| > \varepsilon\} \to 0$$

in probability with respect to $Q$ as $n \to \infty$. $\square$

# Chapter 6

# Remarks on the limiting processes of diffusion processes in semi-selfsimilar random environments

In this chapter, we construct semi-selfsimilar processes as the limiting processes of suitably scaled diffusion processes discussed in Chapter 5 and study some properties of the limiting processes. Hence, we use the same notations as those in Chapter 5. To see properties, firstly we show that the limiting process of a suitably scaled diffusion process along a subsequence is semi-selfsimilar. Secondly we consider the case of a reflecting diffusion process in a Lévy environment. Reflecting at $x = 0$ is needed to show the independence of increments of the limiting process. Moreover in the case of the semi-selfsimilar Lévy environment, we show that marginal distributions of the limiting process are semi-selfdecomposable. Maejima and Sato [MS99] discussed properties of semi-selfdecomposable distributions and showed the way to construct a semi-selfsimilar process whose marginal distributions are semi-selfdecomposable (see Example 2.2.6). Our process is constructed in a different way from theirs.

## 6.1 Semi-selfsimilarity of the limiting processes

The following theorem implies that any finite dimensional distribution of the process $\{X(t), t \geq 0\}$ with the generator (5.1.4) converges to that of a semi-selfsimilar process whose exponent is determined by an environment under a suitable scaling.

**Theorem 6.1.1** *Let $\{W, Q\}$ be an $H$-semi-selfsimilar environment such that $Q(\mathcal{W}_0) = 1$. Assume that for each $r > 0$ almost all sample functions of $\{W, Q\}$ have proper valleys containing $0$ with $A < r < D$. Then we have the following:*

(i) *If $\lambda_0 = 1$ (see (2.1.4) for the definition of $\lambda_0$), then any finite dimensional distribution of the process*

$$\{\lambda^{-1/H} X(e^{\lambda t}, W), t > 0, \mathcal{P}\}$$

*converges to that of the $1/H$-selfsimilar process*

$$\{b_t(W), t > 0, Q\}$$

*as $\lambda \to \infty$, where $b_t(W)$ is a bottom of a valley of $W$ whose depth equals to $t$ (see Section 5.1).*

(ii) *If $\lambda_0 > 1$, then any finite dimensional distribution of the process*

$$\{(\lambda_0^n \xi)^{-1/H} X(e^{(\lambda_0^n \xi)t}, W), t > 0, \mathcal{P}\}$$

*converges to that of the $1/H$-semi-selfsimilar process*

$$\{b_t(W_\xi), t > 0, Q\}$$

*as $n \to \infty$ for any $\xi \in [1, \lambda_0)$.*

We give some examples satisfying the assumptions above.

**Example 6.1.2** Let $Q$ be a probability measure on $\mathcal{W}$ with respect to which $\{W(x),\ x \geq 0\}$ and $\{W(-x),\ x \geq 0\}$ are independent strictly $\alpha$-semi-stable

Lévy processes. As mentioned in Example 2.1.2, we have the following scaling property,

$$\{W(ax), x \in \mathbf{R}\} \stackrel{\mathcal{D}}{=} \{a^{1/\alpha}W(x), x \in \mathbf{R}\}$$

for some $a \in (0,1) \cup (1,\infty)$ and $0 < \alpha \leq 2$. We assume that neither $\{W(x)\}$ nor $\{-W(x)\}$ is a subordinator process. Then for each $r > 0$ almost all sample functions of $\{W, Q\}$ have proper valleys containing 0 with $A < r < D$.

If a Lévy process $\{W(x)\}$ is a strictly $\alpha$-stable Lévy process, then it is also a strictly $\alpha$-semi-stable Lévy process. Hence, this example is an extension of that in p.175 of [KTT88].

