Cycles Containing Specified Vertices and Edges and Trees with Bounded Degree in Graphs

2005

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Preface

This thesis is written on the subject "Cycles Containing Specified Vertices and Edges and Trees with Bounded Degree in Graphs" and is to be submitted for the degree of Doctor of Science at Keio University.

The basis of this thesis is formed by papers written during these four years. After an introductory chapter, the reader will find six chapters. General terminology can be found in Chapter 1. The other chapters can be read independently from one another.

This thesis consists of two parts. In the first part, I will present my work about vertex-disjoint cycles containing specified vertices and edges. In Chapter 2, we study partitions of a graph into vertex-disjoint cycles containing specified vertices and edges. This work is a joint work with H. Enomoto. In Chapters 3 and 4, we will give minimum degree and degree sum conditions for a general graph or a bipartite graph to have vertex-disjoint short cycles containing specified edges.

In the second part, I will present my work about trees with bounded degree. In Chapter 5, we will give two sufficient conditions, an Ore-type condition and a Chvátal-Erdős-type condition, for a graph to have a spanning tree with bounded degree containing the specified leaves. In Chapter 6, we investigate a tree with restrictions on the degrees of the specified vertices. These two works are joint works with H. Matsuda.

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Papers underlying the thesis

[a] H. Enomoto and H. Matsumura, Cycle-partitions with specified vertices and edges, to appear in Ars Combinatoria.

[b] H. Matsumura, Vertex-disjoint short cycles containing specified edges in a graph, to appear in Ars Combinatoria.

[c] H. Matsumura, Vertex-disjoint 4-cycles containing specified edges in a bipartite graph, Discrete Mathematics 297 (2005), 78-90.

[d] H. Matsuda and H. Matsumura, On a k-tree containing specified leaves in a graph, to appear in Graphs and Combinatorics.

[e] H. Matsuda and H. Matsumura, Degree conditions and degree bounded trees, submitted to Discrete Mathematics.

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Introduction

A salesman is to make a tour of n cities and he returns to the head office. The cost of the journey between any two cities is known. The problem asks for an efficient algorithm for finding a least expensive tour. This problem is called the *traveling salesman problem*.

In a version of the traveling salesman problem, the route is required to be a cycle. That is, the salesman is not allowed to visit the same city twice (except the city of the head office). A cycle containing all the vertices of a graph is called a *Hamilton cycle* If there are only two costs, 1 and ∞, then the question is whether or not the graph formed by the edges with cost 1 contains a Hamilton cycle. Even this special case of the traveling salesman problem is difficult to solve. No efficient algorithm is known for constructing a Hamilton cycle. Also, it is not known whether there is such a good algorithm or not. This fact gave rise to a number of sufficient conditions for a graph to have a Hamilton cycle. In particular, the following sufficient condition is well-known.

Theorem 0.1 (Ore [19]) *Suppose that G is a graph of order $n \geq 3$. If the minimum degree sum of nonadjacent vertices is at least n, then G has a Hamilton cycle.*

This degree sum condition is best possible in a sense that we cannot relax the bound n to $n-1$ without destroying the conclusion, however, it seems to be 'strong'. In fact, Brandt et al. proved the following theorem.

Theorem 0.2 (Brandt et al. [1]) *Suppose that $k \geq 1$ is an integer and G is a graph of order $n \geq 4k$. If the minimum degree sum of nonadjacent vertices is at least n, then G can be partitioned into k cycles.*

This theorem says that the condition of Ore's theorem implies not only the existence of a Hamilton cycle but also the existence of a partition into a specified number of cycles. With this result as a starting point, partitions of a graph into a specified number of cycles have been studied.

In [25], Wang considered partitions into cycles passing through specified edges, and conjectured that if $k \geq 2$, n is sufficiently large compared with k, and the minimum degree sum of nonadjacent vertices is at least $n + 2k - 2$, then for any independent edges $e_1, \ldots, e_k$, G can be partitioned into cycles $H_1, \ldots, H_k$ such that $e_i \in E(H_i)$. This conjecture was completely solved by Egawa et al.

Theorem 0.3 (Egawa et al.[10]) *Suppose that $k \geq 2$ is an integer and G is a graph of order $n \geq 4k - 1$. If the minimum degree sum of nonadjacent vertices is at least $n + 2k - 2$, then for any independent edges $e_1, \ldots, e_k$, G can be partitioned into cycles $H_1, \ldots, H_k$ such that $e_i \in E(H_i)$.*

Also, Egawa et al. considered partitions into cycles containing specified vertices, and proved the following theorem.

Theorem 0.4 (Egawa et al. [9]) *Suppose that $k \geq 1$ and G is a graph of order $n \geq 6k - 2$. If the minimum degree is at least $n/2$, then for any distinct vertices $x_1, \ldots, x_k$, G can be partitioned into cycles $H_1, \ldots, H_k$ such that $x_i \in V(H_i)$.*

In Chapter 2, we consider the case where both vertices and edges are specified. We prove the following.

Theorem 0.5 *Suppose that $k \geq p + q$, $p \geq 0$, $q \geq 0$ and G is a graph of order $n \geq 10k$. If either the minimum degree is at least*

$$\max\left\{\frac{n+q}{2}, \frac{n+p+2q-3}{2}\right\}$$

or the minimum degree sum of nonadjacent vertices is at least

$$\max\{n+q, n+2p+2q-2\},$$

then for any distinct vertices $x_1, \ldots, x_p$ and any independent edges $e_{p+1}, \ldots, e_{p+q}$, G can be partitioned into cycles $H_1, \ldots, H_k$ such that $x_i \in V(H_i)$ for $1 \leq i \leq p$ and $e_i \in E(H_i)$ for $p + 1 \leq i \leq p + q$.

Not only partitions into cycles but also the existence of vertex-disjoint cycles has been studied. In [10], Egawa et al. also considered the existence of vertex-disjoint cycles containing specified edges.

Theorem 0.6 (Egawa et al. [10]) *Suppose that $k \geq 1$ and G is a graph of order $n \geq 4k - 1$. If the minimum degree sum of nonadjacent vertices is at least $n + 2k - 2$, then for any independent edges $e_1, \ldots, e_k$, G contains k vertex-disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| \leq 4$.*

In Theorem 0.6, we can replace the assumption 'the minimum degree sum of nonadjacent vertices is at least $n + 2k - 2$' to 'the minimum degree is at least $(n + 2k-2)/2$' to get the same conclusion. The bound '$n+2k-2$' is sharp but '$(n+2k-2)/2$' is not sharp when n is odd. In Chapter 3, we consider this problem and give the sharp minimum degree condition.

Theorem 0.7 *Suppose that $k \geq 1$ and G is a graph of order $n \geq \max\{6k, 4k + 6\}$. If the minimum degree is at least $(n + 2k - 3)/2$, then for any independent edges $e_1, \ldots, e_k$, G contains k vertex-disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ for $1 \leq i \leq k$ and '$|C_i| \leq 4$ for $1 \leq i \leq k$' or '$|C_i| = 5$ for some i and the rest are all triangles'.*

For bipartite graphs, partitions into a specified number of cycles have been also studied. For example, see [4, 5, 15, 17, 27]. Among them, Wang and Chen et al. independently proved the following analogue of Theorem 0.6 for bipartite graphs.

Theorem 0.8 (Wang [26],[29]; Chen et al. [3]) *Suppose that $k \geq 1$ and G is a bipartite graph with partite sets V_1 and V_2 such that $|V_1| = |V_2| = n \geq 2k$. If the minimum degree sum of nonadjacent vertices in the different partite set is at least*

$$\max\left\{n + k, \left\lceil \frac{2n-1}{3} \right\rceil + 2k\right\}$$

or the minimum degree is at least

$$\max\left\{\left\lceil \frac{n+k}{2} \right\rceil, \left\lceil \frac{2n+4k}{5} \right\rceil\right\},$$

then for any independent edges $e_1, \ldots, e_k$, G contains k vertex-disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| \leq 6$.

In Chapter 4, we consider the existence of cycles of length 4 containing specified edges.

