Heavy Paths and Cycles in Weighted Graphs and Related Topics

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Preface

This thesis is the result of almost three years of research on weighted graphs. After an introductory chapter, the reader will find eight chapters. In Chapter 2 I will present my work on heavy fans, which is useful when we discuss about heavy paths and cycles which contain some specified vertices. Through Chapters 3 to 7, we will discuss about heavy cycles, and in Chapter 8 the results on heavy paths are shown. In Chapter 9 the topic changes to the weighted Ramsey problem. The reader may feel some difference between this topic and the topics discussed in Chapters 2 to 8, however this topic is also important to investigate the difference between unweighted graphs and weighted graphs. The basis of Chapters 2 to 9 is formed by papers written during these three years. At the beginning of each chapter the readers can find out which paper in the reference is based on the chapter.

I would like to express my gratitude to the researchers and students who stimulated me to further work. First of all, I am deeply indebted to Professor Katsuhiro Ota. His willingness to share his wealth of knowledge and careful reading of my theses improved my research a great deal. I would never have completed this thesis without his help. I would also like to thank Professor Hikoe Enomoto, for his valuable suggestions and comments. Professor Hajo Broersma, Dr. Daniel Paulusma, Dr. Kiyoshi Yoshimoto and Prof. Shenngui Zhang, the seminars with them were great experience for me. I would like to thank them for the valuable discussions though my English talk is poor. I would also like to mention Dr. Tomoki Yamashita, Dr. Shinya Fujita and Dr. Hajime Matsumura. I was able to spent pleasant time with them, and their attitude toward Graph Theory affected me. Now I am not able to list all the names of the people who contributed to my work, but I am grateful to them; The members of Graph Seminar at Tokyo University of Science, the people who I met at the conference, and so on. Lastly, I would like to thank my parents, who continue to give themselves to me.

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Chapter 1

Introduction

1.1 Problems studied in this thesis

It is told that the Graph Theory began its history with the problem of the Königsberg's bridge. In the ancient Prussian city Königsberg, there was the River Pregel and seven bridges over the river. Peoples guessed that there is a way to go once over each bridge and return back to the home. However, though they tried to find it several times, it always ended in failure, and in 1736 Leonhard Euler showed that this is impossible. To prove it he replaced the map of the city by a diagram (See Figure 1.1), and then he was able to give a general method for all the other problems of the same type. This diagram is a depiction of a graph; a graph G is a pair of two sets (V, E) such that every element of E is a 2-element subset of V. We call $v \in V$ a *vertex* and $e \in E$ an *edge* (In the digram, the points represents vertices and lines represents edges). For a graph G, the vertex set of G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. An edge $\{u, v\}$ is usually written as uv or vu. If there is an edge uv, then we say it *joins* two vertices u and v, u and v are *adjacent*, and u is a *neighbor* of v (also, v is a neighbor of u). In a graph G, $N_G(v)$ denotes the set, and $d_G(v)$ denotes the number, of neighbors of v. When no confusion occurs, we will denote $N_G(v)$ and $d_G(v)$ by $N(v)$ and $d(v)$, respectively. A graph H is called a *subgraph* of a graph G, and written as $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case we say G *contains* H.

In the graph described as Figure 1.1, there exists two pairs of edges which join same two vertices. These pairs are called *multiple edges*. However, in this thesis, we deal with *simple graphs*: the graph in which there is no multiple edge.

As we can see in Königsberg's problem, to analyze a phenomenon, it is sometimes useful to consider an object as a graph. However, there are a lot of situations which need additional informations to a graph. Consider the cities in a country, in which some of the two cities

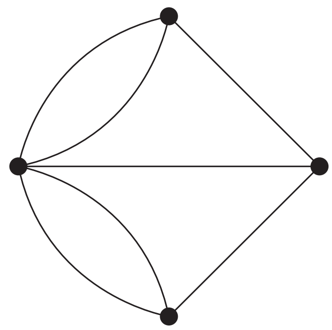

Figure 1.1:

are joined by the airline route. Now the cost of the journey between any joined two cities is known. Then, using simple graphs, we can explain the relations of any two cities whether they are joined or not. However, the cost between them cannot be illustrated; we must add the cost to every edge. Such graphs—the graphs in which every edge is assigned a nonnegative number—are called *weighted graphs*, which are studied in this thesis.

In a weighted graph G, the number assigned on each edge is called *the weight* of the edge, and we denote the weighting function by w_G. For a subgraph H of G, the *weight* of H is defined by

$$w_G(H) = \sum_{e \in E(H)} w_G(e).$$

And, for a vertex v in G, we define the *weighted degree* of v in G by

$$d_G^w(v) = \sum_{u \in N_G(v)} w_G(uv).$$

When no confusion occurs, we will denote w_G, $w_G(H)$ and $d_G^w(v)$ by w, $w(H)$ and $d^w(v)$, respectively.

In this thesis, we mainly discuss about the sufficient condition for the weighted graphs to have heavy paths and cycles. A *path* P is a graph with vertex set $\{v_1, v_2, \ldots, v_p\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{p-1}v_p\}$, which is usually denoted by $v_1v_2 \ldots v_p$. The vertices v_1 and v_p are called *endvertices* of P, and we say P joins v_1 and v_p. A *cycle* is a graph obtained by a path adding an edge joining two endvertices, and it is also denoted by its sequence of vertices, the same as paths. A cycle is called a *hamiltonian cycle* of a graph if it contains all the vertices of the graph, and a graph is said to be *hamiltonian* if it contains a hamiltonian cycle.

In 1989, Bondy and Fan proved the following result, which is the cornerstone of the studies of heavy cycles in weighted graphs.

Theorem 1.1 (Bondy and Fan [4]). *Let G be a 2-connected weighted graph and d a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then either G contains a cycle of weight at least $2d$ or every heaviest cycle in G is a hamiltonian cycle.*

In a weighted graph G with constant weight 1, $d^w_G(v) = d_G(v)$ for every vertex $v \in V(G)$ and $w(H) = |E(H)|$ for every subgraph H of G. Hence, we can regard an unweighted graph as a weighted graph with special property, and it is clear that Theorem 1.1 is a generalization of the following well-known result. The *length* of a path or a cycle is the number of the edges that it contains.

Theorem 1.2 (Dirac [8]). *Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then G contains either a cycle of length at least $2d$ or a hamiltonian cycle.*

Considering the heavy cycle passing through a specified vertex, Theorem 1.1 is extended as follows.

Theorem 1.3 (Zhang et al. [35]). *Let G be a 2-connected weighted graph and d a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then for every vertex y in G, either G contains a cycle of weight at least $2d$ containing y or every heaviest cycle in G is a hamiltonian cycle.*

In the proofs of Theorems 1.1 and 1.3, the existence of heavy paths is used to find heavy cycles. But in fact, the existence of heavy fan (a set of paths joining a vertex and a vertex set) is useful to show the existence of heavy cycles. In Chapter 2 we show the existence of heavy fan. Using this, in the first section of Chapter 3, we give alternative simple proofs of Theorems 1.1 and 1.3, and moreover, we show an extension of Theorem 1.3. All these theorems on heavy cycles are easily shown by using the existence of heavy fan.

The (weighted) degree condition in Theorem 1.2 (1.1) is on every one vertex. Such condition is called *Dirac-type*. There is another well-known weaker (weighted) degree condition, called an *Ore-type* condition, the condition on the degree sum of every two non-adjacent vertices. Using Ore-type condition, Bondy et al. [3] extended a previous result in unweighted graphs to weighted graphs, and proved a theorem on the existence of heavy cycles. In the second section of Chapter 3, using the existence of heavy fan again, an extension of the theorem of Bondy et al. is obtained.

Most of the previous results in weighted graphs are on 2-connected weighted graphs. But in Chapter 4, we deal with 3-connected weighted graphs. In unweighted graphs, it is known that if we enlarge the connectivity, then we can enlarge the number of specified vertices contained in a long cycle. We obtain that the same is true in 3-connected weighted graphs, and the existence of heavy cycles passing through three specified vertices is shown.

There exists some weighted graphs which contain no cycle of weight at least $2d$, though they satisfy the conditions of Theorem 1.1. In Chapter 5 it is shown that we are always be able to find a cycle of weight at least $2d$ if a weighted graph has no cycle of length three and satisfies the conditions of Theorem 1.1.

Fan-type condition is more weaker condition than Ore-type one. There are several results which shows the existence of long cycle in unweighted graphs with Fan-type condition. However, to extend them to the weighted graphs, it is shown in [34] that some extra-condition on the weight of the edges is necessary. In Chapter 6 we weaken both of the weighted degree condition and the extra-condition used in [34] and show the the existence of heavy cycles. And in Chapter 7, using the same extra-condition as in [34] and another weighted degree condition, called σ_k-type condition, we show the existence of heavy cycles.

About the existence of heavy paths in weighted graphs, the following is known. A path joining u and v is denoted by a (u, v)-path.

Theorem 1.4 (Bondy and Fan [4]). *Let G be a 2-connected weighted graph and d be a non-negative real number. Let x and y be distinct vertices of G. If $d^w(v) \geq d$ for all $v \in V(G)\setminus\{x, y\}$, then G contains an (x, y)-path of weight at least d.*

In Chapter 8 we give an extension of Theorem 1.4 by using the existence of heavy fan. And also two Ore-type conditions for the existence of heavy paths are shown.

In Chapter 9, we turn our topic to the graphs in which each edge is colored by two colors, and introduce a *weighted Ramsey problem* (For the basic concepts of Ramsey Theory, I refer the reader to [29]). In this chapter we will discuss about heavy small subgraphs in which every edge has the same color, and show some theorems on Ramsey problem in weighted graphs.

1.2 Terminology and Notation

In this section we give some basic terminology and notation used in this thesis. We call the number of vertices of a graph the *order* of the graph. An edge e is called *incident* with a vertex v if $v \in e$. Now if two edges e_1 and e_2 are incident with a common vertex v, we say e_1 and e_2 are *adjacent*. If every two vertices in a graph are joined by an edge, we call this graph a *complete graph*, and we denote a complete graph of order n by K_n.

Let H be a subgraph of a graph G such that for every pair of vertices $u, v \in V(H)$, $uv \in E(H)$ if and only if $uv \in E(G)$, then H is called an *induced subgraph* of G. In this case, H is denoted by $G[V(H)]$ and we say $V(H)$ *induces* H in G. For a graph G and $U \subseteq V(G)$, we denote $G[V(G) \setminus U]$ by $G - U$. Let E' be a set of 2-element subsets of $V(G)$. The graph $G' = (V(G), E(G) \setminus E')$ is denoted by $G - E'$, and the graph $G'' = (V(G), E(G) \cup E')$ is denoted by $G + E'$. Moreover, for two graphs G_1 and G_2, we define $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ and $G_1 + G_2 = G_1 \cup G_2 + \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Let $e = xy$ be an edge of a graph G. Sometimes we make a new graph regarding two vertices x with y as a new vertex v_e. We call this operation *contraction* of the edge e and the new graph is denoted by G/e. Formally, G/e is a graph such that

- $V(G/e) = \{V(G) \cup v_e\} \setminus \{x, y\}$
- $E(G/e) = E(G - \{x, y\}) \cup \{v_e v \mid xv \in E(G) \setminus \{e\} \text{ or } yv \in E(G) \setminus \{e\}\}$.

The vertex v_e is called the *contracted vertex*.

We call a graph G *connected* if, for any two vertices, there is a path joining them. And G is called *k-connected* if $|V(G)| > k$ and $G - U$ is connected for any $U \subseteq V(G)$ of cardinality $k - 1$. If G is connected and $G - \{v\}$ is not connected for a vertex $v \in V(G)$, then we call v a *cutvertex* of G.

A connected graph which contains no cycle is called a *tree*. A *component* of a graph G is a maximal connected subgraph of G, and a *block* B of G is a maximal connected subgraph of G which does not contain a cutvertex of B itself (That is, if B is a block of a graph, then B is 2-connected or K_2). An endblock of G is a block which contain only one cutvertex of G. For an endblock B in a graph, we denote the cutvertex of the graph in B by c_B, and $V(B) \setminus \{c_B\}$ is denoted by I_B.

A *k-partite graph*, also called a *multi-partite graph*, is a graph which have partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ such that there is no edge in $G[V_1], G[V_2], \ldots, G[V_k]$. We call each V_i a *partite set*. Especially, a 2-partite graph is called a *bipartite graph*. A *complete multi-partite graph* is a multi-partite graph such that all two vertices in the different partite sets are adjacent. A complete k-partite graph whose partite sets contain $n_1, n_2, \ldots, n_k$ vertices is denoted by $K_{n_1,n_2,\ldots,n_k}$.

Chapter 2

Heavy fans in weighted graphs

(The main result of this chapter appears in [19], which is an extension of the result proved in [22].)

In [4] and [35], a heavy path in a weighted graph is used to show the existence of heavy cycles. But in fact, as we can see in the later chapters, the existence of heavy fan is useful to find heavy cycles passing through some specified vertices. In this chapter, we prove the existence of heavy fans in weighted graphs.

To describe the main theorem of this chapter, we need some terminology and notation. Let X, Z be disjoint subsets of $V(G)$. A path P is called an (X, Z)-path if

(i) P is an (x, z)-path, where $x \in X$ and $z \in Z$, and

(ii) $V(P) \cap X = \{x\}$ and $V(P) \cap Z = \{z\}$.

Let X, Z be subsets of $V(G)$ and $y \in V(G) \setminus \{X \cup Z\}$. If every $(\{y\}, Z)$-path contains at least one vertex of X, then we call *X separates y from Z*. A subgraph F of G is called a *(y, Z)-fan of width k* if F is a union of $(\{y\}, Z)$-paths $P_1, \ldots P_k$, where $P_i \cap P_j = \{y\}$ for $i \neq j$. The maximum number of the width of (y, Z)-fans in G is denoted by $k(G; y, Z)$. Note that $k(G; y, Z)$ is equal to the minimum number of vertices separating y from Z in G.

Our main result in this chapter is the following.

Theorem 2.1. *Let G be a connected weighted graph, $L \subset V(G)$, M a component of $G - L$, and $y \in V(M)$. Now assume that, for all $v \in V(M)$,*

- $d^w_G(v) \geq d$*, and*
- *there is no vertex in $V(M) \setminus \{v\}$ which separates v from L.*

Then for every (y, L)*-fan* F_1*, there exists a* (y, L)*-fan* F_2 *such that*

- $w(F_2) \geq d$*,*
- $V(F_1 \cap L) \subseteq V(F_2 \cap L)$*, and*
- *the width of* $F_2 = k(G; y, L)$*.*

Theorem 2.1 is a weighted analogue of the following theorem due to Perfect.

Theorem 2.2 (Perfect [30]). *Let* G *be a connected graph,* L *a subset of* $V(G)$*,* $y \in V(G) \setminus L$*, and* l *an integer such that* $l < k(G; y, L)$*. Then for every* (y, L)*-fan* F_1 *of width* l*, there exists a* (y, L)*-fan* F_2 *of width* $k(G; y, L)$ *such that* $V(F_1 \cap L) \subset V(F_2 \cap L)$*.*

Again we need some terminology and notation used in the proof of Theorem 2.1. A component of a graph which contains a vertex y is called a *y-component*. For two disjoint subsets L and M of $V(G)$, we denote $\bigcup_{v \in L}(N_G(v) \cap M)$ by $N_M(L)$. In this chapter, for a weighted graph G with $u, v \in V(G)$, we define $w_G(uv) = 0$ if $uv \notin E(G)$. Let $e = xy$ be an edge of G. When we contract an edge, there may occur some multiple edges. In this chapter we identify them as a simple edge whose weight is the sum of the two previous edges. So, G/e is a weighted graph such that

- $V(G/e) = \{V(G) \cup \{v_e\}\} \setminus \{x, y\}$,
- $E(G/e) = E(G - \{x, y\}) \cup \{v_e v \mid xv \in E(G) \setminus \{e\}$ or $yv \in E(G) \setminus \{e\}\}$,
- if $uv \in E(G/e - \{v_e\})$, $w_{G/e}(uv) = w_G(uv)$, and
- if $v \in N_{G/e}(v_e)$, $w_{G/e}(vv_e) = w_G(vx) + w_G(vy)$,

The vertex v_e is called the *contracted vertex*.

We often identify a subgraph H of G and its vertex set $V(H)$. For example, $N_G(V(H))$ is often denoted by $N_G(H)$.

Proof of Theorem 2.1. Let $k = k(G; y, L)$. If the width of F_1 is less than k, then Theorem 2.2 shows the existence of a (y, L)-fan $F_1{}'$ of width k such that $V(F_1 \cap L) \subseteq V(F_1{}' \cap L)$. The required fan F_2 for $F_1{}'$ is also a required fan for F_1, hence we may assume that the width of F_1 is k.

We use induction on $|V(M)|$. If $M = \{y\}$, then it is obvious that $F_2 = \bigcup_{v \in N(y)} vy$ is a required fan, since $d^w_G(y) \geq d$. Now suppose $|V(M)| \geq 2$.

Case 1. Every $X \subseteq V(G)$ of cardinality k separating y from L is contained in L.

In this case, we have $|N_L(M)| = k$, so $F \cap L = N_L(M)$ for any (y, L)-fan F of width k. Hence it suffices to show the existence of a (y, L)-fan of width k and weight $\geq d$.

Assume that there exists $x \in N_M(L) \setminus \{y\}$ and $t \in N_L(\{x\})$ such that $k(G/xt; y, L) = k' < k$. When we make G/xt from G, we regard the contracted vertex as t'. Let X' be a vertex set of cardinality k' which separates y from L in G/xt. If $t' \notin X'$, then X' separates y from L in G, which contradicts the fact $k(G; y, L) = k$. Hence we have $t' \in X'$. Then $X' \cup \{x, t\} \setminus \{t'\}$ separates y from L in G. If $|X'| < k - 1$, $|X' \cup \{x, t\} \setminus \{t'\}| < k$, which contradicts the fact $k(G; y, L) = k$. If $|X'| = k - 1$, $|X' \cup \{x, t\} \setminus \{t'\}| = k$ and $X' \cup \{x, t\} \setminus \{t'\} \nsubseteq L$. This contradicts the assumption of this case. Therefore, we have $k(G/xt; y, L) = k$ for every $x \in N_M(L) \setminus \{y\}$ and $t \in N_L(\{x\})$.

Case 1.1. There exists $t \in N_L(M)$ such that $yt \notin E(G)$ or $w(xt) > w(yt)$ for some $x \in N_M(\{t\})$.

Take a vertex $x \in N_M(\{t\})$ such that $w(xt)$ is as large as possible. Now make a new graph $G' = G/xt$, and regard the contracted vertex as t. Let $L' = L$ and M' be the y-component of $G' - L'$. Then $V(M') \subseteq V(M) \setminus \{x\}$, and it is clear that, for all $v \in V(M')$,

- $d^w_{G'}(v) = d^w_G(v) \geq d$ and
- there is no vertex in $V(M') \setminus \{v\}$ which separates v from L'.

Hence, by the induction hypothesis, we can find a (y, L')-fan F' of width $k(G/xt; y, L) = k$ and weight at least d. Since $k(G/xt; y, L) = k$ and $|N_{L'}(M')| \leq |N_L(M)| = k$, we have $|N_{L'}(M')| = k$, which implies $t \in F'$. And the fact $t \neq y$ implies that there is a vertex t^- which is the only neighbor of t in F'. If $t^- x \notin E(G)$, then $tt^- \in E(G)$ and $w_{G'}(tt^-) = w_G(tt^-)$. Therefore, $F = F'$ is a required fan in G. If $t^- x \in E(G)$, let F be a graph obtained by replacing an edge tt^- of F'

with a path $t^- xt$. Then F is a (y, L)-fan of width k in G such that

$$\begin{aligned} w(F) &= w(F') - w_{G'}(tt^-) + w_G(t^- x) + w_G(xt) \\ &= w(F') - (w_G(tt^-) + w_G(xt^-)) + w_G(t^- x) + w_G(xt) \\ &= w(F') - w_G(tt^-) + w_G(xt) \\ &\geq w(F') \\ &\geq d. \end{aligned}$$

Hence F is the required fan.

Case 1.2. For every vertex $t \in N_L(M)$, $yt \in E(G)$ and $w(xt) \leq w(yt)$ for all $x \in N_M(\{t\})$.

First, we prove the following claim.

Claim 1. *There exists a (y, z)-path P in M such that $z \in N_M(L)$ and the weight of P is at least* $\min\{d^w_M(z), d\}$.

Proof. If $|V(M)| = 2$, let z be the vertex of $V(M)$ other than y, then it is obvious that yz is a required path. So assume $|V(M)| \geq 3$. Note that

$$d^w_M(v) = d^w_G(v) \geq d \text{ for all } v \in V(M) \setminus N_M(L). \tag{2.1}$$

In the case where M is 2-connected, let z be a vertex in $N_M(L) \setminus \{y\}$ such that $d^w_M(z) \leq d^w_M(v)$ for all $v \in N_M(L) \setminus \{y\}$. Then with (2.1), we have

$$d^w_M(v) \geq \min\{d^w_M(z), d\} \text{ for all } v \in V(M) \setminus \{y, z\}.$$

Hence, by the induction hypothesis, there exists an $(x, \{y, z\})$-fan F of width $k(M; x, \{y, z\}) = 2$ and weight at least $\min\{d^w_M(z), d\}$ for every $x \in V(M) \setminus \{y, z\}$. Since the width of F is 2, F is a required path.

Assume that M is not 2-connected, and choose an endblock B such that $y \notin I_B = V(B) \setminus \{c_B\}$. Then there exists a (y, c_B)-path P_1 which is internally disjoint with B. Now let z be a vertex in $N_{I_B}(L)$ such that

$$d^w_M(z) \leq d^w_M(v) \text{ for all } v \in N_{I_B}(L). \tag{2.2}$$

If $|V(I_B)| = 1$, we have $w(zc_B) = d^w_M(z)$, hence $P = zc_BP_1y$ is a required path. So we may assume that $|V(I_B)| \geq 2$, then B is 2-connected. It follows from (2.1) and (2.2) that

$$d^w_M(v) \geq \min\{d^w_M(z), d\} \text{ for all } v \in V(B) \setminus \{c_B, z\}.$$

Then, by the induction hypothesis, there exists an $(x, \{c_B, z\})$-fan F of width $k(M; x, \{c_B, z\}) = 2$ and weight at least $\min\{d^w_M(z), d\}$ for every $x \in V(B) \setminus \{c_B, z\}$. Since the width of F is 2, F is a path. Joining P_1 and F, we have a required path. □

Now we are ready to complete the proof of Case 1.2. Choose a vertex z and a path P which satisfy the conditions of Claim 1. Let z' be a neighbor of z in L and

$$F = \bigcup_{v \in N_L(M) \setminus \{z'\}} yv \cup Pzz'.$$

Then F is a (y, L)-fan such that

$$\begin{aligned} w(F) &= \sum_{v \in N_L(M) \setminus \{z'\}} w(yv) + w(P) + w(zz') \\ &\geq \sum_{v \in N_L(z) \setminus \{z'\}} w(zv) + w(P) + w(zz') \\ &\geq d^w_L(z) + \min\{d, d^w_M(z)\} \\ &\geq \min\{d, d^w_G(z)\} \\ &= d. \end{aligned}$$

Now the width of F is $|N_L(M)| = k(G; y, L)$, hence F is the required fan. This completes the proof in Case 1.2 and the proof in Case 1.

Case 2. There exists $X \subseteq V(G)$ of cardinality k such that X separates y from L and $X \not\subseteq L$.

Let M^* be the y-component of $G - X$. Since $M^* \subseteq M$, $d^w_G(v) \geq d$ for every $v \in V(M^*)$. Now it is obvious that there is no vertex in $V(M^*) \setminus \{v\}$ which separates v from X. Hence, by the induction hypothesis, we can find a (y, X)-fan such that $w(F^*) \geq d$ and the width of $F^* = k(G; y, X) = k$. Now adding $\mathcal{P} = F_1 - V(M^*)$ to F^*, we can find a (y, L)-fan F_2 such that $w(F_2) = w(F^*) + w(\mathcal{P}) \geq d$ and $F_2 \cap L = \mathcal{P} \cap L = F_1 \cap L$, which is a required fan. This completes the proof of Theorem 2.1. □

Remark. Theorem 2.1 has the following corollary.