**Example 6.1.3** For a fixed $\lambda_0 > 1$ we consider the case where almost all of the sample functions of $\{W, Q\}$ have bottoms on $\{\pm\lambda_0^n, n \in \mathbf{Z}\}$. Let $\{A_i\}_{i=-\infty}^{\infty}$ be a sequence of finite, independent and identically distributed random variables. We assume that the distribution of $A_i$ is continuous (namely, $Q(A_i = a) = 0$ for any $a \in \mathbf{R}$) and $Q(A_i > 0) > 0$. We set $W(\lambda_0^n) = \lambda_0^{nH}A_n$ and define $W(x)$ for $\lambda_0^n < x < \lambda_0^{n+1}$ by the linear interpolation

$$W(x) = W(\lambda_0^n) + \frac{W(\lambda_0^{n+1}) - W(\lambda_0^n)}{\lambda_0^n(\lambda_0 - 1)}(x - \lambda_0^n).$$

We also construct $\{W(x), x \leq 0\}$ by setting $W(-\lambda_0^n) = \lambda_0^{nH}\tilde{A}_n$, where $\tilde{A}_n$ and $A_n$ are independent and identically distributed random variables. Then the law of large numbers implies that $W(0) = 0$ almost surely with respect to $Q$. This environment $\{W, Q\}$ is $H$-semi-selfsimilar and the continuity of the distribution implies that for each $r > 0$ almost all sample functions of $\{W, Q\}$ have proper valleys containing 0 with $A < r < D$.

**Proof of Theorem 6.1.1**

We give a proof in the case of semi-selfsimilar environments. The proof of the selfsimilar case also works in this case. For a fixed $t > 0$ we set $\lambda_n = \lambda_0^n\xi t$ in (5.2.11). Then we have

$$\mathcal{P}\{|(\lambda_0^n\xi t)^{-1/H}X(e^{\lambda_0^n\xi t}, W) - b_1(W_{\lambda_0^n\xi t})| > \varepsilon\} \to 0 \tag{6.1.1}$$

as $n \to \infty$. The semi-selfsimilarity of environments implies

$$\{b_1(W_{\lambda_0^n\xi t}), t > 0\} \stackrel{\mathcal{D}}{=} \{b_1(W_{\xi t}), t > 0\}.$$

The definition of the valley and (5.1.6) imply

$$\begin{aligned} b_1(W_{\lambda_0^n \xi t}) &= b_1\left(\frac{1}{\xi t}W\left((\xi t)^{1/H}\right)\right) \\ &= b_t\left(\frac{1}{\xi}W\left((\xi t)^{1/H}x\right)\right) \\ &= t^{-1/H}b_t\left(\frac{1}{\xi}W\left(\xi^{1/H}x\right)\right) \\ &= t^{-1/H}b_t(W_\xi). \end{aligned}$$

Therefore we conclude that

$$\left\{(\lambda_0^n \xi)^{-1/H}X(e^{(\lambda_0^n \xi)t}, W), t > 0, \mathcal{P}\right\} \overset{f.d.}{\Longrightarrow} \{b_t(W_\xi), t > 0, Q\}$$

as $n \to \infty$, where $\overset{f.d.}{\Longrightarrow}$ denotes the convergence all finite dimensional distributions.

In order to see the semi-selfsimilarity of the process $\{b_t(W), t > 0, Q\}$, we give another description of $b_t(W)$ which was given in [T87]. For simplicity, we consider the case where $\xi = 1$. For $W \in \mathcal{W}_0$ and $t > 0$, let

$$W^\sharp(x) = \begin{cases} W(x) - \inf\{W(y) : 0 \le y \le x\} & \text{for } x \ge 0, \\ W(x) - \inf\{W(y) : x \le y \le 0\} & \text{for } x < 0, \end{cases}$$

$$c_t^+ = c_t^+(W) = \inf\{x > 0 : W^\sharp(x) \ge t\}, \tag{6.1.2}$$

$$c_t^- = c_t^-(W) = \sup\{x < 0 : W^\sharp(x) \ge t\},$$

$$V_t^+ = V_t^+(W) = \inf\{W(x) : 0 \le x \le c_t^+\}, \tag{6.1.3}$$

$$V_t^- = V_t^-(W) = \inf\{W(x) : c_t^- \le x \le 0\}.$$

We define $b_t^+ = b_t^+(W)$ in $(0, c_t^+)$ by $W(b_t^+) = V_t^+$ and $b_t^- = b_t^-(W)$ in $(c_t^-, 0)$ by $W(b_t^-) = V_t^-$. The definition of $\mathcal{W}_0$ implies that such $b_t^\pm$ are uniquely determined for almost all sample functions of $\{W, Q\}$. Let

$$M_t^+ = M_t^+(W) = \sup\{W(x) : 0 \le x \le b_t^+\},$$

$$M_t^- = M_t^-(W) = \sup\{W(x) : b_t^- \le x \le 0\}.$$

By using the notation above, $b_t(W)$ is given as

$$b_t(W) = \begin{cases} b_t^+ & \text{if} \quad M_t^+ \vee (V_t^+ + t) < M_t^- \vee (V_t^- + t), \\ b_t^- & \text{if} \quad M_t^+ \vee (V_t^+ + t) > M_t^- \vee (V_t^- + t). \end{cases}$$