Theorem 0.9 *Suppose that $k \geq 1$ and G is a bipartite graph with partite sets V_1, V_2 such that $|V_1| = |V_2| = n \geq 2k$. If the minimum degree sum of nonadjacent vertices in the different partite set is at least*

$$\left\lceil \frac{4n+2k-1}{3} \right\rceil$$

or the minimum degree is at least

$$\left\lceil \frac{2n+3k}{4} \right\rceil,$$

then for any independent edges $e_1, \ldots, e_k$, G contains k vertex-disjoint cycles $C_1, \ldots C_k$ such that $e_i \in E(C_i)$ and $|C_i| = 4$.

Besides Hamilton cycles, Hamilton paths in graphs have been studied well. A *Hamilton path* of a graph is a path containing all the vertices of the graph. Same as a Hamilton cycle, no easily verifiable necessary and sufficient condition for a graph to have a Hamilton path is known and many sufficient conditions were obtained. Among them, the following two theorems are well-known.

Theorem 0.10 (Ore [19]) *Suppose that G is a graph of order n and the minimum degree sum of nonadjacent vertices is at least $n-1$. Then G has a Hamilton path.*

Theorem 0.11 (Chvátal and Erdős [6]) *Suppose that G is a t-connected graph and the independence number of G is at most $t+1$. Then G has a Hamilton path.*

Note that a Hamilton path is a spanning tree with the maximum degree two. In this point of view, spanning trees with bounded maximum degree have been considered. For example, see [2, 7, 13, 14, 22]. Most of the results are based on results on a Hamilton path. In this fashion, Win and Neumann-Lara and Rivera-Campo proved the following.

Theorem 0.12 (Win [30]) *Suppose that $k \geq 2$ and G is a connected graph of order n. If the minimum degree sum of pairwise nonadjacent k vertices is at least $n-1$, then G has a spanning tree with the maximum degree at most k.*

Theorem 0.13 (Neumann-Lara and Rivera-Campo [18]) *Suppose that $k \geq 2$ and G is a t-connected graph. If the independence number of G is at most $t(k-1)+1$, then G has a spanning tree with the maximum degree at most k.*

In Chapters 5 and 6, we consider extensions of Theorems 0.12 and 0.13. A *leaf* is a vertex of degree one in a tree. In Chapter 5, we give two sufficient conditions for a graph to have a spanning tree with bounded degree containing the specified leaves.

Theorem 0.14 *Suppose that $k \geq 2$, $0 \leq s \leq k$ and G is an $(s+1)$-connected graph of order n. If the minimum degree sum of pairwise nonadjacent k vertices is at least $n+(k-1)s-1$, then for any s distinct vertices, G has a spanning tree with the maximum degree at most k such that the specified s vertices are contained in the set of its leaves.*

Theorem 0.15 *Suppose that $k \geq 2$, $0 \leq s \leq k$, $t \geq s+1$ and G is a t-connected graph. If the independence number of G is at most $(t-s)(k-1)+1$, then for any s distinct vertices, G has a spanning tree with the maximum degree at most k such that the specified s vertices are contained in the set of its leaves.*

As an extension of Hamilton cycles, cycles passing through all the specified vertices were considered. In fact, the following theorem is known.

Theorem 0.16 (Shi [23], Ota [21]) *Suppose that G is a 2-connected graph of order $n \geq 3$ and A is a vertex subset of G. If the minimum degree sum of nonadjacent vertices of A is at least n, then G has a cycle containing all the vertices of A.*

In Chapter 6, we consider analogues on trees with bounded degree. That is, we investigate a tree with restrictions on the degrees of the specified vertices.

Theorem 0.17 *Suppose that $k \geq 2$, G is a connected graph of order n and A is a vertex subset of G. If the minimum degree sum of pairwise nonadjacent k vertices of A is at least $n-1$, then G has a tree T with the maximum degree at most k such that T contains all the vertices of A.*

Theorem 0.18 *Suppose that $k \geq 2$, G is a connected graph of order n and A is a vertex subset of G. Let t be the number of components of $G-A$. If $t \leq k-1$ and the minimum degree sum of pairwise nonadjacent $k-t$ vertices of A is at least $|A|-1$, then G has a spanning tree T such that the degree of each vertex in A is at most k.*

Both of them are extensions of Theorem 0.12.

Chapter 1

Fundamentals

In this chapter, we shall present basic terminology and notation of graph theory which will be needed in the following chapters.

1.1 Graphs and directed graphs

A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set $V^{(2)}$ of unordered pairs of V. In this thesis, we consider only finite graphs, that is, V and E are always finite. The set V is the set of *vertices* and E is the set of *edges*.

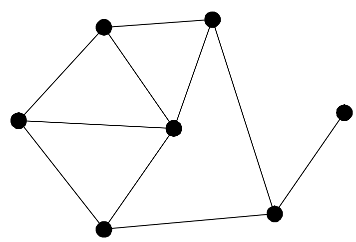

Figure 1.1: A graph.

Given a graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set. An edge $\{x, y\}$ is said to *join* the vertices x and y and is denoted by xy. Thus xy and yx mean exactly the same edge. If $xy \in E(G)$, then x and y are *adjacent* vertices of G, and the vertices x and y are *incident* with the edge xy. Two edges are *adjacent* if they have exactly one common vertex. The *order* of a graph G is the number of vertices in G and is denoted by $|G|$.

For given disjoint subsets U and W of the vertex set of a graph, we write $E(U, W)$ for the set of edges joining a vertex in U to a vertex in W.

A graph is *complete* if every two of its vertices are adjacent. We denote a complete graph of order n by K_n.

If the edges are ordered pair of vertices, we get the notions of a *directed graph.* An ordered pair (a, b) is said to be an *edge* or an *arc directed from a to b* and is denoted by $\overrightarrow{ab}$ or simply ab.

By definition, a graph contains neither a *loop*, an edge joining a vertex to itself, nor *multiple edges*, several edges joining the same two vertices. In a *multigraph*, both multiple edges and loops are allowed.

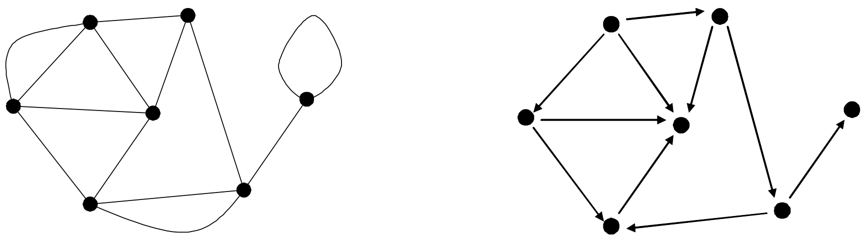

Figure 1.2: A multigraph and a directed graph.

1.2 Subgraphs and operations on graphs

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$ and every edge of E' joins two vertices of V'. If G' contains *all edges* of G that join two vertices in V', then G' is called the subgraph *induced by* V' and is denoted by $\langle V' \rangle$. If $V' = V$, then G' is called a *spanning* subgraph of G.

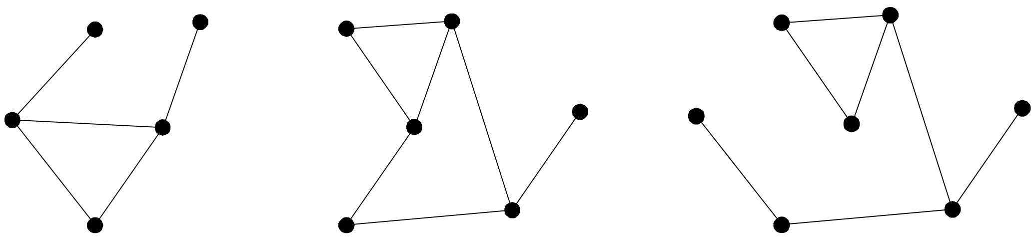

Figure 1.3: A subgraph, an induced subgraph and a spanning subgraph of the graph in Fig. 1.1.

We often construct new graphs from old ones by deleting or adding some vertices and edges. For a subset W of $V(G)$, we define $G - W = \langle V(G) - W \rangle$. Similarly, for a subset E' of $E(G)$, $G - E' = (V(G), E(G) - E')$. If $W = \{w\}$ and $E' = \{xy\}$, then we denote simply by $G - w$ and $G - xy$. If x and y are nonadjacent vertices of G, then $G + xy$ is obtained from G by joining x and y. For a subgraph H of G, we define $G - H = \langle V(G) - V(H) \rangle$.