Corollary 2.3. *Let G be a connected weighted graph and $x, z \in V(G)$. Assume that, for all $v \in V(G) \setminus \{x\}$,*

- $d^w_G(v) \geq d$*, and*
- *there is no vertex in $V(G) \setminus \{v\}$ which separates v from x.*

Now if there exists k disjoint (x, z)-paths in G, then there exists a set of k' disjoint (x, z)-paths $\mathcal{P}$ such that $k' \geq k$ and $w(\mathcal{P}) \geq d$.

Proof. Apply Theorem 2.1 with $L = N_G(x) \setminus \{z\}$, then the assertion is obvious. □

However, the following is false.

False statement. *Let G be a k-connected weighted graph and $X, Z \subset V(G)$. If $d^w_G(v) \geq d$ for all $v \in V(G)$, then there exists a set of k disjoint (X, Z)-paths $\mathcal{P}$ such that $w(\mathcal{P}) \geq d$.*

Let G be a complete tripartite graph $K_{1,t,t}$, where $t \geq 2$, let v be the vertex in the partite set of cardinality 1, and let X and Z be the partite sets of cardinality t. If we assign weight $t/(2t-1)$ to the edges incident to v and weight 1 to all the other edges, then the minimum weighted degree of G is $2t^2/(2t-1)$, while the maximum weight of the set of k disjoint (X, Z)-paths is $t - 1 + 2 \cdot t/(2t-1) < 2t^2/(2t-1)$.

Chapter 3

Heavy cycles passing through some specified vertices in 2-connected weighted graphs

(This chapter is based on the paper [22].)

3.1 A Dirac-type condition for heavy cycles passing through two specified vertices

In 1989, Bondy and Fan began the study on the existence of heavy cycles in weighted graphs, and proved the following.

Theorem 3.1 (Bondy and Fan [4]). *Let G be a 2-connected weighted graph and d a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then either G contains a cycle of weight at least $2d$ or every heaviest cycle in G is a hamiltonian cycle.*

And, in 2000, Zhang et al. proved that we can find a heavy cycle passing through a specified vertex with the same conditions as in Theorem 3.1.

Theorem 3.2 (Zhang et al. [35]). *Let G be a 2-connected weighted graph and d a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then for every vertex y in G, either G contains a cycle of weight at least $2d$ containing y or every heaviest cycle in G is a hamiltonian cycle.*

Theorems 3.1 and 3.2 are generalization of the following two theorems to weighted graphs, respectively.

Theorem 3.3 (Dirac [8]). *Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then G contains either a cycle of length at least $2d$ or a hamiltonian cycle.*

Theorem 3.4 (Grötschel [24]). *Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then for every vertex y in G, G contains either a cycle of length at least $2d$ containing y or a hamiltonian cycle.*

First we give alternative short proofs of Theorems 3.1 and 3.2, using Theorem 2.1. From now we use the following notation. Let $C = v_1v_2 \ldots v_pv_1$ be a cycle with a fixed orientation. The segment $v_iv_{i+1} \cdots v_j$ is denoted by $C[v_i, v_j]$ or v_iCv_j. Let R be a tree or a path and $u, v \in V(R)$ with $u \neq v$, then there is only one (u, v)-path in R. This path is also denoted by $R[u, v]$ or uRv. When S is a cycle, a tree or a path, we denote $S[v_i, v_j] - \{v_i\}$, $S[v_i, v_j] - \{v_j\}$ and $S[v_i, v_j] - \{v_i, v_j\}$ by $S(v_i, v_j]$, $S[v_i, v_j)$ and $S(v_i, v_j)$, respectively.

Proof of Theorem 3.1. Let G be a weighted graph satisfying the conditions of Theorem 3.1. Assume that there exists a heaviest cycle C in G which is not a hamiltonian cycle and $w(C) < 2d$. Now take a vertex $y \in V(G) - V(C)$. Then, from Theorem 2.1, we obtain a (y, C)-fan F of width $= p \geq 2$ and weight $\geq d$. Let $F \cap C = \{v_1, v_2, \ldots, v_p\}$, where v_i are in order around C, and regard the indices as modulo p. Then for all i with $1 \leq i \leq p$, there exists a cycle $C_i = a_iFa_{i+1}Ca_i$, hence we have $w(F[a_i, a_{i+1}]) \leq w(C[a_i, a_{i+1}])$ since C is a heaviest cycle in G. Now $w(C_1) = w(F[a_1, a_2]) + \sum_{2 \leq i \leq p} w(C[a_i, a_{i+1}]) \geq \sum_{i=1}^{p} w(F[a_i, a_{i+1}]) \geq 2w(F) \geq 2d$, contradicting that C is a heaviest cycle in G. □

Proof of Theorem 3.2. Let G be a weighted graph satisfying the conditions of Theorem 3.2. Assume that there exists a heaviest cycle C in G which is not a hamiltonian cycle. Then from Theorem 3.1, $w(C) \geq 2d$. If $y \in V(C)$, there is nothing to prove, so assume that $y \notin V(C)$. It follows from Theorem 2.1 that there is a (y, C)-fan F of width $= p \geq 2$ and weight $\geq d$. Now take C_i as in the proof of Theorem 3.1, then also we obtain $w(F[a_i, a_{i+1}]) \leq w(C[a_i, a_{i+1}])$ for all i with $1 \leq i \leq p$. Hence $w(C_1) = w(F[a_1, a_2]) + \sum_{2 \leq i \leq p} w(C[a_i, a_{i+1}]) \geq \sum_{i=1}^{p} w(F[a_i, a_{i+1}]) \geq 2w(F) \geq 2d$ and C_1 contains y, which is a required cycle. □

Next, we prove the following theorem.

Theorem 3.5. *Let G be a 2-connected weighted graph and d a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then for every two vertices y_1 and y_2 in G, either G contains a cycle of weight at least $2d$ containing y_1 and y_2 or every heaviest cycle in G is a hamiltonian cycle.*

Theorem 3.5 is a generalization of Theorem 3.6 to weighted graphs.

Theorem 3.6 (Locke [28]). *Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then for every two vertices y_1 and y_2 in G, G contains either a cycle of length at least $2d$ containing y_1 and y_2 or a hamiltonian cycle.*

Proof of Theorem 3.5. Let G be a weighted graph satisfying the conditions of Theorem 3.5. If there is a heaviest cycle which is not a hamiltonian cycle, then Theorem 3.2 implies that there exists a cycle of weight $\geq 2d$ which contains either y_1 or y_2. Let C be the heaviest one among these cycles. Without loss of generality, we may assume that C contains y_1. If $y_2 \in C$, there is nothing to prove, so assume that $y_2 \notin C$. It follows from Theorem 2.1 that there is a (y_2, C)-fan F of width $= p \geq 2$ and weight $\geq d$. Now take C_i as in the proof of Theorem 3.1. Then, each C_i contains y_2, hence we have $w(F[a_i, a_{i+1}]) \leq w(C[a_i, a_{i+1}])$ since C is a heaviest cycle which contains either y_1 or y_2. Now, since $p \geq 2$, there exists an index j with $1 \leq j \leq p$ such that $V(C(a_j a_{j+1})) \cap \{y_1\} = \emptyset$. Then C_j contains y_1 and y_2, and $w(C_j) = w(F[a_j, a_{j+1}]) + \sum_{1\leq i\leq p,\ i\neq j} w(C[a_i, a_{i+1}]) \geq \sum_{i=1}^{p} w(F[a_i, a_{i+1}]) \geq 2w(F) \geq 2d$, hence C_j is a required cycle. □

3.2 An Ore-type condition for heavy cycles passing through a specified vertex

The (weighted) degree condition we discussed in Section 3.1 is on every one vertex. In this section, we consider another weighted degree condition, called the *Ore-type* condition, the condition on the degree sum of every two non-adjacent vertices. The following result, which was shown by several authors independently, gives a generalization of Theorem 3.3 in unweighted graphs. For non-complete graph G, let

$$\sigma_2(G) = \min\{d(u) + d(v) \mid u \text{ and } v \text{ are nonadjacent}\},$$

and if G is complete, let $\sigma_2(G) = \infty$.

Theorem 3.7 (Bermond [2], Linial [27], Pósa [31]). *Let G be a 2-connected graph. Then G contains either a cycle of length at least $\sigma_2(G)$ or a hamiltonian cycle.*

Enomoto [10] gave a further generalization of Theorem 3.7 as follows.

Theorem 3.8 (Enomoto [10]). *Let G be a 2-connected graph and y a vertex of G. Then G contains either a cycle of length at least $\sigma_2(G)$ containing y or a hamiltonian cycle.*

And, the following result is due to Bondy et al. [3], which is a weighted generalization of Theorem 3.7. Similar to the notation of σ_2, we denote

$$\sigma_2^w(G) = \min\{d^w(u) + d^w(v) \mid u \text{ and } v \text{ are nonadjacent}\},$$

and if G is complete, let $\sigma_2^w(G) = \infty$.

Theorem 3.9 (Bondy et al. [3]). *Let G be a 2-connected weighted graph. Then G contains either a cycle of weight at least $\sigma_2^w(G)$ or a hamiltonian cycle.*

In this section, we prove the following, which is a weighted generalization of Theorem 3.8. Clearly this also generalizes Theorem 3.9.

Theorem 3.10. *Let G be a 2-connected weighted graph and y a vertex of G. Then G contains either a cycle of weight at least $\sigma_2^w(G)$ containing y or a hamiltonian cycle.*

Now we prepare a lemma which is used in the proof of Theorem 3.10. Modifying the proof of Theorem 3.9, easily we can obtain the following. Let

$$\delta^w(G - C) = \min_{v \in V(G) \setminus V(C)} d_G^w(v).$$

Lemma 3.11. *If G is a 2-connected weighted graph, then there is a cycle C of weight at least $\max\{\sigma_2^w(G), 2(\sigma_2^w(G) - \delta^w(G - C))\}$ or a hamiltonian cycle.*

In our proof of Lemma 3.11, we use the following theorem, which is another version of Theorem 2.2.

Theorem 3.12 (Perfect [30]). *Let G be a k-connected graph, X, Z be disjoint subsets of $V(G)$ such that $|X|, |Z| \geq k$, and l be an integer with $l < k$. If $\mathcal{P}_1$ is a set of l (X, Z)-paths in G, then there exists a set of k (X, Z)-paths $\mathcal{P}_2$ such that $\mathcal{P}_1 \cap (X \cup Z) \subset \mathcal{P}_2 \cap (X \cup Z)$.*

Proof of Lemma 3.11. Let $P = u_1 u_2 \cdots u_p$ be a heaviest path in all longest paths in G. Let $e_l = u_{l-1}u_l$ and $e'_l = u_1 u_l$ for all $u_l \in N(u_1)$, and $f_l = u_l u_{l+1}$ and $f'_l = u_l u_p$ for all $u_l \in N(u_p)$. Suppose G is not hamiltonian. Then $\{u_l \mid u_{l+1} \in N(u_1)\} \cap N(u_p) = \emptyset$ and so $\{e_l \mid u_l \in N(u_1)\} \cap \{f_k \mid u_k \in N(u_p)\} = \emptyset$ as P is longest. Because the weight of P is at least the weights of the paths $P - e_l + e'_l$ and $P - f_l + f'_l$, we have $w(e_l) \geq w(e'_l)$ and $w(f_l) \geq w(f'_l)$.

Let $s = \max\{l \mid u_l \in N(u_1)\}$ and $t = \min\{l \mid u_l \in N(u_p)\}$. If $s > t$, then there exist $u_i \in N(u_1)$ and $u_j \in N(u_p)$ such that neither u_1 nor u_p has neighbors in $P(u_j, u_i)$ (See Figure 3.1). Then the cycle $C = u_1 P u_j u_p P u_i u_1$ contains every edge in:

$$\{e_l \mid u_l \in N(u_1) \setminus u_i\} \cup \{f_l \mid u_l \in N(u_p) \setminus u_j\} \cup \{e'_i, f'_j\} \tag{3.1}$$

and so $N(u_1) \cup N(u_p) \subset V(C)$. Therefore both of $d^w(u_1)$ and $d^w(u_p)$ are at least $\sigma_2^w(G) - \delta^w(G - C)$ and the following inequalities hold because $\{e_l\}_l \cap \{f_k\}_k = \emptyset$.

$$\begin{aligned} w(C) &\geq \sum_{u_l \in N(u_1) \setminus u_i} w(e_l) + w(e'_i) + \sum_{u_l \in N(u_p) \setminus u_j} w(f_l) + w(f'_j) \\ &\geq d^w(u_1) + d^w(u_p) \geq \max\{\sigma_2^w(G), 2(\sigma_2^w(G) - \delta^w(G - C))\}. \end{aligned} \tag{3.2}$$

If $s = t$, then there is a path Q joining $u_{i'} \in P(u_1, u_s)$ and $u_{j'} \in P(u_s, u_p)$ which is internally disjoint to P as G is 2-connected. Let $i = \min\{l > i' \mid u_l \in N(u_1)\}$ and $j = \max\{l < j' \mid u_l \in N(u_p)\}$ (See Figure 3.2). Then the cycle $C = u_1 P u_{i'} Q u_{j'} P u_p u_j P u_i u_1$ contains every edge in (3.1), and so the inequalities (3.2) hold.

Suppose $s < t$. By Theorem 3.12, there are two vertex disjoint paths Q_1 and Q_2 joining $P[u_1, u_s]$ and $P[u_t, u_p]$ such that u_s and u_t are ends of Q_1 or Q_2, and both of Q_1 and Q_2 are internally disjoint to $P[u_1, u_s] \cup P[u_t, u_p]$. Let $\{u_{i'}, u_s, u_t, u_{j'}\}$ be the set of all the ends of Q_1 and Q_2 such that $i' < s$ and $j' > t$. Let $i = \min\{l > i' \mid u_l \in N(u_1)\}$ and $j = \max\{l < j' \mid u_l \in N(u_p)\}$. Then the cycle

$$C = P[u_1, u_{i'}] \cup P[u_i, u_s] \cup P[u_t, u_j] \cup P[u_{j'}, u_p] \cup Q_1 \cup Q_2 \cup \{e'_i, f'_j\}$$

contains every edge in (3.1), and thus the inequalities (3.2) hold. □

Now we are ready to prove Theorem 3.10.

Proof of Theorem 3.10. Assume that G is not hamiltonian. Then by Lemma 3.11, there is a cycle C of weight at least $\max\{\sigma_2^w(G), 2(\sigma_2^w(G) - \delta^w(G - C))\}$. If $y \in C$, there is nothing to prove, so assume that $y \notin C$. Let $d = \delta^w(G - C)$. It follows from Theorem 2.1 that there

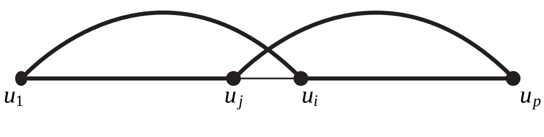

Figure 3.1:

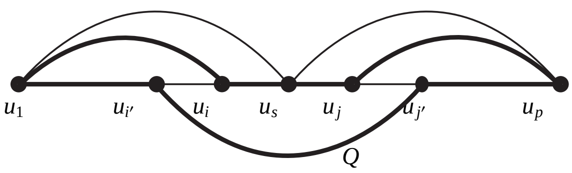

Figure 3.2:

is a (y, C)-fan F of weight $\geq d$. Now take C_i as in the proof of Theorem 3.1. Then, each C_i contains y and

$$\begin{aligned}\sum_{i=1}^{k} w(C_i) &= (k-1)w(C) + 2w(F) \\ &\geq (k-1)w(C) + 2d \\ &= (k-2)w(C) + w(C) + 2d \\ &\geq (k-2)\sigma_2^w(G) + 2(\sigma_2^w(G) - d) + 2d \\ &= k\sigma_2^w(G).\end{aligned}$$

Hence one of them is a cycle of weight at least $\sigma_2^w(G)$ containing y. □

Remark. Let $\delta(G) = \min_{v \in V(G)} d(v)$ and $\delta^w(G) = \min_{v \in V(G)} d^w(v)$. Zhu [37] showed that a 2-connected graph G contains a cycle of length at least $2(\sigma_2(G) - \delta(G))$ or a hamiltonian cycle. However, we can not give its weighted generalization. Let G be the complete bipartite graph $K_{k,k+1}$ with partite set V_1 of order k. Let $u \in V_1$, and we assign weight zero to every edge incident with u, and suppose other edges have weight one. Then $\sigma_2^w(G) = k + 1$ and $\delta^w(G) = 0$, and the weight of a heaviest cycle is $2k - 2 < 2(\sigma_2^w(G) - \delta^w(G))$, though G is not hamiltonian.

Chapter 4

Heavy cycles passing through some specified vertices in 3-connected weighted graphs

(This chapter is based on the paper [19].)

In Chapter 3, some theorems on heavy cycles passing through at most two vertices are shown. What happens when the number of specified vertices becomes 3? In weighted graphs of connectivity 2, there may be three vertices which cannot be contained in a common cycle. Let G_i be a 2-connected graph with $y_i \in V(G_i)$ for $i = 1$, 2 and 3. Consider a graph $G = (G_1 \cup G_2 \cup G_3) + K_2$, then G is 2-connected and there exists no cycle containing all of y_1, y_2 and y_3. Hence, to obtain the similar result to Theorem 3.5, we must enlarge the connectivity of the graphs. Now we prove that it is enough to enlarge the connectivity to 3, and no other extra-condition is necessary.

Theorem 4.1. *Let G be a 3-connected weighted graph and let d be a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then for any given three vertices y_1, y_2 and y_3 in G, either G has a cycle of weight at least $2d$ containing all of y_1, y_2 and y_3 or every heaviest cycle in G is a hamiltonian cycle.*

Theorem 4.1 is a weighted generalization of the following theorem in case of $k = 3$.

Theorem 4.2 (Egawa, Glas and Locke [9]). *Let G be a k-connected graph where $k \geq 2$, and let d be an integer. If $d(v) \geq d$ for every vertex v in G, then for any given vertex set Y with $|Y| = k$, there exists either a cycle of length at least $2d$ containing all the vertices of Y or a hamiltonian cycle.*

In our proof of Theorem 4.1, we call a cycle an *l-cycle* if it contains at least l vertices of $\{y_1, y_2, y_3\}$, where $1 \leq l \leq 3$.

Proof of Theorem 4.1. Assume the contrary. Then, by Theorem 3.5, there exists a 2-cycle of weight at least $2d$. Let C be a heaviest one among these cycles. Without loss of generality, we may assume that C contains y_1 and y_2. Since $w(C) \geq 2d$, $y_3 \notin V(C)$. By Theorem 2.1, we can find a (y_3, C)-fan F of width $k(G; y_3, C) \geq 3$ and weight at least d. Let $V(C) \cap V(F) = \{a_1, a_2, \ldots, a_p\}$ ($p \geq 3$). We may assume $a_1, a_2, \ldots, a_p$ appear in the consecutive order along C.

Claim 1. *There exists an index l with $1 \leq l \leq p$ such that $\{y_1, y_2\} \subseteq C(a_l, a_{l+1})$.*

Proof. Assume the contrary. Then for all i with $1 \leq i \leq p$, the cycle $a_iFa_{i+1}Ca_i$ is a 2-cycle, hence $w(F[a_ia_{i+1}]) \leq w(C[a_ia_{i+1}])$. Now, since $p \geq 3$, there exists j with $1 \leq j \leq p$ such that $V(C(a_ja_{j+1})) \cap \{y_1, y_2, y_3\} = \emptyset$. Hence $C' = a_jFa_{j+1}Ca_j$ is a 3-cycle and

$$
\begin{aligned}
w(C') &= w(F[a_j, a_{j+1}]) + \sum_{1 \leq i \leq p,\ i \neq j} w(C[a_i, a_{i+1}]) \\
&\geq \sum_{i=1}^{p} w(F[a_i, a_{i+1}]) \\
&\geq 2w(F) \\
&\geq 2d,
\end{aligned}
$$

which is a contradiction. □

Note that Claim 1 holds for every (y_3, C)-fan F of width $k(G; y, C)$ and weight at least d. Now, among such fans, take F_1 such that $C[v_l, v_{l+1}]$ is as short as possible. Without loss of generality, we may assume that $l = p$ and a_p, y_1, y_2, a_1 appear in the consecutive order along C. Note that $w(F_1[a_i, a_{i+1}]) \leq w(C[a_i, a_{i+1}])$ for all i with $1 \leq i \leq p - 1$, because the cycle $a_iF_1a_{i+1}Ca_i$ is a 2-cycle.

Claim 2. *$C[a_1, a_p]$ separates y_3 from $\{y_1, y_2\}$.*

Proof. Let H be a y_3-component of $G - C$. Assume that there exists $v \in C(a_p, a_1) \cap N(H)$. Let P be a (v, F_1)-path in $G[V(H) \cup \{v\}]$ and $V(P) \cap V(F_1) = \{v'\}$. Then, there exists j with $1 \leq j \leq p$ such that $v' \notin F_1(y_3, a_j]$. Now $F' = a_jF_1v'Pv$ is a (y_3, C)-fan, hence Theorem 2.1 shows that there exists a (y_3, C)-fan of width $k(G; y_3, C)$ and weight $\geq d$ which contains v and a_j. By Claim 1, we have $v \notin C[y_1, y_2]$. Without loss of generality, we may assume $v \in C[y_2, a_1)$. Now we have $v' \in F_1[y, a_p]$, for otherwise Theorem 2.1 shows that there exists

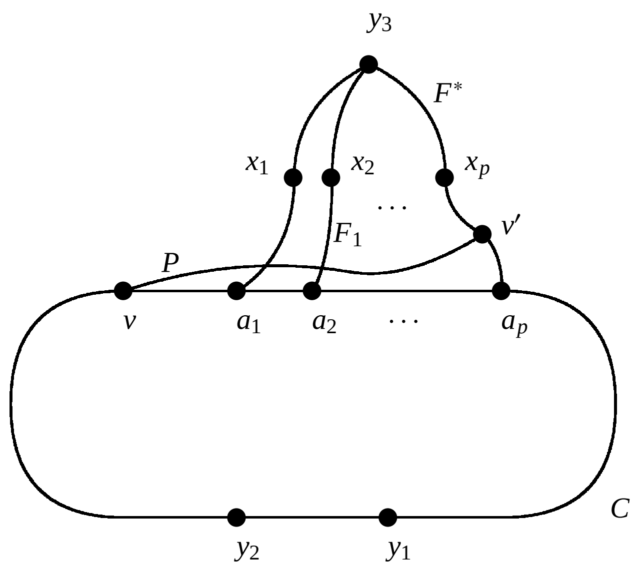

Figure 4.1:

a (y_3, C)-fan F' of width $k(G; y_3, C)$ and weight $\geq d$ such that $v, a_p \in F'$, which contradicts the choice of F_1.

Since $p = k(G; y_3, C)$, there exists a vertex set X in $V(H) \cup N_C(H) \setminus \{y_3\}$ such that $|X| = p$ and X separates y_3 from C. Note that there is one vertex of $F[a_i, y_3] \cap X$ for each i with $1 \leq i \leq p$. Let x_i be such a vertex. Since X separates y_3 from C, we have $x_p \in F[v', y_3]$.

Now Theorem 2.1 shows the existence of (y_3, X)-fan F^* of width $k(G; y_3, X) = p$ and $w(F^*) \geq d$ (See Figure 4.1). We have $w(C[a_i, a_{i+1}]) \geq w(F^*[x_i, x_{i+1}])$ for every i with $1 \leq i \leq p-1$, since otherwise $a_iF_1x_iF^*x_{i+1}F_1a_{i+1}Ca_i$ is a 2-cycle heavier than C. Let $C' = vPv'F_1x_pF^*x_1F_1a_1Cv$. Then C' is a 3-cycle and

$$\begin{aligned}
w(C') &= w(vPv'F_1x_pF^*y_3) + w(y_3F^*x_1F_1a_1) \\
&\quad + \sum_{i=1}^{p-1} w(C[a_i, a_{i+1}]) + w(C[a_p, v]) \\
&\geq w(x_pF^*y_3) + w(y_3F^*x_1) + \sum_{i=1}^{p-1} w(F^*[x_i, x_{i+1}]) \\
&\geq 2w(F^*) \\
&\geq 2d,
\end{aligned}$$

which is a contradiction. Hence we have $N_C(H) \subseteq C[a_1, a_p]$, which implies the assertion. □

Claim 3. $w(C[a_p, a_1]) < w(F_1[a_p, a_1])$.