For stochastic processes $\{X(t)\}$ and $\{Y(t)\}$ and each $t > 0$, we denote by $X(t) \overset{\mathcal{D}}{\sim} Y(t)$ equality of the marginal distribution with respect to $Q$. The semi-selfsimilarity of environments implies for each $t > 0$

$$\begin{aligned} c_{\lambda_0^n t}^+ &= \inf\{x > 0 : W^\sharp(x) \geq \lambda_0^n t\} \\ &= \inf\{x > 0 : \lambda_0^{-n} W^\sharp(x) \geq t\} \\ &\overset{\mathcal{D}}{\sim} \inf\{x > 0 : W^\sharp(\lambda_0^{-n/H} x) \geq t\} \\ &= \lambda_0^{n/H} c_t^+, \end{aligned}$$

$$\begin{aligned} V_{\lambda_0^n t}^+ &= \inf\left\{W(x) : 0 \leq x \leq c_{\lambda_0^n t}^+\right\} \\ &\overset{\mathcal{D}}{\sim} \inf\{W(x) : 0 \leq x \leq \lambda_0^{n/H} c_t^+\} \\ &= \inf\{W(\lambda_0^{n/H} x) : 0 \leq x \leq c_t^+\} \\ &\overset{\mathcal{D}}{\sim} \lambda_0^n V_t^+. \end{aligned}$$

Hence, we have

$$\begin{aligned} b_{\lambda_0^n t}^+ &= \inf\{x > 0 : W(x) = V_{\lambda_0^n t}^+\} \\ &\overset{\mathcal{D}}{\sim} \inf\{x > 0 : W(x) = \lambda_0^n V_t^+\} \\ &\overset{\mathcal{D}}{\sim} \inf\{x > 0 : W(\lambda_0^{-n/H} x) = V_t^+\} \\ &= \lambda_0^{n/H} b_t^+, \end{aligned}$$

and similarly we have $c_{\lambda_0^n t}^- \overset{\mathcal{D}}{\sim} \lambda_0^{n/H} c_t^-$, $V_{\lambda_0^n t}^- \overset{\mathcal{D}}{\sim} \lambda_0^n V_t^-$, $b_{\lambda_0^n t}^- \overset{\mathcal{D}}{\sim} \lambda_0^{n/H} b_t^-$ for each $t > 0$. These equalities imply that for any choice of distinct $t_1, t_2, \ldots, t_n$

$$\left(b_{\lambda_0^n t_1}, b_{\lambda_0^n t_2}, \ldots, b_{\lambda_0^n t_n}\right) \overset{\mathcal{D}}{\sim} \left(\lambda_0^{n/H} b_{t_1}, \lambda_0^{n/H} b_{t_2}, \ldots, \lambda_0^{n/H} b_{t_n}\right),$$

and this proves the semi-selfsimilarity of the process $\{b_t(W), t > 0, Q\}$. $\square$

## 6.2 The case of a reflecting Lévy environment

We next study some properties of the marginal distributions of $\{b_t(W)\}$ in the case of a reflecting diffusion process $\{X(t), t \geq 0\}$ on $[0, \infty)$ in a Lévy environment. Let

$$\widetilde{\mathcal{W}} = \{W \in \mathrm{D}([0, \infty); \mathbf{R}) : W(0) = 0\}$$

and let $\widetilde{Q}$ be a probability measure on $\widetilde{\mathcal{W}}$ such that $\{W(x), x \geq 0\}$ is a Lévy process. Let

$$\widetilde{\Omega} = \{\widetilde{\omega} \in \mathrm{C}([0, \infty); [0, \infty)) : \widetilde{\omega}(0) = 0\}.$$

