Operator semi-selfsimilar processes and one of their constructions as limiting processes of random walks in random sceneries

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Chapter 1

Introduction

Brownian motion is the most important stochastic process in probability theory. This process is obtained by mathematical modeling of irregular movement of pollen, suspended in water. Brownian motion has many mathematical properties such as independent and stationary increments, Gaussianity, or Markovian property, etc. Each of these is a worth concept in probability theory and useful for modeling of random phenomena. Selfsimilarity is one of them. We call a stochastic process “selfsimilar” if for any $a > 0$, time scaling parameter, there exists $b > 0$, space scaling parameter such that laws of $\{X(at)\}$ and $\{bX(t)\}$ are the same in the sense of finite dimensional distributions. If b equals to a^H with some $H > 0$, then this process is said to be H-selfsimilar. For example, Brownian motion is 1/2-selfsimilar.

Selfsimilar processes have been widely studied, and some extensions are considered. One of the extensions is “semi-selfsimilarity”, where $\{X(at)\}$ and $\{bX(t)\}$ have the same laws for some $a \neq 1$. Another extension is “operator selfsimilarity”. In a multidimensional case, space scaling parameter can be taken as linear operator. By these extensions, we can expect more flexible modeling of random phenomena. For example, Maejima and Sato, and Becker-Kern studied operator semi-selfsimilar processes with independent increments in [MaSa03] and [Be04]. However, some problems are left. In this thesis, we consider some aspects of operator semi-selfsimilar processes, namely, its general theory and concrete examples. The organization of this thesis is the following:

In chapter 2, mainly we describe known results. In Section 2.1, we give the definition of selfsimilarity and existence of its exponent. In Section 2.2, definitions of selfsimilarity and stability are given. These distributions have deep relations to selfsimilar processes. These relations are extended in following chapters. In section 2.3, we show new result, local limit theorem for semi-stable Lévy processes.

In chapter 3, we consider processes extended to two directions, semi-selfsimilarities and operator case. Selfsimilar processes are studied widely and many properties are known. Operator semi-selfsimilar processes have similar properties to those of selfsimilar processes. For selfsimilar processes, the space scaling parameter b is represented by time scaling parameter a and an exponent H as $b = a^H$. This representations still valid for the cases of semi-selfsimilar or operator selfsimilar processes. On the other hand, in the case of operator semi-selfsimilar processes, space scaling parameter B is represented by a time scaling parameter a, an exponent matrix H and sign matrix S. Marginal distributions of processes are also effected by the sign matrix. We investigate for operator semi-stable Lévy processes.

In chapter 4, we consider stochastic integrals with respect to the random measures induced by operator semi-stable Lévy processes. The integrated processes have operator semi-selfsimilarity with stationary but not necessarily with independent increments. In [MaM94], they defined integral related to operator stable processes, and in [MaSa99], they dealt semi-stable cases. In Section 4.1, we will give definition of stochastic integral by operator semi-stable. Conditions of integrand are also given. In Section 4.2, we investigate properties of special case of this integrated process as integrand function has selfsimilarity. In [V87], he dealt with the case where integrand function has 1-dimensional selfsimilarity and seek the relation among selfsimilarities of integrand process, integrated one and obtained one. We also inquire operator semi-stable case.

In chapter 5 ,we consider Kesten and Spitzer's problem and construct the example of operator semi-selfsimilar processes. Kesten and Spitzer considered "Random walks in random scenery" in [KS79] as follows: Let $\mathbf{Z}$-valued random variables X_i's and $\mathbf{R}$-valued random variables $\xi(k)$'s belong to the domain of attraction of strictly α-stable ($\alpha \in (1, 2]$) distribution and that of strictly β-stable ($\beta \in (0, 2]$) distribution, respectively. Assume that they are independent and $E[X_1] = 0$. We set $W_l = \sum_{k=0}^{l} \xi(S_k)$, where $S_k = \sum_{i=1}^{k} X_i$ and $S_0 = 0$. Asymptotic behavior of $\{W_n\}$ is determined by two kinds of randomness, random walks $\{S_n\}$ and random scenery $\{\xi(k)\}$, and they imply an interesting selfsimilarity for a scaled random walks. We assume that X_i's and $\xi(k)$'s belong to the domains of partial attraction of strictly semi-stable and that of strictly operator semi-stable distribution, respectively. Semi-stable distributions are infinitely divisible, and to any infinitely divisible distribution there corresponds a Lévy process, which is a process having independent and stationary increments, stochastic continuity, and starting at 0. We show that the scaled $\{W_n\}$ converges weakly to a process, which is defined by stochastic integrals in chapter 4.

Throughout the thesis, we mainly deal two topics of operator semi-selfsimilar processes. One is general theory, especially an exponent of semi-selfsimilarity. In the case of a selfsimilar process, there always exsits an exponent. Namely, a space scaling matrix is "positive". However for the semi-selfsimilar case, a matrix of sign is necessary. This indicates that a semi-selfsimilar process allows "negativity" for space scaling matrices. The other is the example of operator semi-selfsimilar processes. Operator semi-selfsimilar processes are defined formally and have few examples. Thus, we construct the example of operator semi-selfsimilar processes by using some stochastic integral. However, this is still abstract. To realize this stochastic integral, we construct a concrete model by using random walks in random sceneries.

Chapter 2

Definitions of selfsimilarity and stability

2.1 Selfsimilar, semi-selfsimilar and operator selfsimilar processes

As mentioned in Introduction, selfsimilar, semi-selfsimilar and operator selfsimilar processes are already widely studied. In Section 2.1, we introduce some of known results. These are extended to operator semi-selfsimilar cases in the following chapter. In Section 2.2, we explain stable distributions. In the following chapters, there are many relations among operator semi-selfsimilar processes, operator stable distributions.

Most stochastic processes discussed in this chapter are $\mathbf{R}^d$-valued processes and defined on a common probability space $(\Omega, \mathcal{F}, P)$. We give a definition of selfsimilarity and semi-selfsimilarity of processes and operator selfsimilar processes as follows:

Definition 2.1.1

(1) *An $\mathbf{R}^d$-valued stochastic process $\{X(t), t \geq 0\}$ is called wide sense selfsimilar if for any $a > 0$, there exist $b > 0$ and a non-random function $c : [0, \infty) \to \mathbf{R}^d$ such that such that*

$$\{X(at)\} \stackrel{\mathrm{d}}{=} \{bX(t) + c(t)\}, \tag{2.1.1}$$

(2) *an $\mathbf{R}^d$-valued stochastic process $\{X(t), t \geq 0\}$ is called wide sense semi-selfsimilar if there exists a $a \in (0, 1) \cup (1, \infty), b > 0$ and a non-random function $c : [0, \infty) \to \mathbf{R}^d$ such that such that*

$$\{X(at)\} \stackrel{\mathrm{d}}{=} \{bX(t) + c(t)\}, \tag{2.1.2}$$

(3) *an* $\mathbf{R}^d$*-valued stochastic process* $\{X(t), t \geq 0\}$ *is called wide sense operator selfsimilar if for any* $a > 0$*, there exist real* $d \times d$ *matrix* B *and a non-random function* $c : [0, \infty) \to \mathbf{R}^d$ *such that*

$$\{X(at)\} \stackrel{\mathrm{d}}{=} \{BX(t) + c(t)\}, \tag{2.1.3}$$

(4) *an* $\mathbf{R}^d$*-valued stochastic process* $\{X(t), t \geq 0\}$ *is called wide sense operator semi-selfsimilar if for some* $a \in (0, 1) \cup (1, \infty)$*, there exist real* $d \times d$ *matrix* B *and a non-random function* $c : [0, \infty) \to \mathbf{R}^d$ *such that*

$$\{X(at)\} \stackrel{\mathrm{d}}{=} \{BX(t) + c(t)\}, \tag{2.1.4}$$

where $\stackrel{\mathrm{d}}{=}$ *denotes equality in all joint distributions with respect to* P*. The number* b *in* (2) *is called a span of a semi-selfsimilar process.*

We say that $\{X(t), t \geq 0\}$ is stochastically continuous at t if

$$\lim_{h \to 0} P\{|X(t+h) - X(t)| > \varepsilon\} = 0, \quad \text{for any } \varepsilon > 0.$$

We also say that $\{X(t), t \geq 0\}$ is trivial if the distribution of $\{X(t)\}$ is a δ-distribution for every $t > 0$. The following relations between a and b in the above definition are known:

Theorem 2.1.2 **(**[La62]**,** [MaSa99] and [Sa91]**)**

(1) *If a wide sense selfsimilar process* $\{X(t), t \geq 0\}$ *is non-trivial,* X_0 *=const. almost surely and stochastically continuous at* $t = 0$*, then there exists a unique exponent* $H > 0$ *such that* b *in* (2.1.1) *can be expressed as* $b = a^H$*.*

(2) *If a wide sense semi-selfsimilar process* $\{X(t), t \geq 0\}$ *is non-trivial,* X_0 *=const. almost surely and stochastically continuous at any* $t \geq 0$*, then there exists a unique exponent* $H > 0$ *such that* b *in* (2.1.2) *can be expressed as* $b = a^H$*.*

(3) *If a wide sense operator selfsimilar process* $\{X(t), t \geq 0\}$ *is non-trivial,* X_0 *=const. almost surely and stochastically continuous at any* $t \geq 0$*, then there exists a unique exponent real* $d \times d$ *matrix* H *such that* (2.1.4) *can be expressed* $\{X(at)\} \stackrel{\mathrm{d}}{=} \{a^H X(t) + c(t)\}$*.*

Considering the selfsimilar and semi-selfsimilar processes such that each of them has a unique exponent $H > 0$, we use terminology H-selfsimilar and H-semi-selfsimilar process respectively.

2.2 Stable distributions and Lévy processes

We say that a probability distribution μ on $\mathbf{R}^d$ is full if its support is not contained in any proper hyperplane of $\mathbf{R}^d$.

Definition 2.2.1 *For operator semi-stable distributions, Q^* is an transposed matrix of Q and $\langle\cdot,\cdot\rangle$ is an inner product in $\mathbf{R}^d$.*

(1) *A full probability distribution μ on $\mathbf{R}^d$ is called semi-stable, if its characteristic function $\widehat{\mu}$ satisfies*

$$\widehat{\mu}(z)^a = \widehat{\mu}(a^q z)e^{i\langle z,c\rangle}, z \in \mathbf{R}^d, \tag{2.2.1}$$

for some $a > 1$, $q > 0$ and $c \in \mathbf{R}^d$. If $c = 0$, μ is said to be strictly semi-stable. If (2.2.1) *satisfies for any $a > 0$ and some $q > 0$ and $c \in \mathbf{R}^d$, μ is called stable and strictly stable, respectively.*

(2) *A full probability distribution μ on $\mathbf{R}^d$ is called Q-operator semi-stable, if $\widehat{\mu}$ satisfies*

$$\widehat{\mu}(z)^a = \widehat{\mu}(a^{Q^*} z)e^{i\langle z,c\rangle}, z \in \mathbf{R}^d, \tag{2.2.2}$$

for some $a > 1$, real invertible $d \times d$ matrix Q and $c \in \mathbf{R}^d$. If $c = 0$, μ is said to be strictly Q-operator semi-stable. If (2.2.2) *satisfies for any $a > 0$ and some $b > 0$ and $c \in \mathbf{R}^d$, μ is called Q-operator stable and strictly Q-operator stable, respectively.*

Such an matrix Q is not determined uniquely but its eigenvalues are determined. For real parts of eigenvalues of Q we denote by T_Q and τ_Q their maximum one and minimum one, respectively. In the case where (2.2.2) is satisfied, we have $\tau_Q \geq 1/2$, and if $\tau_Q > 1/2$, then μ is purely non-Gaussian. Set

$$r = \inf\{a > 1 : (2.2.2) \text{ holds.}\}.$$

In the case where $r = 1$, μ is nothing but operator stable. We thus assume $r > 1$ and call μ in (2.2.2) and r above an *operator (r, Q)-semi-stable distribution* and its *span*, respectively. When $q = \frac{1}{\alpha}$ in (2.2.1), we call it *(r, α)-semi-stable distribution*. It is known that semi-stable distributions can be characterized as certain limits of normalized partial sums of independent and identically distributed random variables. See Chapter 7 in [MS01] for more details about operator (semi-)stable distributions. For a full semi-stable distribution μ, we define the domain of partial attraction of μ as follows:

Definition 2.2.2 *Let $\{X_i, i \in \mathbf{N}\}$ be independent and identically distributed $\mathbf{R}^d$-valued random variables. We say that X_i's belong to the domain of partial attraction of operator (r, Q)-semi-stable distribution μ with span $r > 1$, if there exist a* *sequence* $\{k_n\}$ *satisfying* $\lim_{n\to\infty} k_{n+1}/k_n = r^{n_0}$ *with some* $n_0 \in \mathbf{N}$, *a sequence of real invertible* $d \times d$ *matrices* $\{A_n\}$ *and a sequence* $\{c_n\}$, $c_n \in \mathbf{R}^d$ *such that*

$$A_n^{-1} \sum_{i=1}^{k_n} X_i + c_n \xrightarrow{d} \mu, \tag{2.2.3}$$

where $\xrightarrow{d}$ *denotes weak convergence.*

We use the following representation of the characteristic function of purely non-Gaussian operator (r, Q)-semi-stable distribution $\widehat{\mu}$ in (2.2.2) given in [Ch87]:

$$\begin{aligned} \widehat{\mu}(z) &= \exp\left\{ \int_{S_Q} \gamma(dx) \int_0^\infty \left[e^{i\langle z, s^Q x\rangle} - 1 \right.\right. \\ &\qquad \left.\left. - i\langle z, s^Q x\rangle I[s^Q x \in D] \right] d\left(-\frac{H_x(s)}{s} \right) + i\langle z, c\rangle \right\}, \end{aligned} \tag{2.2.4}$$

where $S_Q = \{x \in \mathbf{R}^d : \|x\| = 1, \|t^Q x\| > 1 \text{ for any } t > 1\}$ with Euclidean norm $\|\cdot\|$, $D = \{x \in \mathbf{R}^d : \|x\| \le 1\}$, γ is a finite measure on S_Q, and $H_x(s)$ is a non-negative function such that

(1) $H_x(s)/s$ is non-increasing in s for each x,

(2) $H_x(s)$ is right-continuous in s for each x and measurable in x for each s,

(3) $H_x(1) = 1$,

(4) $H_x(rs) = H_x(s)$.

We next give the definition of Lévy processes, that is a very important class of stochastic processes including Brownian motion.

Definition 2.2.3 *A stochastic process* $\{X(t)\}$ *on* $\mathbf{R}^d$ *is called Lévy process if the following conditions are satisfied.*

(1) $X(0) = 0$ *almost surely.*

(2) *For any choice of* $n \ge 1$ *and* $0 \le t_0 < t_1 < t_2 < \cdots < t_n$, *random variables* $X(t_0), X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ *are independent* (*independent increment property*).

(3) *The distribution of* $X(s+t) - X(s)$ *does not depend on* s (*stationary increment property*).

(4) *It is stochastically continuous.*

(5) *Its sample paths are right-continuous and have left limits almost surely.*

The following proposition implies the condition that a Lévy process is selfsimilar and semi-selfsimilar process.

Proposition 2.2.4 (Proposition 13.5 in [Sa99b]) *We assume that* $\{X(t)\}$ *is a Lévy process. Then, the distribution of* $X(1)$ *is* α*-stable* (*resp.* α*-semi-stable*) *if and only if* $\{X(t)\}$ *is wide sense* $1/\alpha$*-selfsimilar* (*resp. wide sense* $1/\alpha$*-semi-selfsimilar*).

This proposition implies that their marginal distributions of selfsimilar and semi-selfsimilar Lévy processes are completely determined by the distribution at time 1.

2.3 Local limit theorems for semi-stable Lévy processes

In this section, we assume the dimension $d = 1$. As mentioned in the previous section, semi-stable distributions can be characterized as a certain subsequential limits of normalized partial sums of independent and identically distributed random variables, and we show that (r, α)-semi-stable Lévy processes can also be constructed from such random variables. In this section, we consider the case of (r, α)-semi-stable distributions.

By using suitable slowly varying functions l_1 and l_2 at ∞, subsequences $\{k_n\}$ and $\{a_n\}$ in (2.2.3) such that

$$k_n = r^n l_1(r^n), \quad \text{and} \quad a_n = k_n^{1/\alpha} l_2(k_n) \tag{2.3.1}$$

can be taken, respectively. In the following, we always assume that $\alpha \in (1, 2)$ unless specified. We also assume that C is an absolute positive constant, which may differ with other C's.

We study properties of asymptotic behavior of $\mathbf{Z}$-valued random walks $\{S_n\}$, which converges weakly to a strictly (r, α)-semi-stable Lévy process $\{Y(t)\}$ under a suitable scaling. The purpose of this subsection is to show the following local limit theorems.

Let X_i's belong to a domain of partial attraction of a strictly (r, α)-semi-stable distributions and $S_n = \sum_{i=1}^n X_i$.

Theorem 2.3.1 *We have the following:*

(1) *Let* $\alpha \in (0, 2]$. *Then,* $P\{S_l = 0\} = O\left(l^{-1/\alpha}\right)$ *for all large* l.