Let G and H be two graphs. If $V(G) \cap V(H) = \emptyset$, then G and H are *vertex-disjoint*. Similarly, if $E(G) \cap E(H) = \emptyset$, then G and H are *edge-disjoint*. We shall write $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$ for the *union* of G and H, and kG for the union of k disjoint copies of G. We obtain the *join* $G + H$ from the disjoint union $G \cup H$ by adding all edges between G and H.

Given an edge xy of a graph G, the graph G/xy is obtained from G by *contracting* the edge xy. To get G/xy, we identify the vertices x and y and remove all resulting loops and multiple edges. A graph obtained by a sequence of edge-contractions is called a *contraction* of G.

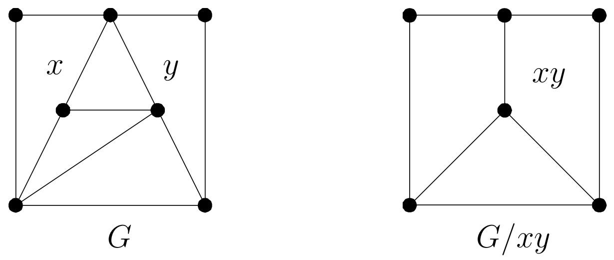

Figure 1.4: A graph G and its contraction G/xy.

1.3 Neighborhoods, degrees and independent sets

The set of vertices adjacent to a vertex $x \in V(G)$ is the *neighborhood* of x and is denoted by $N_G(x)$. Every vertex of $N_G(x)$ is the *neighbor* of x. The *degree* of x is $d_G(x) = |N_G(x)|$. For a subgraph H of a graph G and a vertex $x \in V(G) - V(H)$, we denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. For a subgraph H of G and a subset S of $V(G)$, we define $d_H(S) = \sum_{x \in S} d_H(x)$ and $N_H(S) = \bigcup_{x \in S} N_H(x)$. For a vertex $x \in V(G)$ and a subset S of $V(G)$, we define $N_G[x] = N_G(x) \cup \{x\}$ and $N_G[S] = N_G(S) \cup S$.

The term *independent* will be used in connection with vertices and edges of a graph. A set of *vertices* (*edges*) is *independent* if no two elements of it are adjacent. The *independence number* of G is the maximum size of an independent vertex set of G and is denoted by $\alpha(G)$.

The number $\delta(G) = \min\{d_G(x) \,|\, x \in V(G)\}$ is the *minimum degree* of G. The *maximum degree* of G is defined analogously. For a graph G with $\alpha(G) \geq k$, we define

$$\sigma_k(G) = \min \left\{ \sum_{x \in S} d_G(x) \,\middle|\, S \text{ is an independent subset of } V(G) \text{ with } |S| = k. \right\}$$

and $\sigma_k(G) = \infty$ if $\alpha(G) < k$.

1.4 Paths and cycles

A *walk* W in a graph is an alternating sequence of vertices and edges, say $x_0, e_1, x_1, e_2, \ldots, e_l, x_l$ where $e_i = x_{i-1}x_i$ for $0 \leq i \leq l$. This walk W is denoted by $x_0x_1 \cdots x_l$. The vertices x_0 and x_l are *endvertices* of W and $l = |E(W)|$ is the *length* of W. We say that W is a walk *connecting* x_0 *and* x_l. Also, we say that W is an $x_0 - x_l$ walk. A walk with distinct vertices is called a *path*. If a walk $W = x_0x_1 \cdots x_l$ is such that $l \geq 3$, $x_0 = x_l$ and the vertices x_i, $0 \leq i < l$, are distinct from each other, then W is said to be a *cycle*. We call a cycle of length l an l-cycle. In particular, 3-cycle is called a *triangle*.

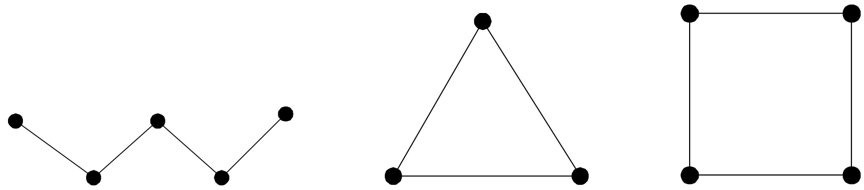

Figure 1.5: A path, a triangle and a 4-cycle.

A cycle containing all the vertices of a graph is called a *Hamilton cycle*. A *Hamilton path* is a path containing all the vertices of a graph.

A collection of paths is called *internally-disjoint* if any two of its elements does not have vertices in common, other than their endvertices.

1.5 Connectivity

A graph is *connected* if any two of its vertices can be joined by a path, and otherwise it is *disconnected*. A *maximal connected subgraph* of a graph G is a *component* of G.

If G is connected and $G - W$ is disconnected for some vertex subset W, then we say that W *separates* G and W is a *separating set* in G. For $t \geq 2$, we say that a graph G is *t-connected* if G has at least $t+2$ vertices and no set of $t-1$ vertices separating it. A connected graph is said to be 1-connected. The maximal value of t for which a connected graph G is t-connected is the *connectivity* of G.

1.6 Trees, matchings and bipartite graphs

A graph without any cycles is a *forest* and a *tree* is a connected forest. We can say that a forest is a graph each of whose components is a tree. A tree with the maximum degree at most k is called a *k-tree*. A *spanning tree* is a tree containing every vertex of a graph. A *leaf* is a vertex of degree one in a tree.

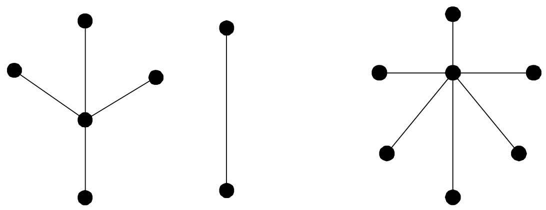

Figure 1.6: A forest.

Sometimes it is convenient to consider one vertex of a tree as a special. Such a vertex is called the *root* of this tree. A tree with a fixed root is a *rooted tree*. An *outdirected tree* $\overrightarrow{T}$ is a rooted tree in which all the edges are directed away from the root. When $\overrightarrow{T}$ is an outdirected tree with the vertex set $V(\overrightarrow{T})$ and the arc set $A(\overrightarrow{T})$ and S is a subset of $V(\overrightarrow{T})$, we denote by $N_T^+(S)$ the set of vertices $w \in V(\overrightarrow{T})$ for which there is an arc $uw \in A(\overrightarrow{T})$ for some $u \in S$.

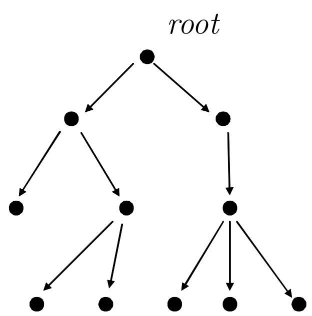

Figure 1.7: An outdirected tree.

A set M of independent edges in a graph G is called a *matching*. We say M *covers* $U \subseteq V(G)$ if every vertex in U is incident with an edge in M.

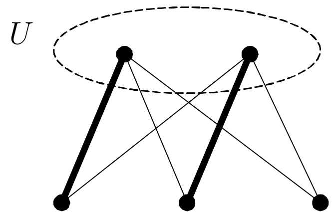

Figure 1.8: A matching which covers U.

A graph G is a *bipartite graph with partite sets* V_1 *and* V_2 if $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and every edge joins a vertex of V_1 to a vertex of V_2. If every pair of a vertex in V_1 and a vertex in V_2 is joined, then G is said to be a *complete bipartite graph*, and is denoted by $K_{m,n}$ if $|V_1| = m$ and $|V_2| = n$.

Chapter 2

Cycle-Partitions with Specified Vertices and Edges

In this chapter, we consider the cycle-partition problems which deal with the case where both vertices and edges are specified and we require that they should belong to different cycles. Minimum degree and degree sum conditions are given, which are best possible.

2.1 Introduction

In this chapter, 'disjoint' means 'vertex-disjoint' since we only deal with partitions of the vertex set, and n always denotes the order of a graph G. Suppose that $C_1, \ldots, C_k$ are disjoint cycles of a graph G. Then $\{C_1, \ldots, C_k\}$ is called a k-cycle-packing of G. Moreover, if $V(G) = \bigcup_{i=1}^{k} V(C_i)$, $\{C_1, \ldots, C_k\}$ is called a k-cycle-partition of G.