For $\widetilde{\omega} \in \widetilde{\Omega}$ we write $\widetilde{X}(t) = \widetilde{X}(t, \widetilde{\omega}) = \widetilde{\omega}(t) =$ the value of $\widetilde{\omega}$ at time $t$. For a fixed $W$, we consider a probability measure $\widetilde{P}_W$ on $\widetilde{\Omega}$ such that $\{\widetilde{X}(t), t \geq 0, \widetilde{P}_W\}$ is a reflecting diffusion process on $[0, \infty)$ with the generator (5.1.4) and starting from 0. We can construct a version of $\{\widetilde{X}(t), t \geq 0, \widetilde{P}_W\}$ from a one-dimensional Brownian motion by a scale-change and a time-change in a similar way to that in Chapter 5. We define a probability measure $\widetilde{\mathcal{P}}$ on $\widetilde{\mathcal{W}} \times \widetilde{\Omega}$ by $\widetilde{\mathcal{P}}(dW d\widetilde{\omega}) = \widetilde{Q}(dW)\widetilde{P}_W(d\widetilde{\omega})$. Then $\{\widetilde{X}(t), t \geq, \widetilde{\mathcal{P}}\}$ is regarded as a process defined on the probability space $(\widetilde{\mathcal{W}} \times \widetilde{\Omega}, \widetilde{\mathcal{P}})$ and called a reflecting diffusion process in a Lévy environment.

Let $\widetilde{\mathcal{W}}_0$ be the set of elements $W \in \widetilde{\mathcal{W}}$ which satisfy the following conditions:

(i$'$) $\limsup_{x\to\infty} W(x) = -\liminf_{x\to\infty} W(x) = +\infty$.

(ii$'$) $W$ does not take the same local maximum (minimum) value of $W$ at different points of local maximum (minimum).

(iii$'$) $\inf\{x > 0 : W(x) > 0\} = \inf\{x > 0 : W(x) < 0\} = 0$.

(iv$'$) If $W$ is discontinuous at $x$, then $\sup\{W(y) : y \in I_\pm\} > W(x\pm)$ and $\inf\{W(y) : y \in I_\pm\} < W(x\pm)$ for any $\varepsilon > 0$.

Then we have the following.

**Lemma 6.2.1** *If almost all sample functions of* $\{W, \widetilde{Q}\}$ *satisfy conditions* (i$'$) *and* (iii$'$)*, then conditions* (ii$'$) *and* (iv$'$) *are satisfied almost surely, namely,* $\widetilde{Q}(\widetilde{\mathcal{W}}_0) = 1$.

**Proof**
From the assumption that condition (iii$'$) holds almost surely, it follows that the distribution of $W(x)$ is continuous for every $x > 0$ (see Theorem 27.4 and Theorem 47.5 of [S99]). Using this fact, we can prove the assertion in the same way as that in showing Lemma 3.1 of [KTT92]. $\square$

For $W \in \widetilde{\mathcal{W}}_0$ and $t > 0$ let $W^\sharp(x) = W(x) - \inf\{W(y) : 0 \leq y \leq x\}, c_t = c_t^+$ as (6.1.2) and $V_t = V_t^+$ as (6.1.3). We define $b_t = b_t(W)$ in $(0, c_t)$ by $W(b_t) = V_t$. We have the following theorem.

**Theorem 6.2.2**

(i) *Let* $\{W, \widetilde{Q}\}$ *be a Lévy environment such that* $\widetilde{Q}(\widetilde{\mathcal{W}}_0) = 1$. *Then the process* $\{b_t, t > 0, \widetilde{Q}\}$ *has independent increments.*

(ii) *Let* $\{W, \widetilde{Q}\}$ *be a strictly* $1/H$*-semi-stable Lévy environment satisfying* $\widetilde{Q}(\widetilde{\mathcal{W}}_0) = 1$ *with* $\lambda_0 > 1$ (*see* (2.1.4) *for the definition of* $\lambda_0$). *Then*

  (1) $\{b_t, t > 0, \widetilde{Q}\}$ *is a* $1/H$*-semi-selfsimilar process with independent increments, hence the distribution of* $b_t$ *is semi-selfdecomposable for each* $t > 0$,

  (2) *any finite dimensional distribution of the process*

$$\{\lambda_0^{-n/H} X(e^{\lambda_0^n t}), t > 0, \widetilde{\mathcal{P}}\}$$

  *converges to that of the process* $\{b_t(W), t > 0, \widetilde{Q}\}$ *as* $n \to \infty$.