(2) *Let* $\alpha \in (1, 2]$. *Then,*

$$\sum_{k=0}^{\infty} \{P\{S_k = 0\} - P\{S_k = u\}\} = O\left(|u|^{\alpha - 1}\right) \quad \textit{for all large } |u| \in \mathbf{N}$$

In the case where $\alpha = 2$, (r, α)-semi-stable distribution is nothing but Gaussian, and this is already known (cf. Chapter 4 in [IL71]). Hence we consider the case where $0 < \alpha < 2$. To prove (1) of Theorem 2.3.1, we firstly calculate a characteristic function of X_i (we denote by λ), which belong to the domain of partial attraction of strictly (r, α)-semi-stable distribution. Secondly, using the characteristic function λ, we prove local limit theorems of random walks along subsequences. Lastly, we prove for full sequence's case. Here we use Lévy-Khinchin representation of characteristic function of strictly (r, α)-semi-stable distribution (here we denote by φ_α) and the distribution function of X_i's (here we denote it by $F(x)$) given in [Me00] as follows:

- For $z \in \mathbf{R}$,

$$\varphi_\alpha(z) = \exp\left\{\int_{-\infty}^{0}\left(e^{izx} - 1 - \frac{izx}{1+x^2}\right) d\left(\frac{M_L(x)}{|x|^\alpha}\right) + \int_{0}^{\infty}\left(e^{izx} - 1 - \frac{izx}{1+x^2}\right) d\left(\frac{M_R(x)}{x^\alpha}\right)\right\},$$

where M_L on $(-\infty, 0)$ and M_R on $(0, \infty)$ are non-negative, bounded, one of them has a strictly positive infimum and the other one either has a strictly positive infimum or is identically 0, and satisfy $M_L(r^{1/\alpha}x) = M_L(x)$ and $M_R(r^{1/\alpha}x) = M_R(x)$.

- For all large $|x|$,

$$F(x) = \begin{cases} (-x)^{-\alpha}\widetilde{l}(-x)\{M_L(x) + h_L(-x)\}, & x < 0, \\ 1 - x^{-\alpha}\widetilde{l}(x)\{M_R(x) + h_R(x)\}, & x > 0, \end{cases}$$

where $\widetilde{l}$ is right continuous and slowly varying at ∞ defined by

$$x^{-\alpha}\widetilde{l}(x) := \sup\{u : u^{-1/\alpha} l_2(u) > x\}, \quad x > 0, \tag{2.3.2}$$

(recall l_2 is the slowly varying function for the subsequence $\{a_n\}$ in (2.3.1)) and error functions h_L and h_R are right continuous and

$$h_L(a_n x_0) \to 0 \text{ and } h_R(a_n x_0) \to 0 \text{ as } n \to \infty \tag{2.3.3}$$

at every continuity point x_0 of each of M_L and M_R, respectively.

Lemma 2.3.2 *If X_i's belong to the domain of the partial attraction of strictly (r, α)-semi-stable distribution with sequences $\{k_n\}$ and $\{a_n\}$, then their characteristic function $\lambda(z)$ in the neighborhood of the origin is represented as*

$$|\lambda(z)| = \exp\{-\eta(z)|z|^\alpha \widetilde{l}(1/|z|)\},$$

where $\eta(z)$ is a nonnegative bounded continuous function satisfying $\eta(r^{1/\alpha}z) = \eta(z)$ and $\widetilde{l}(\cdot)$ is a slowly varying function at ∞, which is determined by a representation of the distribution function of X_i.

Proof of Lemma 2.3.2

We follows the proof of α-stable case in Section 2.6 of [IL71]. In the neighborhood of the origin, we have

$$\log\lambda(z)=\log\{1+(\lambda(z)-1)\}=\{\lambda(z)-1\}+O(|\lambda(z)-1|^2),$$

and we thus need to calculate $\lambda(z)-1$. Set $F_-(x)=F(-x)$. Further calculations depend on the value of α, here we distinguish into three cases. For each case we assume $z>0$, and in the case where $z<0$ we can calculate similarly.

∘ $1<\alpha<2$. Now, X_i's belongs to a domain of partial attraction of strictly semi-stable and $E[X_i]=0$. For a sufficiently small z there exists a $k\in(0,1)$ such that

$$\begin{aligned}
\lambda(z)-1 &= \int_{-\infty}^{\infty}(e^{izx}-1-izx)dF(x)\\
&= -\int_0^{\infty}(e^{izx}-1-izx)d(1-F(x))-\int_0^{\infty}(e^{-izx}-1+izx)dF_-(x)\\
&= iz\int_0^{\infty}(e^{izx}-1)(1-F(x))dx-iz\int_0^{\infty}(e^{-izx}-1)F_-(x)dx\\
&= i\int_0^{\infty}(e^{ix}-1)(1-F(x/z))dx-i\int_0^{\infty}(e^{-ix}-1)F_-(x/z)dx\\
&= i\left\{\int_{z^k}^{\infty}(e^{ix}-1)(1-F(x/z))dx+\int_0^{z^k}(e^{ix}-1)(1-F(x/z))dx\right\}\\
&\quad -i\left\{\int_{z^k}^{\infty}(e^{-ix}-1)(F_-(x/z))dx+\int_0^{z^k}(e^{-ix}-1)(F_-(x/z))dx\right\}\\
&\sim iz^{\alpha}\int_0^{\infty}(e^{ix}-1)\frac{\widetilde{l}(x/z)(M_R(x/z)+h_R(x/z))}{x^{\alpha}}dx\\
&\quad -iz^{\alpha}\int_0^{\infty}(e^{-ix}-1)\frac{\widetilde{l}(x/z)(M_L(-x/z)+h_L(x/z))}{x^{\alpha}}dx,
\end{aligned}$$

as $z\longrightarrow 0$.

∘ $0<\alpha<1$. In the same way as in the case where $1<\alpha<2$, we have

$$\begin{aligned}
\lambda(z)-1 &\sim iz^{\alpha}\int_0^{\infty}e^{ix}\frac{\widetilde{l}(x/z)(M_R(x/z)+h_R(x/z))}{x^{\alpha}}dx\\
&\quad -iz^{\alpha}\int_0^{\infty}e^{-ix}\frac{\widetilde{l}(x/z)(M_L(-x/z)+h_L(x/z))}{x^{\alpha}}dx,
\end{aligned}$$

as $z\longrightarrow 0$.

∘ $\alpha=1$. For some $c\in\mathbf{R}$, we have

$$\begin{aligned}
\lambda(z)-1+i &= iz\widetilde{l}(1/z)M_R(1/z)\int_z^{\infty}\left(\frac{e^{ix}}{x}+o(1)\right)dx\\
&\quad -iz\widetilde{l}(1/z)M_L(-1/z)\int_z^{\infty}\left(\frac{e^{-ix}}{x}+o(1)\right)dx+icz+O(z^2)
\end{aligned}$$

For a general slowly varying function $l(x)$ at ∞, the following fact is known (cf. Section 2.6 in [IL71]).

Proposition 2.3.3 *We assume that $l(x)$ is a positive slowly varying function at ∞ and $x^{-\alpha}l(x)$ is monotone decreasing.*

(1) *If $0 < \alpha < 1$, then*

$$\begin{aligned}\lim_{z\downarrow 0}\int_0^\infty \frac{\sin x}{x^\alpha} l(x/z)dx &= \lim_{z\downarrow 0} l(1/z)\int_0^\infty \frac{\sin x}{x^\alpha}dx\\ &= \lim_{z\downarrow 0} l(1/z)\cos(\alpha\pi/2)\Gamma(1-\alpha),\\ \lim_{z\downarrow 0}\int_0^\infty \frac{\cos x}{x^\alpha} l(x/z)dx &= \lim_{z\downarrow 0} l(1/z)\int_0^\infty \frac{\cos x}{x^\alpha}dx\\ &= \lim_{z\downarrow 0} l(1/z)\sin(\alpha\pi/2)\Gamma(1-\alpha).\end{aligned}$$

(2) *If $\alpha = 1$, then*

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}dx &= \frac{\pi}{2},\\ \int_t^\infty \frac{\cos x}{x}dx &= -\log t + O(1).\end{aligned}$$

(3) *If $1 < \alpha < 2$, then*

$$\begin{aligned}\lim_{z\downarrow 0}\int_0^\infty \frac{e^{\pm ix}-1}{x^\alpha} l(x/z)dx &= \lim_{z\downarrow 0} l(1/z)\int_0^\infty \frac{e^{\pm ix}-1}{x^\alpha}dx\\ &= \lim_{z\downarrow 0} l(1/z)\exp\left\{\pm\frac{1}{2}i\pi(\alpha-1)\right\}\Gamma(1-\alpha).\end{aligned}$$

In our case, $\widetilde{l}(x/z)(M_L(-x/z)+h(x/z))$ and $\widetilde{l}(x/z)(M_R(x/z)+h(x/z))$ satisfy the conditions of Proposition 2.3.3, and use the fact that $\lim_{z\to 0} h(x/z) \to 0$ except on Lebesgue measure 0 set. Thus we have, for $0 < \alpha < 2$ and z in the neighborhood of the origin,

$$|\lambda(z)| = \exp\left\{-\eta(z)|z|^\alpha \widetilde{l}(1/|z|)\right\},$$

where

$$\eta(z) = \begin{cases} (M_L(-1/z)+M_R(1/z))\cos\dfrac{\pi\alpha}{2}\Gamma(1-\alpha), & \alpha\in(0,1)\cup(1,2)\\ \dfrac{\pi}{2}(M_L(-1/z)+M_R(1/z))(1+o(1)), & \alpha = 1.\end{cases} \tag{2.3.4}$$

This proves Lemma 2.3.2 □

We next prove a local limit theorem for random walks along subsequences as follows:

Lemma 2.3.4 *Let $g_{k_n}(x)$ be the density of $\varphi_\alpha(z)$, that is,*

$$g_{k_n}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixz}\varphi_\alpha(z)dz.$$

Then

$$\lim_{n\to\infty}\sup_{u\in\mathbf{Z}} |a_n P\{S_{k_n} = u\} - g_{k_n}(u/a_n)| = 0.$$

Proof.

The characteristic function of S_{k_n} is given by

$$\lambda(z)^{k_n} = \sum_{u\in\mathbf{Z}} e^{iuz} P\{S_{k_n} = u\}.$$

This implies

$$\begin{aligned} P\{S_{k_n} = u\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iuz}\lambda(z)^{k_n}dz \\ &= \frac{1}{2\pi a_n}\int_{-\pi a_n}^{\pi a_n} e^{-iuz/a_n}\lambda(z/a_n)^{k_n}dz. \end{aligned}$$

For any $u \in \mathbf{Z}$ we have

$$|a_n P\{S_{k_n} = u\} - g_{k_n}(u/a_n)| \le \frac{1}{2\pi}(I_1 + I_2 + I_3 + I_4),$$

where

$$\begin{aligned} I_1 &= \int_{-A}^{A} \left|\lambda(z/a_n)^{k_n} - \varphi_\alpha(z)\right| dz, \\ I_2 &= \int_{A\le|z|\le\varepsilon a_n} |\lambda(z/a_n)|^{k_n} dz, \\ I_3 &= \int_{\varepsilon a_n\le|z|\le\pi a_n} |\lambda(z/a_n)|^{k_n} dz, \\ I_4 &= \int_{|z|>A} |\varphi_\alpha(z)| dz, \end{aligned}$$

and constants A and ε are determined later.

We turn now the estimation of each integral.

(I_1): Since X_i's belong to the domain of partial attraction of strictly α-semi-stable distribution, I_1 converges to zero as $n\to\infty$.

(I_3): Since X_i's are $\mathbf{Z}$-valued, Theorem 1.4.2 of [IL71] implies that $|\lambda(z)| < 1$ for $0 < z < 2\pi$, and thereby a positive constant c such that $|\lambda(z)| \le e^{-c}$ for $\varepsilon \le |z| \le 2\pi$ can be taken. This implies

$$\begin{aligned} I_3 &= \int_{\varepsilon a_n}^{\pi a_n} |\lambda(z/a_n)|^{k_n} dz \\ &\le 2\pi e^{-ck_n}a_n \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

(I_4): $|\varphi_\alpha(z)|$ is integrable on $\mathbf{R}$, and this implies $\lim_{A\to\infty} I_4 = 0$.

(I_2): By Karamata's theorem there exists a function $\varepsilon(u) \to 0$ as $u \to \infty$ such that

$$\frac{\widetilde{l}(a_n/|z|)}{\widetilde{l}(a_n)} = \exp\left\{-\int_{a_n}^{a_n/|z|} \frac{\varepsilon(u)}{u} du\right\}(1 + o(1)),$$

and we have

$$\log \frac{\widetilde{l}(a_n/|z|)}{\widetilde{l}(a_n)} = o(\log |z|).$$

(2.3.2) implies that $\lim_{n\to\infty} k_n a_n^{-\alpha} \widetilde{l}(a_n) = 1$, and for sufficiently large k_n and a_n, and for any $\delta < \alpha$ there exists a positive constant $c(\delta)$ not depending on n such that

$$|\lambda(z/a_n)|^{k_n} = \exp\left\{-\frac{\eta(z)k_n}{a_n^\alpha}\widetilde{l}(a_n)|z|^\alpha \frac{\widetilde{l}(a_n/|z|)}{\widetilde{l}(a_n)}\right\} \le \exp\{-c(\delta)|z|^\delta\}.$$

Thus, a sufficiently large k_n such that for sufficient small $\varepsilon > 0$

$$I_2 \le \int_{A\le|z|\le\varepsilon a_n} \exp\{-c(\alpha/2)|z|^{\alpha/2}\}dz \le \int_{|z|\ge A} \exp\{-c(\alpha/2)|z|^{\alpha/2}\}dz$$

can be shown, and this implies that $I_2 \to 0$ as $A \to \infty$. Hence we have shown that each integral can be made arbitrarily small, and (2.3.4) follows. □

To show the full sequence's case, we need the following lemma:

Lemma 2.3.5 *Let μ and $\widetilde{\mu}$ be strictly (r,α)-semi-stable distributions as scaled limit of sums of X_i's with a pair of subsequences in* (2.2.3), *$\{k_n\}, \{a_n\}$ and $\{\widetilde{k}_n\}, \{\widetilde{a}_n\}$, respectively. Denote by $\varphi_\alpha(z)$ and $\widetilde{\varphi}_\alpha(z)$ the characteristic functions of μ and $\widetilde{\mu}$, respectively. If $\lim \widetilde{k}_n/k_n = \theta < \infty$, then there exists a positive constant $\widetilde{\theta} = \lim a_n/\widetilde{a}_n$ such that*

$$\widetilde{\varphi}_\alpha(z) = \varphi_\alpha(\widetilde{\theta}z)^\theta. \tag{2.3.5}$$

Proof.

As mentioned in Section 2.2, to the distribution μ there corresponds a strictly (r,α)-semi-stable Lévy process, which we denote by $\{Y(t)\}$, namely, $a_n^{-1}\sum_{i=1}^{[k_n t]} X_i$ converges weakly to $\{Y(t)\}$ in $D([0,\infty),\mathbf{R})$. Then we have

$$\frac{1}{\widetilde{a}_n}\sum_{i=1}^{\widetilde{k}_n} X_i = \frac{a_n}{\widetilde{a}_n}\frac{1}{a_n}\sum_{i=1}^{\widetilde{k}_n} X_i \xrightarrow{d} \widetilde{\theta}Y(\theta),$$

and this implies $\widetilde{\mu}$ coincides with the distribution of $\widetilde{\theta}Y(\theta)$. Since $Y(t)$ is a Lévy process, whose distribution at each $t > 0$ can be represented by t-convolution of μ. This implies (2.3.5). □

Proof of (1) *of Theorem* 2.3.1.

From Lemma 2.3.4, the following estimation is satisfied:

$$P\{S_{\widetilde{k}_n} = u\} = O\left(\frac{g_{\widetilde{k}_n}(u/\widetilde{a}_n)}{\widetilde{a}_n}\right).$$

From Lemma 2.3.5, for each subsequence $\{\widetilde{k}_n\}$, (2.3.5) holds. Now the characteristic function $\varphi_\alpha(z)$ belongs to $L^1(\mathbf{R})$, and general theory of Fourier transformation implies that $g_{\widetilde{k}_n}(x)$ is uniformly continuous. Here $g_{k_n}(0)$ is bounded, and Lemma 2.3.5 implies that $g_{k_n}(0)$ is also bounded. Thus, for any $\widetilde{k}_n$, $P\{S_{\widetilde{k}_n} = 0\} = O(1/\widetilde{a}_n) = O(1/(\widetilde{k}_n^{1/\alpha} l_2(\widetilde{k}_n)))$ and (1) of Theorem 2.3.1 follows. □

Proof of (2) *of Theorem* 2.3.1.