The following result is the first step of the research on a k-cycle-partition.

Theorem 2.1 (Brandt et al. [1]) *Suppose that $n \geq 4k$ and $\sigma_2(G) \geq n$. Then G has a k-cycle-partition.*

Egawa et al. considered the cycle-partition with specified vertices. When k vertices $x_1, \ldots, x_k$ are specified, a cycle C is called admissible if $|V(C) \cap \{x_1, \ldots, x_k\}| = 1$. A k-cycle-packing $\{C_1, \ldots, C_k\}$ is admissible if each C_i is admissible. They proved the following theorem.

Theorem 2.2 (Egawa et al. [9]) *Suppose that $n \geq 6k - 2$ and $\delta(G) \geq n/2$. Then G has an admissible k-cycle-partition for any k distinct vertices.*

When k independent edges $e_1 = x_1y_1, \dots, e_k = x_ky_k$ are specified, a cycle C is called admissible if $|E(C) \cap \{e_1, \dots, e_k\}| = 1$ and $|V(C) \cap \{x_1, \dots, x_k, y_1, \dots, y_k\}| = 2$. A k-cycle-packing $\{C_1, \dots, C_k\}$ is admissible if each C_i is admissible. In this case, the following result is obtained.

Theorem 2.3 (Egawa et al. [10]) *Suppose that $k \geq 2, n \geq 4k-1$ and $\sigma_2(G) \geq n+2k-2$. Then G has an admissible k-cycle-partition for any k independent edges.*

In this chapter, we consider the case where both vertices and edges are specified. Let $S = \{v_1, \dots, v_p\}$ be a subset of $V(G)$, $F = \{e_1 = x_1y_1, \dots, e_q = x_qy_q\}$ be a subset of $E(G)$, and $V(F) = \{x_1, \dots, x_q, y_1, \dots, y_q\}$. If $|V(F)| = 2q$ (that is, F is independent) and $S \cap V(F) = \emptyset$, then $S \cup F$ is called *feasible*. A cycle C of G is called *admissible* if one of the following holds:

(a) $V(C) \cap (S \cup V(F)) = \emptyset$,

(b) $|V(C) \cap S| = 1$ and $V(C) \cap V(F) = \emptyset$,

(c) $|E(C) \cap F| = 1$ and $|V(C) \cap (S \cup V(F))| = 2$.

If $C_1, \dots, C_k$ are admissible disjoint cycles and $S \cup V(F)$ is contained in $\bigcup_{i=1}^{k} V(C_i)$, $\{C_1, \dots, C_k\}$ is called an admissible k-cycle-packing. An admissible k-cycle-partition is defined similarly.

The main result of this chapter is the following theorem.

Theorem 2.4 *Suppose that $n \geq 10k$, $k \geq p+q$, $p \geq 0$, $q \geq 0$ and either*

$$\delta(G) \geq \max\left\{\frac{n+q}{2}, \frac{n+p+2q-3}{2}\right\},$$

or

$$\sigma_2(G) \geq \max\{n+q, n+2p+2q-2\}.$$

Then for any feasible set $S \cup F$ with $|S| = p$ and $|F| = q$, G has an admissible k-cycle-partition.

To prove Theorem 2.4, we first solve the packing problem.

Theorem 2.5 *Suppose that $n \geq 9k$, $k \geq p+q$, $p \geq 0$, $q \geq 0$ and either $\delta(G) \geq \max\{n/2, (n+p+2q-3)/2\}$ or $\sigma_2(G) \geq n+2p+2q-2$. Then for any feasible set $S \cup F$ with $|S| = p$ and $|F| = q$, G has an admissible k-cycle-packing.*

Note that the assumption $n \geq 9k$ is not sharp, but it cannot be dropped in a sence that we need the assumption $n \geq 3k$ at least. The degree conditions in Theorem 2.5 are sharp when $q \geq 1$ in the following sense.

Example 2.1. Let $m \geq 1$ and $G = 2K_m + K_{p+2q-2}$ with an edge e_1 which joins two K_ms. Take p distinct vertices $v_1, \ldots, v_p$ and $q-1$ independent edges $e_2, \ldots, e_q$ in K_{p+2q-2} so that $\{v_1, \ldots, v_p, e_1, \ldots, e_q\}$ is feasible. Then there is not an admissible k-cycle-packing, while $\delta(G) = (n + p + 2q - 4)/2$.

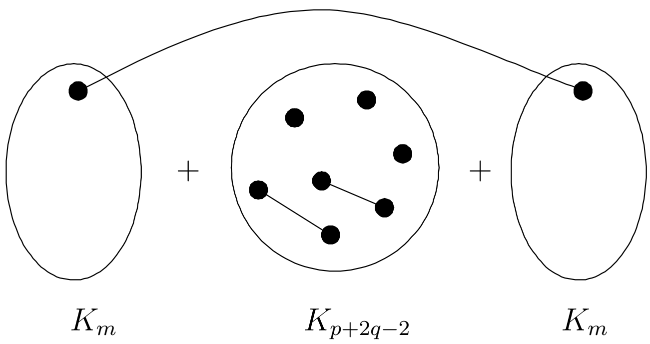

Figure 2.1: The graph G in Example 2.1.

Example 2.2. Let $m \geq 1$ and $G = (K_{p+q} \cup K_m) + K_{2p+2q-1}$. Take p distinct vertices $v_1, \ldots, v_p$ in K_{p+q} and q independent edges $e_1, \ldots, e_q$ between K_{p+q} and $K_{2p+2q-1}$ so that $\{v_1, \ldots, v_p, e_1, \ldots, e_q\}$ is feasible. Then G does not contain an admissible k-cycle-packing, while $\sigma_2(G) = n + 2p + 2q - 3$.

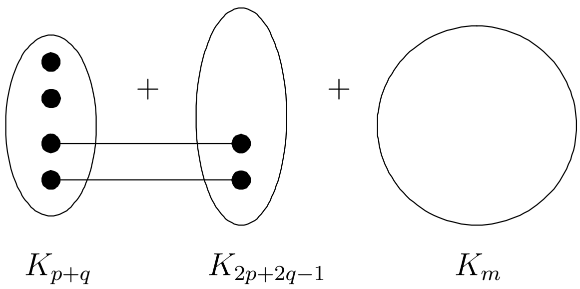

Figure 2.2: The graph G in Example 2.2.

Next, we extend a packing to a partition.

Theorem 2.6 *Let $S \cup F$ be a feasible set with $|S| = p$ and $|F| = q$. Suppose that $n \geq 10k$, $k \geq 1$, $k \geq p+q$, $p \geq 0$, $q \geq 0$, $\delta(G) \geq p+q+1$, $\sigma_2(G) \geq n+q$, and G has an admissible k-cycle-packing. Then G has an admissible k-cycle-partition.*

The assumption $n \geq 10k$ is not sharp, but it cannot be dropped again. The degree conditions in Theorem 2.6 are sharp in the following sense.

Example 2.3. Let $m \geq 2p+2q$ and $G = (K_1 \cup K_m) + K_{p+q}$. Take p distinct vertices in K_{p+q} and q independent edges between K_{p+q} and K_m so that these p vertices and q edges form a feasible set. Then G has an admissible k-cycle-packing but has no admissible k-cycle-partition, while $\delta(G) = p+q$.

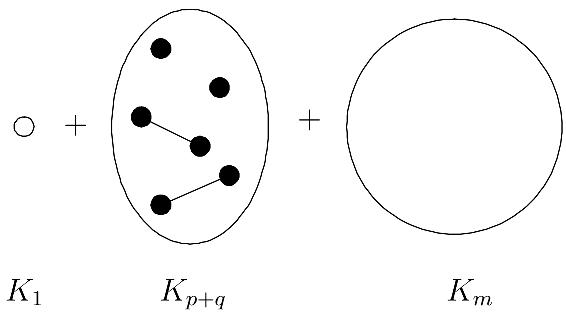

Figure 2.3: The graph G in Example 2.3.

Example 2.4. Let $m \geq 2p+q$ and $G = K_{m+q} + (m+1)K_1$. Take p distinct vertices and q independent edges in K_{m+q} so that these p vertices and q edges form a feasible set. Then G has an admissible k-cycle-packing but does not contain an admissible k-cycle-partition, while $\sigma_2(G) = n+q-1$.