**Remark 6.2.3** It is known that the limit distribution of the reflecting diffusion process in a Brownian environment is selfdecomposable (see Kawazu's example: Example 3.3 of [S91]).

**Example 6.2.4** Let $\{W, \widetilde{Q}\}$ be a strictly $\alpha$-semi-stable Lévy process such that neither $\{W(x)\}$ nor $\{-W(x)\}$ is a subordinator process. Then conditions (i$'$) and (iii$'$) are satisfied almost surely, hence $\widetilde{Q}(\widetilde{\mathcal{W}}_0) = 1$ by Lemma 6.2.1.

**Proof of (i) of Theorem 6.2.2**
Let $t > 0$ be fixed. Since $W^\sharp(b_t) = 0$, we have the independence of $\{W_{c_t}^\sharp(x), 0 \leq x \leq b_t\}$ and $\{W_{c_t}^\sharp(x), b_t \leq x\}$ by an application of the result due to Millar

[Mi78] for the stopped process $W^\sharp_{c_t}(x) = W^\sharp(x \wedge c_t)$ with life time $\infty$. Since $\{W^\sharp(x)\}$ is the strong Markov process, the process $\{W^\sharp(c_t + x), x \geq 0\}$ is a Markov process starting at $W^\sharp(c_t)$. From the independence above, the value of $W^\sharp(c_t)$ is independent of the process $\{W^\sharp(x), 0 \leq x \leq b_t\}$, hence the process $\{W^\sharp(b_t+x), x \geq 0\}$ is also independent of the process before $b_t$. Since the increments $b_u - b_t, u > t$, are functionals of the process $\{W^\sharp(b_t + x), x \geq 0\}$, $\{b_s, s \leq t\}$ and $b_u - b_t, u > t$ are independent. This implies that $\{b_t, t > 0, \widetilde{Q}\}$ has independent increments. $\square$

**Proof of (ii) of Theorem 6.2.2**

We can show the semi-selfsimilarity of the process $\{b_t, t > 0, \widetilde{Q}\}$ in the same way as that in the case of both-sided environments. Assertion (2) is shown in the same way as Theorem 1 of [T00] as follows:

Subtracting some negligible set from $\widetilde{\mathcal{W}_0}$ if necessary, we may assume that for any $W \in \widetilde{\mathcal{W}_0}$ and for any small $\varepsilon > 0$, there exists $c_t^{'} = c_t^{'}(W)$ satisfying the following conditions;

- $c_t < c_t^{'} < c_t + \varepsilon$,
- $W$ is continuous at $c_t^{'}$,
- $W(c_t) - \varepsilon < W(x) < W(c_t^{'})$ for any $x \in [c_t, c_t^{'})$.

Hence, for each $t > 0$ the definition of $b_t$ and $c_t^{'}$ implies

$$W(c_t^{'}) - W(b_t) > t, \tag{6.2.4}$$

$$\sup\{W(y) - W(x) : 0 \leq x < y \leq b_t\} < t. \tag{6.2.5}$$

For each $W \in \widetilde{\mathcal{W}_0}$ we denote by $\widehat{W}$ the function defined for all $x \in \mathbf{R}$ in such a way that

$$\widehat{W}(x) = \begin{cases} W(x) & \text{for } x \geq 0, \\ W(-x) & \text{for } x < 0, \end{cases}$$

and let $\widehat{Q}$ be the probability measure on $\mathcal{W}$ induced by the mapping $W \mapsto \widehat{W}$ under $\widetilde{Q}$. Let $\widehat{\mathcal{P}} = \widehat{Q} \times P$ and consider the diffusion process $\{X(t), t > 0, \widehat{\mathcal{P}}\}$ on $\mathcal{W} \times \Omega$. Then the process $\{|X(t)|, t > 0, \widehat{\mathcal{P}}\}$ is identical in law to the process $\{\widetilde{X}(t), t > 0, \widetilde{\mathcal{P}}\}$. We consider the convergence of $\{X, \widehat{\mathcal{P}}\}$ instead of that with $\{\widetilde{X}, \widetilde{\mathcal{P}}\}$.