Let $a(u)$ be a potential kernel of random walk S_k, defined by

$$a(u) = \sum_{k=0}^{\infty} \left\{P\{S_k = 0\} - P\{S_k = u\}\right\}, \quad \text{for } u \in \mathbf{Z}.$$

Since $\{S_n\}$ is recurrent for $\alpha > 1$, the following is satisfied for all large u (see Section 28 P4 in [Sp76]):

$$a(u) + a(-u) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos zu}{1-\lambda(z)} dz.$$

Arguments in the proof of Lemma 2.3.2 imply that for all sufficient small z, there exists slowly varying function $l'(1/|z|)$ at ∞, which is determined by $\widetilde{l}(1/|z|)$, such that

$$|1-\lambda(z)| = |z|^\alpha |\eta(z) l'(1/|z|)|,$$

where $\eta(z)$ is in (2.3.4). Hence there exist some constants C, C' and C'' such that for a sufficiently small z

$$\begin{aligned} |a(u)+a(-u)| &= \left|\frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} \frac{1-\cos zu}{1-\lambda(z)} dz\right| + C \\ &\le \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} \left|\frac{1-\cos zu}{1-\lambda(z)}\right| dz + C \\ &= \frac{1}{\pi}\int_{-1/u}^{1/u} \frac{z^2u^2}{|z|^\alpha|\eta(z)l'(1/|z|)|} dz + C' \\ &\sim C'' u^{\alpha-1} \quad \text{as } u \to \infty. \end{aligned}$$

This implies (2) of Theorem 2.3.1. □

Chapter 3

Operator semi-selfsimilar processes

In this chapter, we consider operator semi-selfsimilar processes. Selfsimilar processes are studied widely and many properties are known. Wide sense operator semi-selfsimilar processes have similar properties to wide sense selfsimilar processes. We investigate some basic properties of wide sense operator semi-selfsimilar processes guided by [MaSa99] by noting this class of parameters. Especially, we are concerned with B in (2.1.4), which we call a space scaling matrix, and its exponent H if exists. The problem around B is more delicate than the case of $d = 1$ as we will see. Actually, we show that properties of B and H are not the same as for operator selfsimilar processes or semi-selfsimilar processes, and it implies some new mathematical questions.

In Section 3.1, we study the basic property of the classes of scaling parameters and matrices.

In Section 3.2, we give two results about reduction of the wide sense operator semi-selfsimilar process to the operator semi-selfsimilar process. The first one is a reduction for fixed scaling matrix B and the second one is a reduction of the class of scaling matrices under some condition.

In Section 3.3, we treat the connection to scaling limit.

In Section 3.4, first we show the existence of exponents of wide sense operator semi-selfsimilar processes similar to the case of wide sense semi-selfsimilar processes. In this expression we need the matrices of sign. Next we give some characterization of the Lévy measures of the operator semi-stable distributions to construct operator semi-stable Lévy processes given by basic time scaling parameter, basic space scaling parameter matrix and the class of invariant matrices for the distributions of processes.

3.1 The class of space scaling matrices

Let $\mathcal{M}(\mathbf{R}^d)$ be the set of all real $d \times d$ matrices, $\mathcal{M}_I(\mathbf{R}^d)$ be the set of all invertible matrices in $\mathcal{M}(\mathbf{R}^d)$ and $\mathcal{P}(\mathbf{R}^d)$ be the set of all probability measures on $\mathbf{R}^d$ with weak topology.

Let us go back to Definition 2.1.1. A scaling parameter a in (2.1.4) is called an epoch of wide sense operator semi-selfsimilar process $\mathbf{X} = \{X(t), t \geq 0\}$. A probability measure μ on $\mathbf{R}^d$ is called full if its support is not contained in any proper hyperplane in $\mathbf{R}^d$. For $A \in \mathcal{M}(\mathbf{R}^d)$ and $\gamma \in \mathbf{R}^d$, we denote by $\alpha = (A; \gamma)$ an affine mapping on $\mathbf{R}^d$ given by $\alpha x = Ax + \gamma$. We denote by $\mathcal{A}(\mathbf{R}^d)$ the class of all affine mappings on $\mathbf{R}^d$ and set $\mathcal{A}_I(\mathbf{R}^d) = \{\alpha \in \mathcal{A}(\mathbf{R}^d) | \ \alpha = (A; \gamma), A \in \mathcal{M}_I(\mathbf{R}^d)\}$. $\mathcal{A}(\mathbf{R}^d)$ can be regarded as a Banach space with a norm $\|\alpha\| = \|A\|_{op} \vee \|\gamma\|$, where $\| \cdot \|_{op}$ is the operator norm. The following lemma is fundamental (see Corollary 2.1.3 and Lemma 2.2.3 in [JM93]).

Lemma 3.1.1 (1) *Let $\alpha \in \mathcal{A}(\mathbf{R}^d)$ and $\mu \in \mathcal{P}(\mathbf{R}^d)$. The image measure $\mu \circ \alpha^{-1}$ is full if and only if $\alpha \in \mathcal{A}_I(\mathbf{R}^d)$ and μ is full.*
(2) *Suppose $\mu_n \in \mathcal{P}(\mathbf{R}^d), \mu$ is full and $\alpha_n \in \mathcal{A}(\mathbf{R}^d)$. If $\mu_n \stackrel{\mathrm{d}}{\to} \mu$ and $\{\mu_n \circ \alpha_n^{-1}\}$ is tight, then $\{\alpha_n\}$ is relatively compact in $\mathcal{A}(\mathbf{R}^d)$, where $\stackrel{\mathrm{d}}{\to}$ denotes the weak convergence.*

Let $\mathbf{X} = \{X(t), t \geq 0\}$ be a wide sense operator semi-selfsimilar process. We set

$$\begin{aligned} \Gamma(\mathbf{X}) &= \{a > 0 | \ \exists B \in \mathcal{M}(\mathbf{R}^d), \exists c : [0, \infty) \to \mathbf{R}^d \\ &\qquad\qquad \text{s.t. } \{X(at)\} \stackrel{\mathrm{d}}{=} \{BX(t) + c(t)\}\}, \\ \mathcal{B}(\mathbf{X}) &= \{B \in \mathcal{M}(\mathbf{R}^d) | \ \exists a > 0, \exists c : [0, \infty) \to \mathbf{R}^d \\ &\qquad\qquad \text{s.t. } \{X(at)\} \stackrel{\mathrm{d}}{=} \{BX(t) + c(t)\}\}. \end{aligned}$$

We assume the following conditions for each wide sense operator semi-selfsimilar process $\mathbf{X}$ throughout this thesis.

(X-1) $\mathbf{X} = \{X(t), t \geq 0\}$ is stochastically continuous.

(X-2) $X(0)$ is a constant almost surely.

(X-3) There exists a $t_0 > 0$ such that the distribution of $X(t_0)$ is full.

For a wide sense operator semi-selfsimilar process $\mathbf{X}$, by (1) of Lemma 3.1.1, we see that the space scaling matrix B in (2.1.4) is invertible and the function $c(t)$ in (2.1.4) is continuous. Set

$$a_0 = a_0(\mathbf{X}) = \inf\{\Gamma(\mathbf{X}) \cap (1, \infty)\}. \qquad (3.1.1)$$

Then for a class of epochs of $\mathbf{X}$, we obtain the following proposition in a similar way to Theorem 1 in [MaSa99] by using (2) of Lemma 3.1.1.

Proposition 3.1.2 *Let* $\mathbf{X} = \{X(t), t \geq 0\}$ *be a wide sense operator semi-selfsimilar process. Then* $\Gamma(\mathbf{X}) = (0, \infty)$ *if* $a_0 = 1$*, and* $\Gamma(\mathbf{X}) = \{a_0^n |\ n \in \mathbf{Z}\}$ *if* $a_0 > 1$.

Proof. Firstly, we show that Γ is group. From operator semi-selfsimilarity, for any $B \in \mathcal{B}$, there exists an $a \in \Gamma$ and $c : [0, \infty) \to \mathbf{R}^d$ such that

$$X(t_0) = X(aa^{-1}t_0) \stackrel{\mathrm{d}}{=} BX(a^{-1}t_0) + c(a^{-1}t_0).$$

Since the distribution of $X(t_0)$ is full, by Lemma 3.1.1, we see that $B \in \mathcal{M}_I(\mathbf{R}^d)$. Let $a, a' \in \Gamma$. By (2.1.4), there exist $B, B' \in \mathcal{M}(\mathbf{R}^d)$ and $c, c' : [o, \infty) \to \mathbf{R}^d$ such that $\{X(at)\} \stackrel{\mathrm{d}}{=} \{BX(t) + c(t)\}$ and $\{X(a't)\} \stackrel{\mathrm{d}}{=} \{B'X(t) + c'(t)\}$. This implies

$$\{X(aa't)\} \stackrel{\mathrm{d}}{=} \{BB'X(t) + (Bc'(t) + c(a't))\}.$$

We see that $aa' \in \Gamma$. Let $a \in \Gamma$. Then,

$$\{X(t)\} = \{X(aa^{-1}t)\} \stackrel{\mathrm{d}}{=} \{BX(a^{-1}t) + c_a(a^{-1}t)\}$$

Thus, we have

$$\{X(a^{-1}t)\} \stackrel{\mathrm{d}}{=} \{B^{-1}X(t) - B^{-1}c_a(a^{-1}t)\}.$$

This shows $a^{-1} \in \Gamma$ and Γ becomes a group.

Next, we show the closedness of Γ. Let us assume that $a_n \to a_\infty > 0$. Then, there exist $B_n \in \mathcal{M}_I(\mathbf{R}^d)$ and a sequence of function $c_n : [0, \infty) \to \mathbf{R}^d$ such that

$$\{X(a_n t)\} \stackrel{\mathrm{d}}{=} \{B_n X(t) + c_n(t)\}.$$

From stochastic continuity of $\{X(t), t \geq 0\}$, we have

$$\{X(a_n t)\} \stackrel{\mathrm{d}}{\Rightarrow} \{X(a_\infty t)\}, \tag{3.1.2}$$

where $\stackrel{\mathrm{d}}{\Rightarrow}$ denotes the convergence of all finite dimensional distributions. Especially,

$$X(a_n a_\infty^{-1} t_0) \to X(t_0) \quad \text{in law}.$$

By (2) of Lemma 3.1.1, we see that $\{B_n\}$ is relatively compact. Taking subsequence $\{n'\}$ of $\{n\}$ if necessity, there exists a $B \in \mathcal{M}(\mathbf{R}^d)$ such that $B_{n'} \to B$. Since (3.1.2) and

$$\{B_{n'}X(t)\} \stackrel{\mathrm{d}}{\Rightarrow} \{BX(t)\},$$

implies that there exists a function $c_\infty : [0, \infty) \to \mathbf{R}^d$ such that $c_{n'} \to c_\infty$ (pointwise) and

$$\{X(a_\infty t)\} \stackrel{\mathrm{d}}{=} \{BX(t) + c_\infty(t)\}.$$

Furthermore, by Lemma 3.1.1, we see that $B \in \mathcal{M}_I(\mathbf{R}^d)$.

Lastly, put

$$a_0 = \inf(\Gamma \cap (1, \infty)).$$

Then, by the closedness of Γ, $a_0 \in \Gamma$. By the same argument in [MaSa99], we have that $\Gamma = (0, \infty)$ or $\Gamma = \{a_0^n | \ n \in \mathbf{Z}\}$ corresponding to $a_0 = 1$ or $a_0 > 1$. If $a_0 = 1$, then the stochastic process $\{X(t)\}$ becomes a wide-sense operator self similar process. Thus we assume after all the case where $a_0 > 1$. We call this a_0 a basic epoch.

When $a_0 = 1$, the process $\mathbf{X}$ becomes a wide sense operator selfsimilar, on the other hand when $a_0 > 1$, the process $\mathbf{X}$ is purely wide sense operator semi-selfsimilar. Thus from now on we assume that $a_0 > 1$. We call this a_0 basic epoch of $\mathbf{X}$. We set

$$\mathcal{N}(\mathbf{X}) = \{N \in \mathcal{M}(\mathbf{R}^d) | \ \exists c : [0, \infty) \to \mathbf{R}^d \text{ s.t. } \{X(t)\} \stackrel{\mathrm{d}}{=} \{NX(t) + c(t)\}\},$$

and for $n \in \mathbf{Z}$

$$\mathcal{B}^n(\mathbf{X}) = \{B \in \mathcal{M}_I(\mathbf{R}^d) | \ \exists c : [0, \infty) \to \mathbf{R}^d \text{ s.t. } \{X(a_0^n t)\} \stackrel{\mathrm{d}}{=} \{BX(t) + c(t)\}\}.$$

Theorem 3.1.3 *Suppose that $\mathbf{X} = \{X(t), t \geq 0\}$ is a wide sense operator semi-selfsimilar process. Then the following hold.*

(1) *$\mathcal{N}(\mathbf{X})$ is a normal subgroup of $\mathcal{B}(\mathbf{X})$.*

(2) *For each $n \in \mathbf{Z}$, $\mathcal{B}^n(\mathbf{X})$ is a coset of $\mathcal{N}(\mathbf{X})$ in $\mathcal{B}(\mathbf{X})$.*

(3) *$\mathcal{B}^n(\mathbf{X}) = (\mathcal{B}^1(\mathbf{X}))^n$ for $n \in \mathbf{Z}$ and $\mathcal{B}(\mathbf{X}) = \bigcup_{n \in \mathbf{Z}} \mathcal{B}^n(\mathbf{X})$ (disjoint union).*

Proof. (1) Let $N \in \mathcal{N}(\mathbf{X})$ and $B \in \mathcal{B}(\mathbf{X})$. Then for some $a \in \Gamma(\mathbf{X})$ and c_N and $c_B : [0, \infty) \to \mathbf{R}^d$, we have $\{BX(t) + c_B(t)\} \stackrel{\mathrm{d}}{=} \{X(at)\} \stackrel{\mathrm{d}}{=} \{N(BX(t) + c_B(t)) + c_N(at)\}$. Thus,

$$\{X(t)\} \stackrel{\mathrm{d}}{=} \{B^{-1}NBX(t) + (B^{-1}Nc_B(t) + B^{-1}c_N(at) - B^{-1}c_B(t))\}.$$

This implies that $\mathcal{N}(\mathbf{X})$ is a normal subgroup of $\mathcal{B}(\mathbf{X})$.
(2) Let B_1 and $B_2 \in \mathcal{B}^n(\mathbf{X})$. Then $\{B_1X(t) + c_{B_1}(t)\} \stackrel{\mathrm{d}}{=} \{X(a_0^n t)\} \stackrel{\mathrm{d}}{=} \{B_2X(t) + c_{B_2}(t)\}$. This implies

$$\{X(t)\} \stackrel{\mathrm{d}}{=} \{B_1^{-1}B_2X(t) + B_1^{-1}c_{B_2}(t) - B_1^{-1}c_{B_1}(t)\},$$

and then $B_2 = B_1N$ for some $N \in \mathcal{N}(\mathbf{X})$. In a similar way, we have that $B_2 = N'B_1$ for some $N' \in \mathcal{N}(\mathbf{X})$.
(3) is shown by the use of (1) and (2). □

3.2 Reduction to the operator semi-selfsimilar process

It is known that an $\mathbf{R}^d$-valued 1-stable Lévy process can not be reduced to strictly stable Lévy process. But [Sa91] showed that any wide sense selfsimilar process can be reduced to a selfsimilar process with a suitable shift. [MaSa99] extended this result to wide sense semi-selfsimilar processes. First we state the corresponding result for a wide sense operator semi-selfsimilar process in the case where a space scaling matrix is fixed. We can prove the following in a similar way to Theorem 5 in [MaSa99].

Lemma 3.2.1 *Let $\mathbf{X} = \{X(t), t \geq 0\}$ be a wide sense operator semi-selfsimilar process. Let $B_1 \in \mathcal{B}^1(\mathbf{X})$. Let c_n be non-random functions in* (2.1.4) *for $a_0^n \in \Gamma(\mathbf{X})$ and $B_1^n \in \mathcal{B}^n(\mathbf{X}), n \in \mathbf{Z}$. We assume that a non-random continuous function $d_n : [0, \infty) \to \mathbf{R}^d$ satisfies the following two conditions.*

$$B_1 d_n(t) = d_{n+1}(t),$$
$$c_n(a_0) - c_{n+1}(1) = d_{n+1}(1) - d_n(a_0).$$

Put

$$k(t) = c_\ell(a_0^{-\ell}t) + d_\ell(a_0^{-\ell}t), \quad a_0^\ell \leq t < a_0^{\ell+1}, \ell \in \mathbf{Z}.$$

Then the function $k(t)$ is continuous and $\mathbf{Y} = \{X(t) - k(t), t \geq 0\}$ is operator semi-selfsimilar.