Proof. Let $C' = a_1F_1a_2Ca_1$. Since $w(C')$ is a 3-cycle, $w(C') < 2d$. Hence

$$\begin{aligned} w(C[a_p, a_1]) &= w(C') - (w(F_1[a_1, a_2]) + w(C[a_2, a_p])) \\ &< 2d - (w(F_1[a_1, a_2]) + \sum_{i=2}^{p-1} w(F_1[a_i, a_{i+1}])) \\ &= 2d - (2w(F_1) - w(F_1[a_p, a_1])) \\ &\leq w(F_1[a_p, a_1]). \end{aligned}$$

□

Claim 4. *For any* 2*-cycle* D*,* $w(D) < w(C[a_1, a_p]) + w(F_1[a_p, a_1])$.

Proof. Claim 3 shows $w(C) < w(C[a_1, a_p]) + w(F_1[a_p, a_1])$. By the choice of C, we have $w(D) \leq w(C)$, which implies the assertion. □

Claim 5. *Let* v_1, v_2 *be two vertices in* $C[a_1, a_p]$ *such that* a_1, v_1, v_2, a_p *appear in the consecutive order along* C*. If* P *is a* (v_1, v_2)*-path which is internally disjoint with* $C[a_1, a_p]$ *and* $V(P) \cap \{y_1, y_2\} \neq \emptyset$*, then* $w(P) \leq w(C[v_1, v_2])$.

Proof. Let $C' = v_1Pv_2Ca_pF_1a_1Cv_1$. Since C' is a 2-cycle, by Claim 4, $w(C') < w(C[a_1, a_p]) + w(F_1[a_p, a_1])$. Hence $w(P) \leq w(C[v_1, v_2])$. □

Claim 6. *Let* v_1, v_2 *be two vertices in* $C[a_1, a_p]$ *such that* a_1, v_1, v_2, a_p *appear in the consecutive order along* C*. Let* P *be a* (v_1, v_2)*-path which is internally disjoint with* C *with* $\{y_1, y_2\} \subset V(P)$*, and* $P' = v_2Ca_pFa_1Cv_1$*. Then* $w(P) < w(P')$.

Proof. Let $C' = v_1Cv_2Pv_1$, then C' is a 2-cycle. Hence Claim 4 shows that $w(C') < w(C[a_1, a_p]) + w(F_1[a_p, a_1])$, which implies the assertion. □

Note that $C[a_p, a_1]$ is a $(y_1, C[a_1, a_p])$-fan which includes a_1 and a_p. Hence, by Theorem 2.1, there exists a $(y_1, C[a_1, a_p])$-fan F_2 such that $w(F_2) \geq d$ and $a_1, a_p \in F_2$. Note that Claim 2 implies that $V(F_1) \cap V(F_2) \subseteq C[a_1, a_p]$.

Case 1. $w(F_1[a_1, a_p]) \leq w(F_2[a_1, a_p])$.

Let P be a path which satisfies the followings;

- P is an (a_1, a_p)-path which is internally disjoint with $F_1 \cup C[a_1, a_p]$, and
- $V(P) \cap \{y_1, y_2\} \neq \emptyset$.

We may assume that such a path P was chosen so that $w(P)$ is as large as possible, and without loss of generality, we may also assume that $y_1 \in V(P)$. Let $C' = a_1F_1a_2Ca_pPa_1$. Then $y_1, y_3 \in C'$ and

$$\begin{aligned} w(C') &= w(F_1[a_1, a_2]) + w(C[a_2, a_p]) + w(P) \\ &\geq w(F_1[a_1, a_2]) + \sum_{i=2}^{p-1} w(F_1[a_i, a_{i+1}]) + w(P) \\ &\geq 2w(F_1) - w(F_1[a_1, a_p]) + w(F_2[a_p, a_1]) \\ &\geq 2w(F_1) \\ &\geq 2d, \end{aligned}$$

hence $y_2 \notin P$. Since $C[y_1, y_2]$ is a path disjoint with $C[a_1, a_p]$, Theorem 2.1 shows the existence of $(y_2, P \cup C[a_1, a_p])$-fan F_3 of weight at least d, width at least 3 and $F_3 \cap P \neq \emptyset$. By symmetry, we may assume that $F_3 \cap P(a_1, y_1] \neq \emptyset$. Note that Claim 2 implies $V(F_1) \cap V(F_3) \subseteq C[a_1, a_p]$. And if there exists two distinct vertices $u, v \in P \cap F_3$, then by the choice of P, $w(P[u, v]) \geq w(F_3[u, v])$. Now we assume that P has the orientation from a_1 to a_p.

Case 1.1. $F_3 \cap C(a_1, a_p) \neq \emptyset$.

Let $C[a_1, a_p) \cap F_3 = \{b_1, b_2, \ldots, b_l\}$ and $P(a_1, a_p] \cap F_3 = \{b_{l+1}, b_{l+2}, \ldots, b_m\}$. We may assume $b_1, b_2, \ldots, b_l$ and $b_{l+1}, b_{l+2}, \ldots, b_m$ appear in the consecutive order along C and P, respectively (See Figure 4.2). Now we consider three paths $P_1 = b_1Cb_lF_3y_2$, $P_2 = y_2F_3b_{l+1}Pa_p$ and $P_3 = a_pF_1a_1Cb_1$. Then by Claim 5,

$$\begin{aligned} w(P_1) &= \sum_{i=1}^{l-1} w(C[b_i, b_{i+1}]) + w(F_3[b_l, y_2]) \\ &\geq \sum_{i=1}^{l-1} w(F_3[b_i, b_{i+1}]) + w(F_3[b_l, y_2]) \\ &= \sum_{i=1}^{l} 2w(F_3[b_i, y_2]) - w(F_3[b_1, y_2]), \end{aligned}$$

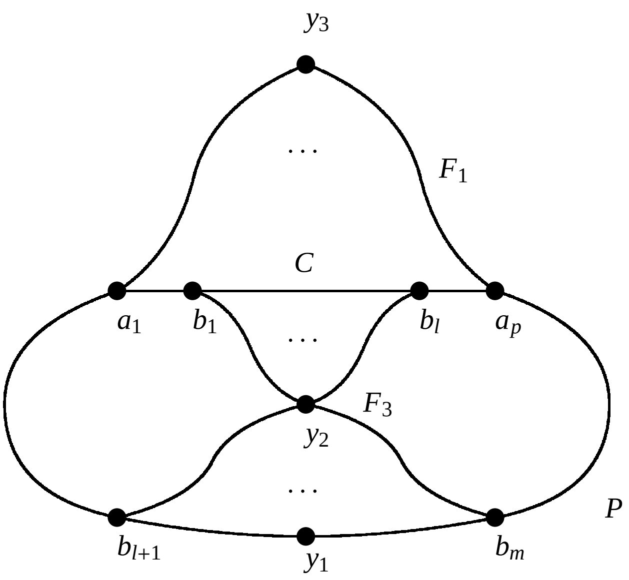

Figure 4.2:

and by the maximality of P,

$$\begin{aligned} w(P_2) &\geq w(F_3[y_2, b_{l+1}]) + \sum_{i=l+1}^{m-1} w(P[b_i, b_{i+1}]) \\ &\geq w(F_3[y_2, b_{l+1}]) + \sum_{i=l+1}^{m-1} w(F_3[b_i, b_{i+1}]) \\ &= \sum_{i=l+1}^{m} 2w(F_3[b_i, y_2]) - w(F_3[b_m, y_2]). \end{aligned}$$

Moreover, let $P' = b_1 F_3 y_2 P_2 a_p$. Then $y_1, y_2 \in P'$, hence by Claim 6,

$$\begin{aligned} w(P_3) &\geq w(P') \\ &\geq w(F_3[b_1, y_2]) + w(P_2) \\ &\geq w(F_3[b_1, y_2]) + w(F_3[b_m, y_2]). \end{aligned}$$

Now $b_1P_1y_2P_2a_pP_3b_1$ is a 3-cycle of weight

$$\begin{aligned} w(P_1)+w(P_2)+w(P_3) &\geq \sum_{i=1}^{l} 2w(F_3[b_i,y_2]) - w(F_3[b_1,y_2]) \\ &\quad + \sum_{i=l+1}^{m} 2w(F_3[b_i,y_2]) - w(F_3[b_m,y_2]) \\ &\quad + w(F_3[b_1,y_2]) + w(F_3[b_m,y_2]) \\ &= 2w(F_3) \\ &\geq 2d, \end{aligned}$$

a contradiction. □

Case 1.2. $F_3 \cap C(a_1,a_p) = \emptyset$.

Let $P \cap F_3 = \{b_1, b_2, \ldots, b_m\}$. We may assume $b_1, b_2, \ldots, b_m$ appear in the consecutive order along P. Since $m \geq 3$, there exists l with $1 \leq l \leq m-1$ such that $y_1 \notin P(b_l, b_{l+1})$. Now we consider two paths $P_1 = a_1Pb_lF_3b_{l+1}Pa_p$ and $P_2 = a_1F_1a_2Ca_p$. Then

$$\begin{aligned} w(P_1) &\geq \sum_{1\leq i\leq m-1,\ i\neq l} w(P[b_i,b_{i+1}]) + w(F_3[b_l,b_{l+1}]) \\ &\geq \sum_{1\leq i\leq m-1,\ i\neq l} w(F_3[b_i,b_{i+1}]) + w(F_3[b_l,b_{l+1}]) \\ &\geq w(F_3) \\ &\geq d \end{aligned}$$

and

$$\begin{aligned} w(P_2) &\geq w(F_1[a_1,a_2]) + \sum_{i=2}^{p-1} w(C[a_i,a_{i+1}]) \\ &\geq w(F_1[a_1,a_2]) + \sum_{i=2}^{p-1} w(F_1[a_i,a_{i+1}]) \\ &\geq w(F_1) \\ &\geq d. \end{aligned}$$

Hence $a_1P_1a_pP_2a_1$ is a 3-cycle of weight $w(P_1) + w(P_2) \geq 2d$, which is a contradiction. □

Case 2. $w(F_1[a_1,a_p]) > w(F_2[a_1,a_p])$.

Let $V(C[a_1, a_p]) \cap V(F_2) = \{a'_1, a'_2, \ldots, a'_q\}$. We may assume that $a'_1, a'_2, \ldots, a'_q$ appear in the consecutive order along C. Note that $a'_1 = a_1$ and $a'_q = a_p$. Now let $P = a'_1 F_2 a'_2 C a_p$ and consider a cycle $C' = a'_1 P a_p F_1 a_1$. Then by Claim 5,

$$\begin{aligned} w(C') &= w(F_2[a'_1, a'_2]) + w(C[a'_2, a_p]) + w(F_1[a_p, a_1]) \\ &\geq w(F_2[a'_1, a'_2]) + \sum_{i=2}^{q-1} w(F_2[a'_i, a'_{i+1}]) + w(F_1[a_p, a_1]) \\ &\geq 2w(F_2) - w(F_2[a'_1, a'_p]) + w(F_1[a_p, a_1]) \\ &\geq 2w(F_2) \\ &= 2d. \end{aligned}$$

Now let Q be an (a_1, a_p)-path such that $Q = Q_1 \cup Q_2$, where

- Q_1 is an (a_1, t)-path with $t \in C(a_1, a_p]$, which is internally disjoint with $F_1 \cup C[a_1, a_p]$ and $Q_1 \cap \{y_1, y_2\} \neq \emptyset$, and
- $Q_2 = C[t, a_p]$.

Note that such Q exists since P satisfies the above conditions. Take Q so that $w(Q)$ is as large as possible, and assume that Q has an orientation from a_1 to a_p. Now consider a cycle $C^* = a_1 Q a_p F_1 a_1$, then $w(C^*) \geq w(C') \geq 2d$ and $y_3 \in C^*$. Hence $\{y_1, y_2\} \not\subseteq V(Q)$. Without loss of generality, we may assume that $y_1 \in Q$ and $y_2 \notin Q$. Since $C[y_1, y_2]$ is a path disjoint with $C[a_1, a_p]$, Theorem 2.1 shows that there exists a $(y_2, Q \cup C[a_1, t])$-fan F_4 of weight at least d, width at least 3 and $F_4 \cap Q(a_1, t) \neq \emptyset$.

Now assume that there exists a vertex $s_1 \in F_4 \cap C(a_1, t)$. If there exists $s_2 \in F_4 \cap Q(a_1, y_1]$, then $\tilde{C} = a_1 C s_1 F_4 s_2 Q a_p F_1 a_1$ is a 3-cycle and, by Claim 5,

$$\begin{aligned} w(\tilde{C}) &= w(C[a_1, s_1]) + w(F_4[s_1, s_2]) + w(Q[s_2, t]) \\ &\quad + w(Q[t, a_p]) + w(F_1[a_p, a_1]) \\ &\geq w(Q[a_1, s_2]) + w(F_4[s_2, s_1]) + w(F_4[s_1, s_2]) \\ &\quad + w(Q[s_2, t]) + w(Q[t, a_p]) + w(F_1[a_p, a_1]) \\ &\geq w(Q) + w(F_1[a_p, a_1]) \\ &\geq w(C^*) \\ &\geq 2d, \end{aligned}$$

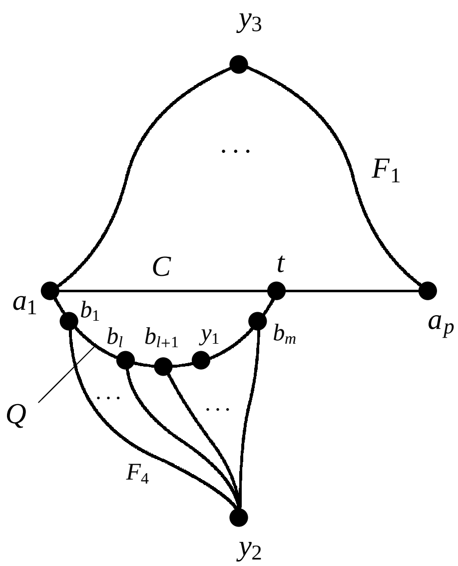

Figure 4.3:

which is a contradiction. Otherwise, there exists $s_2 \in F_4 \cap Q(y_1, t)$. Then $\hat{C} = a_1Qs_2F_4s_1Ca_pF_1a_1$ is a 3-cycle and Claim 5 implies that

$$\begin{aligned} w(\hat{C}) &= w(Q[a_1, s_2]) + w(F_4[s_2, s_1]) + w(C[s_1, a_p]) + w(F_1[a_p, a_1]) \\ &\geq w(Q[a_1, s_2]) + w(F_4[s_2, s_1]) + w(F_4[s_1, s_2]) \\ &\quad + w(Q[s_2, t]) + w(Q[t, a_p]) + w(F_1[a_p, a_1]) \\ &\geq w(Q) + w(F_1[a_p, a_1]) \\ &\geq w(C^*) \\ &\geq 2d, \end{aligned}$$

which is a contradiction. Hence we have $F_4 \cap C(a_1, t) = \emptyset$, which shows $F_4 \cap (Q \cup C[a_1, t]) \subseteq Q$.

Let $F_4 \cap Q = \{b_1, b_2, \ldots, b_m\}$. We may assume that $b_1, b_2, \ldots, b_m$ appear in the consecutive order along Q. It follows from the choice of Q and Claim 5 that $w(Q[b_i, b_j]) \geq w(F_4[b_i, b_j])$ for every i, j with $1 \leq i < j \leq m$.

Case 2.1. $F_4 \cap Q(t, a_p] = \emptyset$.

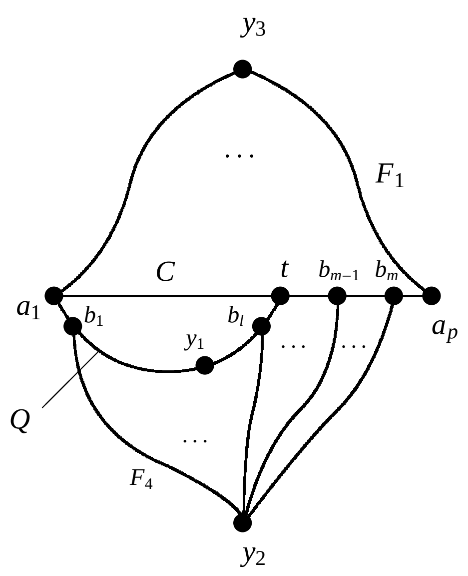

Figure 4.4:

Since $m \geq 3$, there exists l with $1 \leq l \leq m-1$ such that $y_1 \notin Q(b_l, b_{l+1})$ (See Figure 4.3). Now consider two paths $Q_1 = a_1 Q b_l F_4 b_{l+1} Q t$ and $Q_2 = t Q a_p F_1 a_1$. Then

$$\begin{aligned} w(Q_1) &\geq \sum_{1 \leq i \leq m-1,\ i \neq l} w(Q[b_i, b_{i+1}]) + w(F_4[b_l, b_{l+1}]) \\ &\geq \sum_{1 \leq i \leq m-1,\ i \neq l} w(F_4[b_i, b_{i+1}]) + w(F_4[b_l, b_{l+1}]) \\ &\geq w(F_4) \\ &\geq d. \end{aligned}$$

And, by Claim 6 and the fact that $y_1, y_2 \in Q_1$,

$$w(Q_2) \geq w(Q_1) \geq d.$$

Hence $a_1 Q_1 t Q_2 a_1$ is a 3-cycle of weight $w(Q_1) + w(Q_2) \geq 2d$, which is a contradiction. □

Case 2.2. $F_4 \cap Q(t, a_p] \neq \emptyset$ and $P[y_1, t) \cap F_4 \neq \emptyset$.

Let $b_l \in Q[y_1, t) \cap F_4$ and consider three paths $Q_1 = a_1 Q b_{m-1} F_4 b_m$, $Q_2 = a_1 Q b_l F_4 b_m$ and $Q_3 = b_m Q a_p F_1 a_1$ (See Figure 4.4). Note that both of Q_1, Q_2 contains y_1 and y_2. Now we have

$$\begin{aligned} w(Q_1) &\geq \sum_{i=1}^{m-2} w(Q[b_i, b_{i+1}]) + w(F_4[b_{m-1}, b_m]) \\ &\geq \sum_{i=1}^{m-2} w(F_4[b_i, b_{i+1}]) + w(F_4[b_{m-1}, b_m]) \\ &\geq 2w(F_4) - w(F_4[b_1, b_m]) \end{aligned}$$

and

$$\begin{aligned} w(Q_2) &\geq \sum_{i=1}^{l-1} w(Q[b_i, b_{i+1}]) + w(F_4[b_l, b_m]) \\ &\geq \sum_{i=1}^{l-1} w(F_4[b_i, b_{i+1}]) + w(F_4[b_l, b_m]) \\ &\geq w(F_4[b_1, b_m]). \end{aligned}$$

Moreover, by Claim 6, we have $w(Q_3) \geq w(Q_2)$. Hence $a_1Q_1b_mQ_3a_1$ is a 3-cycle of weight $w(Q_1) + w(Q_2) \geq 2w(F_4) \geq 2d$, a contradiction. □

Case 2.3. $F_4 \cap Q(t, a_p] \neq \emptyset$ and $P[y_1, t) \cap F_4 = \emptyset$.

Note that $Q(a_1, y_1) \cap F_4 \neq \emptyset$ in this case. Let l, l' be integers with $1 \leq l, l' \leq m$ such that $b_l \in Q(a_1, y_1]$, $b_{l+1} \notin Q(a_1, y_1]$, $b_{l'} \in Q(t, a_p]$ and $b_{l'-1} \notin Q(t, a_p)$ (See Figure 4.5). Now consider three cycles $C_1 = a_1CtQb_lF_4b_{l'}Qa_pF_1a_1$, $C_2 = b_1Qb_mF_4b_1$ and $C_3 = a_1Ca_pF_1a_1$. Note that C_1 is a 3-cycle and C_2 is a 2-cycle. By Claim 4, we have $w(C_2) \leq w(C_3)$. Hence,

$$\begin{aligned} w(C_1) &\geq w(C_1) + w(C_2) - w(C_3) \\ &= w(C[a_1, t]) + w(Q[t, b_l]) + w(F_4[b_l, b_{l'}]) + w(Q[b_{l'}, a_p]) + w(F_1[a_p, a_1]) \\ &\quad + w(Q[b_1, b_m]) + w(F_4[b_m, b_1]) - w(C[a_1, a_p]) - w(F_1[a_p, a_1]) \\ &= w(C[a_1, t]) + w(Q[t, b_l]) + w(F_4[b_l, b_{l'}]) + w(Q[b_{l'}, a_p]) + w(F_1[a_p, a_1]) \\ &\quad + w(Q[b_1, b_l]) + w(Q[b_l, t]) + w(Q[t, b_{l'}]) + w(Q[b_{l'}, b_m]) + w(F_4[b_m, b_1]) \\ &\quad - w(C[a_1, t]) - w(C[t, b_{l'}]) - w(C[b_{l'}, a_p]) - w(F_1[a_p, a_1]) \\ &= w(Q[t, b_l]) + w(F_4[b_l, b_{l'}]) + w(Q[b_1, b_l]) + w(Q[b_l, t]) + w(Q[b_{l'}, b_m]) \\ &\quad + w(F_4[b_m, b_1]) \\ &= w(Q[b_1, b_l]) + w(F_4[b_1, b_l]) + 2w(Q[b_l, t]) + w(Q[b_{l'}, b_m]) + w(F_4[b_{l'}, b_m]). \end{aligned}$$

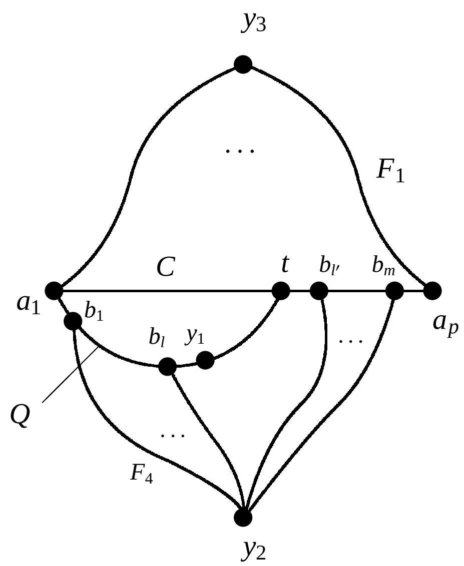

Figure 4.5:

In case of $t \notin F_4$, $l' = l + 1$. Hence,

$$\begin{aligned} & w(Q[b_1, b_l]) + w(F_4[b_1, b_l]) + 2w(Q[b_l, t]) + w(Q[b_{l'}, b_m]) + w(F_4[b_{l'}, b_m]) \\ \geq & \sum_{i=1}^{l-1} w(F_4[b_i, b_{i+1}]) + w(F_4[b_1, b_l]) + \sum_{i=l+1}^{m-1} w(F_4[b_i, b_{i+1}]) + w(F_4[b_{l+1}, b_m]) \\ \geq & 2w(F_4) \\ \geq & 2d, \end{aligned}$$

a contradiction. Otherwise, $l' = l + 2$ and $t = b_{l+1}$. Hence,

$$\begin{aligned} & w(Q[b_1, b_l]) + w(F_4[b_1, b_l]) + 2w(Q[b_l, t]) + w(Q[b_{l'}, b_m]) + w(F_4[b_{l'}, b_m]) \\ \geq & \sum_{i=1}^{l-1} w(F_4[b_i, b_{i+1}]) + w(F_4[b_1, b_l]) + 2w(F_4[b_l, b_{l+1}]) \\ & + \sum_{i=l+2}^{m-1} w(F_4[b_i, b_{i+1}]) + w(F_4[b_{l+2}, b_m]) \\ \geq & 2w(F_4) \\ \geq & 2d, \end{aligned}$$

a contradiction. This completes the proof of Theorem 4.1.