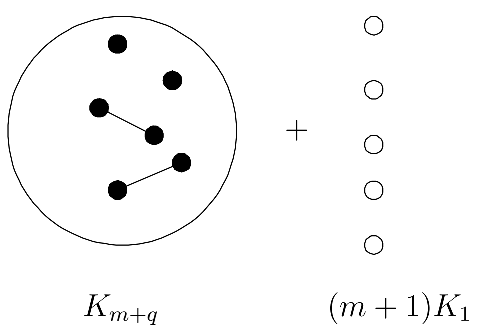

Figure 2.4: The graph G in Example 2.4.

By Theorem 2.5 and Theorem 2.6, we get Theorem 2.4 as a corollary.

If we put $p = 0$ and $q = k$ in Theorem 2.4, we get the following.

Corollary 2.7 *Suppose that $n \geq 10k$, $k \geq 2$, and either*

$$\sigma_2(G) \geq n + 2k - 2$$

or

$$\delta(G) \geq \frac{n + 2k - 3}{2}.$$

Then G has an admissible k-cycle-partition for any k independent edges.

This corollary gives an improvement of Theorem 2.3 on the minimum degree condition when n is odd.

For a path $P = x_1 x_2 \cdots x_l$, we use the notation $P[x_i, x_j]$, $1 \leq i < j \leq l$, for a subpath of P from x_i to x_j. We also use $C[x, y]$ to denote the segment of the cycle C from x to y (including u and v) under some orientation of C, and $C[x, y) = C[x, y] - \{y\}$ and $C(x, y) = C[x, y] - \{x, y\}$. Given a cycle C with an orientation, we let v^+ (resp. v^-) denote the successor (resp. the predecessor) of v along C according to this orientation.

2.2 Proof of Theorem 2.5

To prove Theorem 2.5, we first prove the following two theorems.

Theorem 2.8 *Suppose that $n \geq 9p + 8q - 2$, $p + q \geq 1$ and $\delta(G) \geq (n + p + 2q - 3)/2$. Then for any feasible set $S \cup F$ with $|S| = p$ and $|F| = q$, G has an admissible $(p + q)$-cycle-packing such that all $p + q$ cycles are length at most* 5.

Theorem 2.9 *Suppose that $n \geq 4p + 4q - 1$, $p + q \geq 1$ and $\sigma_2(G) \geq n + 2p + 2q - 2$. Then for any feasible set $S \cup F$ with $|S| = p$ and $|F| = q$, G has an admissible $(p + q)$-cycle-packing such that all $p + q$ cycles are length at most* 4.

The sharpness of the assumptions in Theorems 2.8 and 2.9 is already shown in Section 2.1.

In this section, we will use the following results to prove the above theorems.

Theorem 2.10 (Egawa et al. [10]) *Suppose that $k \geq 1$, $n \geq 4k - 1$ and $\sigma_2(G) \geq n + 2k - 2$. Then for any k independent edges, G has an admissible k-cycle-packing such that each cycle is length at most* 4.

Theorem 2.11 (Enomoto [11], Wang [28]) *Suppose that $k \geq 1$, $n \geq 3k$ and $\sigma_2(G) \geq 4k-1$. Then G has a k-cycle-packing.*

Let $S \cup F$ be a feasible set with $S = \{v_1 \dots, v_p\} \subseteq V(G)$ and $F = \{e_1, \dots, e_q\} \subseteq E(G)$. If $C_1, \dots, C_h$ are admissible disjoint cycles and $S \cup V(F) - \{v_i\}$ for some $v_i \in S$ or $S \cup V(F) - V(e_j)$ for some $e_j \in F$ is contained in $\bigcup_{l=1}^{h} V(C_l)$, $\{C_1, \dots, C_h\}$ is called a *semi-admissible* h-cycle-packing.

2.2.1 Proof of Theorem 2.8

Let G be an edge-maximal counterexample to Theorem 2.8, $S \cup F$ be a feasible set with $S = \{v_1, \dots, v_p\} \subseteq V(G)$ and $F = \{e_{p+1}, \dots, e_{p+q}\} \subseteq E(G)$, and $e_i = x_i y_i$ for $p+1 \leq i \leq p+q$. In the rest of the proof, a cycle is called *short* if its length is at most 5. Since if G is a complete graph, G contains an admissible $(p+q)$-cycle-packing, G is not complete. Let x and y be nonadjacent vertices of G and define $G' = G + xy$, the graph obtained from G by adding the edge xy. Then G' is not a counterexample by the maximality of G, and so G' contains an admissible $(p+q)$-cycle-packing $\{C_1, \dots, C_{p+q}\}$. Since $xy \in E(C_i)$ for some i, $1 \leq i \leq p+q$, G has a semi-admissible $(p+q-1)$-cycle-packing. We take these $p+q-1$ cycles so that admissible cycles which contain specified edges are as many as possible. Subject to this, we take these cycles so that the sum of the length of cycles is as small as possible.

We consider the following two cases.

Case 1 Some specified edge is not contained in the admissible cycles.

We may assume that G contains a semi-admissible $(p+q-1)$-cycle-packing $\{C_1, \dots, C_{p+q-1}\}$ such that $v_i \in V(C_i)$ for $1 \leq i \leq p$, $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q-1$ and $|C_i| \leq 5$ for $1 \leq i \leq p+q-1$. Let $L = \langle \bigcup_{i=1}^{p+q-1} V(C_i) \rangle$, $M = G - L$, and $D = M - \{x_{p+q}, y_{p+q}\}$.

Claim 2.2.1.1 *For any $z \in V(D)$, $d_{C_i}(z) \leq 3$ for $1 \leq i \leq p+q-1$.*

Proof. If $d_{C_i}(z) \geq 4$, $\langle V(C_i) \cup \{z\} \rangle$ contains a cycle passing through v_i or e_i which is shorter than C_i. □

Claim 2.2.1.2 *$d_D(x_{p+q}) \geq 2$ and $d_D(y_{p+q}) \geq 2$.*

Proof. Suppose that $d_D(x_{p+q}) \leq 1$. Then

$$\frac{n+p+2q-3}{2} \leq d_G(x_{p+q}) \leq |L| + 2 \leq 5(p+q-1) + 2.$$

Hence we get

$$n \le 9p + 8q - 3.$$

This is a contradiction. $\square$

Take any $z_1, z_2 \in N_D(x_{p+q})$ and $z_1', z_2' \in N_D(y_{p+q})$ and let $S = \{x_{p+q}, y_{p+q}, z_1, z_2, z_1', z_2'\}$. Since M has no short cycle passing through e_{p+q}, $d_S(y) \le 3$ for any $y \in V(M) - S$. Then,

$$d_M(S) \le 3(|M| - 6) + 14 = 3|M| - 4.$$

Therefore,

$$\begin{aligned} d_L(S) &\ge 6\delta(G) - (3|M| - 4) \\ &= 3n + 3p + 6q - 9 - 3|M| + 4 \\ &= 3|L| + 3p + 6q - 5 \\ &> \sum_{i=1}^{p}(3|C_i| + 3) + \sum_{i=p+1}^{p+q-1}(3|C_i| + 6). \end{aligned}$$

Hence $d_{C_i}(S) \ge 3|C_i| + 4$ for some i, $1 \le i \le p$, or $d_{C_i}(S) \ge 3|C_i| + 7$ for some i, $p + 1 \le i \le p + q - 1$.

Case 1.1 $d_{C_i}(S) \ge 3|C_i| + 4$ for some i, $1 \le i \le p$.

Suppose that $d_{C_i}(\{a, b\}) \ge |C_i| + 2$ for $a \in \{x_{p+q}, z_1, z_2\}$ and $b \in \{y_{p+q}, z_1', z_2'\}$. Then we can find some $c \in N_{C_i}(a) \cap N_{C_i}(b) - \{v_i\}$ and this makes an admissible short cycle passing through e_{p+q}. Hence $d_{C_i}(\{a, b\}) \le |C_i| + 1$ and $d_{C_i}(S) \le 3|C_i| + 3$. This is a contradiction.

Case 1.2 $d_{C_i}(S) \ge 3|C_i| + 7$ for some i, $p + 1 \le i \le p + q - 1$.