Theorem 3.2.2 *Let $\mathbf{X} = \{X(t), t \geq 0\}$ be a wide sense operator semi-selfsimilar process. Then for each $B_1 \in \mathcal{B}^1(\mathbf{X})$ there exists a non-random continuous function $k = k_{B_1} : [0, \infty) \to \mathbf{R}^d$ such that $\mathbf{Y} = \{X(t) - k(t), t \geq 0\}$ is operator semi-selfsimilar.*

Proof. Let c_n be a function in (2.1.4) corresponding to an epoch a_0^n and a space scaling matrix B_1^n for $n \in \mathbf{Z}$. Set

$$d_n(t) = \frac{t-1}{a_0 - 1} B_1^n (c_1(1) - 1), \quad n \in \mathbf{Z}. \tag{3.2.1}$$

Then d_n, satisfy the conditions in Lemma 3.2.1, and we have our assertion. □

For an operator semi-selfsimilar process $\mathbf{X} = \{X(t), t \geq 0\}$, we use corresponding notation $\Gamma_0(\mathbf{X}), \mathcal{B}_0(\mathbf{X}), \mathcal{N}_0(\mathbf{X})$ and $\mathcal{B}_0^n(\mathbf{X})$ to $\Gamma(\mathbf{X})$, $\mathcal{B}(\mathbf{X})$, $\mathcal{N}(\mathbf{X})$ and $\mathcal{B}^n(\mathbf{X})$, $n \in \mathbf{Z}$, for $c(t) \equiv 0$, respectively. For the process $\mathbf{Y}$ in Theorem 3.2.2, $\Gamma(\mathbf{X}) = \Gamma_0(\mathbf{Y})$ but $\mathcal{B}(\mathbf{X}) \neq \mathcal{B}_0(\mathbf{Y})$ in general. Then we have a new question: Is there any reduced operator semi-selfsimilar process $\mathbf{Y} = \{Y(t), t \geq 0\}$ satisfying $\mathcal{B}(\mathbf{X}) = \mathcal{B}_0(\mathbf{Y})$? We give such a shift function $\{k(t)\}$ under some conditions. Let $\mathbf{X} = \{X(t), t \geq 0\}$ be a wide sense operator semi-selfsimilar process. For $N \in \mathcal{N}(\mathbf{X})$, we denote by c_N a function with $\{X(t)\} \stackrel{\mathrm{d}}{=} \{NX(t) + c_N(t)\}$.

Theorem 3.2.3 *Let* $\mathbf{X} = \{X(t), t \geq 0\}$ *be a wide sense operator semi-selfsimilar process. Suppose that there exists an* $N_1 \in \mathcal{N}(\mathbf{X})$*, which satisfies* $I - N_1 \in \mathcal{M}_I(\mathbf{R}^d)$ *and*

$$(I - N)(I - N_1)^{-1} c_{N_1}(t) = c_N(t) \quad \text{for any} \quad N \in \mathcal{N}(\mathbf{X}). \tag{3.2.2}$$

Then there exists a non-random continuous function $k : [0, \infty) \to \mathbf{R}^d$ *such that* $\mathbf{Y} = \{X(t) - k(t), t \geq 0\}$ *is an operator semi-selfsimilar process with* $\Gamma(\mathbf{X}) = \Gamma_0(\mathbf{Y})$ *and* $\mathcal{B}^n(\mathbf{X}) = \mathcal{B}_0^n(\mathbf{Y}), n \in \mathbf{Z}$.

Proof. Let $B_1 \in \mathcal{B}^1(\mathbf{X})$. Then by Theorem 3.1.3, there exists an $N_2 \in \mathcal{N}(\mathbf{X})$, which satisfies $N_1 B_1 = B_1 N_2$. Then we have next two relations:

$$\begin{aligned} \{X(a_0 t)\} &\stackrel{\mathrm{d}}{=} \{N_1 X(a_0 t) + c_{N_1}(a_0 t)\} \stackrel{\mathrm{d}}{=} \{N_1 B_1 X(t) + (N_1 c_{B_1}(t) + c_{N_1}(a_0 t))\}, \\ \{X(a_0 t)\} &\stackrel{\mathrm{d}}{=} \{B_1 X(t) + c_{B_1}(t)\} \stackrel{\mathrm{d}}{=} \{B_1 N_2 X(t) + (B_1 c_{N_2}(t) + c_{B_1}(t))\}. \end{aligned}$$

Thus we have

$$c_{N_1}(a_0 t) = (I - N_1) c_{B_1}(t) + B_1 c_{N_2}(t).$$

From this, if we take $k(t) = (I - N_1)^{-1} c_{N_1}(t)$ and $Y(t) = X(t) - k(t)$, we have

$$\begin{aligned} \{Y(a_0 t)\} &\stackrel{\mathrm{d}}{=} \{B_1 X(t) + c_{B_1}(t) - (I - N_1)^{-1} c_{N_1}(a_0 t)\} \\ &= \{B_1 X(t) - (I - N_1)^{-1} B_1 c_{N_2}(t)\}. \end{aligned} \tag{3.2.3}$$

Note that if $I - N_1 \in \mathcal{M}_I(\mathbf{R}^d)$, then $I - N_2 \in \mathcal{M}_I(\mathbf{R}^d)$. By the definition of N_2 and (3.2.2), we have

$$\begin{aligned} (I - N_1)^{-1} B_1 c_{N_2}(t) &= B_1 (I - N_2)^{-1} c_{N_2}(t) \\ &= B_1 (I - N_1)^{-1} c_{N_1}(t). \end{aligned} \tag{3.2.4}$$

By induction, (3.2.3) and (3.2.4) imply

$$\{Y(a_0^n t)\} \stackrel{\mathrm{d}}{=} \{B_1^n Y(t)\}, \quad \text{for any } n \in \mathbf{Z}.$$

This implies our assertion. □

Corollary 3.2.4 *If* $-I \in \mathcal{N}(\mathbf{X})$*, then a wide sense operator semi-selfsimilar process* $\mathbf{X}$ *can be reduced to an operator semi-selfsimilar process* $\mathbf{Y}$ *satisfying* $\mathcal{B}^n(\mathbf{X}) = \mathcal{B}_0^n(\mathbf{Y}),\ n \in \mathbf{Z}$.

Corollary 3.2.5 *If* $\mathcal{N}(\mathbf{X})$ *is commutative and there exists an* $N_1 \in \mathcal{N}(\mathbf{X})$ *such that* $I - N_1 \in \mathcal{M}_I(\mathbf{R}^d)$*, then a wide sense operator semi-selfsimilar process* $\mathbf{X}$ *can be reduced to an operator semi-selfsimilar process* $\mathbf{Y}$ *satisfying* $\mathcal{B}^n(\mathbf{X}) = \mathcal{B}_0^n(\mathbf{Y}),\ n \in \mathbf{Z}$.

Proof. Let $N \in \mathcal{N}(\mathbf{X})$. Then we have $\{X(t), t \geq 0\} \stackrel{\mathrm{d}}{=} \{N_1 X(t) + c_{N_1}(t)\}$ $\stackrel{\mathrm{d}}{=} \{N_1 N X(t) + (N_1 c_N(t) + c_{N_1}(t))\}$ and therefore

$$c_{N_1 N}(t) = N_1 c_N(t) + c_{N_1}(t). \tag{3.2.5}$$

In the same way, we have $c_{N N_1}(t) = N c_{N_1}(t) + c_N(t)$. Then we have $(I - N_1) c_N(t) = (I - N) c_{N_1}(t)$. Using commutativity of N_1 and N, we have (3.2.2), which implies the assertion. □

3.3 Connection to scaling limit

Analogues to [MaSa99], we can prove the following theorem.

Theorem 3.3.1 (1) *Let* $\mathbf{X} = \{X(t)\}$ *be a stochastic process satisfying the conditions* (X1), (X2) *and* (X3)*. Let us assume that there exist a stochastic process* $\{Y(t), t \geq 0\}, a_n \uparrow \infty, B_n \in \mathcal{M}_I(\mathbf{R}^d)$ *and non-random functions* $c_n : [0, \infty) \to \mathbf{R}^d$ *such that*

(i) *there exists* $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$,

(ii) $B_n^{-1}(Y(a_{n+1}t) - Y(a \cdot a_n t)) \to 0$ *in probability.*

(iii) $\{B_n^{-1} Y(a_n t) + c_n(t)\} \stackrel{\mathrm{d}}{\Rightarrow} \{X(t)\}$.

Then, $\{X(t)\}$ *is wide sense operator semi-selfsimilar process.*

(2) *Conversely, if* $\{X(t)\}$ *satisfies conditions* (X1), (X2) *and* (X3) *and is wide sense operator semi-selfsimilar, then* $\{X(t)\}$ *is such a limit.*

The proof can be carried out exactly by the same way as in [MaSa99] by noting Lemma 3.1.1. Let a be a limiting value in (i) of Theorem 3.3.1 and B be an arbitrary accumulating matrix of $\{B_{n+1}^{-1} B_n, n \in \mathbf{N}\}$. Then, we see that $\Gamma(\mathbf{X}) \supseteq \{a^n, n \in \mathbf{Z}\}$ and $\mathcal{B}(\mathbf{X}) \supset \{B^n, n \in \mathbf{Z}\}$. If there exist an $N \in \mathcal{M}_I(\mathbf{R}^d)$ and non-random function $C_{n,N} : [0, \infty) \to \mathbf{R}^d$ such that

$$\{Y(t)\} \stackrel{\mathrm{d}}{=} \{NY(t) + c_{n,N}(t)\},$$

and $B_n \mathcal{N}(\mathbf{X}) = \mathcal{N}(\mathbf{X}) B_n$, then $N \in \mathcal{N}(\mathbf{X})$.

3.4 Exponents of operator semi-selfsimilar processes

[Sa91] showed the existence of an exponent of a wide sense operator selfsimilar process. In this section we consider the problem for a wide sense operator semi-selfsimilar process. For a wide sense operator selfsimilar process, there always exists its exponent. However in the case of wide sense operator semi-selfsimilar process, we need a matrix of signs for such an expression. [C94] treated a similar problem for a semi-stable distribution. Set $\mathcal{M}_{\pm 1}(\mathbf{R}^d) = \{S \in \mathcal{M}_I(\mathbf{R}^d) |\ S^2 = I\}$ and $\mathcal{M}_+(\mathbf{R}^d)$ be the class of matrices, all of whose eigenvalues have positive real parts. We remark that A is belongs to $\mathcal{M}_+(\mathbf{R}^d)$ if and only if $\lim_{t\to-\infty} \exp(tA) = O$.

Lemma 3.4.1 *For each invertible matrix B, there exist an $S \in \mathcal{M}_{\pm 1}(\mathbf{R}^d)$ and an $H \in \mathcal{M}(\mathbf{R}^d)$ such that S and H are commutative and $B = S \exp H$.*

Proof. B can be represented in a Jordan canonical form J. Namely, $B \sim J = J_+ \oplus J_- \oplus J_{IM}$, where $J+$ (resp. $J-$) is a direct sum of Jordan blocks corresponding to positive eigenvalues (resp. negative ones), J_{IM} is a direct sum of real Jordan blocks corresponding to complex eigenvalues and $A \sim B$ means $P^{-1}AP = B$ for some $P \in \mathcal{M}_I(\mathbf{R}^d)$. We see that $J+$ can be expressed as an exponential form by using that $J(\lambda, k) \sim \exp(J(\log \lambda, k))$ for any $\lambda > 0$ and $k \in \mathbf{N}$, where $J(\lambda, k)$ is a Jordan block of k-th degree. For J_{IM}, by using

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \exp \begin{pmatrix} \log\sqrt{a^2+b^2} & -\arctan\dfrac{b}{a} \\ \arctan\dfrac{b}{a} & \log\sqrt{a^2+b^2} \end{pmatrix},$$

we see that any Jordan blocks in J_{IM} can be expressed in an exponential form by a similar way to J_+. A Jordan block in J_- cannot be expressed in an exponential form of real exponent matrix in general, but $-J_-$ has an exponential expression. Let $D = I \oplus (-I) \oplus I$. Then there exists an $H_1 \in \mathcal{M}(\mathbf{R}^d)$ such that $J = D \exp H_1$. Since the signs of the elements of D corresponding to each of Jordan blocks are definite, H_1 and D are commutative. This concludes the lemma. □

Theorem 3.4.2 *Let $\mathbf{X} = \{X(t), t \geq 0\}$ be a wide sense operator semi-selfsimilar process. Let a_0 be the basic epoch of $\mathbf{X} = \{X(t), t \geq 0\}$. Then there exist an $H \in \mathcal{M}_+(\mathbf{R}^d)$, an $S \in \mathcal{M}_{\pm 1}(\mathbf{R}^d)$ and a function $c_n : [0, \infty) \to \mathbf{R}^d, n \in \mathbf{Z}$ such that S and H are commutative and*

$$\{X(at), t \geq 0\} \stackrel{\mathrm{d}}{=} \{S^n a^H X(t) + c_n(t), t \geq 0\} \tag{3.4.1}$$

for any $a \in \Gamma(\mathbf{X})$, where n is an integer such that $a = a_0^n$.

We call H an exponent matrix of a wide sense operator semi-selfsimilar process. and the process $\mathbf{X}$ is called an (a_0, H, S)-wide sense operator semi-selfsimilar process $\mathbf{X}$. If we can take $S = I$, we omit S and call $\mathbf{X}$ an (a_0, H)-wide sense operator semi-selfsimilar process .

Remark 3.4.3 *S and H are not necessarily unique. If $\mathbf{X}$ is also expressed with an exponent matrix H_2 and a sign matrix S_2, then the real parts of eigenvalues of*

H_2 are same as those of H and by Theorem 3.1.3 *there exists an $N_n \in \mathcal{N}(\mathbf{X}), n \in \mathbf{Z}$, such that*

$$S_2 a_0^{nH_2} = N_n S^n a_0^{nH}.$$

Proof of Theorem 3.4.2. We take an arbitrary $B_1 \in \mathcal{B}^1(\mathbf{X})$ and fix it. By Proposition 3.1.2, $a = a_0^n$ for some $n \in \mathbf{Z}$. Then by Lemma 3.4.1, there exist an $S \in \mathcal{M}_{\pm 1}(\mathbf{R}^d)$ and an $H \in \mathcal{M}(\mathbf{R}^d)$ such that $SH = HS$ and $B_1 = Sa_0^H$. Then we have our exponential expression of $B_1^n, n \in \mathbf{Z}$. Further, using Theorem 3.2.2, we see that $H \in \mathcal{M}_+(\mathbf{R}^d)$, since the reduced operator semi-selfsimilar process $\mathbf{Y}$ is vanishing almost surely at $t = 0$, which completes our assertion. □

As we have seen in Theorem 3.1.3, for each wide sense operator semi-selfsimilar process, there exists an epoch $a > 1$, a space scaling matrix B and an normal subgroup $\mathcal{N}(\mathbf{X})$ of $\mathcal{B}(\mathbf{X})$. A new question arises. Let $\mathcal{M}_{ID}^1(\mathbf{R}^d) = \{N \in \mathcal{M}_I(\mathbf{R}^d) \mid |\lambda| = 1$ for any eigenvalue λ of N and $N \sim D$, D is a diagonal matrix$\}$. For any given $a > 1$, $H \in \mathcal{M}_+(\mathbf{R}^d)$, $S \in \mathcal{M}_{\pm 1}(\mathbf{R}^d)$ and a normal subgroup $\mathcal{N}$ of $\mathcal{M}_{ID}^1(\mathbf{R}^d)$ such that $SH = HS$ and $(Sa^H)\mathcal{N} = \mathcal{N}(Sa^H)$, is there a wide sense operator semi-selfsimilar process with an epoch a, an exponent matrix H, a sign matrix S and $\mathcal{N}(\mathbf{X}) = \mathcal{N}$? We construct examples of wide sense operator semi-selfsimilar processes as operator semi-stable processes for any given parameters. Recall Definition 2.2.1 and we redefine the semi-stablity.

Definition 3.4.4 *Let $\mu \in \mathcal{P}(\mathbf{R}^d)$ be an infinitely divisible distribution. μ is called operator semi-stable if its characteristic function $\widehat{\mu}(z)$ satisfies that there exist $a \in (0,1) \cup (1,\infty), B \in \mathcal{M}(\mathbf{R}^d)$ and $c \in \mathbf{R}^d$ such that*

$$\widehat{\mu}(z)^a = \widehat{\mu}(B^* z) e^{i\langle c,z \rangle}, \quad z \in \mathbf{R}^d, \tag{3.4.2}$$

where B^* is the adjoint matrix of B. If (3.4.2) *holds for $c \equiv 0$, it is called strictly operator semi-stable.*

We have the following proposition by the same way as in [Sa99b]

Proposition 3.4.5 *Let $\mathbf{X} = \{X(t), t \geq 0\}$ be an $\mathbf{R}^d$-valued Lévy process. Then $\mathbf{X} = \{X(t), t \geq 0\}$ is wide sense operator semi-selfsimilar* (*resp. operator semi-selfsimilar*) *if and only if the distribution of $X(1)$ is operator semi-stable* (*resp. strictly operator semi-stable*).

Remark 3.4.6 *By using Theorem* 3.4.2*, B in* (3.4.2) *is represented by*

$$B = Sa^H, \quad \text{where } S \in \mathcal{M}_{\pm 1}(\mathbf{R}^d), H \in \mathcal{M}_+(\mathbf{R}^d),$$

and all real parts of eigenvalues of H is greater than or equal to 1/2*. Further if a real part of eigenvalue is* 1/2*, the corresponding component is Gaussian* (see [Lu81]).