Chapter 5

Heavy cycles in triangle-free weighted graphs

(This chapter is based on the paper [20].)

Again, Bondy and Fan proved the following theorem in [4].

Theorem 5.1 (Bondy and Fan [4]). *Let G be a 2-connected weighted graph and let d be a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then*

(a) G has a cycle of weight at least $2d$, or

(b) every heaviest cycle in G is a hamiltonian cycle.

If we consider the weighted complete graph in which every edge has weight 1, we know that conclusion (b) of Theorem 5.1 cannot be dropped. However, there are a lot of graphs in which both (a) and (b) of Theorem 5.1 hold. In such weighted graphs, though it contains a cycle of weight at least $2d$, we cannot guarantee the weight of a heaviest cycle of a graph by this theorem. In this chapter, we prove the following theorem, by which we can always find a heavy cycle. A *triangle-free* graph is one which contains no cycle of length 3.

Theorem 5.2. *Let G be a 2-connected triangle-free weighted graph and let d be a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then G has a cycle of weight at least $2d$.*

In our proof of Theorem 5.2, we call a path P a *longest heaviest path* of G if

(i) $w(P)$ is maximum, and

(ii) P is a longest path of G subject to (i).

Now we prepare the following lemma.

Lemma 5.3. *Let G be a weighted graph and let P be a longest heaviest path of G with endvertices x and y. Assume that*

$$d(x) + d(y) - \varepsilon(xy) \leq |E(P)|,$$

where

$$\varepsilon(xy) = \begin{cases} 0 & \text{if } xy \notin E(G) \\ 1 & \text{if } xy \in E(G). \end{cases}$$

Then

- *if $xy \notin E(G)$, then P has weight at least $d^w_G(x) + d^w_G(y)$, and*
- *if $xy \in E(G)$, then the cycle $xPyx$ has weight at least $d^w_G(x) + d^w_G(y)$.*

Proof. Let $P = a_1a_2 \ldots a_p$ be a longest heaviest path of G where $a_1 = x$ and $a_p = y$. Then we have $N(a_1) \subseteq V(P)$ and $N(a_p) \subseteq V(P)$. Let

- $N_1 = \{a_i \mid a_i \in N_G(a_1),\ a_{i-1} \notin N_{G-a_1}(a_p)\}$,
- $N_2 = \{a_i \mid a_i \in N_G(a_1),\ a_{i-1} \in N_{G-a_1}(a_p)\}$,
- $N_3 = \{a_i \mid a_i \in N_{G-a_1}(a_p),\ a_{i+1} \notin N_G(a_1)\}$ and
- $N_4 = \{a_i \mid a_i \in N_{G-a_1}(a_p),\ a_{i+1} \in N_G(a_1)\}$.

Moreover, let

$$E_1 = \{a_1v \mid v \in N_1\},\ E_2 = \{a_1v \mid v \in N_2\},\ E_3 = \{va_p \mid v \in N_3\} \text{ and } E_4 = \{va_p \mid v \in N_4\}.$$

Now we define a mapping φ_1 of $\bigcup_{i=1}^{3} E_i$ to $E(P)$ such that

- for $e = a_1a_i \in E_1 \cup E_2$, $\varphi_1(e) = a_{i-1}a_i$ and
- for $e = a_ia_p \in E_3$, $\varphi_1(e) = a_ia_{i+1}$,

and let $F_i = \{\varphi_1(e) \mid e \in E_i\}$ for $i = 1, 2, 3$. Now it is easy to see that $F_1 \cap F_2 = \emptyset$. And by the definition of E_3, $a_{i+1} \notin N(a_1)$ if $a_ia_p \in E_3$, hence F_1, F_2 and F_3 are disjoint.

It follows from the fact $d(x) + d(y) - \varepsilon(xy) \leq |E(P)|$ that

$$\begin{aligned}\sum_{i=1}^{3} |F_i| &= |E_1| + |E_2| + |E_3| \\ &= |N_1| + |N_2| + |N_3| \\ &= |N(a_1)| + |N(a_p) \setminus \{a_1\}| - |N_4| \\ &\leq |E(P)| - |N_4| \\ &= |E(P)| - |E_4|.\end{aligned}$$

Thus $|E(P) \setminus \bigcup_{i=1}^{3} F_i| \geq |E_4|$. Let φ_2 be an injection of E_4 to $E(P) \setminus \bigcup_{i=1}^{3} F_i$ and let $F_4 = \{\varphi_2(e) \mid e \in E_4\}$. Note that F_1, F_2, F_3 and F_4 are disjoint.

Assume that $a_1a_i \in E_1$ and $Q_1 = a_{i-1}a_{i-2} \ldots a_1a_ia_{i+1} \ldots a_p$. Then, since $w(Q_1) \leq w(P)$, $w(a_1a_i) \leq w(\varphi_1(a_1a_i))$. By the similar argument as above, we have $w(e) \leq w(\varphi_1(e))$ for all $e \in E_1 \cup E_3$. Suppose $a_j \in N_2$. Then we have $a_{j-1}a_p \in E_4$. Let C be a cycle $a_ja_1a_2 \ldots a_{j-1}a_pa_{p-1} \ldots a_j$ and $e = \varphi_2(a_{j-1}a_p)$. Since $e \in E(C)$, $Q_2 = C - \{e\}$ is a path in G. Then it follows from the fact $w(Q_2) \leq w(P)$ that $w(a_1a_j) + w(a_{j-1}a_p) \leq w(\varphi_1(a_1a_j)) + w(\varphi_2(a_{j-1}a_p))$ for all $a_j \in N_2$. Therefore, if $a_1a_p \notin E(G)$,

$$\begin{aligned}d^w(a_1) + d^w(a_p) &= \sum_{v\in N_1} w(a_1v) + \sum_{v\in N_2} w(a_1v) + \sum_{v\in N_3} w(va_p) + \sum_{v\in N_4} w(va_p) \\ &= \sum_{e\in E_1} w(e) + \sum_{e\in E_3} w(e) + \sum_{a_j\in N_2} (w(a_1a_j) + w(a_{j-1}a_p)) \\ &\leq \sum_{e\in F_1} w(e) + \sum_{e\in F_3} w(e) + \sum_{e\in F_2} w(e) + \sum_{e\in F_4} w(e) \\ &\leq w(P),\end{aligned}$$

which implies the assertion. And in case of $a_1a_p \in E(G)$,

$$\begin{aligned}d^w(a_1) + d^w(a_p) &= \sum_{v\in N_1} w(a_1v) + \sum_{v\in N_2} w(a_1v) + \sum_{v\in N_3} w(va_p) + \sum_{v\in N_4} w(va_p) + w(a_1a_p) \\ &= \sum_{e\in E_1} w(e) + \sum_{e\in E_3} w(e) + \sum_{a_j\in N_2} (w(a_1a_j) + w(a_{j-1}a_p)) + w(a_1a_p) \\ &\leq \sum_{e\in F_1} w(e) + \sum_{e\in F_3} w(e) + \sum_{e\in F_2} w(e) + \sum_{e\in F_4} w(e) + w(a_1a_p) \\ &\leq w(P) + w(a_1a_p).\end{aligned}$$

Hence the cycle $a_1Pa_pa_1$ has weight at least $d^w(a_1) + d^w(a_p)$, which implies the assertion. □

Now we prove Theorem 5.2 by using Lemma 5.3 and the following lemma.

Lemma 5.4 (Bondy and Fan [5]). *Let G be a 2-connected weighted graph and let P be a heaviest path in G with endvertices x and y. Then there exists a cycle C in G such that $w(C) > w(P)$ or $w(C) \geq d^w(x) + d^w(y)$.*

Proof of Theorem 5.2. Let P be a longest heaviest path in G, and let x, y be endvertices of P. Since G is triangle-free and $N(x), N(y) \subseteq V(P)$, $|N(x)| \leq |V(P)|/2$ and $|N(y)| \leq |V(P)|/2$. Moreover, if $xy \notin E(G)$, $|N(x)| \leq (|V(P)| - 1)/2$ and $|N(y)| \leq (|V(P)| - 1)/2$. Hence, whether x and y are adjacent or not, we have $d(x) + d(y) - \varepsilon(xy) \leq |E(P)|$. In case of $xy \in E(G)$, Lemma 5.3 implies the existence of a cycle of weight at least $d^w(x) + d^w(y) \geq 2d$, which is a required cycle. Thus we may assume $xy \notin E(G)$, then Lemma 5.3 implies that $w(P) \geq d^w(x) + d^w(y) \geq 2d$. Now it follows from Lemma 5.4 that there exists a cycle C in G such that $w(C) > w(P) \geq 2d$ or $w(C) \geq d^w(x) + d^w(y) \geq 2d$, which is a required cycle. □

Chapter 6

Claw conditions for heavy cycles in weighted graphs

(This chapter is based on the paper [18].)

6.1 Fan-type condition and Claw conditions

About the existence of long cycles in unweighted graphs, Fan introduced weaker degree condition than Ore-type one. The *distance* of two vertices u and v is a minimal length of (u, v)-paths (if there is no such path, then we define the distance as ∞), and we will denote it by $d(u, v)$. Fan's theorem is the following.

Theorem 6.1 (Fan [16]). *Let G be a 2-connected graph. If $\max\{d(u), d(v)\} \geq c/2$ for each pair of vertices u and v in $V(G)$ such that $d(u, v) = 2$, then G contains either a hamiltonian cycle or a cycle of length at least c.*

And, Theorem 6.1 is weakened as the following theorem. We call the graph $K_{1,3}$ a *claw*, and the graph $K_{1,3} + e$ (e is an edge) a *modified claw*. A modified claw can also be described as the graph obtained by joining a pendant edge to some vertex of a K_3.

Theorem 6.2 (Bedrossian et al. [1]). *Let G be a 2-connected graph. If $\max\{d(u), d(v)\} \geq c/2$ for each pair of non-adjacent vertices u and v, which are vertices of an induced claw of G or an induced modified claw of G, then G contains either a hamiltonian cycle or a cycle of length at least c.*

In this chapter, we discuss about two weighted degree conditions of the same type as above two theorems. To extend Theorem 6.1 to the weighted graphs, the following problem may naturally be considered.

Problem 6.3. *Let G be a 2-connected weighted graph. If $\max\{d^w(u), d^w(v)\} \geq c/2$ for each pair of vertices u and v in $V(G)$ such that $d(u, v) = 2$, does G contain either a hamiltonian cycle or a cycle of weight at least c?*

However, in [34], Zhang et al. gave the negative answer to Problem 6.3, and alternatively they suggested the following problem.

Problem 6.4. *Let G be a 3-connected weighted graph. If $\max\{d^w(u), d^w(v)\} \geq c/2$ for each pair of vertices u and v in $V(G)$ such that $d(u, v) = 2$, does G contain either a hamiltonian cycle or a cycle of weight at least c?*

Enomoto [11] proved that the answer to Problem 6.4 is also negative, even if we enlarge the connectivity of G more than 3. Let k, l and m be integers satisfying $k \geq 3$, $l \geq k+1$, $m > k^2 - k$ and $kl - 1 \geq m$. Let $V_x = \{x_i \mid 1 \leq i \leq k\}$, $V_y = \{y_{i,j} \mid 1 \leq i \leq l,\ 1 \leq j \leq k\}$, $V_z = \{z_{i,j} \mid 1 \leq i \leq l,\ 1 \leq j \leq m\}$, $E_x = \{uv \mid u, v \in V_x\}$, $E_{xy} = \{uv \mid u \in V_x \textit{ and } v \in V_y\}$, $E_{yz} = \{y_{i,j}z_{i,j'} \mid 1 \leq i \leq l,\ 1 \leq j \leq k,\ 1 \leq j' \leq m\}$, and $E_z = \{z_{i,j}z_{i,j'} \mid 1 \leq i \leq l,\ 1 \leq j < j' \leq m\}$. Now we consider a graph G with $V(G) = V_x \cup V_y \cup V_z$ and $E(G) = E_x \cup E_{xy} \cup E_{yz} \cup E_z$, then G is a k-connected non-hamiltonian graph (See Figure 6.1). However, if we assign weight $r > 0$ to the edges in $E_x \cup E_{xy} \cup E_{yz}$ and weight 0 to the edges in E_z, G satisfies the condition of Problem 6.4 with $c = (m+k) \cdot r$, though the weight of a heaviest cycle of G is $2k^2 \cdot r < 2c$.

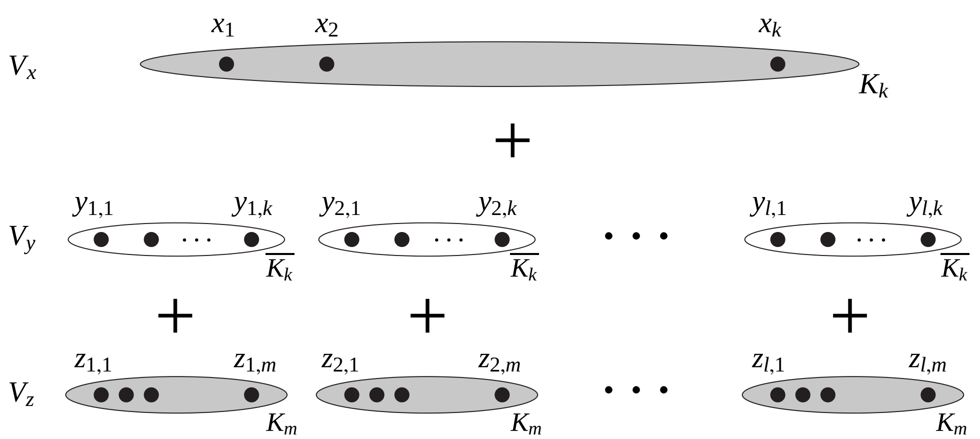

Figure 6.1:

Hence, to obtain a positive consequence to Problem 6.3, it is no use to add the condition of the connectivity of graphs, and we must add the other condition. In [34], the following theorem is shown.

Theorem 6.5 (Zhang et al. [34]). *Let G be a 2-connected weighted graph which satisfies the following conditions:*

(1) $\max\{d^w(u), d^w(v)\} \geq c/2$ *for each pair of vertices u and v in $V(G)$ such that $d(u, v) = 2$.*

(2) $w(xz) = w(yz)$ *for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$.*

(3) In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.

Then G contains either a hamiltonian cycle or a cycle of weight at least c.

Also it is shown that neither of the conditions (2) nor (3) of Theorem 6.5 can be dropped. The aim of this chapter is to obtain a more general result for larger classes of weighted graphs, extending Theorem 6.2 to weighted graphs. Corresponding to the conditions of Theorem 6.5, we consider the following conditions, which are said to be Claw Conditions, for a weighted graph G.

(CC1) For each induced claw and each induced modified claw of G, all its non-adjacent pairs of vertices x and y satisfy $\max\{d^w(x), d^w(y)\} \geq c/2$.

(CC2) For each induced claw and each induced modified claw of G, all of its edges have the same weight.

Then we can prove the following theorem.

Theorem 6.6. *Let G be a 2-connected weighted graph which satisfies Claw Conditions (CC1) and (CC2). Then G contains either a hamiltonian cycle or a cycle of weight at least c.*

Note that if a graph satisfies the condition (1) of Theorem 6.5, it satisfies (CC1). Also, if a graph satisfies the conditions (2) and (3) of Theorem 6.5, it satisfies (CC2). Thus, Theorem 6.6 weakened the conditions of Theorem 6.5. An example is shown in Figure 6.2. Let H be a complete graph with $l \geq 2$ vertices such that all edges in H have the same weight r_1, and $u, v \in V(H)$. Now make a new graph G adding m paths of length 3 $\{us_it_iv \mid 1 \leq i \leq m\}$ to H and assigning weight r_1 to the edges in $\{us_i \mid 1 \leq i \leq m\} \cup \{t_iv \mid 1 \leq i \leq m\}$, weight $r_2 \neq r_1$ to the edges in $\{s_it_i \mid 1 \leq i \leq m\}$. Then G does not satisfy the condition (2) of Theorem 6.5, although G satisfies the condition (CC2) of Theorem 6.6.

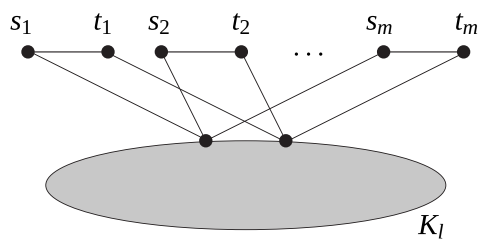

Figure 6.2:

(CC2) in Theorem 6.6 cannot be dropped even in the sense of only claw or only modified claw. The examples are shown in Figures 6.3 and 6.4. In both graphs, we define $w(v_4v_5) = w(v_5v_6) = w(v_7v_8) = w(v_8v_9) = 4$, and define $w(e) = 5$ for all the other edges and let $c = 38$. Then, the resulting weighted graph in Figure 6.3 satisfies (CC1), and for each induced modified claw, all of its edges have the same weight. But, this graph does not contain a hamiltonian cycle and the weight of its heaviest cycle is $36 < c$. Similarly, the weighted graph in Figure 6.4 satisfies (CC1), and for each induced claw, all of its edges have the same weight. But, this graph also does not contain a hamiltonian cycle and the weight of its heaviest cycle is $36 < c$.

In the proof of Theorem 6.6, we call a path P a *heaviest longest path* if P satisfies the followings;

(a) P is a longest path of G, and

(b) $w(P)$ is maximum, subject to (a).

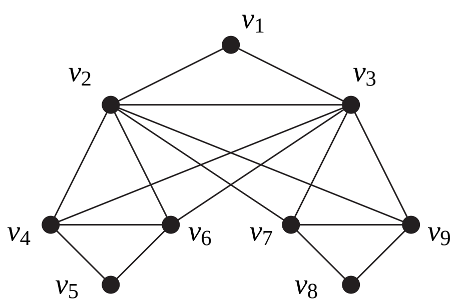

Figure 6.3:

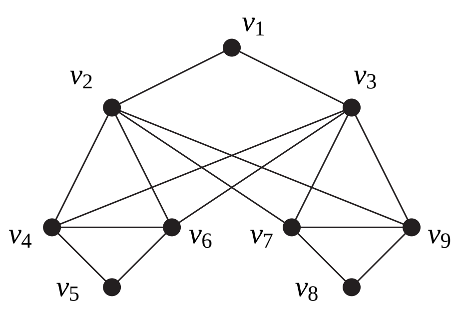

Figure 6.4:

6.2 Key Lemma

In our proof of Theorem 6.6, the following lemma is essential.

Lemma 6.7. *Let G be a non-hamiltonian 2-connected weighted graph satisfying Claw Conditions (CC1) and (CC2). Suppose that v is an end vertex of a heaviest longest path in G. Then, there exists a heaviest longest path with an end vertex v such that the other end vertex has weighted degree at least $c/2$.*

We prove this lemma in the next section. Theorem 6.6 can be proved by combining Lemma 6.7 and the following lemma. The proof of the lemma is implicit in [34] (Case 2 in the proof of Theorem 1). See also [3, Lemma 5] and Lemma 5.4 [5, Lemma 2.1].

Lemma 6.8. *Let G be a non-hamiltonian 2-connected weighted graph and $P = v_1v_2 \cdots v_p$ be a heaviest longest path in G. Then there is a cycle C in G of weight $w(C) \geq d^w(v_1) + d^w(v_p)$.*
□

Proof of Theorem 6.6. Suppose that G does not contain a hamiltonian cycle. By using Lemma 6.7 twice, we obtain a heaviest longest path with both end vertices having weighted degree $\geq c/2$. Then by Lemma 6.8, we can find a cycle of weight at least c. □

6.3 Proof of Lemma 6.7

Before proving Lemma 6.7, we prepare the following lemmas.

Lemma 6.9. *Let G be a weighted graph satisfying (CC2). If x_1yx_2 is an induced path with $w(x_1y) \neq w(x_2y)$ in G, then each vertex $x \in N(y) \setminus \{x_1, x_2\}$ is adjacent to both x_1 and x_2.*

Proof. By (CC2), $\{x, y, x_1, x_2\}$ cannot induce a claw or a modified claw. Thus we obtain the conclusion. □

Lemma 6.10. *Let G be a weighted graph satisfying (CC2). Suppose x_1yx_2 is an induced path such that $w_1 = w(x_1y)$ and $w_2 = w(x_2y)$ with $w_1 \neq w_2$, and yz_1z_2 is a path such that $\{z_1, z_2\} \cap \{x_1, x_2\} = \phi$ and $x_2z_2 \notin E(G)$. Then the following (i) and (ii) hold:*

(i) $\{z_1x_1, z_1x_2, z_2x_1\} \subseteq E(G)$, and $yz_2 \notin E(G)$. Moreover, all edges in the subgraph induced by $\{x_1, x_2, y, z_1, z_2\}$, other than x_1y, have the same weight w_2 (See Figure 6.5).

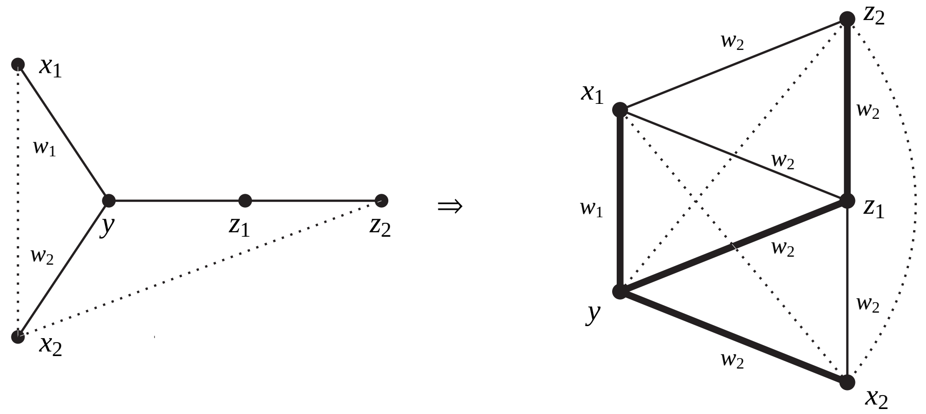

Figure 6.5:

(ii) Let Y be the component of $G - \{x_2, z_1, z_2\}$ with $y \in V(Y)$. For each vertex $v \in V(Y) \setminus \{x_1, y\}$, v is adjacent to all of x_1, x_2, y and z_2. Furthermore, $w(vx_1) = w(vx_2) = w(vy) = w(vz_2) = w_2$.

Proof. By Lemma 6.9, we have $z_1x_1 \in E(G)$, $z_1x_2 \in E(G)$ and $z_2y \notin E(G)$. Thus, $\{z_2, z_1, y, x_2\}$ induces a modified claw, and hence $w(z_1x_2) = w(z_1y) = w(z_1z_2) = w_2$. Then, since $w(x_1y) \neq w(yz_1)$, $\{z_2, z_1, y, x_1\}$ cannot induce a modified claw. This implies $z_2x_1 \in E(G)$. Now, we have a modified claw induced by $\{x_2, z_1, x_1, z_2\}$. Hence $w(z_1x_1) = w(z_2x_1) = w_2$. This proves (i).

For the proof of (ii), suppose first that $v \in V(Y) \setminus \{x_1, y\}$ is adjacent to y. Then, by Lemma 6.9, we have $vx_1, vx_2 \in E(G)$. Applying Lemma 6.9 again to the induced path yx_1z_2 and $v \in N(x_1)$, we have $vz_2 \in E(G)$. The modified claw induced by $\{z_2, v, y, x_2\}$ implies that $w(vx_2) = w(vy) = w(vz_2) = w_2$. Also, the modified claw induced by $\{x_2, v, x_1, z_2\}$ implies that $w(vx_1) = w_2$. This proves (ii) for $v \in N(y) \cap (V(Y) \setminus \{x_1, y\})$. Since y and x_1 are symmetric (by the structure obtained in (i)), the conclusion of (ii) holds also for each vertex $v \in N(x_1) \cap (V(Y) \setminus \{x_1, y\})$.