Almost all sample functions of $\{\widehat{W}, \widehat{Q}\}$ do not satisfy condition (ii) of $\mathcal{W}_0$, hence we need to change the notion of a valley according to [KTT89]. For $W \in \widetilde{\mathcal{W}_0}$ and $t > 0$, let $M_t = M_t(W) = \sup\{W(x) : 0 \le x \le b_t\}$ and define $a_t = a_t(W)$ in $(0, b_t)$ by $W(a_t) = M_t$. Then we have the following cases:

(a) If $M_t < W(c_t)$, then $\{\widehat{W}(x), -c_t \le x \le c_t\}$ is a valley with two bottoms of the same level (see [II-2] in p.512 of [KTT89]).

(b) If $M_t > W(c_t)$, then $\{\widehat{W}(x), -a_t \le x \le c_t\}$ and $\{\widehat{W}(x), -c_t \le x \le a_t\}$ are two valleys that are connected at 0 (see [II-4] in p.512 of [KTT89]).

(6.2.4) and (6.2.5) imply that for any $\widehat{W}$ and each $t > 0$ there exists a valley (exist valleys) with $A < t < D$ for almost all environments. Since $\widehat{W} \notin \mathcal{W}_0$, Theorem 6.1.1 must be slightly modified. In either case, by using Theorem 1 of [KTT89] instead of Theorem I-A-I of [KTT88], we can show that for a fixed $W \in \widetilde{\mathcal{W}_0}$ and any $\varepsilon$-neighborhood $U$ of the two point set $\{-b_t(W), b_t(W)\}$,

$$\widehat{\mathcal{P}}\left\{\lambda_0^{-n/H}\widehat{X}(e^{\lambda_0^n t}, W) \notin U\right\} \to 0 \tag{6.2.6}$$

as $n \to \infty$. If we replace $\{X, \widehat{\mathcal{P}}\}$ by $\{\widetilde{X}, \widetilde{\mathcal{P}}\}$, then the relation (6.2.6) yields

$$\widetilde{\mathcal{P}}\left\{\left|\lambda_0^{-n/H}X(e^{\lambda_0^n t}, W) - b_t(W)\right| > \varepsilon\right\} \to 0 \tag{6.2.7}$$

as $n \to \infty$. $\square$

# Chapter 7

# Recurrence of a diffusion process in a multi-dimensional random environment

In this chapter, we consider multi-dimensional diffusion processes whose components are $d$ independent diffusion processes in random environments considered in Chapter 5 and study their recurrence or transience problem. Hence, we use similar notation to those of Chapter 5. However, we treat selfsimilar environments in this chapter, because (i) recurrence or transience of diffusion processes in semi-selfsimilar environments seems to be similar to those in selfsimilar ones except few singular semi-selfsimilar environments and (ii) such a problem is seemed unsettled and for a preparation for semi-selfsimilar environments' case, at first we consider selfsimilar cases to grasp the essence of this problem. The environments we treat are (1) $d$ independent non-negative reflecting Brownian environments, (2) $d$ non-positive reflecting Brownian environments and (3) $d$ selfsimilar environments determined by the distance from the origin. The third case is an extension of a Brownian environment discussed by Fukushima, Nakao and Takeda [FNT87] to $d$-dimensional selfsimilar ones. In all cases above we use the criterion for recurrence or transience of diffusion processes developed by Ichihara [I78].

## 7.1 The models and the results

In this chapter, we regard $\{W, Q\}$ as an environment. For a fixed $W$ we consider a multi-dimensional diffusion process $X_{\mathbf{W}}$ with a generator

$$\sum_{k=1}^{d} \frac{1}{2} e^{W_k(x_k)} \frac{\partial}{\partial x_k} \left\{ e^{-W_k(x_k)} \frac{\partial}{\partial x_k} \right\}, \tag{7.1.1}$$

where $W_1, W_2, \ldots, W_d$ are $d$ independent copies of $\{W(x), x \in \mathbf{R}\}$ and $\mathbf{W} = (W_1, W_2, \ldots, W_d)$. The process $X_{\mathbf{W}}$ is constructed by $d$ independent scale-changed and time-changed Brownian motions. A diffusion process $\{X(t)\}$ is said to be recurrent if for any open subset $U \in \mathbf{R}^d$

$$P(X(t) \in U \text{ for some } t > 0) = 1,$$

and transient otherwise. In the case of reflecting Brownian environments, we have the following theorem.