Let $a > 1$ and $H \in \mathcal{M}_+(\mathbf{R}^d)$ such that the real parts of eigenvalues of H are greater than 1/2, $S \in \mathcal{M}_{\pm 1}(\mathbf{R}^d)$ satisfying $SH = HS$ and let $\mathcal{N}$ be a normal subgroup of $\mathcal{M}^1_{ID}(\mathbf{R}^d)$ such that $B\mathcal{N} = \mathcal{N}B$ where $B = Sa^H$. In order to construct an operator semi-stable process $\mathbf{X}$ with an epoch a, a space scaling matrix Sa^H and $\mathcal{N}(\mathbf{X}) = \mathcal{N}$, we give a slightly modified characterization of the Lévy measure of an operator semi-stable distribution in [Lu81]. Let $D = \bigcup_{N \in \mathcal{N}} NU$, where $U = \{x \in \mathbf{R}^d | \ \|x\| \leq 1\}$. Then D is bounded and $ND \subset D$ for any $N \in \mathcal{N}$ and $0 \in D$. Since absolute values of eigenvalues of B are greater than 1, there exists an $n_0 \in \mathbf{N}$ such that $B^{-n_0}D \subset D$. Since $B\mathcal{N} = \mathcal{N}B$, $N(BD) \subset BD$ for any $N \in \mathcal{N}$. Set $Z_B = D \setminus (B^{-1}D \cup B^{-2}D \cup \cdots \cup B^{-n_0}D)$. Then we see the following lemma.

Lemma 3.4.7

(1) $NZ_B \subset Z_B$ *for any* $N \in \mathcal{N}(\mathbf{X})$,

(2) *if* $m \neq n$, *then* $B^m Z_B \cap B^n Z_B = \emptyset \quad m, n \in \mathbf{Z}$,

(3) $\mathbf{R}^d \setminus \{0\} = \bigcup_{n \in \mathbf{Z}} B^n Z_B$.

Theorem 3.4.8 *Let* μ *be an infinitely divisible distribution without any Gaussian parts.* μ *is an operator semi-stable distribution which is invariant under* $\mathcal{N}$ *with an epoch* $a > 1$ *and a space scaling matrix* $B \in \mathcal{M}_I(\mathbf{R}^d)$ *if and only if its Lévy measure* ν *has the following form:*

$$\nu(E) = \sum_{n \in \mathbf{Z}} a^n \sigma(B^n E \cap Z_B), \quad \textit{for } E \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}), \tag{3.4.3}$$

where σ *is an* $\mathcal{N}$*-invariant finite measure on* Z_B.

Proof. Let $1/\alpha_1$ (resp. $1/\alpha_2$) be the maximum (resp. minimum) of real parts of eigenvalues H. Then by Remark 3.4.6, $0 < \alpha_1 \leq \alpha_2 < 2$. First we notice that ν in (3.4.3) is a Lévy measure. Actually,

$$\begin{aligned} \int_{\cup_{n=0}^{\infty} B^n Z_B} \nu(dx) &\leq \sum_{n=0}^{\infty} a^{-n\alpha_1} \sigma(Z_B) < \infty \qquad \text{and} \\ \int_{\cup_{n=-\infty}^{0} B^n Z_B} |x|^2 \nu(dx) &\leq C \sum_{n=0}^{\infty} a^{-2n} a^{n\alpha_2} \sigma(Z_B) < \infty \quad \text{for some } C > 0. \end{aligned}$$

It is clear that μ is an operator semi-stable distribution with an epoch $a > 1$ and a space scaling matrix B without Gaussian components if and only if the Lévy measure ν of μ satisfies

$$a\nu(E) = \nu(B^{-1}E). \tag{3.4.4}$$

Lévy measure defined by (3.4.3) satisfies (3.4.4) for any $a^n, n \in \mathbf{Z}$ and thus μ is operator semi-stable. If σ is $\mathcal{N}$-invariant, so are ν and μ. Conversely, if

μ is operator semi-stable without Gaussian components whose Lévy measure ν satisfies (3.4.4), then by using Lemma 3.4.7, we have

$$\nu(E) = \sum_{n\in\mathbf{Z}} \nu(E \cap B^n Z_B) = \sum_{n\in\mathbf{Z}} a_0^{-n}\nu(B^{-n}E \cap Z_B).$$

Therefore, we set $\sigma(C) = \nu(C)$ for $C \subset Z_B$ to have (3.4.3). If μ is $\mathcal{N}$-invariant, then ν is so. Thus, we have conclusion. □

Put a measure σ on Z_B such that $\sigma \circ N^{-1} = \sigma$ for any $N \in \mathcal{N}(\mathbf{X})$ and $\sigma \circ S^{-1} \neq \sigma$. We define a Lévy measure ν by (3.4.3) and put $\mu(t) = \mu^{*t}$. Then $(\mu(t), t \geq 0)$ is the law of an operator semi-stable process $\mathbf{X}$ with an epoch a, an exponent matrix H, a sign matrix S and $\mathcal{N}(\mathbf{X}) = \mathcal{N}$.

Chapter 4

Operator semi-stable integral stochastic processes

In this chapter, we consider stochastic integrals with respect to the random measure induced by operator semi-stable Lévy processes. Namely, we integrate some functions by semi-stable Lévy processes. The integrated process has operator semi-selfsimilarity with stationary but not necessarily with independent increments. In [MaM94], they define integral related to operator stable processes and in [MaSa99], they deal semi-stable case. Firstly, we consider the case where integrand is general function. Next, we consider the case where integrand is special case, some "semi-selfsimilar" function, and investigate integrated process. In [V87], he deal with the case integrand function has 1-dimensional selfsimilarity and seek the relation among the selfsimilarities of integrand and integrated process and obtained process. We inquire semi-selfsimilar case.

4.1 Operator semi-self similar processes with stationary increments

In this section, we will give definition of stochastic integral by operator semi-stable Lévy process. The sufficient condition of integrand which is easily applied is also given.

Let μ be Q-operator semi-stable distributions without Gaussian component, and $\Psi(z) := \log \widehat{\mu}(z)$ and $\mathbf{Q}(\mu)$ be the set of all exponents of the operator semi-stable distribution μ, and define

$$\mathrm{Com}(Q) = \{A \in \mathcal{M}(\mathbf{R}^d) | AQ = QA\}.$$

Theorem 4.1.1 *Let $\{Y(x), x \in \mathbf{R}\}$ be an Q-operator semi-stable Lévy process with $\mathcal{L}(Y(1)) = \mu$ and $\{A(u), u \in \mathbf{R}\}$ be real $d \times d$ matrices. If $A(u)$ is measurable and*

$$\int_{-\infty}^{\infty} |\Psi(A(u)^*)z| du < \infty, \tag{4.1.1}$$

where $A(u)^*$ *is transposed matrix of* $A(u)$, *then*

$$I(A) := \int_{-\infty}^{\infty} A(u)dY(u)$$

can be defined. Furthermore, if for some $Q \in \mathbf{Q}(\mu), A(\mu) \in \mathrm{Com}(Q)$ *for all* u, *then* $I(A)$ *is an operator semi-stable vector with characteristic function*

$$E[e^{i\langle z,I(A)\rangle}] = \exp\left\{\int_{-\infty}^{\infty} \Psi(A(u)^*z)du\right\}.$$

Next lemma is shown in a similar way to that for showing Lemma 5.1 in [MaM94]

Lemma 4.1.2 *Let* ξ_1 *and* ξ_2 *are independent* $\mathbf{R}^d$*-valued operator semi-stable random vectors such that* $E[e^{i\langle z,\xi_j\rangle}] = \widehat{\mu}(z)^{c_j}, c_j > 0, z \in \mathbf{R}, j = 1, 2$. *If* $A_1, A_2 \in \mathrm{Com}(Q)$ *for some* $Q \in \mathbf{Q}(\mu)$, *then* $A_1\xi_1 + A_2\xi_2$ *is* Q*-operator semi-stable.*

Proof of Theorem 4.1.1 When $A(\nu)$ is a $\mathcal{M}(\mathbf{R}^d)$-valued step function with the form

$$A(u) = \sum_{j=1}^{k} A_j I_{(x_{j-1},x_j]}(u), \quad A_j \in \mathrm{Com}(Q), x_{j-1} < x_j,$$

we define

$$I(A) = \sum_{j=1}^{k} A_j\{Y(x_j) - Y(x_{j-1})\}.$$

Recall that for $0 < x < y$,

$$E[e^{i\langle z,Y(y)-Y(x)\rangle}] = E[e^{i\langle z,Y(y-x)\rangle}] = \widehat{\mu}(z)^{y-x}.$$

Hence by Lemma 4.1.2, $I(A)$ is Q-operator semi-stable and we have

$$\begin{aligned} E[e^{i\langle z,I(A)\rangle}] &= \prod_{j=1}^{k} \widehat{\mu}(A_j^*z)^{x_j-x_{j-1}} = \prod_{j=1}^{k} \exp\{(x_j - x_{j-1})\Psi(A_j^*z)\} \\ &= \exp\left\{\sum_{j=1}^{k}\{(x_j - x_{j-1})\}\Psi(A_j^*z)\right\} \\ &= \exp\left\{\int_{-\infty}^{\infty} \Psi(A(u)^*z)du\right\}. \end{aligned} \tag{4.1.2}$$

For any $\{A(u)\}$ satisfying (4.1.1), choose a sequence of simple functions $\{A^{(n)}(u)\}_{n=1}^{\infty}$ satisfying

$$\int_{-\infty}^{\infty} \Psi((A^{(n)}(u)^* - A(u)^*)z)du \to 0 \quad \text{as } n \to \infty.$$

Then we can define $I(A)$ as the limit of $I(A^{(n)}$ which does not depend on the choice of $\{A^{(n)}\}$. Let

$$\chi_n(z) := E[e^{i\langle z,I(A^{(n)})\rangle}].$$

Since $I(A^{(n)})$ is Q-operator semi-stable,

$$\chi_n(z)^t = \chi_n(t^{Q^*}z) \quad \text{for each } t > 0. \tag{4.1.3}$$

On the other hand, we have now proved

$$\chi_n(z) \to \chi(z) := E[e^{i\langle z, I(A)\rangle}].$$

By letting $n \to \infty$ in (4.1.3), we have $\chi(z)^t = \chi(t^{Q^*}z)$. Therefore, $I(A)$ is Q-operator semi-stable. The characteristic function of the limit $I(A)$ is the same as in (4.1.2). This completes the proof.

Here, operator semi-stable integral is well-defined and we give definition.

Definition 4.1.3 *Let $\{Y(x), x \in \mathbf{R}\}$ be the same process as in Theorem 4.1.1 and let $\{A_t(u), u \in \mathbf{R}\}_{t\geq 0}$ be real $d \times d$ matrices. such that for some $Q \in \mathbf{Q}(Y(1)), A_t(u) \in \mathrm{Com}(Q)$ for all $u \in \mathbf{R}, t \geq 0$. Then the process $\{X(t), t \geq 0\}$ defined by*

$$X(t) = \int_{-\infty}^{\infty} A_t(u)dY(u)$$

is called operator semi-stable integral process or simply operator stable process.

The integrability condition (4.1.1) is not of the form to be checked easily. Thus, we give some sufficient condition in the following theorem.

Theorem 4.1.4 *Let $\{Y(x), x \in \mathbf{R}\}$ be a symmetric operator semi-stable Lévy process such that $E[e^{i\langle z, Y(1)\rangle}] = \widehat{\mu}(z), z \in \mathbf{R}^d$, where*

$$\widehat{\mu}(z) = \exp\left\{\int_{S_Q} \gamma(dx) \int_0^{\infty} [\cos\langle z, s^Q x\rangle - 1] d\left(-\frac{H_x(s)}{s}\right)\right\}$$

for some $Q \in \mathcal{M}(\mathbf{R}^d)$ with $\tau_Q > 1/2$. If matrix $\{A(u), u \in \mathbf{R}\}$ satisfy that $A(u) \in \mathrm{Com}(Q)$ for all u, $A(u)$ is measurable and that

$$\int_{-\infty}^{\infty} (\|A(u)\|^{1/\tau_Q+\varepsilon} + \|A(u)\|^{1/T_Q-\varepsilon})du < \infty$$

for some ε with $0 < \varepsilon < \min\{2 - 1/\tau_Q, 1/T_Q\}$, then the integrability condition (4.1.1) is satisfies.

Proof. Note that any $0 < p < 2$, $|\cos x - 1| \leq C|x|^p, x \in \mathbf{R}$ for some positive constant C. Let ε satisfies $0 < 1/T_Q - \varepsilon < 1/\tau_Q + \varepsilon < 2$ and we have

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} |\log \widehat{\mu}(A(u)^* z)| du \\ &= \int_{-\infty}^{\infty} du \int_{S_Q} \gamma(dx) \int_0^{\infty} |\cos\langle A(u)^* z, s^Q x\rangle - 1| d\left(-\frac{H_x(s)}{s}\right) \end{aligned}$$

$$
\begin{aligned}
&= C\int_{-\infty}^{\infty} du \int_{S_Q} \gamma(dx) \int_0^1 |\cos\langle A(u)^* z, s^Q x\rangle|^{1/\tau_Q+\varepsilon} d\left(-\frac{H_x(s)}{s}\right) \\
&\qquad C\int_{-\infty}^{\infty} du \int_{S_Q} \gamma(dx) \int_1^{\infty} |\cos\langle A(u)^* z, s^Q x\rangle|^{1/T_Q-\varepsilon} d\left(-\frac{H_x(s)}{s}\right) \\
&\leq C\int_{-\infty}^{\infty} du \int_{S_Q} \gamma(dx) \int_0^1 (\|A(u)\| \, \|z\| \, \|s^Q\|)^{1/\tau_Q+\varepsilon} d\left(-\frac{H_x(s)}{s}\right) \\
&\qquad C\int_{-\infty}^{\infty} du \int_{S_Q} \gamma(dx) \int_1^{\infty} (\|A(u)\| \, \|z\| \, \|s^Q\|)^{1/T_Q-\varepsilon} d\left(-\frac{H_x(s)}{s}\right) \\
&\leq C'\int_{-\infty}^{\infty} du \int_{S_Q} \gamma(dx) \int_0^1 (\|A(u)\| \, \|z\| \, \|s^Q\|)^{1/\tau_Q+\varepsilon} \frac{1}{s^2} ds \\
&\qquad C'\int_{-\infty}^{\infty} du \int_{S_Q} \gamma(dx) \int_1^{\infty} (\|A(u)\| \, \|z\| \, \|s^Q\|)^{1/T_Q-\varepsilon} \frac{1}{s^2} ds,
\end{aligned}
$$

for some $C' > 0$. Here, we use $H_x(s)$ is nonnegative and bounded. And we use next lemma. (This is (i) of Proposition 2.1 in [MaM94].)

Lemma 4.1.5 *For any $\delta > 0$ there exists $C > 0$ such that*

$$
\|s^Q\| \leq \begin{cases} Cs^{\tau_Q-\delta} & s \leq 1, \\ Cs^{T_Q+\delta} & s > 1. \end{cases}
$$

With this lemma, we have

$$
\begin{aligned}
I &\leq C'\|z\|^{1/\tau_Q+\varepsilon} \int_{-\infty}^{\infty} \|A(u)\|^{1/\tau_Q+\varepsilon} du \int_0^1 s^{-1+\varepsilon\tau_Q-\delta(\frac{1}{\tau_Q}+\varepsilon)} ds \\
&\qquad C'\|z\|^{1/T_Q-\varepsilon} \int_{-\infty}^{\infty} \|A(u)\|^{1/T_Q-\varepsilon} du \int_1^{\infty} s^{-1-\varepsilon T_Q-\delta(\frac{1}{T_Q}-\varepsilon)} ds
\end{aligned}
$$

The condition of this theorem, we have $\int_{-\infty}^{\infty} \|A(u)\|^{1/\tau_Q+\varepsilon} du$ and $\int_{-\infty}^{\infty} \|A(u)\|^{1/T_Q-\varepsilon} du$ are finite. For sufficiently small δ, the integrals with respect to s are finite. Therefore, we have conclusion.

4.2 Semi-selfsimilarity of integral stochastic processes

In section 2, we investigate the properties of special case of this integrated process. Similar way to [V87], we inquire semi-selfsimilar case: integrand functions have some semi-selfsimilarity for example local time of semi-stable Lévy process.

As mentioned in [V87], many selfsimilar processes with stationary increments are obtained from basic $\mathbf{R}^d$-valued selfsimilar processes $\{Y(s)\}$ by the following integral:

$$
D(t) = \int_{-\infty}^{\infty} K(t,x) dY(x) \quad \text{for } t \geq 0, \tag{4.2.1}
$$

where $\{K(t,s)\}$ is a deterministic or random function on $[0,\infty)\times\mathbf{R}$ with values in $[-\infty,\infty]$, provided that the integral can be defined in some stochastic integration. If K is random, it is usually assumed to be independent of X.