In order to complete the proof of (ii), we shall show that every vertex $v \in V(Y) \setminus \{x_1, y\}$ is adjacent to x_1 or y. Assume not. Then there exists a vertex $v \in V(Y) \setminus \{x_1, y\}$ which is distance two apart from y and x_1. Let $v' \in V(Y) \setminus \{x_1, y\}$ be a vertex such that $v' \in N(v) \cap N(y)$. Then, v' is adjacent to both y and x_1, and $w(v'y) = w(v'x_1) = w_2$. Therefore, $\{v, v', y, x_1\}$ cannot induce a modified claw. This implies that $vx_1 \in E(G)$ or $vy \in E(G)$, which contradicts the choice of v. This completes the proof of (ii). □

Proof of Lemma 6.7. Suppose that there is no heaviest longest path with end vertex v such that the other end vertex has weighted degree $\geq c/2$. Let $P = v_1v_2\cdots v_p(v_p = v)$ be a heaviest longest path. From the choice of P, we can see that $N(v_1) \subseteq V(P)$. Next, let $k(P) = \max\{i \mid v_1v_i \in E(G)\}$. Since G is 2-connected, v_1 is adjacent to at least one vertex on P other than v_2. Note that, since P is a longest path in a non-hamiltonian graph G, G does not contain a cycle of length p. So $k(P)$ satisfies $3 \leq k(P) < p$. Assume the heaviest longest path $P = v_1v_2\cdots v_p$ is chosen among all heaviest longest paths ending at v such that $k(P)$ is as large as possible, and let $k = k(P)$.

Since G is 2-connected, there exists a path Q such that

- Q has end vertices v_{s_1} and v_{s_2} such that $s_1 < k < s_2$, and
- $V(P) \cap V(Q) = \{v_{s_1}, v_{s_2}\}$.

We assume that such a path Q was chosen so that

(i) s_2 is as large as possible;

(ii) s_1 is as large as possible, subject to (i).

Case 1. $v_1v_i \in E(G)$ for every i with $s_1 \leq i \leq k$.

Claim 1. $v_{s_1}v_{s_2} \in E(G)$.

Proof. Since $s_1 < k$, we have $v_1v_{s_1+1} \in E(G)$. Recall that there exists a path Q from v_{s_1} to v_{s_2} with $V(Q) \cap V(P) = \{v_{s_1}, v_{s_2}\}$. If there exists a vertex $q \notin \{v_{s_1}, v_{s_2}\}$ on Q, then a path $P' = q\cdots v_{s_1}v_{s_1-1}\cdots v_1v_{s_1+1}v_{s_1+2}\cdots v_p$ satisfies $|V(P')| > |V(P)|$, contradicting the fact that P is a longest path. So we have $v_{s_1}v_{s_2} \in E(G)$. □

Claim 2. $w(v_1v_{s_1+1}) \neq w(v_{s_1}v_{s_1+1})$.

Proof. If $w(v_{s_1}v_{s_1+1}) = w(v_1v_{s_1+1})$, $P' = v_{s_1}v_{s_1-1}\cdots v_1v_{s_1+1}v_{s_1+2}\cdots v_p$ is a heaviest longest path with $k(P') = s_2 > k$, a contradiction. □

Let w_1 and w_2 denote the weight of $v_1v_{s_1+1}$ and $v_{s_1}v_{s_1+1}$, respectively.

Claim 3. $v_{s_1+1}v_{s_2} \in E(G)$.

Proof. Suppose $v_{s_1+1}v_{s_2} \notin E(G)$. By the maximality of k, $v_1v_{s_2} \notin E(G)$. So $\{v_{s_2}, v_{s_1}, v_{s_1+1}, v_1\}$ induces a modified claw, and then we get $w(v_1v_{s_1+1}) = w(v_{s_1}v_{s_1+1})$, contrary to Claim 2. □

By the maximality of s_1, Claim 3 implies that $s_1 = k - 1$ and $v_kv_{s_2} \in E(G)$. Note that $s_2 \neq p$. (Otherwise, $v_{s_2}v_{s_2-1} \cdots v_kv_1v_2 \cdots v_{s_1}v_{s_2}$ becomes a cycle of length p, a contradiction.)

Claim 4. $s_2 = k + 1$.

Proof. Suppose $s_2 > k + 1$. Then v_1v_k, v_kv_{k+1}, $v_kv_{s_2} \in E(G)$ and by the maximality of k, $v_1v_{k+1} \notin E(G), v_1v_{s_2} \notin E(G)$. So $\{v_1, v_k, v_{k+1}, v_{s_2}\}$ induces a claw or a modified claw. From (CC2), we have $w(v_kv_{s_2}) = w(v_1v_k) = w_1$. On the other hand, $w(v_{s_1}v_k) = w_2$, so $\{v_{s_2+1}, v_{s_2}, v_k, v_{s_1}\}$ cannot induce a modified claw. Since $v_{s_1}v_{s_2+1} \notin E(G)$ by the maximality of s_2, we have $v_kv_{s_2+1} \in E(G)$. But then $\{v_{s_2+1}, v_k, v_{s_1}, v_1\}$ induces a modified claw, and we get $w(v_1v_{s_1+1}) = w(v_{s_1}v_{s_1+1})$. This contradicts Claim 2. □

Now we have $s_1 = k-1$ and $s_2 = k+1$, and so $v_{k-1}v_{k+1} \in E(G)$. Then $v_kv_1v_2 \cdots v_{k-1}v_{k+1}v_{k+2} \cdots v_p$ is a longest path. Therefore, we get $N(v_k) \subset V(P)$. By the 2-connectedness of G and the choice of s_2, there must be an edge $v_kv_{s_3} \in E(G)$ such that $s_3 > k + 2$. From the choice of k and s_2, we have $v_1v_{s_3}, v_{s_1}v_{s_3} \notin E(G)$, and so $\{v_{s_3}, v_k, v_{s_1}, v_1\}$ induces a modified claw. This implies $w(v_{s_1}v_{s_1+1}) = w(v_1v_{s_1+1})$, contradicting Claim 2. This completes the proof of Case 1.
□

Case 2. $v_1v_i \notin E(G)$ for some i with $s_1 \leq i \leq k$.

Choose $v_l \notin N(v_1)$ with $s_1 \leq l \leq k$ so that l is as large as possible. It is clear that $3 \leq l < k$ and $v_1v_i \in E(G)$ for every i with $l < i \leq k$.

Claim 5. $d^w(v_l) \geq c/2$.

Proof. Let j be the smallest index such that $j > l$ and $v_j \notin N(v_1) \cap N(v_l)$. Since $v_{l+1} \in N(v_1) \cap N(v_l)$, we have $j \geq l + 2$. Also, it is obvious that $j \leq k + 1$. Then, $\{v_l, v_{j-1}, v_j, v_1\}$ induces a claw or a modified claw. Since $d^w(v_1) < c/2$, by (CC1), we have $d^w(v_l) \geq c/2$. □

We have assumed that there exists no heaviest longest path with end vertex v_p such that the other end vertex has weighted degree at least $c/2$. Hence, by Claim 5, we have the following.

Claim 6. *There is no heaviest longest path with end vertices v_l and v_p.* □

Claim 7. $w(v_l v_{l+1}) \neq w(v_1 v_{l+1})$.

Proof. If $w(v_l v_{l+1}) = w(v_1 v_{l+1})$, then $v_l v_{l-1} \cdots v_1 v_{l+1} v_{l+2} \cdots v_p$ is a heaviest longest path, contradicting Claim 6. □

Let w_1 and w_2 denote the weights of $v_1 v_{l+1}$ and $v_l v_{l+1}$, respectively.

Claim 8. *$vv_1 \in E(G)$ and $vv_l \in E(G)$ for all $v \in N(v_{l+1}) \setminus \{v_1, v_l\}$.*

Proof. Now $w(v_1 v_{l+1}) \neq w(v_l v_{l+1})$. Applying Lemma 6.9 to the induced path $v_1 v_{l+1} v_l$ and $v \in N(v_{l+1}) \setminus \{v_1, v_l\}$, we obtain the conclusion. □

Claim 9. $k \neq l + 1$.

Proof. If $k = l + 1$, then by Claim 8, $v_1 v_{k+1}$ must be in $E(G)$. This contradicts the choice of k. □

Claim 10. *$v_{l+1} v_k \notin E(G)$. In particular, $k \geq l + 3$.*

Proof. Suppose $v_{l+1} v_k \in E(G)$. Now we have $v_1 v_{k+1} \notin E(G)$. Applying Lemma 6.10 to the induced path $v_l v_{l+1} v_1$ and the path $v_{l+1} v_k v_{k+1}$, we get $w(v_1 v_k) = w(v_k v_{k+1}) = w(v_1 v_{l+1}) = w_1$. Moreover, since v_{l-1} is adjacent to v_l, we have $v_{l-1} v_{k+1} \in E(G)$ and $w(v_{l-1} v_l) = w(v_{l-1} v_{k+1}) = w_1$ (See Figure 6.6). Then, the path $v_l v_{l+1} \cdots v_k v_1 v_2 \cdots v_{l-1} v_{k+1} v_{k+2} \cdots v_p$ becomes a heaviest longest path, contradicting Claim 6. □

Claim 11. *If $v_l v_k \in E(G)$, then $v_i v_{k+1} \notin E(G)$ for each i with $l + 1 \leq i \leq k - 1$.*

Proof. Suppose $v_l v_k \in E(G)$. We assume that there exists an edge $v_{t_1} v_{t_2}$ for some t_1 and t_2 with $l + 1 \leq t_1 < k < t_2$. We may assume that t_1 and t_2 were chosen so that

(i) t_1 is as large as possible;

(ii) t_2 is as large as possible, subject to (i).

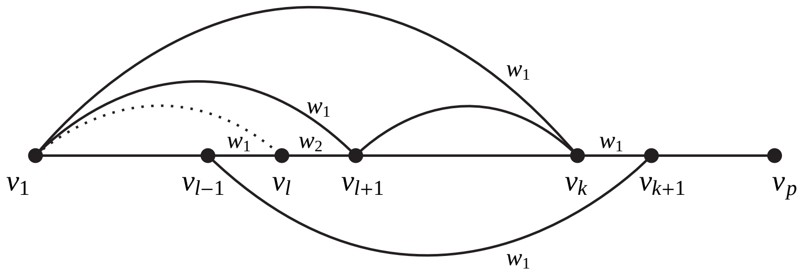

Figure 6.6:

Note that $v_1v_{t_1+1} \in E(G)$, so v_{t_1} is an end vertex of a longest path

$$v_{t_1}v_{t_1-1}\cdots v_1v_{t_1+1}v_{t_1+2}\cdots v_p.$$

Then, we have $v_{t_1}v_{t_2} \in E(G)$, for otherwise there exists a path longer than P.

Claim 11.1. $w(v_1v_{t_1+1}) \neq w(v_{t_1}v_{t_1+1})$.

Proof. If $w(v_1v_{t_1+1}) = w(v_{t_1}v_{t_1+1})$, then $P' = v_{t_1}v_{t_1-1}\cdots v_1v_{t_1+1}v_{t_1+2}\cdots v_p$ becomes a heaviest longest path with $k(P') = t_2 > k$, which contradicts the choice of P. □

Let w_3 and w_4 denote the weight of $v_1v_{t_1+1}$ and $v_{t_1}v_{t_1+1}$, respectively.

Claim 11.2. $v_{t_1+1}v_{t_2} \in E(G)$.

Proof. Suppose that $v_{t_1+1}v_{t_2} \notin E(G)$. Then $\{v_{t_2}, v_{t_1}, v_{t_1+1}, v_1\}$ induces a modified claw, and we obtain $w(v_{t_1}v_{t_1+1}) = w(v_1v_{t_1+1})$. This contradicts Claim 11.1. □

By the maximality of t_1, we have $t_1 = k-1$ and $v_kv_{t_2} \in E(G)$. Note that $t_2 \neq p$. (Otherwise, $v_{t_2}v_{t_2-1}\cdots v_kv_1v_2\cdots v_{t_1}v_{t_2}$ becomes a cycle of length p, a contradiction.)

Claim 11.3. $t_2 = k + 1$.

Proof. Suppose that $t_2 \neq k + 1$. Since $v_1v_{k+1} \notin E(G)$ and $v_1v_{t_2} \notin E(G)$, $\{v_1, v_k, v_{k+1}, v_{t_2}\}$ induces a claw or a modified claw, and we get $w(v_kv_{t_2}) = w_3 \neq w_4 = w(v_{t_1}v_k)$. So $\{v_{t_1}, v_k, v_{t_2}, v_{t_2+1}\}$ cannot induce a modified claw. Now, by the maximality of t_2, $v_{t_1}v_{t_2+1} \notin E(G)$. This implies $v_k v_{t_2+1} \in E(G)$. But then, $\{v_{t_2+1}, v_k, v_{t_1}, v_1\}$ induces a modified claw, and we get $w(v_1 v_k) = w(v_{t_1} v_k)$. This contradicts Claim 11.1. □

Using the above claims, we shall prove Claim 11.

Since $\{v_1, v_k, v_{t_2}, v_l\}$ induces a claw or a modified claw, $w(v_k v_{t_2}) = w(v_1 v_k) = w_3 \neq w_4 = w(v_{t_1} v_k)$. This implies that $\{v_{t_2+1}, v_{t_2}, v_k, v_{t_1}\}$ cannot induce a modified claw. And, by the maximality of t_2, $v_{t_1} v_{t_2+1} \notin E(G)$. So $v_k v_{t_2+1}$ must be in $E(G)$. But then $\{v_{t_2+1}, v_k, v_{t_1}, v_1\}$ induces a modified claw and we have $w(v_1 v_k) = w(v_{t_1} v_k)$. Now $k = t_1 + 1$, so this contradicts Claim 11.1. This completes the proof of Claim 11. □

Now we continue the proof in Case 2.

Claim 12. *For each i with $l + 2 \leq i \leq k, v_l v_i \in E(G)$.*

Proof. By Claim 8, we have $v_l v_{l+2} \in E(G)$. Now suppose that there exists some v_i with $l + 3 \leq i \leq k$ such that $v_l v_i \notin E(G)$. Let $r = \min\{i \mid l + 3 \leq i \leq k, v_l v_i \notin E(G)\}$.

Claim 12.1. $v_{l+1} v_{r-1} \in E(G)$.

Proof. If $r = l + 3$, it is clear that $v_{l+1} v_{r-1} \in E(G)$. So we can assume $r \geq l + 4$. Now suppose $v_{l+1} v_{r-1} \notin E(G)$. By the choice of r and the fact $r \geq l + 4$, we have $v_l v_{r-2}, v_l v_{r-1} \in E(G)$ and $v_l v_r \notin E(G)$. And Claim 8 shows $v_{l+1} v_r \notin E(G)$. Then $\{v_{l+1}, v_1, v_{r-1}, v_r\}$ induces a modified claw, and we get $w(v_1 v_{r-1}) = w(v_1 v_r) = w(v_{r-1} v_r) = w(v_1 v_{l+1}) = w_1$. On the other hand, $\{v_l, v_{r-1}, v_r, v_1\}$ also induces a modified claw. This implies that $w(v_l v_{r-1}) = w(v_{r-1} v_r) = w_1$. Now the fact $w(v_l v_{l+1}) = w_2$ shows that $\{v_{l+1}, v_l, v_{r-2}, v_{r-1}\}$ cannot induce a modified claw, and we have $v_{l+1} v_{r-2} \in E(G)$.

Next, we prove $v_{r-2} v_r \notin E(G)$. If $v_{r-2} v_r \in E(G)$, each of $\{v_{l+1}, v_{r-2}, v_{r-1}, v_r\}$ and $\{v_r, v_{r-2}, v_{l+1}, v_l\}$ induces a modified claw. The first one implies that $w(v_{l+1} v_{r-2}) = w(v_{r-1} v_r) = w_1$. But the second one shows $w(v_{l+1} v_{r-2}) = w(v_l v_{l+1}) = w_2$, a contradiction. Therefore, $v_{r-2} v_r \notin E(G)$.

Recall that $w(v_l v_{r-1}) = w_1 \neq w_2 = w(v_l v_{l+1})$. Applying Lemma 6.9 to the induced path $v_{l+1} v_l v_{r-1}$, we have $v_{l-1} v_{l+1}, v_{l-1} v_{r-1} \in E(G)$. Now we consider two cases, $v_{l-1} v_r \in E(G)$ or $v_{l-1} v_r \notin E(G)$.

If $v_{l-1} v_r \in E(G)$, $\{v_r, v_{l-1}, v_l, v_{l+1}\}$ induces a modified claw. This shows $w(v_{l-1} v_l) = w(v_{l-1} v_r)$. Then $v_l v_{l+1} \cdots v_{r-1} v_1 v_2 \cdots v_{l-1} v_r v_{r+1} \cdots v_p$ becomes a heaviest longest path, contradicting Claim 6.

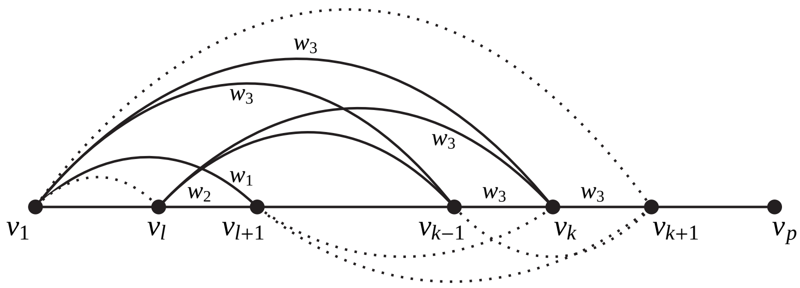

Figure 6.7:

If $v_{l-1}v_r \notin E(G)$, $\{v_r, v_{r-1}, v_l, v_{l-1}\}$ induces a modified claw. This shows $w(v_{l-1}v_l) = w(v_{l-1}v_{r-1})$. Then $v_lv_{l+1}\cdots v_{r-1}v_{l-1}v_{l-2}\cdots v_1v_rv_{r+1}\cdots v_p$ becomes a heaviest longest path, contradicting Claim 6. □

Now we shall complete the proof of Claim 12. Applying Lemma 6.10 to the induced path $v_1v_{l+1}v_l$ and the path $v_{l+1}v_{r-1}v_r$, we get $w(v_1v_{r-1}) = w(v_{r-1}v_r) = w(v_lv_{l+1}) = w_2$. Moreover, since v_{l-1} and v_1 are in the same component of $G - \{v_l, v_{r-1}, v_r\}$, we have $v_{l-1}v_r \in E(G)$ and $w(v_{l-1}v_l) = w(v_{l-1}v_r) = w_2$. Then, the path $v_lv_{l+1}\cdots v_{r-1}v_1v_2\cdots v_{l-1}v_rv_{r+1}\cdots v_p$ becomes a heaviest longest path, contradicting Claim 6. □

By Claim 12, we have v_lv_k and $v_lv_{k-1} \in E(G)$. And by Claim 11, we get $v_{l+1}v_{k+1}, v_{k-1}v_{k+1} \notin E(G)$. Now $\{v_{k+1}, v_k, v_{k-1}, v_1\}$ induces a modified claw. Let w_3 denote the weight of the edges of this modified claw. Now $\{v_1, v_k, v_{k+1}, v_l\}$ induces a claw or a modified claw. So $w(v_lv_k) = w(v_1v_k) = w_3$ (See Figure 6.7).

Claim 13. $v_lv_{k+1} \notin E(G)$.

Proof. Suppose that $v_lv_{k+1} \in E(G)$. Then $\{v_{l+1}, v_l, v_k, v_{k+1}\}$ induces a modified claw. This implies $w_3 = w(v_lv_k) = w(v_lv_{k+1}) = w(v_lv_{l+1}) = w_2$. Hence $w(v_1v_{k-1}) = w(v_1v_k) = w(v_{k-1}v_k) = w(v_kv_{k+1}) = w_2$.

Next, we remark $w(v_1v_{l+1}) = w_1 \neq w_2 = w(v_1v_k)$. Applying Lemma 6.9 to the induced path $v_{l+1}v_1v_k$ and $v_2 \in N(v_1)$, we have $v_2v_{l+1}, v_2v_k \in E(G)$. Then Claim 8 implies $v_2v_l \in E(G)$.

Now, we claim that $v_2v_{k+1} \notin E(G)$. If $v_2v_{k+1} \in E(G)$, each of $\{v_{l+1}, v_2, v_k, v_{k+1}\}$ and $\{v_{k+1}, v_2, v_1, v_{l+1}\}$ induces a modified claw. The first one shows $w(v_2v_{l+1}) = w(v_kv_{k+1}) = w_2$.

But on the other hand, the second one shows $w(v_2v_{l+1}) = w(v_1v_{l+1}) = w_1$, a contradiction.

Thus $v_2v_{k+1} \notin E(G)$. Then the modified claw induced by $\{v_{k+1}, v_k, v_2, v_1\}$ shows $w(v_1v_2) = w(v_kv_{k+1}) = w_2$. And the modified claw induced by $\{v_{k+1}, v_l, v_{l+1}, v_2\}$ implies $w(v_2v_{l+1}) = w(v_lv_{l+1}) = w_2$. Consequently, the path

$$v_lv_{l-1}\cdots v_2v_{l+1}v_{l+2}\cdots v_{k-1}v_1v_kv_{k+1}\cdots v_p$$

becomes a heaviest longest path, which contradicts Claim 6. □

Claim 14. $v_{l-1}v_{l+1} \notin E(G)$.

Proof. Suppose that $v_{l-1}v_{l+1} \in E(G)$. By Claim 8 we have $v_1v_{l-1} \in E(G)$.

First, we prove $v_{l-1}v_k \in E(G)$. Suppose $v_{l-1}v_k \notin E(G)$. Then each of $\{v_k, v_1, v_{l-1}, v_{l+1}\}$ and $\{v_k, v_l, v_{l-1}, v_{l+1}\}$ induces a modified claw. The first one shows $w(v_{l-1}v_{l+1}) = w(v_1v_{l+1}) = w_1$. But the other one shows $w(v_{l-1}v_{l+1}) = w(v_lv_{l+1}) = w_2$, a contradiction. This shows $v_{l-1}v_k \in E(G)$.

Next, we prove $v_{l-1}v_{k+1} \notin E(G)$. Otherwise, each of $\{v_{k+1}, v_{l-1}, v_{l+1}, v_1\}$ and $\{v_{k+1}, v_{l-1}, v_l, v_{l+1}\}$ induces a modified claw. Then the first one shows $w(v_{l-1}v_{l+1}) = w(v_1v_{l+1}) = w_1$ while the second one implies $w(v_{l-1}v_{l+1}) = w(v_lv_{l+1}) = w_2$, a contradiction. This shows $v_{l-1}v_{k+1} \notin E(G)$.

Then, we obtain that $\{v_{k+1}, v_k, v_l, v_{l-1}\}$ induces a modified claw. So $w(v_{l-1}v_l) = w(v_{l-1}v_k) = w(v_kv_{k+1}) = w_3$. This means that the path

$$v_lv_{l+1}\cdots v_{k-1}v_1v_2\cdots v_{l-1}v_kv_{k+1}\cdots v_p$$

is a heaviest longest path, contradicting Claim 6. □

Using the above claims, we shall complete the proof of Case 2.

Claim 13 says $v_lv_{k+1} \notin E(G)$, and by Claim 14, we get $v_{l+1}v_{l-1} \notin E(G)$. Now $\{v_{k+1}, v_k, v_{k-1}, v_l\}$ induces a modified claw. This implies $w(v_lv_{k-1}) = w(v_lv_k) = w(v_{k-1}v_k) = w_3$. And $\{v_{l+1}, v_l, v_{l-1}, v_k\}$ induces a claw or a modified claw. It implies $w_3 = w_2, w_3 \neq w_1$ and $w(v_{l-1}v_l) = w_3$. Then, applying Lemma 6.9 to the induced path $v_kv_1v_{l+1}$ and $v_2 \in N(v_1)$, we have v_2v_{l+1} and $v_2v_k \in E(G)$. If $l-1 = 2$, this contradicts Claim 14. Hence we have $l-1 > 2$. By Claim 8, we have $v_2v_l \in E(G)$.