Since $d_{C_i}(\{z_1, z_1', z_2, z_2'\}) \le 12$, $d_{C_i}(\{x_{p+q}, y_{p+q}\}) \ge 10$ if $|C_i| = 5$ and $d_{C_i}(\{x_{p+q}, y_{p+q}\}) \ge 7$ if $|C_i| = 4$. These mean that there is an admissible triangle passing through e_{p+q}.

If $|C_i| = 3$, $d_{C_i}(S) \ge 16$. Suppose that $d_{C_i}(x_{p+q}) = d_{C_i}(y_{p+q}) = 3$. Then $d_{C_i}(a) = 3$ for some $a \in \{z_1, z_1', z_2, z_2'\}$, but this means that there are two admissible triangles passing through e_i and e_{p+q}. Otherwise, since $d_{C_i}(\{z_1, z_1', z_2, z_2'\}) \ge 11$, we may assume that $d_{C_i}(z_1) = d_{C_i}(z_1') = d_{C_i}(z_2) = 3$. Then there are two admissible cycles passing through e_i and e_{p+q}. This completes the proof of Case 1.

Case 2 Some specified vertex is not contained in the admissible cycles.

We may assume that G has a semi-admissible $(p+q-1)$-cycle-packing $\{C_2, \ldots, C_{p+q}\}$ such that $v_i \in V(C_i)$ for $2 \le i \le p$, $e_i \in E(C_i)$ for $p + 1 \le i \le p + q$ and $|C_i| \le 5$ for $2 \le i \le p + q$. Let $L = \langle \bigcup_{i=2}^{p+q} V(C_i) \rangle$ and $M = G - L$.

Claim 2.2.2.3 *$d_{C_i}(x) \leq 3$ for $x \in V(M)$ and $2 \leq i \leq p$. Moreover, if $x \neq v_1$, $d_{C_i}(x) \leq 3$ for $p+1 \leq i \leq p+q$.*

Proof. If $x \neq v_1$, the proof is similar to that of Claim 2.2.1.1. Suppose that $d_{C_i}(v_1) \geq 4$ for $2 \leq i \leq p$. Then, $\langle V(C_i) \cup \{v_1\} - \{v_i\} \rangle$ contains a cycle passing through v_i and shorter than C_i. □

Claim 2.2.2.4 *$d_M(v_1) \geq 3$.*

Proof. Suppose that $d_M(v_1) \leq 2$. Then,

$$\frac{n+p+2q-3}{2} \leq d_G(v_1) \leq 3(p-1) + 5q + 2 = 3p + 5q - 1$$

by Claim 2.2.2.3. Hence we get

$$n \leq 5p + 8q + 1.$$

This is a contradiction. □

Take $z_1, z_2, z_3 \in N_M(v_1)$ and let $S = \{v_1, z_1, z_2, z_3\}$. Since M has no short cycle passing through v_1, $d_S(y) \leq 1$ for any $y \in V(M) - S$. Then

$$d_M(S) \leq (|M| - 4) + 6 = |M| + 2.$$

Hence

$$\begin{aligned} d_L(S) &\geq 4\delta(G) - (|M| + 2) \\ &= 2n + 2p + 4q - 6 - |M| - 2 \\ &= 2|L| + 2p - 2 + 4q + |M| - 6 \\ &> 2|L| + 2p - 2 + 4q + 4(p-1) \\ &= 2|L| + 6p - 6 + 4q \\ &= \sum_{i=2}^{p}(2|C_i| + 6) + \sum_{i=p+1}^{p+q}(2|C_i| + 4) \end{aligned} \tag{2.1}$$

since

$$\begin{aligned} |M| - 6 &\geq n - 5p - 5q + 5 - 6 \geq 9p + 8q - 2 - 5p - 5q - 1 \\ &= 4p + 3q - 3 > 4(p-1). \end{aligned}$$

Claim 2.2.2.5 *$d_{C_i}(S) \leq 2|C_i| + 4$ for $p+1 \leq i \leq p+q$.*

Proof. Suppose that $d_{C_i}(S) \geq 2|C_i| + 5$ for some i, $p+1 \leq i \leq p+q$.

If $|C_i| = 5$, $d_{C_i}(S) \geq 15$. But this contradicts Claim 2.2.2.3.

If $|C_i| = 4$, $d_{C_i}(S) \geq 13$. Then, $d_{C_i}(v_1) = 4$ and $d_{C_i}(z_1) = d_{C_i}(z_2) = d_{C_i}(z_3) = 3$. This means that there are two admissible short cycles passing through v_1 and e_i.

If $|C_i| = 3$, $d_{C_i}(S) \geq 11$. In this case, we may assume that $d_{C_i}(z_1) = d_{C_i}(z_2) = 3$. Then, $d_{C_i}(z_3) \leq 1$. But this is a contradiction. $\square$

By (2.1) and Claim 2.2.2.5, we may assume that $d_{C_i}(S) \geq 2|C_i| + 7$ for some i, $2 \leq i \leq p$. Clearly, this contradicts Claim 2.2.2.3. This completes the proof of Theorem 2.8.

2.2.2 Proof of Theorem 2.9

Let $S \cup F$ be a feasible set with $S = \{v_1, \ldots, v_p\} \subseteq V(G)$ and $F = \{e_{p+1}, \ldots, e_{p+q}\} \subseteq E(G)$. Since $\sigma_2(G) \geq n+2p+2q-2$, $\delta(G) \geq 2p+2q$. Then we can take p independent edges $e_1, \ldots, e_p$ such that $v_i \in V(e_i)$ for $1 \leq i \leq p$ and $\{e_1, \ldots, e_{p+q}\}$ is also a set of independent edges. Therefore, we can apply Theorem 2.10 and obtain a required $(p+q)$-cycle-packing. $\square$

2.2.3 Proof of Theorem 2.5

The case $p = q = 0$ follows from Theorem 2.11. Thus we may assume that $p + q \geq 1$. Let $S \cup F$ be a feasible set with $|S| = p$ and $|F| = q$. By Theorem 2.8 and Theorem 2.9, G has an admissible $(p+q)$-cycle-packing $\{C_1, \ldots, C_{p+q}\}$ such that $|C_i| \leq 5$ for $1 \leq i \leq p+q$. If $k = p+q$, this is a required k-cycle-packing. Hence we may assume that $k > p+q$. Then we take these cycles so that $|\bigcup_{i=1}^{p+q} V(C_i)|$ is as small as possible. Let $L = \langle \bigcup_{i=1}^{p+q} V(C_i) \rangle$ and $H = G - L$. Note that $d_{C_i}(x) \leq 3$ for any $x \in V(H)$ and $1 \leq i \leq p+q$. Then $|H| \geq n - 5(p+q) \geq 3(k-p-q)$ and

$$\sigma_2(H) \geq n + 2p + 2q - 3 - 6(p+q) \geq 4(k-p-q) - 1.$$

Therefore, we can apply Theorem 2.11 and we get a $(k-p-q)$-cycle-packing of H. Hence we get an admissible k-cycle-packing of G. This completes the proof of Theorem 2.5.

2.3 Proof of Theorem 2.6

2.3.1 Preliminary Lemmas

Before proving the theorem, we prepare several definitions and lemmas.

Let D be a cycle (resp. a path) of G and $x \in V(G-D)$. We say x *can be inserted into* D if $\langle V(D) \cup \{x\} \rangle$ has a cycle (resp. a path) D' such that $V(D') = V(D) \cup \{x\}$. Moreover, if D contains a specified edge e, D' has to contain e, and if D is a u-v path, then D' also has to be a u-v path.