An $\mathbf{R}$-valued stochastic process $\{K(t,x)\}$, regarded as random functions of s and t, is called (h_1,h_2)-semi-selfsimilar if there exists $a>1$ such that

$$\{K(at,a^{h_2}x)\} \stackrel{\mathrm{d}}{=} \{a^{h_1}K(t,x)\}. \tag{4.2.2}$$

Set

$$s=\inf\{a>1:(4.2.2)\text{ satisfies.}\},$$

and we use a notation $(s,(h_1,h_2))$-*semi-selfsimilar process.* Following theorem corresponds to Theorem 7.1 in [V87], which is the case of selfsimilar processes. Since we consider 1-dimensional random function $K(t,x)$ and d-dimensional process $Y(x)$, we define a d-dimensional process $D(t)$ in (4.2.1) by its component,

$$D^{(i)}(t)=\int_{-\infty}^{\infty}K(t,x)dY^{(i)}(x).$$

Theorem 4.2.1 *We assume the following*:

(1) $\{K(t,x)\}$ *is* $(s_1,(h_1,h_2))$*-semi-selfsimilar,*

(2) $\{Y(x)\}$ *is operator* (s_2,H_3)*-semi-selfsimilar,*

(3) $h_2\log s_1/\log s_2\in\mathbf{Q}$,

(4) $\{K(t,x)\}$ *and* $\{Y(x)\}$ *are independent.*

Then there exists $s_0=s_0(h_2,s_1,s_2)$ *such that* $\{D(t)\}$ *in* (4.2.1) *is an operator* $(s_0,(h_1I+h_2H_3))$*-semi-selfsimilar. We say that* $\{K(t,x)\}$ *has stationary increments if there are random variables* $w(b)$ *and* t *such that for* $b,t\geq 0$ *and* $s\in\mathbf{R}$,

$$K(b+t,x)-K(b,x)\stackrel{d}{\sim}K(t,x+w(b)), \tag{4.2.3}$$

where $\stackrel{d}{\sim}$ *means the equality of the marginal distribution. If we also assume that* $\{K\}$ *and* $\{Y\}$ *have stationary increments, then* $\{D\}$ *has stationary increments.*

Proof.

Semi-selfsimilarity Definition of K implies the following semi-selfsimilarity:

$$\{K(t,x)\}\stackrel{\mathrm{d}}{=}\left\{s_1^{-h_1}K(s_1t,s_1^{h_2}x)\right\}.$$

Then we have

$$\{D^{(i)}(t),t\geq 0\}\stackrel{\mathrm{d}}{=}\left\{s_1^{-h_1}\int_{-\infty}^{\infty}K(s_1t,s_1^{h_2}x)dY^{(i)}(x),t\geq 0\right\}.$$

Assumption (3) implies that there exists an irreducible fraction q/p such that $s_2^q = s_1^{ph_2}$, and it can be shown that

$$\begin{aligned}\{Y(x), x \in \mathbf{R}\} &\stackrel{\mathrm{d}}{=} \{s_2^{-qH_3} Y(s_2^q x), x \in \mathbf{R}\} \\ &= \{s_1^{-ph_2 H_3} Y(s_1^{ph_2} x), x \in \mathbf{R}\}.\end{aligned}$$

Setting $s_0 = s_1^p$, we obtain

$$\begin{aligned}\{D(t), t \geq 0\} &\stackrel{\mathrm{d}}{=} \left\{\int_{-\infty}^{\infty} s_1^{-ph_1} K(s_1^p t, s_1^{ph_2} x) s_1^{-ph_2 H_3} dY(s_1^{ph_2} x), t \geq 0\right\} \\ &= \left\{s_0^{-(h_1 I + h_2 H_3)} D(s_0 t), t \geq 0\right\}.\end{aligned}$$

Statinary increments From increments of $K(t, x)$ and $Y(x)$ are stationary, we prove stationary increments of $\{D(t)\}$,

$$\begin{aligned}\{D(b+t) - D(b)\} &= \left\{\int_{-\infty}^{\infty} (K(b+t, x) - K(b)) dY(x)\right\} \\ &\stackrel{\mathrm{d}}{=} \left\{\int_{-\infty}^{\infty} K(t, x + w(b)) dY(x)\right\} \quad \text{(by (4.2.3))} \\ &= \left\{\int_{-\infty}^{\infty} K(t, x') dY(-w(b) + x')\right\} \quad \text{(setting } x' = x + w(b)\text{)} \\ &\stackrel{\mathrm{d}}{=} \left\{\int_{-\infty}^{\infty} K(t, x') dY(x')\right\} \quad \text{(stationary increment of } \{Y(x)\}\text{)} \\ &= \{D(t)\}.\end{aligned}$$

This implies the process $\{D(t)\}$ has stationary increments. □

An example of random function in (4.2.1) is a local time of a strictly α-semi-stable process. In the next section, we consider a problem for the case.

Chapter 5

Random walks in random sceneries

In this chapter, we construct an example of operator semi-selfsimilar processes in the way of random walks in random sceneries. Kesten and Spitzer considered random walks in random sceneries and selfsimilar processes as a scaled limiting process of that in [KS79]. We extend this problem to operator semi-stable case. In the way of limiting, the limiting process turn to be stochastic integral process defined in chapter 4. Thus, we have an example of semi-stable integrated process and can inquire the limiting process by using the properties proves in the previous chapter.

5.1 Local times of semi-stable Lévy processes

In this section, we prepare some propositions of local time of semi-stable processes for dealing operator semi-selfsimilar processes. We study properties of local times for (r, α)-semi-stable Lévy processes, and consider the case where $d = 1$ in this subsection. It is known that a strictly α-stable Lévy process has a local time at x, $L(t, x)$ with $\alpha > 1$, and we can take a version of $L(t, x)$ which is jointly continuous in (x, t) with $\alpha > 1$ (see [GK72]). In the case of (r, α)-semi-stable Lévy processes, we have the following.

Theorem 5.1.1 *An α-semi-stable Lévy process $\{Y(t), t \geq 0\}$ has a local time at x, $L(t, x)$ with $\alpha > 1$, and there exists a jointly continuous version of $L(t, x)$. If $\alpha = 1$ and $\{Y(t), t \geq 0\}$ is not strictly 1-semi-stable, then it has a local time but does not have its continuous version.*

From now we denote by $L_t(x)$ a continuous version of such a local time.

Proof.

Theorem 7.5 in [Sa99a] implies that almost all sample functions of $\{Y(t)\}$ have the following properties:

(i) 0 is regular for $\{0\}$ (see Section 7 in [Sa99a] or Section 43 in [Sa99b] for the definition of "*regular*"),
(ii) for all x, y we have $P^x\{Y(t) = y \text{ for some } t \geq 0\} > 0$, where P^x is the law of $Y(t)$ starting at x.
They ensure that, for each x, a local time $L(t, x)$ exists (see [GK72]), and at any fixed point it is continuous as a function of t almost surely.

We next show its joint continuity. Proposition 14.9 and 24.20 in [Sa99b] imply that for each $t > 0$ $Y(t)$ is determined as follows:

$$E[\exp\{izY(t)\}] = e^{-t\phi(z)}, \quad \phi(z) = |z|^\alpha\{\eta_1(z) + i\eta_2(z)\} - icz, \tag{5.1.1}$$

where $z \in \mathbf{R}$, $\eta_1(z)$ is bounded from below by a positive constant and continuous in $\mathbf{R} \setminus \{0\}$ satisfying $\eta_1(r^{1/\alpha}z) = \eta_1(z)$, $\eta_2(z)$ is a real function continuous in $\mathbf{R} \setminus \{0\}$ satisfying $\eta_2(r^{1/\alpha}z) = \eta_2(z)$. By Theorem 4 in [GK72], it is enough to show that

$$\sum_{n=1}^{\infty}\{\delta(2^{-n})\}^{1/2} < \infty, \tag{5.1.2}$$

where

$$\delta(u) = \sup_{|x|\leq u}\frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos xz)\mathrm{Re}\left\{\frac{1}{1+\phi(z)}\right\}dz.$$

To show (5.1.2), we use a similar way to that for proving Theorem 7.4 in [Sa99a]. It is known that there exist positive constants k_1 and k_2 such that $k_1 \leq \eta_1(z) \leq k_2$. Using them, we obtain

$$\begin{aligned}\mathrm{Re}\left\{\frac{1}{1+\phi(z)}\right\} &= \frac{1+|z|^\alpha\eta_1(z)}{\{1+|z|^\alpha\eta_1(z)\}^2 + \{|z|^\alpha\eta_2(z) - cz\}^2} \\ &\leq \frac{1+k_2|z|^\alpha}{k_1^2|z|^{2\alpha}}.\end{aligned}$$

This implies that for each $x > 0$

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos xz)\mathrm{Re}\left\{\frac{1}{1+\phi(z)}\right\}dz \\ \leq\ & \frac{x^2}{\pi}\int_{-1/x}^{1/x} z^2\frac{1+k_2|z|^\alpha}{k_1^2|z|^{2\alpha}}dz + \frac{2}{\pi}\int_{|z|>1/x}\frac{1+k_2|z|^\alpha}{k_1^2|z|^{2\alpha}}dz \\ =\ & O(x^{\alpha-1}) \qquad \text{as } x \to 0,\end{aligned}$$

and we can take a continuous version of a local time.

In the case of not strictly 1-semi-stable Lévy process $\eta_1(z)$ in (5.1.1) is same, but $\eta_2(z)$ is not; $\eta_2(z)$ is given by

$$\eta_2(r^{1/\alpha}z) = \eta_2(z) + \mathrm{sgn}z\int_{1<|x|\leq r^{1/\alpha}} x\nu(dx),$$

where $\nu(dx)$ is Lévy measure and $M := \int_{1<|x|\le r^{1/\alpha}} x\nu(dx) \neq 0$. Remark that the constant which is larger than $\eta_2(z)$ for any $z \in \mathbf{R}$ does not exist. In this case, we have $|\eta_2(z)| \sim \frac{|M|}{\log r^{1/\alpha}} \log|z|$ as $|z| \to \infty$ (see page 312 in [Sa99b]).

To show the assertion, it is enough to show that

$$\limsup_{\kappa\to\infty}(\log\kappa)\int_{-\infty}^{\infty} \mathrm{Re}\left\{\frac{1}{\kappa+\phi(z)}\right\}dz > 0 \tag{5.1.3}$$

(see Theorem 4 in [GK72]). Hence

$$\begin{aligned}\mathrm{Re}\left\{\frac{1}{\kappa+\phi(z)}\right\} &\ge \frac{k_1|z|}{(\kappa+|z|k_2)^2+(|z|\eta_2(z)-cz)^2}\\ &\sim \frac{k_1(\log r^{1/\alpha})^2}{M^2}\frac{1}{z(\log|z|)^2},\end{aligned}$$

and we have

$$\begin{aligned}\limsup_{\kappa\to\infty}\int_{-\infty}^{\infty}\mathrm{Re}\left\{\frac{1}{\kappa+\phi(z)}\right\}dz &\ge \int_{|z|\ge\kappa}\frac{k_1|z|}{(\kappa+|z|k_2)^2+(|z|\eta_2(z)-cz)^2}\\ &\sim \frac{k_1(\log r^{1/\alpha})^2}{M^2}\frac{1}{\log\kappa},\end{aligned}$$

which concludes (5.1.3). □

5.2 Random walks in random scenery

In this section, we assume that slowly varying functions in (2.3.1) $l_i \equiv 1, i = 1, 2$. Let $\{S_k, k = 0, 1, 2, \ldots\}$ be a $\mathbf{Z}$-valued random walks such that $\{r_1^{-n/\alpha} S_{[r_1^n t]}, t \ge 0\}$ converges to a strictly (r_1, α)-semi-stable Lévy process $\{Y(t), t \ge 0\}$ with $1 < \alpha \le 2$. We also let $\{\xi(u), u \in \mathbf{Z}\}$ independent identically distributed $\mathbf{R}^d$-valued random variables, independent of $\{S_k\}$, belonging to the domain of partial attraction of strictly operator (r_2, Q)-semi-stable random variable Z_Q, namely $r_2^{-nQ}\sum_{k=1}^{[r_2^n]}\xi(k)$ converges weakly to Z_Q. Since if Z_Q is Gaussian, the problem can be handled similarly to [KS79] and [Bo89], we assumed that Z_Q is purely non-Gaussian and use the representation (2.2.2). When for real parts of eigenvalues of Q satisfy $\tau_Q \le 1 \le T_Q$, we need *symmetry condition*, that is, the distribution of $\xi(0)$ is same as that of $-\xi(0)$.

"Random walks in random scenery" in [KS79] as follows: Let $\mathbf{Z}$-valued random variables X_i's and $\mathbf{R}$-valued random variables $\xi(k)$'s belong to the domain of attraction of strictly α-stable ($\alpha \in (1, 2]$) distribution and that of strictly β-stable ($\beta \in (0, 2]$) distribution, respectively. Assume that they are independent and $E[X_1] = 0$. We set

$$W_l = \sum_{k=0}^{l}\xi(S_k), \tag{5.2.1}$$

where $S_k = \sum_{i=1}^{k} X_i$ and $S_0 = 0$. We define $W(t)$ by

$$W(t) = W_l + (t-l)(W_{l+1} - W_l), \quad l < t < l+1. \tag{5.2.2}$$

Asymptotic behavior of $\{W_n\}$ is determined by two kinds of randomness, random walks $\{S_n\}$ and random scenery $\{\xi(k)\}$, and they imply an interesting selfsimilarity for a scaled limiting process of $\{W(t)\}$. For two kinds of randomness, we consider a strongly dependent sequence $\{\xi(S_n)\}$, its partial sum W_l and the process $\{W(t), t \geq 0\}$. Let $\{Z(x), x \in \mathbf{R}\}$ be an $\mathbf{R}^d$-valued strictly operator (r_2, Q)-semi-stable Lévy process, whose distribution of $Z(1)$ is the same as that of Z_Q, independent of strictly (r_1, α)-semi-stable process $\{Y(t)\}$. By Theorem 5.1.1, we can take a version of local time of $\{Y(t)\}$, which is continuous in (t, x), and denote by $L_t(x)$. Hence we can define a stochastic integral

$$\Delta(t) = \int_{-\infty}^{\infty} L_t(x) dZ(x).$$

Then we have the following theorem.

Theorem 5.2.1 *Let $H = (1 - \frac{1}{\alpha})I + \frac{1}{\alpha}Q$. If $\log r_1/(\alpha \log r_2) \in \mathbf{Q}$, then there exists $r_0 = r_0(r_1, r_2, \alpha)$ such that $\{r_0^{-nH} W(r_0^n t), t \geq 0\}$ converges weakly in* $\mathrm{C}([0,\infty); \mathbf{R^d})$ *to the operator (r_0, H)-semi-selfsimilar process $\{\Delta(t), t \geq 0\}$, which has stationary increments.*

Remark 5.2.2 *In the case where $0 < \alpha < 1$, $\{S_n\}$ is transient* (*see Theorem* 3.1 *in* [C94]), *and we omit the case* (*see page 9 in* [KS79]). *In the case where $\alpha = 1$ $\{L_t(x)\}$ does not have its continuous version and we can not define $\Delta(t)$ through a stochastic integral. On the other hand $\{S_n\}$ is recurrent* (*see Theorem* 3.2 *in* [C94]), *and this is the case for* [Bo89].

Since $\{Y(t)\}$ is an (r_1, α)-semi-stable Lévy process $\{Y(r_1 - nt)\} \stackrel{\mathrm{d}}{=} \{r_1^{n/\alpha} Y(t)\}$, we have the following semi-selfsimilarity of its occupation time
$\Gamma_t(a,b) = \int_0^t \mathbf{1}_{[a,b)}(Y(s))ds$ and local time: For each $n \in \mathbf{N}$,

$$\begin{aligned} \{\Gamma_{r_1^n t}(a,b), t \geq 0\} &\stackrel{d}{=} \left\{ r_1^n \Gamma_t(r_1^{-n/\alpha} a, r_1^{-n/\alpha} b), t \geq 0 \right\}, \\ \{L_t(x), t \geq 0\} &\stackrel{d}{=} \left\{ r_1^{-n(1-1/\alpha)} L_{r_1^n t}(r_1^{n/\alpha} x), t \geq 0 \right\}, \end{aligned}$$

where $\stackrel{d}{=}$ denotes the equality of all finite dimensional distributions with respect to the probability measure of $\{Y(t)\}$ on $\mathrm{D}([0,\infty); \mathbf{R})$, and this implies that $\{L_t(x)\}$ has $(r_1, (1-1/\alpha, 1/\alpha))$-semi-selfsimilarity. Hence we can take an r_0 in the same way to that of taking s_0 in Theorem 4.2.1. Namely, if we denote the irreducible fraction $q/p := \log r_1/(\alpha \log r_2)$ and set $r_0 = r_1^p = r_2^{\alpha q}$, then $\{\Delta(t)\}$ has $(r_0, (1 - \frac{1}{\alpha})I + \frac{1}{\alpha}Q)$-semi-selfsimilarity. Moreover, since $\{Y(t)\}$ has stationary independent increments and a spatially homogeneous transition function, we can show that

$\{L_t(x)\}$ has stationary increments (see (4.2.3) for its definition) in a similar way to that of showing for the case of α-stable Lévy process in [L85]. They imply that $\{\Delta(t)\}$ has stationary increments.

We prove the rest of Theorem 5.2.1 by showing the following propositions under the same assumption as those in the theorem.