**Theorem 7.1.1**

(i) *If $\{W, Q\}$ is a non-negative reflecting Brownian environment, then $X_{\mathbf{W}}$ is recurrent for almost all environments and any dimension $d$.*

(ii) *If $\{W, Q\}$ is a non-positive reflecting Brownian environment, then $X_{\mathbf{W}}$ is transient for almost all environments and $d = 2, 3, 4, \ldots$.*

These behaviors are quite different from those of $d$-dimensional Brownian motion. In one-dimensional case Tanaka [T87] studied diffusion processes in a non-negative and non-positive reflecting Brownian environments. He showed that they are recurrent, $(\log t)^{-2} X(t)$ converges weakly as $n \to \infty$ and calculated their limit distributions.

We next give a brief survey about multi-dimensional diffusion processes in random environments. We set $\boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d$. Fukushima, Nakao and Takeda [FNT87] obtained recurrence of the diffusion process $X_W$ with a generator

$$\frac{1}{2} e^{W(|\boldsymbol{x}|)} \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left( e^{-W(|\boldsymbol{x}|)} \frac{\partial}{\partial x_k} \right), \tag{7.1.2}$$

where $|\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ and $\{W, Q\}$ is a one-dimensional Brownian environment. In the case of a diffusion process in a multi-parameter Brownian environment, namely, $\{W, Q\}$ is a Lévy's Brownian motion with multi-dimensional time and the diffusion process $X_W$ corresponds to a generator

$$\frac{1}{2} e^{W(\boldsymbol{x})} \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ e^{-W(\boldsymbol{x})} \frac{\partial}{\partial x_k} \right\},$$

Mathieu [M94] obtained some results of the long time behavior of $X_W$, and Tanaka [T93] proved that $X_W$ is recurrent for almost all environments.

These environments are functions of the distance from the origin. We next consider a similar environment to those. Let $W \in \mathcal{W}$ (see (5.1.2) for the definition of $\mathcal{W}$) and let $Q$ be the probability measure on $\mathcal{W}$ with respect to which $\{W(x), x \geq 0\}$ and $\{W(x), x \leq 0\}$ are independent and $H$-selfsimilar process, namely, for any $\lambda > 0$ there exists $H > 0$ such that

$$\{W(\lambda x), x \in \mathbf{R}\} \overset{\mathcal{D}}{=} \{\lambda^H W(x), x \in \mathbf{R}\}. \tag{7.1.3}$$

In the present case, we consider a multi-dimensional diffusion process $\widehat{X}_{\mathbf{W}}$ with a generator

$$\sum_{k=1}^{n} \frac{1}{2} e^{W_k(|\boldsymbol{x}|)} \frac{\partial}{\partial x_k} \left\{ e^{-W_k(|\boldsymbol{x}|)} \frac{\partial}{\partial x_k} \right\}. \tag{7.1.4}$$

For each $t \in \mathbf{R}$ and $W \in \mathcal{W}$ we define a scaling transformation $T_t$ as

$$T_t(W(x)) = e^{-Ht} W(e^t x), \qquad x \in \mathbf{R}.$$

We have the following.

**Theorem 7.1.2** *Let $\{W, Q\}$ be an $H$-selfsimilar environment. Assume that $\{W, Q\}$ satisfies*

(1) $W(1) > 0$ *with positive probability,*

(2) *any scaling transformation $T_t, t > 0$ is ergodic.*

*Then $\widehat{X}_{\mathbf{W}}$ is recurrent for almost all environments.*

**Remark 7.1.3** If $\{W, Q\}$ is a Brownian (also strictly symmetric $\alpha$-stable Lévy) environment, then the conditions above are satisfied. See [Ta89] for other examples of Gaussian and stable selfsimilar processes with ergodic scaling transformations.

## 7.2 Proofs of theorems

**Proof of (i) of Theorem 7.1.1**
From a result in [I78] we need to study recurrence or transience of the diffusion process with a generator

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ e^{-W_k(x_k)} \frac{\partial}{\partial x_k} \right\}. \tag{7.2.5}$$