Next, we prove $v_2v_{k+1} \notin E(G)$. Suppose $v_2v_{k+1} \in E(G)$. Then each of $\{v_{k+1}, v_2, v_1, v_{l+1}\}$ and $\{v_{k+1}, v_2, v_l, v_{l+1}\}$ induces a modified claw. The first one implies that $w(v_2v_{k+1}) = w(v_1v_{l+1}) =$

w_1, and the second one implies that $w(v_2v_{k+1}) = w(v_lv_{l+1}) = w_2$, a contradiction. Therefore, $v_2v_{k+1} \notin E(G)$.

Now each of $\{v_{k+1}, v_k, v_2, v_1\}$ and $\{v_{k+1}, v_k, v_l, v_2\}$ induces a modified claw. They imply $w(v_1v_2) = w(v_2v_k) = w(v_1v_k) = w_3$, and $w(v_2v_l) = w(v_lv_k) = w_3$. Then, the path $v_{l-1} \cdots v_2v_lv_{l+1} \cdots v_{k-1}v_1v_kv_{k+1} \cdots v_p$ is a heaviest longest path with an end vertex v_p. So we can see that another end vertex v_{l-1} satisfies $d^w(v_{l-1}) < c/2$. On the other hand, since $\{v_{l+1}, v_l, v_{l-1}, v_k\}$ induces a claw or a modified claw with $v_{l-1}v_{l+1} \notin E(G)$, we have $d^w(v_{l+1}) \geq c/2$. Now, there is another heaviest longest path $v_{l+1}v_{l+2} \cdots v_{k-1}v_1v_2 \cdots v_lv_kv_{k+1} \cdots v_p$. This is a contradiction. This completes the proof of Case 2 and the proof of Lemma 6.7. □

Chapter 7

σ_k type condition for heavy cycles in weighted graphs

(This chapter is based on the paper [12].)

7.1 Previous result and the new result

Using the degree condition on three independent vertices, and the condition on the weights of edges which is the same as the condition appeared in Chapter 6, Zhang et al. proved the existence of heavy cycles in weighted graphs. Here, we say a vertex set U is *independent* if every vertex in U has no neighbor in U. And the number of vertices in a maximum independent set of a graph G is denoted by $\alpha(G)$. For a positive integer $k \leq \alpha(G)$, $\sigma_k(G)$ and $\sigma_k^w(G)$ denotes the minimum value of the degree sum of any k independent vertices and the minimum value of the weighted degree sum of any k independent vertices, respectively (For $k > \alpha(G)$, we define $\sigma_k(G), \sigma_k^w(G)$ as ∞.) The theorem of Zhang et al. is the following.

Theorem 7.1 (Zhang et al. [36]). *Let G be a 2-connected weighted graph which satisfies the following conditions:*

(1) $\sigma_3^w(G) \geq m$.

(2) $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x, y) = 2$.

(3) In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.

Then G contains either a hamiltonian cycle or a cycle of weight at least $2m/3$.

Theorem 7.1 is an extension of the following theorem to the weighted graphs in the case $k = 2$.

Theorem 7.2 (Fournier and Fraisse [17]). *Let G be a k-connected graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}(G) \geq m$. Then G contains either a hamiltonian cycle or a cycle of length at least $2m/(k+1)$.*

In this chapter, we extend Theorem 7.2 to the weighted graphs for all k.

Theorem 7.3. *Let G be a k-connected weighted graph where $k \geq 2$. Suppose that G satisfies the following conditions.*

(1) $\sigma^w_{k+1}(G) \geq m$.

(2) $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x, y) = 2$.

(3) In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.

Then G contains either a hamiltonian cycle or a cycle of weight at least $2m/(k+1)$.

7.2 The conditions of Zhang et al. 's theorem

To prove Theorem 7.3, we need the following lemma, which shows that the class of weighted graphs satisfying Conditions (2) and (3) of Theorem 7.3 is limited.

Lemma 7.4. *Let G be a connected weighted graph satisfying Conditions (2) and (3) of Theorem 7.3. Then G satisfies one of the following:*

(a) all edges of G have the same weight, or

(b) G is a complete multi-partite graph.

Proof. Let G be a connected weighted graph satisfying Conditions (2) and (3) of Theorem 7.3. Suppose that there exists $e_1, e_2 \in E(G)$ such that $w(e_1) \neq w(e_2)$. Then what we need to prove is that G is a complete multi-partite graph.

Since G is connected, we can choose a vertex x so that there exist $u, v \in N(x)$ such that $w(ux) \neq w(vx)$. Let $\bigcup_{i=1}^{n} V_i$ be a partition of $N(x)$ such that for $u \in V_i$ and $v \in V_j$, $w(ux) = w(vx)$ if and only if $i = j$. Now we denote the weight of the edges joining x and V_i by w_i for $1 \leq i \leq n$.

Claim 1. *Let $1 \leq i, j \leq n$ and $v_i \in V_i$, $v_j \in V_j$. If $i \neq j$, $v_i v_j \in E(G)$.*

Proof. Since $w(xv_i) \neq w(xv_j)$, Condition (2) of Theorem 7.3 implies $d(v_i, v_j) \neq 2$. Hence $v_iv_j \in E(G)$. □

Claim 2. *If there exists a vertex y such that $d(x, y) = 2$, then $vy \in E(G)$ for all $v \in N(x)$.*

Proof. The fact $d(x, y) = 2$ shows that there is a neighborhood v_1 of y in $N(x)$. Without loss of generality, we may assume $v_1 \in V_1$. And Condition (2) of Theorem 7.3 implies $w(yv_1) = w_1$.

Now suppose that there exists a vertex $v \in \bigcup_{i=2}^{n} V_i$ with $yv \notin E(G)$. Then Claim 1 implies $v_1v \in E(G)$, and Condition (2) of Theorem 7.3 shows $w(v_1v) = w(yv_1) = w_1$. Hence, applying Condition (3) of Theorem 7.3 to the triangle xv_1v, we have $w(xv) = w_1$. This contradicts the definition of the partition $\bigcup V_i$. So we must have $yv \in E(G)$ for all $v \in \bigcup_{i=2}^{n} V_i$.

Applying the same argument to $v_2 \in V_2 \cap N(y)$ and $v \in V_1$, we have $yv \in E(G)$ for every $v \in V_1$. □

If there exists a vertex y such that $d(x, y) = 2$, Condition (2) of Theorem 7.3 implies $w(v_iy) = w_i$ for all $v_i \in V_i$.

Claim 3. *There is no vertex z such that $d(x, z) = 3$.*

Proof. Suppose that there exists a vertex z such that $d(x, z) = 3$. Then z has a neighbor y such that $d(x, y) = 2$. Now Claim 2 implies that we have $v_1 \in N(y) \cap V_1$ and $v_2 \in N(y) \cap V_2$ with $w(yv_1) = w_1$ and $w(yv_2) = w_2$.

Since $d(z, v_1) = d(z, v_2) = 2$, Condition (2) of Theorem 7.3 shows $w(zy) = w(yv_1) = w_1$ and $w(zy) = w(yv_2) = w_2$. So we have $w_1 = w_2$, a contradiction. □

Let $V_0 = \{x\} \cup \{y \mid d(x, y) = 2\}$. Then $\bigcup_{i=0}^{n} V_i$ is a partition of $V(G)$.

Claim 4. *Let $0 \leq i < j \leq n$ and $v_i \in V_i$, $v_j \in V_j$. Then $v_iv_j \in E(G)$.*

Proof. If $i \neq 0$ and $j \neq 0$, Claim 1 implies $v_iv_j \in E(G)$. So we may assume $i = 0$. If $v_i = x$, the definition of $\bigcup_{i=1}^{n} V_i$ shows $v_iv_j \in E(G)$, and if $v_i \neq x$, Claim 2 implies $v_iv_j \in E(G)$. □

Note that for all $v_0 \in V_0$, $v_0 = x$ or $d(x, v_0) = 2$. Hence for all $v_i \in V_i (1 \leq i \leq n)$, $w(v_0v_i) = w_i$.

Claim 5. *$v_0v_0' \notin E(G)$ for all $v_0, v_0' \in V_0$.*

Proof. If $v_0 = x$, $d(x, v_0') = 2$ for all vertices $v_0' \in V_0 \setminus \{v_0\}$. Hence $v_0v_0' \notin E(G)$. So we may assume $v_0, v_0' \neq x$. Now we suppose $v_0v_0' \in E(G)$. Claim 2 implies that there exists $v_1 \in V_1, v_2 \in V_2$ such that $v_1, v_2 \in N(v_0) \cap N(v_0')$. Now we have $w(v_0v_1) = w(v_0'v_1) = w_1$, $w(v_0v_2) = w(v_0'v_2) = w_2$. So applying Condition (3) of Theorem 7.3 to the triangles $v_0v_0'v_1$ and $v_0v_0'v_2$, we have $w_1 = w_2$, a contradiction. □

Claim 6. *Let $0 \leq i \leq n$ and $t, u, v \in V_i$. If $tu, uv \notin E(G)$, then $tv \notin E(G)$.*

Proof. If $i = 0$, Claim 5 implies that $tv \notin E(G)$. So we assume that $1 \leq i \leq n$. Suppose $tv \in E(G)$. Without loss of generality, we may assume $i = 1$. Let $v_2 \in V_2$. Now, since $t, u, v \in V_1$, $w(xt) = w(xu) = w(xv) = w_1$. Then applying Condition (3) of Theorem 7.3 to the triangle xtv, we have $w(tv) = w(xt) = w_1$. On the other hand, Claim 4 implies $v_2t, v_2u, v_2v \in E(G)$. Since tu and $uv \notin E(G)$, Condition (2) of Theorem 7.3 shows $w(v_2t) = w(v_2u) = w(v_2v)$. Then applying Condition (3) of Theorem 7.3 to the triangle v_2tv, we have $w(tv) = w(v_2t)$. Hence, $w(v_2t) = w(tv) = w(xt) = w_1$. So applying Condition (3) of Theorem 7.3 to the triangle xtv_2, we have $w_2 = w(xv_2) = w(xt) = w_1$, a contradiction. □

Now on every V_i $(0 \leq i \leq n)$, nonadjacency is an equivalence relation. Let $V_{i,1}, \ldots, V_{i,m_i}$ be the equivalence classes of V_i. Then, for all vertices $u \in V_{i,j}$ and $v \in V_{i',j'}$, $uv \in E(G)$ if and only if $(i, j) \neq (i', j')$. Hence, G is a complete multi-partite graph with partite sets $V_0, V_{i,j}(1 \leq i \leq n, 1 \leq j \leq m_i)$. This completes the proof of Lemma 7.4. □

7.3 Proof of Theorem 7.3

Let G be a weighted graph satisfying the conditions of Theorem 7.3. If $k \geq \alpha(G)$, the following theorem implies the assertion.

Theorem 7.5 (Chvátal and Erdős [7]). *Let G be a k-connected graph with at least three vertices. If $k \geq \alpha(G)$, then G contains a hamiltonian cycle.*

So we may assume $2 \leq k < \alpha(G)$. Now Lemma 7.4 implies that all edges of G have the same weight or G is a complete multi-partite graph.

Assume that all edges of G have the same weight w_1. If $w_1 = 0$, the assertion is obvious. If $w_1 \neq 0$, $d^w(v) = w_1 d(v)$ for all $v \in V(G)$. Hence $\sigma_{k+1}(G) = \sigma^w_{k+1}(G)/w_1 \geq m/w_1$. Then Theorem 7.2 implies that G contains either a hamiltonian cycle or a cycle C of length at least $2m/(k+1)w_1$. Now $w(C) = w_1|E(C)| \geq 2m/(k+1)$.

Therefore, we may assume that G is a complete multi-partite graph. Let $n = |V(G)|$ and $V_1, \cdots, V_l$ be the partite sets of G.

Claim 1. *If* $x, y \in V_i$, *then* $w(xz) = w(yz)$ *for every* $z \in V(G) \setminus V_i$. *In particular,* $d^w(x) = d^w(y)$.

Proof. Since x and y are in the same partite set V_i, $xy \notin E(G)$. Hence, Condition (2) implies $w(xz) = w(yz)$. And hence, the assertion $d^w(x) = d^w(y)$ is obvious. □

Claim 2. *If* G *is not hamiltonian, then* $|V_i| > n/2$ *for some* i *such that* $1 \le i \le l$.

Proof. Suppose that $|V_i| \le n/2$ for all i $(1 \le i \le l)$. Then for each $v \in V_j$ $(1 \le j \le l)$,

$$d(v) = \sum_{r \neq j} |V_r| = n - |V_r| \ge n/2.$$

Hence, Theorem 7.2 implies that G has a hamiltonian cycle, a contradiction. □

Without loss of generality, we can assume that $|V_1| > n/2$. Let $p = |V_1|$ and $q = n - p$. Then, since G is k-connected, it is obvious that $k \le q < p$. And let $V_1 = \{v_1, v_2, \ldots, v_p\}$, $V(G) \setminus V_1 = \{u_1, u_2, \ldots u_q\}$.

Claim 3. $d^w(v) \ge m/(k+1)$ *for all* $v \in V_1$.

Proof. Since $k < p$, we can choose $v_1, v_2, \ldots, v_{k+1}$ in V_1. Now, $\{v_1, v_2, \ldots, v_{k+1}\}$ is independent, hence $\sum_{i=1}^{k+1} d^w(v_i) \ge m$. Then Claim 1 implies $d^w(v_1) = d^w(v_2) = \cdots = d^w(v_{k+1})$, so $d^w(v_1) \ge m/(k+1)$. Using Claim 1 again, we have $d^w(v) \ge m/(k+1)$ for all $v \in V_1$. □

Now we consider the cycle $C = v_1 u_1 v_2 u_2 \cdots v_{q-1} u_{q-1} v_q u_q v_1$. Then Claim 1 implies

$$\begin{aligned}
w(C) &= w(v_1u_1) + w(u_1v_2) + w(v_2u_2) + \cdots \\
&\quad + w(v_{q-1}u_{q-1}) + w(u_{q-1}v_q) + w(v_qu_q) + w(u_qv_1) \\
&= w(v_1u_1) + w(u_1v_1) + w(v_1u_2) + \cdots \\
&\quad + w(v_1u_{q-1}) + w(u_{q-1}v_1) + w(v_1u_q) + w(u_qv_1) \\
&= 2\sum_{i=1}^{q} w(v_1u_i) \\
&= 2d^w(v_1).
\end{aligned}$$

Hence, by Claim 3, $w(C) \ge 2m/(k+1)$. This completes the proof of Theorem 7.3.

Chapter 8

Heavy paths in weighted graphs

(The result in the first section appears in [19], and the other results in this chapter appear in [13].)

The topic of this chapter is the existence of heavy paths joining two specified vertices. We consider two weighted degree conditions, Dirac-type and Ore-type.

8.1 A Dirac-type condition

The following theorem shows the existence of heavy paths joining two specified vertices, which is a motivation of the results in this section.

Theorem 8.1 (Bondy and Fan [4]). *Let G be a 2-connected weighted graph and d be a nonnegative real number. Let x and y be distinct vertices of G. If $d^w(v) \geq d$ for all $v \in V(G) \setminus \{x, y\}$, then G contains an (x, y)-path of weight at least d.*

Zhang et al. extended Theorem 8.1 as follows. If an (x, z)-path contains all vertices in $Y \subseteq V(G)$, we call it an (x, Y, z)-*path*.

Theorem 8.2 (Zhang, Li and Broersma [35]). *Let G be a 2-connected weighted graph, let d be a nonnegative real number, and let $x, z \in V(G)$ such that $x \neq z$. If $d^w(v) \geq d$ for every vertex $v \in V(G) \setminus \{x, z\}$, then for any given vertex y, G has an $(x, \{y\}, z)$-path of weight at least d.*

Note that Theorem 2.1 immediately implies Theorem 8.2. In this section, we prove the following theorem, which is an extension of Theorem 8.2.

Theorem 8.3. *Let G be a 2-connected weighted graph, let d be a nonnegative real number, $x, z \in V(G)$ such that $x \neq z$, and $y_1, y_2 \in V(G)$. Now assume that there exists an $(x, \{y_1, y_2\}, z)$-path P in G. If $d^w(v) \geq d$ for every vertex $v \in V(G) \setminus \{x, z\}$, then there exists an $(x, \{y_1, y_2\}, z)$-path of weight at least d.*

Proof. In case of $y_1 = y_2$ or $\{y_1, y_2\} \cap \{x, z\} \neq \emptyset$, Theorem 8.2 implies the assertion, so we may assume that x, y_1, y_2 and z are distinct vertices. We use induction on $|V(G)|$. Let $|V(G)| = 4$, then without loss of generality we may assume that $P = xy_1y_2z$. If $zy_1 \notin E(G)$ or $xy_2 \notin E(G)$, it is obvious that $w(P) \geq d$, hence P is a required path. If zy_1 and $xy_2 \in E(G)$, let $P' = xy_2y_1z$. Then $w(P) + w(P') \geq d^w(y_1) + d^w(y_2) \geq 2d$, hence P or P' is weight at least d, which is a required path. Now assume that $|V(G)| \geq 5$.

By Theorem 8.2, there exists an (x, z)-path Q of weight at least d such that $V(Q) \cap \{y_1, y_2\} \neq \emptyset$. Take Q so that $w(Q)$ is as large as possible. If $\{y_1, y_2\} \subseteq Q$, there is nothing to prove, so without loss of generality we may assume that $y_1 \in Q$ and $y_2 \notin Q$. Then by Theorem 2.1, there exists a (y_2, Q)-fan F of weight at least d and width $k(G; y_2, Q)$.

In case of $k(G; y_2, Q) \geq 3$, let $F \cap Q = \{a_1, a_2, \ldots, a_m\}$. We may assume that $x, a_1, a_2, \ldots, a_m, z$ appear in the consecutive order along Q. By the choice of Q, $w(Q[a_i, a_{i+1}]) \geq w(F[a_i, a_{i+1}])$ for every i with $1 \leq i \leq m - 1$. Since $m \geq 3$, there exists l with $1 \leq l \leq m - 1$ such that $y_1 \notin Q(a_l, a_{l+1})$. Let $Q' = xQa_lFa_{l+1}Qz$. Then $\{y_1, y_2\} \subseteq Q'$ and

$$
\begin{aligned}
w(Q') &\geq w(Q'[a_1, a_m]) \\
&\geq \sum_{1 \leq i \leq m-1,\ i \neq l} w(Q[a_i, a_{i+1}]) + w(F[a_l, a_{l+1}]) \\
&\geq \sum_{i=1}^{m-1} w(F[a_i, a_{i+1}]) \\
&\geq w(F) \\
&\geq d.
\end{aligned}
$$

Hence Q' is a required path.

If $k(G; y_2, Q) = 2$, there exists $b_1, b_2 \in V(G) \setminus \{y_2\}$ such that $\{b_1, b_2\}$ separates y_2 from Q. Note that b_1, b_2 also separates y_2 from $\{x, z\}$. Since P is an (x, y_2, z)-path, $\{b_1, b_2\} \subset V(P)$. Without loss of generality, we may assume that x, b_1, b_2, z appear in the consecutive order along P. Let H be the y_2-component of $G - \{b_1, b_2\}$ and $G' = G[H \cup \{b_1, b_2\}]$. If $b_1b_2 \notin E(G)$, we add the edge b_1b_2 of weight zero to G', then G' is 2-connected. In G', by the induction hypothesis, there exists a $(b_1, \{y_1, y_2\}, b_2)$-path P' of weight at least d in case of $y_1 \in H$, and otherwise there exists a $(b_1, \{y_2\}, b_2)$-path P' of weight at least d. In both cases, $Q' = xPb_1P'b_2Pz$ contains y_1 and y_2 and $w(Q') \geq d$, hence Q' is a required path. This completes the proof of Theorem 8.3. □

8.2 An Ore-type condition

And the another aim of this chapter is to weaken the condition of Theorem 8.1 to Ore-type degree condition. The following problem may naturally be suggested.

Problem 8.4. *Let G be a 2-connected weighted graph and d a nonnegative real number. Let x and z be distinct vertices of G. If $d^w(u) + d^w(v) \geq 2d$ for every pair of nonadjacent vertices u and v in $V(G) \setminus \{x, z\}$, is it true that G contains an (x, z)-path of weight at least d?*

However, the answer to Problem 8.4 is negative. Let G_1 be the weighted complete graph of order n such that all of its edges are assigned the same weight $r > 0$, and x and z be any distinct vertices of G_1. Then G_1 satisfies the condition of Problem 8.4 for all $d > 0$, but the weight of the heaviest path in G is $(n - 1)r$. Hence if $d > (n - 1)r$, G_1 does not have any (x, z)-path of weight d or more.

There is another counterexample which is not a complete graph. Let G_2 be the weighted graph such that an edge pq is removed from a complete graph of order $n \geq 7$, and let x and z be any distinct vertices in $V(G_2) \setminus \{p.q\}$. Now we assign weight r to all the edges incident with p, and weight 0 to all the other edges. Then G_2 satisfies the condition of Problem 8.4 for $d = (n - 2)r/2$, but the weight of the heaviest path in G is $2r$. Hence G_2 does not have any (x, z)-path of weight d or more.

In each of the above examples, x and z are not connected by a heavy path, but they are connected by a *hamiltonian path*, a path containing all the vertices in a graph. Considering this fact, we prove the following theorem.

Theorem 8.5. *Let G be a 2-connected weighted graph and d a nonnegative real number. Let x and z be distinct vertices of G. If $d^w(u) + d^w(v) \geq 2d$ for every pair of nonadjacent vertices u and v in $V(G) \setminus \{x, z\}$, then G contains an (x, z)-path of weight at least d or a hamiltonian (x, z)-path.*

Moreover, extending Theorem 8.5, we prove the following.

Theorem 8.6. *Let G be a 2-connected weighted graph and d a nonnegative real number. Let x and z be distinct vertices of G, and W be a subset of $V(G)\setminus\{x,z\}$. If $d^w_{G-W}(u)+d^w_{G-W}(v) \geq 2d$ for every pair of nonadjacent vertices u and v in $V(G)\setminus(W\cup\{x,z\})$, then G contains an (x,z)-path of weight at least d or an (x,z)-path which contains all the vertices of $V(G)\setminus W$.*

In our proofs of Theorems 8.5 and 8.6, we use the following notation: For $U \subseteq V(G)$, we denote $\delta^w_G(U) = \min\{d^w_G(v) \mid v \in U\}$, and For $H \subseteq G$, we denote $\delta^w_G(H) = \delta^w_G(V(H))$.

8.3 Proof of Theorem 8.5

If $d = 0$, the assertion is obvious. Hence we may assume $d > 0$. Let $|V(G)| = n$. We use induction on n.

If $n = 3$, let y be the third vertex other than x, z. From the 2-connectedness of G, there is a path xyz, which is a hamiltonian (x,z)-path. Suppose now that $n \geq 4$ and the theorem is true for all graphs of k vertices such that $3 \leq k \leq n-1$. Let $H = G - x$.

Case 1. H is 2-connected.

Since G is 2-connected, we have $d(x) \geq 2$. Choose $x' \in N(x)\setminus\{z\}$ such that $w(xx') = \max\{w(xv) \mid v \in N(x)\setminus\{z\}\}$. Then for every $v \in V(H)\setminus\{z\}$, $d^w_H(v) \geq d^w_G(v) - w(xx')$. Hence $d^w_H(u) + d^w_H(v) \geq 2(d - w(xx'))$ for every pair of non-adjacent vertices $u, v \in V(H)\setminus\{z\}$. By the induction hypothesis, there is an (x',z)-path Q in H such that $w(Q) \geq d - w(xx')$, or Q is a hamiltonian path of H. Then the path $P = xx'Q$ is a required path.

Case 2. H is separable.

In this case, H has at least two endblocks, say B_1 and B_2. First we prove that there is a required path in case of $I_{B_1} = \{z\}$. In this case, let $G'' = G[V(G)\setminus\{z\}]$. If $xc_{B_1} \notin E(G'')$, we add the edge xc_{B_1} of weight zero to G''. Then, the resulting graph is 2-connected, and $d^w(u)+d^w(v) \geq 2d$ for every $u, v \in V(G'')\setminus\{x, c_{B_1}\}$ such that $uv \notin E(G)$. Then by the induction hypothesis, there is an (x, c_{B_1})-path Q such that $w(Q) \geq d$ or Q contains all the vertices in G''. It is obvious that Q is not the added edge xc_{B_1} itself, hence we can take the path $P = Qc_{B_1}z$ in G, which is a required path. By the same argument, we can obtain a required path in case of $I_{B_2} = \{z\}$. So we can assume that $I_{B_1} \neq \{z\}$ and $I_{B_2} \neq \{z\}$.