Lemma 2.1 *Let C be a cycle of G and $x \in V(G-C)$. Suppose that C does not contain a specified edge and $d_C(x) \geq (|C|+1)/2$. Then x can be inserted into C.*

Proof. Since $d_C(x) \geq (|C|+1)/2$, $N_C(x)$ contains two consecutive vertices of C. Hence x can be inserted into C. $\square$

Lemma 2.2 *Let $P = u_1u_2\cdots u_l$ be a path of G and $x \in V(G-P)$. Suppose that P does not contain a specified edge and $d_P(x) \geq |P|/2+1$. Then x can be inserted into P.*

Proof. Since $d_P(x) \geq |P|/2+1$, $N_P(x)$ contains two consecutive vertices of P. Hence x can be inserted into P. $\square$

Lemma 2.3 *Let C be a cycle of G and $x \in V(G-C)$. Suppose that $e \in E(C)$ is a specified edge and $d_C(x) \geq |C|/2+1$. Then x can be inserted into C.*

(*Proof.*) Let $e = aa^+$. Since $d_C(x) \geq |C|/2+1$, $N_G(x) \cap C[a^+, a^-]$ contains two consecutive vertices of C. Then x can be inserted into C. $\square$

Lemma 2.4 *Let $P = u_1u_2\cdots u_l$ be a path of G and $x \in V(G-P)$. Suppose that $e \in E(P)$ be a specified edge and $d_P(x) \geq (|P|+3)/2$. Then x can be inserted into P.*

Proof. Let $e = u_iu_{i+1}$, $1 \leq i \leq l-1$. Since $d_P(x) \geq (|P|+3)/2$, $N_G(x) \cap P[u_1, u_i]$ or $N_G(x) \cap P[u_{i+1}, u_l]$ contains two consecutive vertices of P. Hence x can be inserted into P. $\square$

Let $C_1, \ldots, C_k$ be disjoint subgraphs such that C_h is a u-v path for some h, $1 \leq h \leq p+q$, the rest are all cycles, and $v_i \in V(C_i)$ for $1 \leq i \leq p$ and $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q$. Let also $L = \langle \bigcup_{i=1}^{k} V(C_i) \rangle$ and $M \subseteq V(G-L)$, $M \neq \emptyset$. Then we say M *can be inserted into* L if $\langle V(L) \cup M \rangle$ contains disjoint subgraphs $C'_1, \ldots, C'_k$

such that C'_h is a u-v path, the rest are all cycles, $v_i \in V(C_i)$ for $1 \leq i \leq p$ and $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q$, and $\bigcup_{i=1}^{k} V(C'_i) = V(L) \cup M$.

Lemma 2.5 *Let L be a subgraph of G defined in the above definition, $M \subseteq V(G-L)$ and $M \neq \emptyset$. Suppose that $N_G(M) \subseteq V(L) \cup M$ and*

$$d_G(x) \geq \frac{|L|+q}{2} + (|M|-1) + \frac{3}{2}$$

for any $x \in V(M)$. Then M can be inserted into L.

Proof. Take any $x \in V(M)$. Then

$$\begin{aligned} d_L(x) &\geq \frac{|L|+q}{2} + (|M|-1) + \frac{3}{2} - (|M|-1) = \frac{|L|+q}{2} + \frac{3}{2} \\ &= \sum_{i=1}^{p} \frac{|C_i|}{2} + \sum_{i=p+1}^{p+q} \frac{|C_i|+1}{2} + \sum_{i=p+q+1}^{k} \frac{|C_i|}{2} + \frac{3}{2}. \end{aligned}$$

Hence one of the following holds.

(a) $1 \leq h \leq p$ and $d_{C_h}(x) \geq \frac{|C_h|}{2} + 1$.

(b) $p+1 \leq h \leq p+q$ and $d_{C_h}(x) \geq \frac{|C_h|+3}{2}$.

(c) $d_{C_i}(x) \geq \frac{|C_i|+1}{2}$ for some $i \neq h$, $1 \leq i \leq p$ or $p+q+1 \leq i \leq k$.

(d) $d_{C_i}(x) \geq \frac{|C_i|}{2} + 1$ for some $i \neq h$, $p+1 \leq i \leq p+q$.

Then, by Lemmas 2.1, 2.2, 2.3, and 2.4, x can be inserted into C_h or C_i.

Let $L' = \langle V(L) \cup \{x\} \rangle$ and $M' = M - \{x\}$, and suppose that $M' \neq \emptyset$. Then $N_G(M') \subseteq V(L') \cup M'$ and for any $y \in V(M')$,

$$\begin{aligned} d_G(y) &\geq \frac{|L|+q}{2} + (|M|-1) + \frac{3}{2} \\ &= \frac{|L'|-1+q}{2} + (|M'|+1-1) + \frac{3}{2} \\ &= \frac{|L'|+q}{2} + (|M'|-1) + 2. \end{aligned}$$

Again, y can be inserted into L'. By repeating this operation, M can be inserted into L. □

2.3.2 Proof of Theorem 2.6

Suppose that $\mathcal{C} = \{C_1, \ldots, C_k\}$ and $\mathcal{C}' = \{C'_1, \ldots, C'_k\}$ are two admissible k-cycle-packing. We say $\mathcal{C}$ *is larger than* $\mathcal{C}'$ if $|\bigcup_{i=1}^{k} V(C_i)| > |\bigcup_{i=1}^{k} V(C'_i)|$.

In the rest of this section, $N(x)$ and $N(H)$ will be used instead of $N_G(x)$ and $N_G(H)$ for $x \in V(G)$ and a subgraph H of G.

Let $S \cup F$ be a feasible set with $S = \{v_1, \ldots, v_p\} \subseteq V(G)$ and $F = \{e_{p+1}, \ldots e_{p+q}\} \subseteq E(G)$, and $e_i = x_i y_i$ for $p+1 \leq i \leq p+q$. Since G contains an admissible k-cycle-packing, we take an admissible k-cycle-packing $\{C_1, \ldots, C_k\}$ such that $|\bigcup_{i=1}^{k} V(C_i)|$ is as large as possible. We may assume that $v_i \in V(C_i)$ for $1 \leq i \leq p$ and $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q$. Let $L = \langle \bigcup_{i=1}^{k} V(C_i) \rangle$ and $H = G - L$. If $H = \emptyset$, we have nothing to prove. Hence we may assume that $H \neq \emptyset$.

By Lemmas 2.1 and 2.3, the next claim holds.

Claim 2.3.1 *For $x \in V(H)$, $d_{C_i}(x) \leq |C_i|/2$ for $1 \leq i \leq p$ and $p+q+1 \leq i \leq k$, and $d_{C_i}(x) \leq (|C_i|+1)/2$ for $p+1 \leq i \leq p+q$.*

Claim 2.3.2 *H is connected.*

Proof. Let H_0 be a connected component of H, $x \in V(H_0)$ and $y \in V(H - H_0)$. Then,

$$\begin{aligned} n + q &\leq d_G(x) + d_G(y) \\ &\leq |H_0| - 1 + \sum_{i=1}^{k} d_{C_i}(x) + |H - H_0| - 1 + \sum_{i=1}^{k} d_{C_i}(y) \\ &\leq |H| - 2 + \sum_{i=1}^{k} |C_i| + q = n + q - 2 \end{aligned}$$

by Claim 2.3.1. But this is a contradiction. □

Claim 2.3.3 *Suppose that $b_1, b_2 \in N(H) \cap V(C_i)$, $b_1 \neq b_2$, and $v_i \notin V(C_i(b_1, b_2))$ if $1 \leq i \leq p$ and $e_i \notin E(C_i[b_1, b_2])$ if $p+1 \leq i \leq p+q$. Then $V(C_i(b_1, b_2)) \neq \emptyset$.*

Proof. Take $a_1, a_2 \in V(H)$ such that $a_1 b_1, a_2 b_2 \in E(G)$ (possibly $a_1 = a_2$) and suppose that $b_2 = b_1^+$. Then we can get an admissible cycle $b_1 a_1 P a_2 b_2 C_i(b_2, b_1) b_1$ which is longer than C_i, where P is a path in H connecting a_1 and a_2. This contradicts the maximality of L. □

Claim 2.3.4 $|N(H) \cap V(C_i)| \leq 1$ *for $1 \leq i \leq k$.*

Proof. Suppose that $|N(H)\cap V(C_i)|\geq 2$ for some i, $1\leq i\leq k$. Choose two vertices $b_1,b_2\in V(C_i)$ and vertices $a_1,a_2\in V(H)$ (possibly $a_1=a_2$) such that $a_jb_j\in E(G)$ for $j=1,2$, $v_i\notin V(C_i(b_1,b_2))$ if $1\leq i\leq p$, $e_i\notin E(C_i[b_1,b_2])$ if $p+1\leq i\leq p+q$ and $N(H)\cap V(C_i(b_1,b_2))=\emptyset$. Take $x\in V(H)$ and $y\in V(C_i(b_1,b_2))$. Then,

$$\begin{aligned} n+q &\leq d_G(x)+d_G(y)\\ &\leq |H|-1+\sum_{h=1}^{p}\frac{|C_h|}{2}+\sum_{h=p+1}^{p+q}\frac{|C_h|+1}{2}+\sum_{h=p+q+1}^{k}\frac{|C_h|}{2}-\frac{|C_i(b_1,b_2)|}{2}+\frac{1}{2}+d_G(y)\\ &\leq |H|-\frac{1}{2}+\frac{|L|}{2}+\frac{q}{2}-\frac{|C_i(b_1,b_2)|}{2}+d_G(y). \end{aligned}$$