According to Ichihara's recurrent test, it is enough to show that for almost all environments

$$\int_1^{\infty} r^{1-d} \left\{ \int_{S^{d-1}} \sum_{k=1}^{d} \sigma_k^2 e^{-|W_k(r\sigma_k)|} d\sigma \right\}^{-1} dr = \infty, \tag{7.2.6}$$

where $d\sigma$ is the normalized uniform measure on $S^{d-1}$. Hence, it suffices to show

$$E[\text{meas}\left\{x \in [0,1] : x^2 e^{-|W_1(rx)|} \geq r^{2-d}\right\}] \to 0, \tag{7.2.7}$$

as $r \to \infty$. For any $\varepsilon \in (0,1)$ we have

$$\begin{aligned} & E\left[\text{meas}\left\{x \in [0,1] : x^2 e^{-|W_1(rx)|} \geq r^{2-d}\right\}\right] \\ = \quad & E\left[\text{meas}\left\{x \in [r^{1-2/d}, 1] : \log x^2 - \sqrt{r}|W_1(x)| \geq \log r^{2-d}\right\}\right] \\ \leq \quad & E\left[\text{meas}\left\{x \in [r^{1-2/d}, 1] : |W_1(x)| \leq r^{-1/2} \log r^{d-2}\right\}\right] \\ = \quad & \int_{r^{1-d/2}}^{1} Q\left(|W_1(x)| \leq r^{-1/2} \log r^{d-2}\right) dx \\ \leq \quad & \int_{r^{1-d/2}}^{r^{-1+\varepsilon}} dx + \sqrt{\frac{2}{\pi}} \int_{r^{-1+\varepsilon}}^{1} dx \int_0^{\frac{\log r^{d-2}}{\sqrt{xr}}} e^{-u^2/2} du \\ \to \quad & 0 \end{aligned}$$

as $n \to \infty$, which proves (7.2.7). $\square$

**Proof of (ii) of Theorem 7.1.1**

From Ichihara's transient test it is enough to show that there exists $\bar{\sigma} \subset S^{d-1}$ with a positive uniform measure such that

$$\int_1^\infty r^{1-d} \left\{ \int_{\bar{\sigma}} \left\{ \sum_{k=1}^d \sigma_k^2 e^{|W_k(r\sigma_k)|} \right\}^{-1} d\sigma \right\}^{-1} dr < \infty \tag{7.2.8}$$

for almost all environments. Hence, it suffice to show

$$E \left[ \text{meas} \left\{ x \in [0,1] : e^{|W(rx)|} < r^{2-d} \right\} \right] \to 0$$

as $r \to \infty$. This convergence is shown in the same manner as in the proof of (i), hence (7.2.8) is shown. $\square$

**Proof of Theorem 7.1.2**

For almost all $W \in \mathcal{W}$ the Dirichlet space theory guarantees that there exists a diffusion process $\widehat{X}^0_{\mathbf{W}}$ with a generator

$$\sum_{k=1}^d \frac{\delta}{\delta x_k} \left\{ e^{-W_k(|\boldsymbol{x}|)} \frac{\delta}{\delta x_k} \right\},$$

and $\widehat{X}_{\mathbf{W}}$ is constructed from $\widehat{X}^0_{\mathbf{W}}$ by a scale-changed and a time-change. From Ichihara's recurrence test, it is enough to show that for almost all environments

$$\int_1^\infty r^{1-d} \left\{ \sum_{k=1}^d e^{-W_k(r)} \right\}^{-1} dr = \infty. \tag{7.2.9}$$

We show the above in the same way as [T93] as follows:
Let $M(t) = \min\{T_t(W_k(1)) : k = 1, 2, \ldots, d\}$. We have

$$\int_1^\infty r^{1-d} \left\{ \sum_{k=1}^d e^{-W_k(r)} \right\}^{-1} dr$$

$$
\begin{aligned}
&= \int_0^\infty (e^t)^{2-d} \left\{ \sum_{k=1}^{d} \exp\left\{-e^{Ht} T_t(W_k(1))\right\} \right\}^{-1} dt \\
&\geq \int_0^\infty (e^t)^{2-d} \exp\left\{e^{Ht} M(t)\right\} dt \\
&\geq \int_0^\infty 1_{(a,\infty)}\left(M(t)\right) dt,
\end{aligned}
$$

where $a$ satisfies $e^{Ht}a - (d-2)t \geq 0$ for any $t \geq 0$. The ergordicity implies

$$
\lim_{T \to \infty} T^{-1} \int_0^T 1_{(a,\infty)}(M(t))dt = E[1_{(a,\infty)}(M(0))]
$$

almost surely with respect to $Q$. Hence, $Q(W(1) > 0) > 0$ implies (7.2.9). $\square$

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