For $i \in \{1, 2\}$, let $d_i = \delta^w_G(I_{B_i} \setminus \{z\})$. Since $v_1 v_2 \notin E(G)$ for any vertices $v_1 \in I_{B_1} \setminus \{z\}$ and $v_2 \in I_{B_2} \setminus \{z\}$, $d_1 + d_2 \geq 2d$.

Case 2.1. $z \notin I_{B_1} \cup I_{B_2}$.

Since $d_1 + d_2 \geq 2d$, $\max\{d_1, d_2\} \geq d$. Without loss of generality, we can suppose $d_1 \geq d$. Now, let B^*_1 denote the graph obtained from $G[V(B_1) \cup \{x\}]$ by adding the edge xc_{B_1} of weight zero if $xc_{B_1} \notin E(G)$. Then B^*_1 is 2-connected and for every $v \in V(B^*_1) \setminus \{x, c_{B_1}\}$, $d^w_{B^*_1}(v) = d^w_G(v) \geq d_1$. Hence, Theorem 8.1 implies that B^*_1 has an (x, c_{B_1})-path Q_1 of weight at least $d_1 \geq d$. It is obvious that Q_1 is not the added edge xc_{B_1} itself, so Q_1 is a path in G. On the other hand, there exists a (c_{B_1}, z)-path Q_2 in $H - I_{B_1}$. Then $P = Q_1 Q_2$ is an (x, z)-path of weight at least d. □

Case 2.2. $z \in I_{B_1} \cup I_{B_2}$.

Without loss of generality, we can suppose that $z \in I_{B_1}$. If there exists $v \in I_{B_1}$ such that $w(xv) \geq d$, we can obtain an (x, z)-path of weight at least d by joining xv and any (v, z)-path in B_1. So suppose that $\max\{w(xv) \mid v \in I_{B_1}\} < d$. Then, $d^w_{B_1}(v) > d^w_G(v) - d \geq d_1 - d$ for all $v \in V(B_1) \setminus \{z, c_{B_1}\}$. Now we already have $I_{B_1} \neq \{z\}$. This implies that B_1 is 2-connected. Hence, by Theorem 8.1, there exists a (c_{B_1}, z)-path Q_1 in B_1 of weight at least $d_1 - d$. Now applying the same argument as used in Case 1 to B_2, we can obtain an (x, c_{B_2})-path Q_2 in $G[V(B_2) \cup \{x\}]$ of weight at least d_2. It is easy to see that there exists a (c_{B_1}, c_{B_2})-path Q_3 in $H - I_{B_1} - I_{B_2}$, so G has the (x, z)-path $P = Q_2 Q_3 Q_1$ of weight at least $d_1 - d + d_2 \geq 2d - d = d$.
□

8.4 Proof of Theorem 8.6

Before we prove Theorem 8.6, we prepare some terminology and lemmas. From a given graph G, we can make a new graph H whose vertices are the blocks and cutvertices of G, and two vertices of H is adjacent if and only if one is a cutvertex of G and the other is a block of G containing the cutvertex. By the definition, H contains no cycle. We call H a *block-cutvertex tree* of G.

Now we define an operation, which is edge contraction of weighted graphs, but the weight function for the resulting graph is different from the one appeared in Chapter 2. Let G be a weighed graph and $ab \in E(G)$. The graph $G_{b\to a}$ is equal to the graph G/ab, but the contracted vertex is regarded as a. To assign weights to every edge of $G_{b\to a}$, we consider a mapping φ of $E(G_{b\to a})$ to $E(G)$ such that

- for $u, v \in V(G_{b\to a}) \setminus \{a\}$, $\varphi(uv) = uv$;
- for $v \in V(G_{b\to a}) \setminus \{a\}$, if $av \in E(G)$ then $\varphi(av) = av$;
- for $v \in V(G_{b\to a}) \setminus \{a\}$, if $av \notin E(G)$ then $\varphi(av) = bv$.

Note that φ is an injection. For every edge $e \in E(G_{b\to a})$, we assign $w_{G_{b\to a}}(e) = w_G(\varphi(e))$. Next, we prepare some lemmas.

Lemma 8.7. *Let G be a weighted graph and ab be an edge of G. Then for any path P in $G_{b\to a}$, there exists a path Q in G such that*

(a) $w(P) \leq w(Q)$*;*

(b) *P and Q have the same endvertices.*

Moreover, if $d_G(b) = 2$, we can find Q which also satisfies the following.

(c) $V(P) \subseteq V(Q)$*.*

Proof. We give an orientation to P. If a is an endvertex of P, we regard a as the first vertex. For a vertex $v \in V(P)$, v^+ denotes the next vertex of v on P, and v^- denotes the last vertex before v on P.

Now we define a path Q in G. We distinguish four cases.

(i) a is an endvertex of P and $a^+ \notin N_G(a)$.

(ii) a is an endvertex of P and $a^+ \in N_G(a)$.

(iii) a is a vertex of P which is not an endvertex, and $\left|\{a^+, a^-\} \cap N_G(a)\right| = 1$.

(iv) Otherwise.

In case of (i) or (iii), let

$$Q = \bigcup_{e \in P} \varphi(e) \cup \{ab\}.$$

And, in case of (ii) or (iv), let

$$Q = \bigcup_{e \in P} \varphi(e).$$

Then Q induces a path satisfying (a) and (b). Also, if $V(P) \nsubseteq V(Q)$, then a lies on P and $\{a^+, a^-\} \cap N_G(a) = \phi$. In this case, we have $N_G(b) \supseteq \{a^+, a^-, a\}$, and hence $d_G(b) \geq 3$. This shows (c). □

Lemma 8.8. *Let G be a weighted graph, ab be an edge of G, and L be a subset of $V(G)$. If $b \notin L$, then for every $v \in L$, $d^w_{G[L]}(v) = d^w_{G_{b\to a}[L]}(v)$.*

Proof. Let $v \in L$. For every $u \in N_G(v) \cap L$, since $u \neq b$, we have $uv \in E(G_{b\to a})$ and $w_G(uv) = w_{G_{b\to a}}(uv)$. Hence we obtain $d^w_{G[L]}(v) = d^w_{G_{b\to a}[L]}(v)$. □

And, the following two lemmas are obvious.

Lemma 8.9. *Let G be a weighted graph and ab be an edge of G. For any path P in $G - ba$, there exists a path Q in G such that*

(a) $w(P) \leq w(Q)$;

(b) $V(P) \subseteq V(Q)$;

(c) P and Q have the same endvertices. □

Lemma 8.10. *Let G be a weighted graph, ab be an edge of G, and L be a subset of $V(G)$. If $b \notin L$, then $d^w_{G[L]}(v) = d^w_{G-ba[L]}(v)$ for every $v \in L$.* □

Next, we define the *elimination of $uv \in E(G)$ keeping weights in L*. Let G be a 2-connected graph of order at least 4 and $uv \in E(G)$ such that $u \notin L$. By the following theorem, we obtain that $G - uv$ or G/uv is 2-connected.

Theorem 8.11 (Tutte [33]). *Let G be a 2-connected graph of order at least 4, and e be an edge of G. Then either $G - e$ or G/e is 2-connected.*

If $G - uv$ is 2-connected, we make a new graph $G' = G - uv$. If $G - uv$ is not 2-connected, we make a new graph $G' = G_{u\to v}$. We call this operation from G to G' the *elimination of uv keeping the weight of L*. Note that the resulting graph G' is still 2-connected.

Proof of Theorem 8.6. Let G be a weighted graph satisfying the conditions of Theorem 8.6. If $W = \phi$, the assertion follows from Theorem 8.5. So we may assume $W \neq \phi$. Also in case of $V(G) \setminus (W \cup \{x, z\}) = \phi$ or $d = 0$, the assertion is obvious. So we can assume that $V(G) \setminus (W \cup \{x, z\}) \neq \phi$ and $d > 0$. Hence, $|V(G)|$ is at least 4.

Next, if $xz \notin E(G)$, let $G^* = G + xz$ and assign weight 0 to xz. Then G^* also satisfies the conditions of Theorem 8.6. If G^* has a required path P, P is not xz itself (note that $V(G) \setminus (W \cup \{x, z\}) \neq \phi$ and $d > 0$). Hence we can obtain a required path P in G. So we can assume that $xz \in E(G)$.

Let H be a component of $G - W$. We call H *trivial* if $V(H) \subseteq \{x, z\}$. Now let $\mathcal{H}$ be the set of all the non-trivial components of $G - W$.

Case 1. $|\mathcal{H}| \geq 2$.

Suppose $H_1, H_2 \in \mathcal{H}$. For $i \in \{1, 2\}$, let $d_i = \delta^w_{G-W}(H_i - \{x, z\})$. If $v_1 \in H_1$ and $v_2 \in H_2$, v_1 and v_2 are non-adjacent. Hence $d_1 + d_2 \geq 2d$. In particular, $\max\{d_1, d_2\} \geq d$. Without loss of generality, we may assume that $d_1 \geq d$. Let $L = V(H_1) \cup \{x, z\}$. Now we eliminate all the edges incident with vertices in $V(G) \setminus L$ keeping weights in L. Then the resulting graph G^* is 2-connected, and Lemmas 8.8 and 8.10 imply $d^w_{G^*}(v) \geq d$ for every v in $V(G^*) \setminus \{x, z\}$ (note that $d^w_{G-W}(v) = d^w_{H_1}(v)$ for every vertex $v \in H_1$). Hence Theorem 8.1 implies that there is an (x, z)-path of weight at least d in G^*. And by Lemmas 8.7 and 8.9, we obtain an (x, z)-path of weight at least d in G. □

Case 2. $|\mathcal{H}| = 1$.

Suppose H be the unique non-trivial component. Note that for every vertex v in H, $d^w_{G-W}(v) = d^w_H(v)$.

Case 2.1. H is 2-connected.

If $\{x, z\} \subseteq V(H)$, Let $x' = x$ and $z' = z$. If $\{x, z\} \nsubseteq V(H)$, from the 2-connectedness of G, we can take two disjoint paths Q_1 and Q_2 such that Q_1 is an (x, H)-path and Q_2 is a (z, H)-path. We denote the endvertex of Q_1 in H by x', and the endvertex of Q_2 in H by z'. In each case, Theorem 8.5 implies that there is an (x', z')-path P in H such that $w(P) \geq d$, or P contains all the vertices of H. If necessary, adding Q_1 and Q_2 to P, we obtain a required path. □

Case 2.2. H is not 2-connected.

Case 2.2.1. H has three or more endblocks.

If $\{x, z\} \subseteq V(H)$, there are at least two endblocks of H such that their internal vertices don't contain x or z (note that $xz \in E(G)$ implies that x and z are in the same block of H). Let B_1 and B_2 be such endblocks and let $x' = x$. If $\{x, z\} \not\subseteq V(H)$, take an (x, H)-path Q_1 and let x' be the endvertex of this path in H. Then there are at least two blocks such that their internal vertices don't contain x', and let B_1 and B_2 be such endblocks. In each case, $\max\{\delta^w_H(I_{B_1}), \delta^w_H(I_{B_2})\} \geq d$. Without loss of generality, we can assume that $\delta^w_H(I_{B_1}) \geq d$. Now there is an (x', c_{B_1})-path in $G - I_{B_1}$, hence Theorem 3.12 implies that there are two disjoint paths Q_2 and Q_3 such that Q_2 is an (x', B_1)-path, Q_3 is a (z, B_1)-path, and an endvertex of Q_2 or Q_3 is c_{B_1}. Let z be another endvertex of Q_2 and Q_3 in B_1. Then Theorem 8.1 implies that there is a (c_{B_1}, z)-path P of weight at least $\delta^w_H(B_1) \geq d$ in B_1. Now let $P' = Q_2 P Q_3$. If necessary, adding Q_1 to P', we obtain a required path. □

Case 2.2.2. H has only two endblocks.

We denote $L = V(H) \cup \{x, z\}$. First, we eliminate all the edges of $\{uv \mid uv \in E(G),\ u, v \in V(G) \setminus L\}$. Next, we remove all the edges e if

(a) e is incident with a vertex in $V(G) \setminus L$;

(b) $G - e$ is 2-connected.

We denote the resulting graph G'. Since the block-cutvertex tree of H is a path, $d_{G'}(v) = 2$ for every $v \in V(G') \setminus L$. Then for every $v \in V(G') \setminus L$, choose an edge incident with v and eliminate it keeping weights in L. Now we obtain a graph G'' which is 2-connected and $V(G'') = L$. Moreover, Lemmas 8.8 and 8.10 imply that for all non-adjacent vertices $u, v \in V(G'') \setminus \{x, z\}$, $d^w_{G''}(u) + d^w_{G''}(v) \geq d$. Hence, Theorem 8.5 implies that there exists an (x, z)-path P in G'' such that $w(P) \geq d$ or P is a hamiltonian path in G''. Then, Lemmas 8.7 and 8.9 imply that there exists an (x, z)-path Q in G such that $w(Q) \geq d$ or Q contains all the vertices of $V(G'') \supseteq V(H)$, which is a required path. □

8.5 Remarks

Theorem 8.5 is sharp in the following sense. Let G be a complete 3-partite graph with partite sets V_1, V_2 and V_3 such that $|V_3| > |V_1| + |V_2|$ and $|V_1| = 2$. And let $V_1 = \{x, z\}$ (See Figure 8.1). Now assign weight $d/2$ to all the edges incident with x or z, and assign weight 0 to all the other edges. Then $d^w(u) + d^w(v) \geq 2d$ for every pair of nonadjacent vertices u and v, but G contains no (x, z)-path of weight more than d or hamiltonan (x, z)-path.

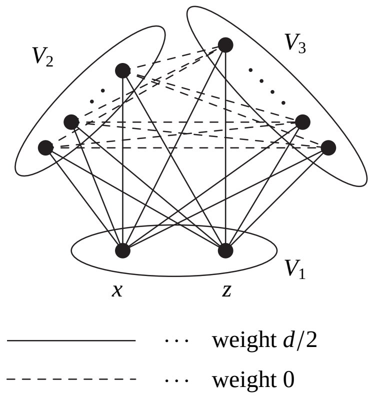

Figure 8.1:

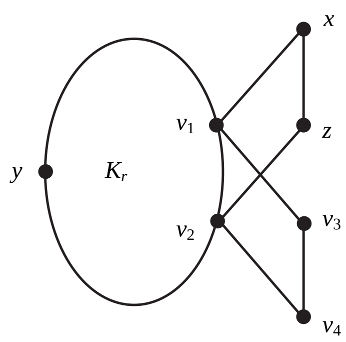

Figure 8.2:

Theorem 8.2 suggests the following problem.

Problem 8.12. *Let G be a 2-connected weighted graph and d a nonnegative real number. Let x and z be distinct vertices of G. If $d^w(u) + d^w(v) \geq 2d$ for every pair of nonadjacent vertices u and v in $V(G) \setminus \{x, z\}$, is it true that for any given vertex z of G, G contains an $(x, \{y\}, z)$-path of weight at least d or a hamiltonian $(x, \{y\}, z)$-path?*

The answer to this problem is negative. Let $H = K_r$ $(r \geq 3)$ such that $\{y, v_1, v_2\} \subseteq V(H)$. Now let G be a graph such that $V(G) = V(H) \cup \{x, z, v_3, v_4\}$, and

$$E(G) = E(H) \cup \{v_1x, v_1v_3, v_2z, v_2v_4, xy, v_3v_4\}$$

Now we assign weight d to all the edges incident with v_3 or v_4, d' to all the edges incident with y, and 0 to all the other edges (See Figure 8.2). Then G satisfies the conditions of Problem 8.12, but if $d > 2d'$, G contains no $(x, \{y\}, z)$-path of weight $\geq d$ or hamiltonan $(x, \{y\}, z)$-path.

In this example, we can enlarge the weighted degree of y by adding sufficiently many vertices to H. Hence it is no use for Problem 8.12 to add weighted degree condition for y.

We conclude this chapter with the following open problem.

Problem 8.13. *Let G be a 3-connected weighted graph and d a nonnegative real number. Let x and z be distinct vertices of G. If $d^w(u) + d^w(v) \geq 2d$ for every pair of nonadjacent vertices u and v in $V(G) \setminus \{x, z\}$, is it true that for any given vertex y of G, G contains an $(x, \{y\}, z)$-path of weight at least d?*

Chapter 9

Weighed Ramsey problem

(This chapter is based on the paper [21].)

9.1 Introduction

In 2-edge-colored complete graph, by using Ramsey-type theorems, we obtain the existence of monochromatic subgraph which have many edges compared with its order. In this chapter, we extend the concept of Ramsey problem to the weighted graphs, and we show the existence of a heavy monochromatic subgraph in 2-edge-colored graph with small order.

We say that a graph G can be *decomposed* into graphs $H_1, H_2, \dots, H_l$ if and only if there is a set $\{G_1, G_2, \dots, G_l\}$ of subgraphs of G such that each G_i is isomorphic to H_i and each edge of G is contained in exactly one of the graphs in $\{G_1, G_2, \dots, G_l\}$. In this case we also say that $\{H_1, H_2, \dots, H_l\}$ is a *decomposition* of G. Especially, we call a decomposition of a graph G into two weighted graphs R and B *a* 2-*edge-coloring* of G, so that the edges in R are colored red, and the edges in B are colored blue. For any subgraph H of a 2-edge-colored weighted graph G, we define

$$w_R(H) = \sum_{e \in E(R) \cap E(H)} w(e), \; w_B(H) = \sum_{e \in E(B) \cap E(H)} w(e).$$

In [15], some Turán-Ramsey theorems for weighted graphs in which every edge has weight 0, $1/2$ or 1, are considered. And in [6] and [23], there are some results of Turán problems for weighted graphs, in which the weight of every edge is rational number. In this chapter we deal with more general weighted graphs, i.e. every nonnegative real number is allowed for the weights of the edges. And, the aim is to introduce the Weighted Ramsey Problem, the extension of the Ramsey Problem to the weighted graphs.

Definition 9.1. *Let n and s be two integers with $n > s \geq 3$. We define $WR(s;n)$ to be the supremum value c such that for any weighting function w of K_n, and for any 2-edge-coloring R and B of K_n, there exists an induced subgraph H of order s satisfying* $\max\{w_R(H), w_B(H)\} \geq c \cdot w(K_n)$.

The following proposition shows the relation between the Ramsey number $R(s,s)$ and the weighted Ramsey number $WR(s;n)$.

Proposition 9.2. *$R(s,s) \leq n$ if*

$$WR(s;n) > \frac{s(s-1)-2}{n(n-1)}. \tag{9.1}$$

Proof. Consider a weighted complete graph G of order n such that $w(e) = 1$ for every edge e in G. By (9.1) and the fact $w(G) = n(n-1)/2$, we can find $H \simeq K_s$ such that

$$\max\{w_R(H), w_B(H)\} > \frac{s(s-1)-2}{n(n-1)} \cdot w(G) = \frac{s(s-1)-2}{2} = \frac{s(s-1)}{2} - 1.$$

Since $w(e) = 1$ for every edge in G, H is a monochromatic K_s, which implies $R(s) \leq n$. □

Since $\max\{w(R), w(B)\} \geq w(G)/2$ for any 2-edge-coloring of weighted complete graph G with order n, we easily obtain the following proposition from the straightforward averaging argument.

Proposition 9.3.

$$WR(s;n) \geq \frac{1}{2} \cdot \frac{s(s-1)}{n(n-1)}.$$

□

On the other hand, the Turán graph and its complement give an upper bound of $WR(s,n)$.

Proposition 9.4.

$$WR(s;n) < \frac{s^2-1}{s^2+1} \cdot \frac{s(s-1)}{n(n-1)}.$$

Proof. Let $T_r(n)$ be the *Turán graph*, the complete r-partite graph with n vertices whose partite sets differ in size by at most 1. Consider the 2-edge-coloring of K_n where $R \simeq T_{s-1}(n)$ and B is the complement of R. Now we assign weight

$$\frac{1}{\binom{s}{2} - 1}$$

for every red edge and weight

$$\frac{1}{\binom{s}{2}}$$

for every blue edge. Then $\max\{w_R(H), w_B(H)\} \leq 1$ for every induced subgraph H of order s and there are

$$\left(1 - \frac{1}{s-1} + f(s,n)\right)\binom{n}{2}$$

red edges and

$$\left(\frac{1}{s-1} - f(s,n)\right)\binom{n}{2}$$

blue edges, where $f(s,n)$ is a function such that $f(s,n) > 0$ for every s, n and $f(s,n) \to 0$ as $n \to \infty$. Hence

$$\begin{aligned} w(G) &= \frac{2}{s(s-1)-2} \cdot \left(1 - \frac{1}{s-1} + f(s,n)\right)\binom{n}{2} + \frac{2}{s(s-1)} \cdot \left(\frac{1}{s-1} - f(s,n)\right)\binom{n}{2} \\ &> \left(\frac{s^2+1}{(s-1)^2 s(s+1)}\right) \cdot 2 \cdot \binom{n}{2} \\ &= \frac{s^2+1}{(s-1)(s+1)} \cdot \frac{n(n-1)}{s(s-1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} WR(s;n) &\leq \frac{1}{w(G)} \\ &< \frac{s^2-1}{s^2+1} \cdot \frac{s(s-1)}{n(n-1)}. \end{aligned}$$

□

In this chapter, we determine exact value of $WR(3;n)$ for $n = 5$ and 6.

Theorem 9.5. $WR(3;5) = 1/5$.

Theorem 9.6. $WR(3;6) = 1/7$.

We prove Theorems 9.5 and 9.6 in the later section. By Proposition 9.2, we obtain that Theorem 9.6 implies the fact $R(3,3) \leq 6$. Note that Theorem 9.5 implies that the equality

$$WR(s;n) = \frac{s(s-1)-2}{n(n-1)}$$

holds for $s = 3$ and $n = 5$. In this sense, we can say that the fact $R(3,3) > 5$ is optimal even for weighted graphs.

By using Theorem 9.6, we can improve the lower bound of $WR(3;n)$ in Proposition 9.3.

Proposition 9.7. *If $n \geq 6$, then*

$$WR(3; n) \geq \frac{30}{7} \cdot \frac{1}{n(n-1)}.$$

Proof. Let G be a weighted complete graph of order n. By the straightforward averaging argument, we obtain the existence of a subgraph $G' \simeq K_6$ in G such that

$$w(G') \geq \frac{30}{n(n-1)}.$$

Then, it follows from Theorem 9.6 that there exists an induced subgraph $H \simeq K_3$ satisfying

$$\max\{w_R(H), w_B(H)\} \geq w(G')/7 \geq \frac{30}{7} \cdot \frac{1}{n(n-1)},$$

which implies the assertion. □

We shall discuss the value $WR(3; n)$ further in Section 9.5.

9.2 Lemmas

For a graph B, we say $E(B)$ is *connected* if $E(B)$ induces a connected graph. A path with r vertices is denoted by P_r, and the graph $K_{1,r}$ is called a *star*. In a star $K_{1,r}$, the vertex of degree r is called its *center*, and degree 1 its *leaf*. The star with the center u and the leaves $v_1, v_2, \ldots, v_r$ is denoted by $u\text{-}v_1 v_2 \ldots v_r$. A graph is called *claw-free* if it contains no $K_{1,3}$ as an induced subgraph.

To prove Theorems 9.5 and 9.6, for the technical reason, we consider the following weighting functions for a given graph B;

$$\mathcal{W}(B) = \{w : E(B) \to \mathbb{R}^+ \mid w(B') \leq 6 \text{ for any subgraph } B' \text{ of } B \text{ with } |B'| \leq 3\},$$

and investigate the following invariant.

$$W(B) = \sup\{w(B) \mid w \in \mathcal{W}(B)\}.$$

Now we prepare some facts and lemmas, which determine the values of $\mathcal{W}(B)$ for several graphs B. The following fact is obvious, so we omit the proof.