Proposition 5.2.3

$$\left\{r_0^{-nH}W(r_0^n t), t \geq 0\ \right\} \stackrel{\mathcal{L}}{\Longrightarrow} \{\Delta(t), t \geq 0\} \qquad \textit{as } n \to \infty,$$

where $\stackrel{\mathcal{L}}{\Longrightarrow}$ *denotes convergence of all finite dimensional distributions with respect to the product measure between the probability measure of* $\{Y(t)\}$ *and* $\{Z_Q(x)\}$.

Proposition 5.2.4
The family $\left\{r_0^{-nH}W(r_0^n t), t \geq 0,\ n \in \mathbf{N}\right\}$ *is tight in* $\mathrm{C}([0,\infty);\mathbf{R}^d)$.

5.3 Proof of Proposition 5.2.3

Let $N(l,u)$ be the number of visits of the random walk $\{S_k\}$ to the point $u \in \mathbf{Z}$ in the time interval $[0,l]$. Using this, we can represent W_l as

$$W_l = \sum_{k=0}^{l} \xi(S_k) = \sum_{u \in \mathbf{Z}} N(l,u)\xi(u). \tag{5.3.1}$$

For the occupation time $N(l,u)$ we consider their linear interpolation:

$$N_t(u) = N(l,u) + (t-l)(N(l+1,u) - N(l,u)), \quad l < t < l+1.$$

For $-\infty < a < b < \infty$ we set

$$T_t^n(a,b) = r_0^{-n} \sum_{a \leq r_0^{-n/\alpha} u < b} N_{r_0^n t}(u) \quad \textit{and} \quad \Gamma_t(a,b) = \int_a^b L_t(x)dx. \tag{5.3.2}$$

Then for each $k \in \mathbf{N}$ and $t_1, t_2, t_3, \ldots, t_k > 0$, Theorem 2.1 and 2.2 imply the following convergence with respect to the measure of $\{Y(t)\}$ (see Section 2 in [KS79]):

$$\{T_{t_j}^n(a_j,b_j), j \in \mathbf{N}\} \stackrel{d}{\longrightarrow} \{\Gamma_{t_j}(a_j,b_j), j \in \mathbf{N}\}. \tag{5.3.3}$$

Using (1) of Theorem 2.3.1, we can show the following lemmas in the same way as that in [KS79]:

Lemma 5.3.1 *There exist C_i's satisfying the following:*

(1) *For each* $p \geq 1$ *and all large* t, *we have*

$$\sup_{u \in \mathbf{Z}} E[N_t(u)^p] = O(t^{p(1-1/\alpha)}),$$

(2) *For all large* t, *we have*

$$\sum_{u \in \mathbf{Z}} E[N_t(u)^2] = O(t^{2-1/\alpha}),$$

Lemma 5.3.2

$$P\{N_t(u) > 0 \text{ for some } u \text{ with } |u| > A(s+1)^{1/\alpha}\} \leq \varepsilon(A) \quad \text{for any } t > 0,$$

where $\varepsilon(A) \to 0$ *as* $A \to \infty$ *and* $\varepsilon(A)$ *is independent of* t.

To prove Proposition 5.2.3, we need more lemmas. In this section we consider two kinds of randomness, hence redenote $\widehat{\mu}$ by $\varphi_Q(z)$, and let $f(z) = \log \varphi_Q(z)$. Since $\{Z_Q(x)\}$ is a Lévy process, we can show the following lemma about joint distribution of $\Delta(t)$ in the same way as that for showing Lemma 3 in [Ma96]:

Lemma 5.3.3 *For any* $k \in \mathbf{N}, t_1, t_2, t_3, \ldots, t_k > 0$ *and* $z_1, z_2, z_3, \ldots, z_k \in \mathbf{R}^d$,

$$E\left[\exp\left\{i\sum_{j=1}^{k}\langle z_j, \Delta(t_j)\rangle\right\}\right] = E\left[\exp\left\{\int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u) z_j\right) du\right\}\right].$$

In the same way as that for showing Lemma 4 in [Ma96] or Lemma 3.1 in [A01], we obtain the following lemma.

Lemma 5.3.4 *Let* $\beta = 1$ *when* $T_Q < 1$, *and let* $0 < \beta < 1/T_Q$ *when* $T_Q \geq 1$. *Then for any* z_1 *and* $z_2 \in \mathbf{R}^d$, *there exists some constant* $K > 0$ *such that*

$$|f(z_1) - f(z_2)| \leq K\left\{\|z_1 - z_2\|(1 + \|z_1\| + \|z_2\|) + \|z_1 - z_2\|^\beta\right\}.$$

We next prepare for showing convergence of all finite dimensional distribution.

Lemma 5.3.5 *For any* $k \in \mathbf{N}$, $t_1, t_2, t_3, \ldots, t_k > 0$ *and* $z_1, z_2, z_3, \ldots, z_k \in \mathbf{R}^d$,

$$\sum_{u \in \mathbf{Z}} f\left(r_0^{-nH^*} \sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j\right) \xrightarrow{d} \int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u) z_j\right) du,$$

Proof.

Since $\varphi_Q(z)^{r_0^n} = \varphi_Q(r_0^{nQ^*} z)$ and $r_0^{-nH^*} = r_0^{-n(1-1/\alpha)} \cdot r_0^{-\frac{n}{\alpha}Q^*}$ for any $z \in \mathbf{R}$, we have

$$\begin{aligned}
\sum_{u \in \mathbf{Z}} f\left(r_0^{-nH^*} \sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j\right) &= \sum_{u \in \mathbf{Z}} f\left(r_0^{-n(1-1/\alpha)} \sum_{j=1}^{k} N_{r_0^n t_j}(u) r_0^{-\frac{n}{\alpha}Q^*} z_j\right) \\
&= \sum_{u \in \mathbf{Z}} r_0^{-n/\alpha} f\left(r_0^{-n(1-1/\alpha)} \sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j\right).
\end{aligned}$$

Hence it is enough to show that

$$\sum_{u\in\mathbf{Z}} {r_0}^{-n/\alpha} f\left(r_0^{-n(1-1/\alpha)}\sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j\right) \xrightarrow{d} \int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u) z_j\right) du. \quad (5.3.4)$$

The following argument is similar to those in [KS79],[M96] and [A01]. Fixing small $\tau > 0$ and large M, we define

$$\begin{aligned}
A_{n,l} &:= \{u\in\mathbf{Z} : l\tau {r_0}^{n/\alpha} \le u < (l+1)\tau {r_0}^{n/\alpha}\}, \quad l\in\mathbf{Z},\\
U(\tau,M,n) &:= \sum_{|x|>M\tau {r_0}^{n/\alpha}} r_0^{-n/\alpha} f\left(r_0^{-n(1-1/\alpha)}\sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j\right),\\
V(\tau,M,n) &:= \sum_{-M\le l<M} |A_{n,l}| {r_0}^{-n/\alpha}\\
&\qquad f\left(r_0^{-n(1-1/\alpha)}\frac{1}{\tau {r_0}^{n/\alpha}}\sum_{y\in A_{n,l}}\sum_{j=1}^{k} N_{r_0^n t_j}(y) z_j\right), \quad (5.3.5)
\end{aligned}$$

where $|A_{n,l}|$ is the number of integers in $A_{n,l}$, and we set

$$\begin{aligned}
I &:= \sum_{u\in\mathbf{Z}} {r_0}^{-n/\alpha} f\left(r_0^{-n(1-1/\alpha)}\sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j\right) - U(\tau,M,n) - V(\tau,M,n)\\
&= \sum_{-M\le l<M}\sum_{u\in A_{n,l}} {r_0}^{-n/\alpha}\left\{ f\left(r_0^{-n(1-1/\alpha)}\sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j\right)\right.\\
&\qquad \left. - f\left(r_0^{-n(1-1/\alpha)}\frac{1}{\tau {r_0}^{n/\alpha}}\sum_{y\in A_{n,l}}\sum_{j=1}^{k} N_{r_0^n t_j}(y) z_j\right)\right\}.
\end{aligned}$$

We denote

$$g_j = N_{r_0^n t_j}(u) \quad \text{and} \quad h_j = \frac{1}{\tau {r_0}^{n/\alpha}}\sum_{y\in A_{n,l}} N_{r_0^n t_j}(y).$$

By Lemma 5.3.4, Hölder's inequality and Minkowski's inequality, we obtain

$$\begin{aligned}
&E[|I|]\\
&\le C(2M+1)\max_{-M\le l<M}|A_{n,l}|{r_0}^{-n/\alpha}\max_{-M\le l<M}\max_{u\in A_{n,l}}\left\{E\left[r_0^{-n(1-1/\alpha)}\|\sum_{j=1}^{k}(g_j-h_j)z_j\|\right.\right.\\
&\quad \left.\left(1+r_0^{-n(1-1/\alpha)}\|\sum_{j=1}^{k} g_j z_j\| + r_0^{-n(1-1/\alpha)}\|\sum_{j=1}^{k} h_j z_j\|\right)\right]\\
&\qquad \left. +E\left[r_0^{-n\beta(1-1/\alpha)}\|\sum_{j=1}^{k}(g_j-h_j)z_j\|^{\beta}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le C^{'} M\tau \max_{-M\le l<M}\max_{u\in A_{n,l}} \left\{ r_0^{-n(1-1/\alpha)} E\left[\| \sum_{j=1}^{k}(g_j-h_j)z_j\|^2 \right]^{1/2} \right. \\
&\qquad E\left[\left(1 + r_0^{-n(1-1/\alpha)} \| \sum_{j=1}^{k} g_j z_j \| + r_0^{-n(1-1/\alpha)} \| \sum_{j=1}^{k} h_j z_j \| \right)^2 \right]^{1/2} \\
&\qquad\qquad \left. + r_0^{-n\beta(1-1/\alpha)} E\left[\| \sum_{j=1}^{k}(g_j-h_j)z_j\|^2 \right]^{\beta/2} \right\} \\
&\le C^{'} M\tau \max_{-M\le l<M}\max_{u\in A_{n,l}} \left\{ r_0^{-n(1-1/\alpha)} \sqrt{ E\left[|\sum_{j=1}^{k}(g_j-h_j)|^2 \right] \sum_{j=1}^{k} \|z_j\|^2 } \right. \\
&\qquad \left(1 + r_0^{-n(1-1/\alpha)} \sqrt{\sum_{j=1}^{k}\|z_j\|^2} \left(E\left[|\sum_{j=1}^{k} g_j|^2 \right]^{1/2} + E\left[|\sum_{j=1}^{k} h_j|^2 \right]^{1/2} \right) \right) \\
&\qquad\qquad \left. + r_0^{-n\beta(1-1/\alpha)} \left(E\left[|\sum_{j=1}^{k}(g_j-h_j)|^2 \right] \sum_{j=1}^{k}\|z_j\|^2 \right)^{\beta/2} \right\}.
\end{aligned}
$$

The following lemma is shown in the same way as that for showing (3.9) in [KS79] by using (2) of Theorem 2.3.1, and we omit its proof:

Lemma 5.3.6

$$
\max_{-M\le l<M}\max_{u\in A_{n,l}} E[|g_j-h_j|^2] \le C\tau^{\alpha-1}(r_0^n)^{2-2/\alpha}.
$$

Using lemma above and (2) of Lemma 5.3.1, we obtain

$$
\begin{aligned}
E[|I|] &\le CM\tau \left\{ r_0^{-n(1-1/\alpha)} \sqrt{C\tau^{\alpha-1}(r_0^n)^{2-2/\alpha}} \right. \\
&\qquad \left(1 + r_0^{-n(1-1/\alpha)} \sqrt{(r_0^n)^{2-2/\alpha}} + r_0^{-n(1-1/\alpha)} \sqrt{(r_0^n)^{2-2/\alpha}} \right) \\
&\qquad \left. + r_0^{-n\beta(1-1/\alpha)} C\tau^{\beta(\alpha-1)/2} r_0^{2n(1-1/\alpha)\beta/2} \right\} \\
&= CM\tau \left(\tau^{(\alpha-1)/2} + \tau^{\beta(\alpha-1)/2} \right).
\end{aligned}
$$

Using (1) of Lemma 5.3.1, for large n and any $\zeta > 0$, we can take $M\tau$ so large that

$$
P\{U(\tau, M, n) \neq 0\} \le \zeta.
$$

Recall $\alpha > 1$, and if we choose τ so small (and thus M so large) that

$$
CM\tau \left(\tau^{(\alpha-1)/2} + \tau^{\beta(\alpha-1)/2} \right) \le \zeta^2,
$$

then we obtain

$$P\left\{\left|\sum_{u\in\mathbf{Z}} f\left(r_0^{-nH^*}\sum_{j=1}^{k} N_{r_0^n t_j}(u)z_j\right) - V(\tau,M,n)\right| > \zeta\right\} \leq 2\zeta. \tag{5.3.6}$$

We next show the convergence of $V(\tau, M, n)$. By using (5.3.2), (5.3.5) can be rewritten as

$$V(\tau,M,n) = \sum_{|l|\leq M} \frac{|A_{n,l}|}{{r_0}^{n/\alpha}} f\left(\frac{1}{\tau}\sum_{j=1}^{k} T_{t_j}^n(l\tau,(l+1)\tau)z_j\right),$$

and (5.3.3) implies that $V(\tau, M, n)$ converges weakly to

$$\tau\sum_{|l|\leq M} f\left(\sum_{j=1}^{k}\frac{1}{\tau}\int_{l\tau}^{(l+1)\tau} L_{t_j}(y)dyz_j\right) \tag{5.3.7}$$

as $n \to \infty$, where we have used $|A_{n,l}|{r_0}^{-n/\alpha} \to \tau$.

Finally, it follows from the continuity and the compact support property of local times of strictly α-semi-stable Lévy processes with $\alpha > 1$ that (5.3.7) converges to

$$\int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u)z_j\right) du$$

as $\tau \to 0$ (and thus $M \to \infty$). This together with (5.3.6) shows (5.3.4) and completes the proof of Lemma 5.3.5. □

The following lemmas is shown in a similar way to that for showing Lemma 6 in [Ma96] by using (1) of Lemma 5.3.1 and Lemma 5.3.2:

Lemma 5.3.7 *For any* $z \in \mathbf{R}^d$,

$$\lim_{s\to\infty}\sup_{u\in\mathbf{Z}} N_s(u)s^{-H^*}z = 0 \textit{ in probability},$$

where H^* *is a transposed matrix of* H.

Recall that for any $z \in \mathbf{R}^d$, $\varphi_Q(z)$ denotes the characteristic function of Z_Q and we denote by $\lambda_Q(z)$ the characteristic function of $\xi(x)$. Then we have

Lemma 5.3.8

$$\lim_{z\to\mathbf{0}}\frac{\log\lambda_Q(z)}{\log\varphi_Q(z)} = 1.$$

Proof.

Respectively replacing r and n/β in Lemma 2.6 in [A01] with r_2^{-1} and nQ, we can show this lemma in a similar way to that for showing Lemma 2.6 in [A01] by taking appropriate norm. □

Proof of Proposition 5.2.3.