Fact 9.1. *Let B be a graph with at most 6 vertices. If $|E(B)| \geq 8$, then $E(B)$ induces a connected graph.* □

Lemma 9.8. *Let B' be a subgraph of B, then $W(B') \leq W(B)$.*

Proof. Assume that $w'(B') > W(B)$ for some $w' \in \mathcal{W}(B')$. Consider the weighting function w such that $w(e) = w'(e)$ if $e \in B'$ and $w(e) = 0$ if $e \notin B'$. Then it is clear that $w(B) = w'(B') > W(B)$ and $w \in \mathcal{W}(B)$, which contradicts the definition of $W(B)$. □

Lemma 9.9. *If B is an edge-disjoint union of the graphs B_1 and B_2, then $W(B) \leq W(B_1) + W(B_2)$.*

Proof. If $w \in \mathcal{W}(B)$ and $w(B) > W(B_1) + W(B_2)$, then $w(B_i) > W(B_i)$ for $i = 1$ or 2, which contradicts the definition of $W(B_i)$. □

Lemma 9.10. *If B is a star $K_{1,r}$ with $r \geq 2$, then $W(B) = 3r$.*

Proof. Let u be the center of B, let $v_1, v_2, \ldots, v_r$ be the leaves of B and let $v_{r+1} = v_1$. For any $w \in \mathcal{W}(B)$, we have $w(v_i u v_{i+1}) \leq 6$ for every i, where the index i is taken as modulo r. Hence

$$w(B) = \frac{1}{2}\sum_{i=1}^{r} w(v_i u v_{i+1}) \leq \frac{1}{2} \cdot 6r = 3r.$$

The constant weight with $w(e) \equiv 3$ shows that $W(B) = 3r$. □

Lemma 9.11. *If B is a cycle with length at least 4, then $W(B) = 3|E(B)|$.*

Proof. Let $B = v_1 v_2 \ldots v_r$. For any $w \in \mathcal{W}(B)$, we have $w(v_i v_{i+1} v_{i+2}) \leq 6$ for every i, where the index i is taken as modulo r. Hence

$$w(B) = \frac{1}{2}\sum_{i=1}^{r} w(v_i v_{i+1} v_{i+2}) \leq \frac{1}{2} \cdot 6r = 3|E(B)|.$$

The constant weight with $w(e) \equiv 3$ shows that $W(B) = 3|E(B)|$. □

Lemma 9.12. *If B is a complete graph of order $n \geq 3$, then $W(B) = n(n-1)$.*

Proof. Let $\mathcal{T} = \{T \mid T$ is a triangle in $B\}$ and let $w \in \mathcal{W}(B)$. Then $w(T) \leq 6$ for all $T \in \mathcal{T}$. Hence

$$\begin{aligned} w(B) &= \frac{1}{n-2}\sum_{T\in\mathcal{T}} w(T) \\ &\leq \frac{1}{n-2} \cdot \binom{n}{3} \cdot 6 \\ &= n(n-1). \end{aligned}$$

The constant weight with $w(e) \equiv 2$ shows that $W(B) = n(n-1)$. □

Lemma 9.13. *If C is a cycle of length $r \geq 4$ and $B = K_1 + C$, then $W(B) = 9r/2$.*

Proof. Let $C = v_1 v_2 \dots v_r$ and let u be the vertex of $V(B) \setminus V(C)$. Moreover, let T_i be the triangle uv_iv_{i+1}, where the indices i and j are taken as modulo r. If $w \in \mathcal{W}(B)$, then by Lemma 9.11, we have $w(C) \leq 3r$. Hence,

$$\begin{aligned} W(B) &\leq w(B) \\ &= \frac{1}{2}\Big(\sum_{i=1}^{r} w(T_i) + w(C)\Big) \\ &\leq \frac{1}{2} \cdot (6r + 3r) \\ &\leq \frac{9}{2}r. \end{aligned}$$

On the other hand, there is $w \in \mathcal{W}(B)$ such that $w(e) = 3$ for every $e \in E(C)$ and $w(e) = 3/2$ for all the other edges. This shows $W(B) = 9r/2$. □

Lemma 9.14. *If $B \simeq K_6 - E(3K_2)$, then $W(B) = 24$.*

Proof. Let $E(\overline{B}) = \{a_1b_1, a_2b_2, a_3b_3\}$. Then B can be decomposed into four triangles $a_1a_2a_3$, $a_1b_2b_3$, $b_1a_2b_3$ and $b_1b_2a_3$. For any $w \in \mathcal{W}(B)$, each of them has weight at most 6, hence we have $w(B) \leq 24$. The constant weight with $w(e) \equiv 2$ shows that $W(B) = 24$. □

Las Vergnas [26] and Sumner [32] proved that every connected claw-free graph of even order has a 1-factor. Since the line graph of any graph is claw-free, we obtain that if B is a connected graph with $|E(B)|$ even, then its line graph has a 1-factor. This implies the following fact.

Fact 9.2. *Let B be a connected graph with $|E(B)|$ even. Then B can be partitioned into $|E(B)|/2$ pairs of adjacent edges.* □

And, the following fact is easily obtained from Fact 9.2.

Fact 9.3. *Let B be a connected graph with $|E(B)|$ odd. Then B can be partitioned into an edge and $(|E(B)| - 1)/2$ pairs of adjacent edges.* □

Using these facts, we obtain the following lemma.

Lemma 9.15. *Suppose that B is a connected graph. If B is a tree with a perfect matching, then $W(B) = 3|E(B)| + 3$. Otherwise, $W(B) \leq 3|E(B)|$.*

Proof. If $|E(B)|$ is even, then B can be decomposed into $|E(B)|/2$ edge-disjoint P_3s. Hence, by Lemma 9.9, $W(B) \leq (|E(B)|/2)W(P_3) = 3|E(B)|$.

Suppose that $|E(B)|$ is odd. Since B can be decomposed into $(|E(B)| - 1)/2$ edge-disjoint P_3s and one K_2, then by Lemma 9.9 again, we have $W(B) \leq ((|E(B)| - 1)/2)W(P_3) + W(K_2) = 3(|E(B)| - 1) + 6 = 3|E(B)| + 3$. In fact, when B is a tree with a perfect matching M, if we assign $w(e) = 6$ for $e \in M$ and $w(e) = 0$ for $e \notin M$, then $w \in \mathcal{W}(B)$ and $w(B) = 3|V(B)| = 3|E(B)| + 3$, which shows that $W(B) = 3|E(B)| + 3$.

Suppose next that B is a tree without perfect matchings. Recall that $|V(B)| = |E(B)| + 1$ is even. Then, there exists a vertex v such that $B - v$ contains at least three odd components B_1, B_2 and B_3. Let v_i be the neighbor of v in B_i for $i = 1, 2, 3$. It is easy to see that each component of $B - \{vv_1, vv_2, vv_3\}$ has even number of edges. This implies that B can be decomposed into $(|E(B)| - 3)/2$ edge-disjoint P_3s and one $K_{1,3}$. By Lemmas 9.9 and 9.10, we have

$$W(B) \leq ((|E(B)| - 3)/2)W(P_3) + W(K_{1,3}) = 3(|E(B)| - 3) + 9 = 3|E(B)|.$$

Suppose that B contains a cycle. We use induction on $|E(B)|$ to prove that $W(B) \leq 3|E(B)|$. Let C be a cycle in B. If $B = C$ itself, then by Lemma 9.11, we have $W(B) \leq 3|E(B)|$. Let T be a unicyclic spanning subgraph of B such that $C \subseteq T$. Since $B \neq C$, we can take a leaf u of T which is farthest from C in T. If there are two edges uu_1 and uu_2 in $E(B) \setminus E(T)$, then let $B' = B - \{uu_1, uu_2\}$. Since B' is connected and $C \subseteq B'$, by the induction hypothesis, we have $W(B') \leq 3|E(B')|$. Then, it follows that $W(B) \leq W(B') + W(P_3) \leq 3|E(B')| + 6 = 3|E(B)|$. Similarly, if we can take a $P \simeq P_3$ containing u such that $E(P) \cap E(C) = \emptyset$ and $E(B) \setminus E(P)$ induces a connected subgraph, then by induction, we have $W(B) \leq W(B - E(P)) + W(P) \leq 3(|E(B)| - 2) + 6 = 3|E(B)|$. This is the case unless $d_B(u) = 1$ and the unique neighbor v of u in B is in C and $d_T(v) = 3$. In this case, let v_1 and v_2 be the neighbor of v in C. It is easy to see that $B - \{vu, vv_1, vv_2\}$ is connected. This implies that B can be decomposed into $(|E(B)| - 3)/2$ edge-disjoint P_3s and one $K_{1,3}$. By Lemmas 9.9 and 9.10, we have

$$W(B) \leq ((|E(B)| - 3)/2)W(P_3) + W(K_{1,3}) = 3(|E(B)| - 3) + 9 = 3|E(B)|.$$

□

9.3 Proof of Theorem 9.5

Let G be a 2-edge-colored complete graph with 5 vertices. If each edge of G has weight 3 and $R \simeq B \simeq C_5$, then $w(G) = 30$ and $\max\{w_R(H), w_B(H)\} \leq 6 = w(G)/5$ for every triangle H in G, hence we have $WR(3; 5) \leq 1/5$. To prove the lower bound, we assume $\max\{w_R(T), w_B(T)\} \leq 6$ for every triangle T in G. Namely, the restricted weightings $w|_{E(R)}$ and $w|_{E(B)}$ are contained in $W(R)$ and $W(B)$, respectively. Then it suffices to prove that $w(G) \leq 30$. If G has no monochromatic K_3, It is easy to see that $R \simeq B \simeq C_5$. Then Lemma 9.11 implies that $w(R) \leq 15$ and $w(B) \leq 15$, hence we obtain $w(G) = w(R) + w(B) \leq 30$. So we may assume that there exists a monochromatic triangle in G.

Now we consider the case where G has a monochromatic $2K_2 + K_1$. Without loss of generality we may assume that $2K_2 + K_1 \subseteq B$. Note that $R \subseteq C_4$. If $R \simeq K_2$ or P_3, we have $w(R) \leq 6$, and Lemmas 9.8 and 9.12 imply $w(B) \leq 20$. Hence $w(G) = w(R) + w(B) \leq 30$. Otherwise, $2K_2 \subseteq R$. Then $B \subseteq C_4 + K_1$, so Lemmas 9.8 and 9.13 imply that $w(B) \leq 18$. On the other hand, since $R \subseteq C_4$, we have $w(R) \leq 12$ by Lemmas 9.8 and 9.11. Thus $w(G) = w(R) + w(B) \leq 30$.

Therefore, we may assume that G has a monochromatic K_3 but no monochromatic $2K_2 + K_1$. Without loss of generality we may assume that $K_3 \subseteq B$. Since $2K_2 + K_1 \nsubseteq B$, we have $|E(R)| \geq 3$.

In case of $|E(R)| = 3$, $R \simeq P_4, K_{1,3}, K_3$ or $P_3 \cup K_2$. By Lemmas 9.9 and 9.15, we obtain $w(R) \leq 12$ in each case. Let B' be a graph obtained from B by deleting the edges of a triangle in B. Then $|E(B')| = 4$ and $E(B')$ must be connected, hence Lemma 9.15 shows $w(B') \leq 12$, which implies $w(B) \leq 18$. Thus $w(G) = w(R) + w(B) \leq 30$. In case of $|E(R)| = 4$, $E(R)$ must be connected, hence Lemma 9.15 implies that $w(R) \leq 12$. Since $|E(B)| = 6$, $E(B)$ is also connected, hence it follows from Lemma 9.15 that $w(B) \leq 18$. Therefore, we have $w(G) = w(R) + w(B) \leq 30$.

If $|E(R)| = 5$, then $3K_2 \nsubseteq R$ and $3K_2 \nsubseteq B$. Hence Lemma 9.15 implies $w(R), w(B) \leq 15$, thus we have $w(G) = w(R) + w(B) \leq 30$. If $|E(R)| = 6$, since $E(R)$ is connected, we have $w(R) \leq 18$ by Lemma 9.15. If $E(B)$ is connected, Lemma 9.15 implies $w(B) \leq 12$, and otherwise $B \simeq K_2 \cup K_3$, so $w(B) \leq 12$. Thus we have $w(G) \leq 30$. And if $|E(R)| = 7$, Lemma 9.15 implies $w(R) \leq 21$. Now the fact $B \simeq K_3$ shows $w(B) \leq 6$, hence we have $w(G) \leq 30$. This completes the proof of Theorem 9.5. □

9.4 Proof of Theorem 9.6

Let G be a 2-edge-colored complete graph with 6 vertices and $R \simeq 3K_2$. If each edge of R has weight 6 and each edge of B has weight 2, then $w(G) = 42$ and $\max\{w_R(H), w_B(H)\} \leq 6 = w(G)/7$ for every triangle H in G. Hence we have $WR(3;5) \leq 1/7$. To prove the lower bound, as in the proof of Theorem 9.5, we assume $w|_{E(R)} \in W(R)$ and $w|_{E(B)} \in W(B)$, and then it suffices to prove $w(G) \leq 42$. Without loss of generality, we may assume that $|E(R)| < 8 \leq |E(B)|$.

Case 1. $|E(R)| \leq 2$.

In this case it is obvious that $w(R) \leq 12$, and Lemmas 9.8 and 9.12 imply that $w(B) \leq 30$, hence $w(G) \leq 42$.

Case 2. $|E(R)| = 3$.

In this case, $R \simeq P_4$, $K_{1,3}$, K_3, $P_3 \cup K_2$ or $3K_2$. If $R \not\simeq 3K_2$, then we obtain $w(R) \leq 12$ by Lemmas 9.9 and 9.15. On the other hand, Lemmas 9.8 and 9.12 imply that $w(B) \leq 30$, thus we have $w(G) \leq 42$. If $R \simeq 3K_2$, then $w(R) \leq 18$. Since Lemma 9.14 implies $w(B) \leq 24$, we obtain $w(G) \leq 42$.

Case 3. $|E(R)| = 4$.

Since $|E(B)| = 11$, there exists a triangle in B, say T. Let $B' = B - E(T)$, then it follows from Fact 9.1 that $E(B')$ is connected. Hence Lemma 9.15 implies $w(B') \leq 24$. Thus we have $w(B) = w(B') + w(T) \leq 30$.

Now suppose that $E(R)$ is connected. Then by Lemma 9.15, we have $w(R) \leq 12$, which implies $w(G) \leq 42$. Hence we may assume that $E(R)$ is not connected, then $R \simeq 2P_3$, $K_2 \cup K_3$, $K_2 \cup K_{1,3}$, or $K_2 \cup P_4$. If $R \simeq 2P_3$ or $K_2 \cup K_3$, then Lemmas 9.9 and 9.15 imply $w(R) \leq 12$, hence $w(G) \leq 42$. If $R \simeq K_2 \cup K_{1,3}$, we have $w(R) \leq 15$ by Lemma 9.10. Let v_1 and v_2 be the vertices of K_2, let v_3 be the center of $K_{1,3}$, and let v_4, v_5, v_6 be leaves of $K_{1,3}$ in R. Then B can be decomposed into two triangles $v_1v_4v_6$, $v_2v_5v_6$ and a cycle $v_1v_3v_2v_4v_5$. Hence by Lemma 9.11, we obtain $w(B) \leq 27$, which implies $w(G) \leq 42$. If $R \simeq K_2 \cup P_4$, by Lemma 9.15, we have $w(R) \leq 18$. Since $B \subseteq K_6 - E(3K_2)$, we have $w(B) \leq 24$ by Lemmas 9.8 and 9.14, therefore $w(G) \leq 42$. □

Case 4. $|E(R)| = 5$.

Since $|E(B)| = 10$, there exists a triangle in B, say T. Let $B' = B - E(T)$. Since $|E(B')| = 7$ and B' is a graph obtained by deleting a triangle from B, B' is connected. Hence we have $w(B') \leq 21$ by Lemma 9.15. Thus $w(B) = w(B') + w(T) \leq 27$. If $w(R) \leq 15$, we are done, so we assume that $w(R) > 15$. Now Lemma 9.15 implies that one of the component of R is a tree with a perfect matching. Considering $|E(R)| = 5$, we have $w(R) \leq 18$ and $3K_2 \subseteq R$ by Lemma 9.15. Then $B \subseteq K_6 - E(3K_2)$, hence by Lemmas 9.8 and 9.14 we have $w(B) \leq 24$, which implies $w(G) \leq 42$. □

Case 5. $|E(R)| = 6$.

First assume that $E(R)$ is not connected, then $R \simeq K_2 \cup K_4^-$ (K_4^- is the graph obtained from K_4 by deleting just one edge) or $2K_3$. If $R \simeq K_2 \cup K_4^-$, then the fact $K_4^- \subseteq K_4$ and Lemmas 9.8 and 9.12 imply that $w(R) \leq 6 + 12 = 18$. Let v_1 and v_2 be vertices of K_2, and v_3, v_4, v_5, v_6 be vertices of K_4^- in R such that $v_3v_4 \notin E(R)$. Then B has a triangle $T = v_1v_3v_4$. Let $B' = B - E(T)$, then $E(B')$ is connected and $|E(B')| = 6$, hence Lemma 9.15 implies $w(B') \leq 18$. Thus $w(B) \leq 24$, which implies $w(G) \leq 42$. In case of $R \simeq 2K_3$, we have $w(R) \leq 12$. Now $B \simeq K_{3,3}$, hence Lemma 9.15 implies $w(B) \leq 27$. Therefore we have $w(G) \leq 42$.

In the case where $E(R)$ is connected, by Lemma 9.15, we have $w(R) \leq 18$. Since $B \not\simeq K_{3,3}$ and $|E(B)| = 9$, there exists a triangle T in B. Let $B' = B - E(T)$, then $|E(B')| = 6$. So if we change B' into R and use the same argument as above, we obtain $w(B') \leq 18$. Hence $w(B) \leq 24$, this implies $w(G) \leq 42$. □

Case 6. $|E(R)| = 7$.

In case of R is not connected, $R \simeq K_2 \cup K_4$. Hence Lemmas 9.9 and 9.15 imply $w(R) \leq 18$. And if R is connected, Lemma 9.15 implies that $w(R) \leq 21$. Now suppose that B has a triangle T and let $B' = B - E(T)$. If $w(B') \leq 15$, we have $w(B) \leq 21$, this implies $w(G) \leq 42$. Hence we may assume that $w(B') > 15$, then Lemma 9.15 implies that $w(B') \leq 18$ and B' contains $3K_2$. Let F_1, F_2 and F_3 be the components of $3K_2$ in B', then each of them must contain just one vertex of T. Let $F_1 = a_1b_1$, $F_2 = a_2b_2$, $F_3 = a_3b_3$. Without loss of generality, we may assume that $T = a_1a_2a_3$. Let H be a graph such that $V(H) = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and $E(H) = E(T) \cup E(F_1) \cup E(F_2) \cup E(F_3)$, then $R \subseteq \overline{H}$. Since $\overline{H}$ can be decomposed into three triangles $a_1b_2b_3$, $b_1a_2b_3$ and $b_1b_2a_3$, we have $w(R) \leq 18$ by Lemma 9.8. Now $w(B) = w(B') + w(T) \leq 18 + 6 = 24$. Hence $w(G) \leq 42$, therefore we may assume that B is triangle-free.

Next, suppose that B has a C_5, say C. Since B is triangle-free, there is no chord in C. Hence the vertex which is not in C must adjacent to three vertices of the C, however this makes triangle in B, a contradiction.

Therefore, we may assume that B is bipartite. It follows from the fact $E(B)$ is connected and Lemma 9.15 that $w(B) \leq 24$. If $B \subseteq K_{3,3}$, then R can be decomposed into two triangles and a K_2. Hence $w(R) \leq 18$, which implies $w(G) \leq 42$. Otherwise, $B \simeq K_{2,4}$. Then R can be decomposed into a K_4 and a K_2. Hence by Lemma 9.12 we have $w(R) \leq 18$, which implies $w(G) \leq 42$. This completes the proof of Theorem 9.6. □

9.5 Weighted Ramsey number for large graphs

In this section, we observe the relation between the value $WR(3; n)$ and the number of edge-disjoint monochromatic triangles in 2-edge-colored graphs with n vertices, for sufficiently large n. Let $N(n, k)$ be the minimum number of pairwise edge-disjoint monochromatic complete subgraphs K_k in any 2-edge-coloring of a K_n.

Proposition 9.16.

$$WR(3; n) \geq \frac{4}{n^2 - 2N(n, 3) + n}.$$

Proof of Proposition 9.16. Let G be a 2-colored graph with n vertices and set $m = N(n, 3)$. As in the proofs of Theorems 9.5 and 9.6, we assume $\max\{w_R(T), w_B(T)\} \leq 6$ for every triangle T in G and prove that

$$w(G) \leq 3n^2/2 - 3m + 3n/2.$$

Let $\mathcal{T}$ be a set of edge-disjoint monochromatic triangles of cardinality m, $E(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} E(T)$, R' be the graph induced by $E(R) \setminus E(\mathcal{T})$ and B' be the graph induced by $E(B) \setminus E(\mathcal{T})$. Since both of R' and B' have at most $n/2$ components, using Facts 9.2 and 9.3, we can find $(|E(R')| + |E(B')| - l)/2$ pairwise edge-disjoint monochromatic paths of length two in $R' \cup B'$, where $l \leq 2 \cdot n/2 = n$. Let $\mathcal{P}$ be the set of such paths, $E(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} E(P)$, and $I = E(G) \setminus (E(\mathcal{T}) \cup E(\mathcal{P}))$.

Then $|I| = l \leq n$ and

$$\begin{aligned} |\mathcal{P}| &= \frac{|E(G)| - |E(\mathcal{T})| - |I|}{2} \\ &\geq \frac{\frac{n(n-1)}{2} - 3m - l}{2} \\ &\geq \frac{n^2 - 6m - 3n}{4}. \end{aligned}$$

Therefore,

$$\begin{aligned} w(G) &= \sum_{T \in \mathcal{T}} w(T) + \sum_{P \in \mathcal{P}} w(P) + \sum_{e \in I} w(e) \\ &\leq 6|\mathcal{T}| + 6|\mathcal{P}| + 6|I| \\ &\leq 6m + 6 \cdot \frac{n^2 - 6m - 3n}{4} + 6n \\ &= \frac{3}{2}n^2 - 3m + \frac{3}{2}n. \end{aligned}$$

Then,

$$WR(3; n) \geq \frac{6}{w(G)} = \frac{4}{n^2 - 2N(n; 3) + n}.$$

□

In [14], considering the Turán graph $T_2(n)$ and its complement, the following conjecture is given.

Conjecture 9.17 (Erdős).

$$N(n, 3) = \left(\frac{1}{12} + o(1)\right) n^2.$$

If this conjecture is true, then Proposition 9.16 implies

$$\begin{aligned} WR(3; n) &\geq \frac{4}{n^2 - 2\left(\frac{1}{12} + o(1)\right) n^2} \\ &= \left(\frac{24}{5} + o(1)\right) \frac{1}{n^2}. \end{aligned}$$

The coefficient of n^{-2} in this lower bound is the same as the coefficient of n^{-2} in the upper bound of Proposition 9.4 for $s = 3$. Considering this, we state the following conjecture.

Conjecture 9.18.

$$WR(3; n) = \left(\frac{24}{5} + o(1)\right) \frac{1}{n^2}.$$

Recently, about the lower bound of $N(n,3)$, Keevash and Sudakov showed the following result.

Theorem 9.19 (Keevash and Sudakov [25]).

$$N(n,3) \geq \left(\frac{1}{12.89} + o(1)\right) n^2.$$

By using Proposition 9.16, we have

$$\begin{aligned} WR(3;n) &\geq \frac{4}{n^2 - 2\left(\frac{1}{12.89} + o(1)\right) n^2} \\ &= \left(\frac{51.56}{10.89} + o(1)\right) \frac{1}{n^2} \\ &\geq (4.73 + o(1)) \frac{1}{n^2}. \end{aligned}$$

which improves the lower bound in Proposition 9.7.

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