By (5.3.1) we have

$$\begin{aligned} I_n &:= E\left[\exp\left\{i\sum_{j=1}^{k}\langle z_j, r_0^{-nH}W_{r_0^n t_j}\rangle\right\}\right] \\ &= E\left[\exp\left\{i\sum_{j=1}^{k}\left\langle z_j, r_0^{-nH}\sum_{u\in\mathbf{Z}}N_{r_0^n t_j}(u)\xi(u)\right\rangle\right\}\right] \\ &= E\left[\prod_{u\in\mathbf{Z}}\lambda_Q\left(r_0^{-nH^*}\sum_{j=1}^{k}N_{r_0^n t_j}(u)z_j\right)\right], \end{aligned}$$

and by Lemmas 5.3.7 and 5.3.8

$$\begin{aligned} \lim_{n\to\infty} I_n &= \lim_{n\to\infty} E\left[\prod_{u\in\mathbf{Z}}\varphi_Q\left(r_0^{-nH^*}\sum_{j=1}^{k}N_{r_0^n t_j}(u)z_j\right)\right] \\ &= \lim_{n\to\infty} E\left[\exp\left\{\sum_{u\in\mathbf{Z}}f\left(r_0^{-nH^*}\sum_{j=1}^{k}N_{r_0^n t_j}(u)z_j\right)\right\}\right] \\ &= E\left[\exp\left\{\int_{-\infty}^{\infty}f\left(\sum_{j=1}^{k}L_{t_j}(u)z_j\right)du\right\}\right] \quad \text{(by Lemma 5.3.5)} \\ &= E\left[\exp\left\{i\sum_{j=1}^{k}\langle z_j, \Delta(t_j)\rangle\right\}\right] \quad \text{(by Lemma 5.3.3).} \end{aligned}$$

This completes the proof of Proposition 5.2.3. □

5.4 Proof of Proposition 5.2.4

Recall (5.3.1), and for each $t \ge 0$ and any $n \in \mathbf{N}$ we set

$$D_n(t) := r_0^{-nH}W(r_0^n t) = r_0^{-nH}\sum_{u\in\mathbf{Z}}N_{r_0^n t}(u)\xi(u).$$

To show the tightness of $\{D_n(t), t \ge 0, n \in \mathbf{N}\}$ in $\mathrm{C}([0,\infty);\mathbf{R}^d)$, we need to show the following estimation: For each $T < \infty$ and any $\eta > 0$,

$$\lim_{\delta\to\infty}\limsup_{n\to\infty} P\left\{\sup_{\substack{0\le t_1,t_2\le T\\ |t_2-t_1|\le\delta}}\{\|D_n(t_2)-D_n(t_1)\| \ge \eta\}\right\} = 0. \tag{5.4.1}$$

The following lemma is shown in a similar way to that for showing (3.19) in [KS79] by using (1) of Lemma 5.3.1 and Lemma 5.3.2:

Lemma 5.4.1 *For any $\varepsilon > 0$, there exists an $A = A(\varepsilon)$ such that*

$$P\{N_{r_0^n t}(u) > 0 \textit{ for some } |u| > A r_0^{n/\alpha} \textit{ and } t \le T\} \le \frac{\varepsilon}{4}.$$

To simplify notation, we use ξ instead of $\xi(0)$ (recall that ξ's are identically distributed). Let

$$\begin{aligned}
c_n(G) &:= r_2^n P\{\|r_2^{-nQ}\xi\| \in G\}, \quad G \in \mathcal{B}((0,\infty)), \\
M(F) &:= \int_{S_Q} \gamma(dx) \int_0^\infty \mathbf{1}_F(s^Q x) d\left(-\frac{H_x(s)}{s}\right), \quad F \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}), \\
c(G) &:= M(\{x : \|x\| \in G\}), \quad G \in \mathcal{B}((0,\infty)).
\end{aligned}$$

By using Theorem 3.3.8 in [MS01], which is a general central limit theorem for independent and infinitely divisible distributed random variables, it is shown that

$$r_2^n P\{r_2^{-nQ}\xi \in F\} \longrightarrow M(F) \tag{5.4.2}$$

for every Borel set F, which is bounded away from zero and $M(\partial F) = 0$. Since we consider purely non-Gaussian case for random scenery, by (5.4.2) we obtain

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} r_2^n \int_{\|x\| < \varepsilon} \langle z, x \rangle^2 P\{r_2^{-nQ}\xi \in dx\} = 0, \quad z \in \mathbf{R}^d. \tag{5.4.3}$$

Remark that $c(\{y\}) = 0$ for each $y > 0$ except Lebesgue measure zero set, and thus by (5.4.2) and (5.4.3), we obtain that

$$c_n([y,\infty)) \longrightarrow c([y,\infty)) \tag{5.4.4}$$

for any $y > 0$ such that $c(\{y\}) = 0$. Then we have the following:

Lemma 5.4.2 *We can find a ρ such that for all large n*

$$(2Ar_0^{n/\alpha} + 1) P\{\|r_0^{-\frac{n}{\alpha}Q}\xi\| > \rho\} \le \frac{\varepsilon}{4}.$$

Proof.
Recall $r_0 = r_2^{\alpha q}$ for some $q \in \mathbf{N}$. By (5.4.4) we have for ρ with $c(\{\rho\}) = 0$ such that

$$\begin{aligned}
r_0^{n/\alpha} P\{\|r_0^{-\frac{n}{\alpha}Q}\xi\| > \rho\} &= r_2^{qn} P\{\|r_2^{-qnQ}\xi\| > \rho\} \\
&= c_{qn}([\rho,\infty)) \longrightarrow c([\rho,\infty)),
\end{aligned}$$

which concludes the lemma. □

Using ρ above, we let

$$\xi_n(u) = \xi(u) I[\|r_0^{-\frac{n}{\alpha}Q}\xi(u)\| \le \rho], \tag{5.4.5}$$

and next estimate its expectation. We prepare properties of measures, $c_n(\cdot)$ and $c(\cdot)$.

Lemma 5.4.3 *Let $\rho > 0$. Then the following are satisfied:*

(1) $\sup_n \int_0^\rho y^2 c_n(dy) < \infty$,

(2) $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \int_0^\varepsilon y^2 c_n(dy) = 0$.

Proof.

(1) By a property of Lévy measure, we have a conclusion.

(2) Suppose $\{\theta_1, \ldots, \theta_d\}$ is an orthonormal basis for $\mathbf{R}^d$. Then $\|x\|^2 = \sum_{i=1}^d \langle \theta_i, x\rangle^2$. Since

$$\int_0^\varepsilon y^2 c_n(dy) = r_2^n \int_{\|x\|<\varepsilon} \|x\|^2 P\{r_2^{nQ}\xi \in dx\},$$

we conclude the lemma by (5.4.3) with $\theta = \theta_1, \ldots, \theta_d$. □

Lemma 5.4.4 *Let $\rho > 0$*

(1) *If $\tau_Q > 1$, then*

$$\int_0^\rho y c(dy) < \infty.$$

(2) *If $T_Q < 1$, then*

$$\int_\rho^\infty y c(dy) < \infty.$$

Proof.

We have

$$c((y,\infty)) = M(\{x : \|x\| > y\}) = \int_{S_Q} \gamma(dx) \int_0^\infty I[\|s^Q x\| > y] d\left(-\frac{H_x(s)}{s}\right).$$

Note that for any $\delta > 0$ there exists $C > 0$ such that

$$\|s^Q\| \le \begin{cases} Cs^{\tau_Q - \delta} & s \le 1, \\ Cs^{T_Q+\delta} & s > 1. \end{cases}$$

Using this, for some $C > 0$ and $x_0 \in S_Q$, we have and set

$$\begin{aligned} c((y,\infty)) &\le \gamma(S_Q)\left\{\int_0^1 I[s > Cy^{1/(\tau_Q-\delta)}]d\left(-\frac{H_{x_0}(s)}{s}\right)\right. \\ &\qquad \left. + \int_1^\infty I[s > Cy^{1/(T_Q+\delta)}]d\left(-\frac{H_{x_0}(s)}{s}\right)\right\} \\ &=: \gamma(S_Q)(J_1(y) + J_2(y)). \end{aligned}$$

○ In the case of (1), we have $J_2(y) = O(1)$ as $y \to 0$ and an inequality,

$$J_1(y) \le \frac{H_{x_0}(Cy^{1/(\tau_Q-\delta)})}{Cy^{1/(\tau_Q-\delta)}}.$$

Since $H_{x_0}(s)$ is bounded for any $s \in \mathbf{R}^d$, we obtain $J_1(y) = O(y^{-1/(\tau_Q-\delta)})$ as $y \to 0$. If $\tau_Q > 1$, then we can find $\delta > 0$ such that $1/(\tau_Q - \delta) < 1$. They imply that $\int_0^\rho c((y,\infty))dy < \infty$.

∘ In the case of (2), we also have $J_1(y) = o(1)$ as $y \to \infty$, an inequality

$$J_2(y) \leq \frac{H_{x_0}(Cy^{1/(T_Q+\delta)})}{Cy^{1/(T_Q+\delta)}},$$

and $J_2(y) = O(y^{-1/(T_Q+\delta)})$ as $y \to \infty$. If $T_Q < 1$, we can find $\delta > 0$ such that $1/(T_Q + \delta) > 1$. They imply that $\int_\rho^\infty c((y,\infty))dy < \infty$.
They conclude the lemma. □

Lemma 5.4.5 *Let $\rho > 0$ with $c(\{\rho\}) = 0$. If $\tau_Q > 1$, then*

$$\sup_n \int_0^\rho y c_n(dy) < \infty.$$

Proof.

Definition of c_n and Lemma 5.4.4 imply that

$$\int_0^\rho y c_n(dy) < \infty, \quad \text{and} \quad \int_0^\rho y c(dy) < \infty,$$

respectively. Note that $c_n(\cdot)$ and $c(\cdot)$ are Lévy measures on $(0,\rho)$, that is, $\int_0^\rho (y^2 \wedge 1)c_n(dy) < \infty$ and $\int_0^\rho (y^2 \wedge 1)c(dy) < \infty$. Hence by (5.4.4), (2) of Lemma 5.4.3 and a convergence theorem of infinitely divisible distributions imply that

$$\exp\left\{\int_0^\rho (e^{izy} - 1)c_n(dy)\right\} \longrightarrow \exp\left\{\int_0^\rho (e^{izy} - 1)c(dy)\right\}, \quad z \in \mathbf{R},$$

and thus

$$\lim_{n\to\infty} \int_0^\rho (e^{izy} - 1)c_n(dy), \quad z \in \mathbf{R}$$

exists. This together with (1) of Lemma 5.4.3 concludes the lemma. □

In the case $T_Q < 1$, we have the following:

Lemma 5.4.6 *Let $\rho > 0$. If $T_Q < 1$, then*

$$\sup_n \int_\rho^\infty y c_n(dy) < \infty.$$

Proof.

Replacing n in Lemma 12 of [Ma96] by r_2^n, firstly we consider the case that ξ is symmetric and next nonsymmetric case. Then we have the conclusion. □
For notational simplicity, we write ξ_n for $\xi_n(0)$ again. We have the following:

Lemma 5.4.7

$$\left\|E\left[r_0^{-\frac{n}{\alpha}Q}\xi_n\right]\right\| = O(r_0^{-n/\alpha}),$$

provided that ξ *is symmetric when* $\tau_Q \le 1 \le T_Q$.

Proof.

◦ When $\tau_Q \le 1 \le T_Q$, we assume that ξ is symmetric and the expectation of ξ_n equals to zero.

◦ In the case where $T_Q < 1$, using the facts $E[\|\xi\|] < \infty$ and $E[\xi] = 0$, and Lemma 5.4.6, we have

$$\begin{aligned}\left\|\sup_n r_0^{\frac{n}{\alpha}} E\left[r_2^{-\frac{n}{\alpha}Q}\xi_n\right]\right\| &= \sup_n r_0^{n/\alpha}\left\|E\left[r_0^{-\frac{n}{\alpha}Q}\xi I[\|r_0^{-\frac{n}{\alpha}Q}\xi\| \le \rho]\right]\right\| \\ &= \sup_n r_0^{n/\alpha}\left\|E\left[r_0^{-\frac{n}{\alpha}Q}\xi I[\|r_0^{-\frac{n}{\alpha}Q}\xi\| > \rho]\right]\right\| \\ &\le \sup_n \int_0^\rho y c_{qn}(dy) < \infty.\end{aligned}$$

◦ In the case where $\tau_Q > 1$. by Lemma 5.4.5 we obtain

$$\begin{aligned}\sup_n r_0^{n/\alpha}\left\|E\left[r_0^{-\frac{n}{\alpha}Q}\xi_n\right]\right\| &= \sup_n r_0^{n/\alpha}\left\|E\left[r_0^{-\frac{n}{\alpha}Q}\xi I[\|r_0^{-\frac{n}{\alpha}Q}\xi\| \le \rho]\right]\right\| \\ &\le \sup_n \int_0^\rho y c_{qn}(dy) < \infty,\end{aligned}$$

and they conclude Lemma 5.4.7. □

Proof of Proposition 5.2.4.

To show (5.4.1), we introduce the following notation:

$$\begin{aligned}E_n &:= r_0^{-nH} E\left[\sum_{u\in\mathbf{Z}} N_{r_0^n}(u)\xi_n(u)\right].\\ D_n'(t) &:= r_0^{-nH}\sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)\{\xi_n(u) - E[\xi_n(u)]\}.\end{aligned}$$

Using them, we divide $\{D_n(t)\}$ as following:

$$\begin{aligned}||D_n(t_2) - D_n(t_1)|| \le\ & ||D_n(t_2) - D_n'(t_2) - E_n t_2|| + ||D_n(t_1) - D_n'(t_1) - E_n t_1|| \\ & + \|E_n\||t_2 - t_1| + ||D_n'(t_2) - D_n'(t_1)||,\end{aligned}$$

and estimate each part. Lemma 5.4.7 implies that

$$\begin{aligned}\|E_n\| &= \left\|r_0^{-n(1-1/\alpha)} r_0^{-\frac{n}{\alpha}Q} E\left[\sum_{u\in\mathbf{Z}} N_{r_0^n}(u)\xi_n(u)\right]\right\| \\ &= \left\|r_0^{-n(1-1/\alpha)} E\left[r_0^{-\frac{n}{\alpha}Q}\xi_n\right] E\left[\sum_{u\in\mathbf{Z}} N_{r_0^n}(u)\right]\right\| \\ &= r_0^{-n(1-1/\alpha)} O(r_0^{-n/\alpha})(r_0^n + 1) = O(1).\end{aligned}$$

We have and set

$$
\begin{aligned}
& D_n(t) - D'_n(t) - E_n t \\
= \; & r_0^{-nH} \sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)\left\{\xi(u) - (\xi_n(u) - E[\xi_n(u)])\right\} - r_0^{-nH} E\left[\sum_{u\in\mathbf{Z}} N_{r_0^n}(u)\xi_n(u)\right] t \\
= \; & r_0^{-nH} \sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)\left\{\xi(u) - \xi_n(u)\right\} \\
& \qquad + r_0^{-nH}\left\{\sum_{u\in\mathbf{Z}} N_{r_0^n t}(u) E[\xi_n(u)] - E\left[\sum_{u\in\mathbf{Z}} N_{r_0^n}(u)\xi_n(u)\right] t\right\} \\
=: \; & r_0^{-nH} \sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)\left\{\xi(u) - \xi_n(u)\right\} + Q_n(t).
\end{aligned}
$$

By using Lemma 5.4.7, it is shown that for each $t \le T$,

$$
\begin{aligned}
\|Q_n(t)\| &= \|r_0^{-nH} E[\xi_n]\{r_0^n t + 1 - (r_0^n + 1)t\}\| \\
&\le T r_0^{-n(1-1/\alpha)} \left\| E\left[r_0^{-\frac{n}{\alpha}Q} \xi_n\right]\right\| \\
&= O(r_0^{-n}),
\end{aligned}
$$

and by using Lemma 5.4.1 and 5.4.2

$$
\begin{aligned}
& P\left\{\sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)\{\xi(u) - \xi_n(u)\} \neq 0 \text{ for some } t \le T\right\} \\
\le \; & P\left\{\xi(u) \neq \xi_n(u) \text{ for some } |u| \le A r_0^{n/\alpha}\right\} + P\left\{N_{r_0^n t}(u) > 0 \text{ for some } |u| > A r_0^{n/\alpha}\right\} \\
\le \; & (2A r_0^{n/\alpha} + 1) P\left\{\left\| r_0^{-\frac{n}{\alpha}Q} \xi\right\| > \rho\right\} + \frac{\varepsilon}{4} \\
\le \; & \frac{\varepsilon}{2}.
\end{aligned}
$$

Hence for any $\eta > 0$ we have

$$
\limsup_{n\to\infty} P\left\{\sup_{t\le T} \|D_n(t) - D'_n(t) - E_n t\| \ge \frac{1}{2}\eta\right\} \le \frac{\varepsilon}{2}, \tag{5.4.6}
$$

and need to show

$$
E[\|D'_n(t) - D'_n(s)\|^2] \le C(t-s)^{2-1/\alpha}. \tag{5.4.7}
$$

If (5.4.7) is satisfied, with the respective replacements of $D_n(t)$ and η by $D'_n(t)$ and $\eta/2$, the relation (5.4.1) is also satisfied, and this together with (5.4.6) imply (5.4.1). We have

$$
E[\|D'_n(t) - D'_n(s)\|^2]
$$

$$
\begin{aligned}
&= E\left[\left\| r_0^{-nH} \sum_{u\in\mathbf{Z}} (N_{r_0^n t}(u) - N_{r_0^n s}(u))(\xi_n(u) - E(\xi_n(u)))\right\|^2\right] \\
&= r_0^{-2n(1-1/\alpha)} E\left[\left\| r_0^{-\frac{n}{\alpha}Q}\{\xi_n - E[\xi_n]\}\right\|^2\right] \sum_{u\in\mathbf{Z}} E\left[\left\{N_{r_0^n t}(u) - N_{r_0^n s}(u)\right\}^2\right] \\
&\leq r_0^{-2n(1-1/\alpha)} E\left[\left\| r_0^{-\frac{n}{\alpha}Q}\xi_n\right\|^2\right] \sum_{u\in\mathbf{Z}} E\left[\left\{N_{r_0^n t}(u) - N_{r_0^n s}(u)\right\}^2\right]. \qquad (5.4.8)
\end{aligned}
$$

Here using (1) of Lemma 5.4.3, we obtain that

$$
\begin{aligned}
& \sup_n r_0^{n/\alpha} E\left[\left\| r_0^{-\frac{n}{\alpha}Q}\xi_n\right\|^2\right] \\
&= \sup_n r_0^{n/\alpha} E\left[\left\| r_0^{-\frac{n}{\alpha}Q}\xi\right\|^2 I\left[\left\| r_0^{-\frac{n}{\alpha}Q}\xi\right\| \leq \rho\right]\right] \\
&= \sup_n \int_0^\rho y^2 c_{qn}(dy) < \infty. \qquad (5.4.9)
\end{aligned}
$$

On the other hand, (2) of Lemma 5.3.1 implies

$$
E\left[\sum_{u\in\mathbf{Z}} \{N_{r_0^n t_2}(u) - N_{r_0^n t_1}(u)\}^2\right] \leq C\{r_0^n(t_2 - t_1)\}^{2-1/\alpha}. \qquad (5.4.10)
$$

Thus (5.4.7) is shown by (5.4.8), (5.4.9) and (5.4.10), and the proof is completed.
□

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