Hence

$$d_G(y)=d_L(y)\geq\frac{|L|+q+|C_i(b_1,b_2)|+1}{2}. \tag{2.2}$$

Let $L'=\langle V(C_i[b_2,b_1])\cup(\bigcup_{h=1}^{k}V(C_h)-V(C_i))\rangle$. Then by (2.2),

$$\begin{aligned} d_G(y) &\geq \frac{|L|+q+|C_i(b_1,b_2)|+1}{2}=\frac{|L'|+|C_i(b_1,b_2)|+q+|C_i(b_1,b_2)|+1}{2}\\ &= \frac{|L'|+q}{2}+(|C_i(b_1,b_2)|-1)+\frac{3}{2}. \end{aligned}$$

Hence by Lemma 2.5, $V(C_i(b_1,b_2))$ can be inserted into L'. By adding $b_1a_1Pa_2b_2$ where P is a path in H connecting a_1 and a_2, we get a larger admissible k-cycle-packing. This is a contradiction. □

Claim 2.3.5 *$|N(H)\cap V(C_i)|=\emptyset$ for $p+q+1\leq i\leq k$.*

Proof. Suppose that $|N(H)\cap V(C_i)|\neq\emptyset$ for some i, $p+q+1\leq i\leq k$. Without loss of generality, we may assume that $i=k$. Take $y\in N(H)\cap V(C_k)$.

Subclaim 2.3.5.1 *$|N(H)\cap V(C_i)|\neq\emptyset$ and $d_{C_i}(y^+)+d_{C_i}(y^-)\geq 2|C_i|-1$ for some i, $1\leq i\leq p$ or $p+q+1\leq i\leq k-1$.*

Proof. Suppose that the subclaim does not hold. Let $r=|\{h|N(H)\cap V(C_h)\neq\emptyset, 1\leq h\leq p, p+q+1\leq h\leq k\}|$, $r'=|\{h|N(H)\cap V(C_h)\neq\emptyset, p+1\leq h\leq p+q\}|$. Then

$$d_L(y^+)+d_L(y^-)\leq\sum_{h=1}^{k}2|C_h|-2r=2|L|-2r.$$

Without loss of generality, we may assume that $d_L(y^+) = d_G(y^+) \le |L| - r$. Take any $x \in V(H)$, then

$$\begin{aligned} n + q &\le d_G(x) + d_G(y^+) \le |H| - 1 + r + r' + |L| - r \\ &= n + r' - 1. \end{aligned}$$

Hence we get $q \le r' - 1$, but this is a contradiction. □

We may assume that $N(H) \cap V(C_i) \neq \emptyset$ and $d_{C_i}(y^+) + d_{C_i}(y^-) \ge 2|C_i| - 1$ for some i, $1 \le i \le p$ or $p + q + 1 \le i \le k - 1$. Take $z \in N(H) \cap V(C_i)$. By symmetry, we may assume that $y^+z^-, y^+z^+, y^-z \in E(G)$. Let $ya_1, za_2 \in E(G)$, $a_1, a_2 \in V(H)$ (possibly $a_1 = a_2$). We replace C_i to $C_i' = y^+z^+C_i(z^+, z^-)z^-y^+$ and, let $P = yy^-z$, $L' = \langle (\bigcup_{h=1}^{k} V(C_h) - V(C_i \cup C_k)) \cup V(C_i' \cup P) \rangle$ and $M = V(C_k) - \{y, y^+, y^-\}$. For any $x \in M$, since $d_G(a_1) \le |H| - 1 + k$ and $xa_1 \notin E(G)$,

$$\begin{aligned} d_G(x) &\ge n + q - (|H| - 1 + k) = |L| + q - k + 1 \\ &= |L'| + |M| + q - k + 1 \\ &\ge \frac{|L'| + q}{2} + (|M| - 1) + \frac{3(k-1)}{2} + \frac{q}{2} - k + 2 \\ &= \frac{|L'| + q}{2} + (|M| - 1) + \frac{k + q + 3}{2} \\ &> \frac{|L'| + q}{2} + (|M| - 1) + \frac{3}{2}. \end{aligned}$$

Then by Lemma 2.5, M can be inserted into L'. By adding $za_2P'a_1y$ where P' is a path in H connecting a_1 and a_2, we get a larger admissible k-cycle-packing. □

Let $N(H) \cap V(C_h) = \{u_h\}$ for $1 \le h \le r_1$ and $p + 1 \le h \le r_2$ and $N(H) \cap V(C_h) = \emptyset$ for $r_1 + 1 \le h \le p$ and $r_2 + 1 \le h \le p + q$. Since $\sigma_2(G) \ge n + q$, G is $(q + 2)$-connected. Hence $r_1 \ge 2$. Let also $|N(u_h) \cap V(H)| \ge 2$ for $1 \le h \le s_1$, $|N(u_h) \cap V(H)| = 1$ for $s_1 + 1 \le h \le r_1$ and $r = r_1 + r_2 - p$. Let $U_1 = \{u_1, \ldots u_{s_1}\}$ and $U = \{u_1, \ldots, u_{r_1}, u_{p+1}, \ldots, u_{r_2}\}$. If r_2 does not exist, let $r = r_1$ and $U = \{u_1, \ldots, u_{r_1}\}$.

Claim 2.3.6 *$u_i \neq v_i$ for $u_i \in U_1$.*

Proof. Suppose that $u_i = v_i$ for some $i \in U_1$. Without loss of generality, we may assume that $i = 1$. Let $a_1, a_2 \in N(v_1) \cap V(H)$ and $L' = \langle \bigcup_{i=2}^{k} V(C_i) \rangle$. Since

$d(x) \leq |H| - 1 + k$ and $xv \notin E(G)$ for any $x \in V(H)$ and $v \in V(C_1) - \{v_1\}$,

$$\begin{aligned} d_G(v) &\geq n + q - (|H| - 1 + k) \\ &= |L| + q - k + 1 \\ &= |L'| + |C_1| + q - k + 1 \\ &\geq \frac{|L'| + q}{2} + \frac{3(k-1)}{2} + \frac{q}{2} + (|C_1| - 1) - k + 2 \\ &= \frac{|L'| + q}{2} + (|C_1| - 1) + \frac{k}{2} + \frac{q}{2} + \frac{1}{2} \\ &\geq \frac{|L'| + q}{2} + (|C_1| - 1) + \frac{3}{2} \end{aligned}$$

Since $N(v) \subseteq V(L)$, $V(C_1) - \{v_1\}$ can be inserted into L' by Lemma 2.5. Let $C_1' = v_1 a_1 P a_2 v_1$, where P is a path in H connecting a_1 and a_2. Then we get a larger admissible k-cycle-packing. □

Claim 2.3.7 *For $v \in V(H)$, $|N(v) \cap L| \geq q + 2$.*

Proof. Take $v \in V(H)$ and $y \in V(C_i) - \{u_i\}$ for $1 \leq i \leq r_1$. Then $vy \notin E(G)$, and

$$\begin{aligned} n + q &\leq d_G(v) + d_G(y) \leq |H| - 1 + |N(v) \cap L| + |L| - 1 \\ &= n - 2 + |N(v) \cap L|. \end{aligned}$$

Therefore, $|N(v) \cap L| \geq q + 2$. □

Claim 2.3.8 $s_1 \geq 2$.

Proof. Suppose that $s_1 \leq 1$. Then $|H| \leq r - (q+1) \leq r_1 - 1$ by Claim 2.3.7. Note that $|H|(p+q+1-(|H|-1)) \leq |E(H,L)| \leq s_1|H| + (r_1 - s_1) + q|H|$. (This inequality will be used several times.) Then $|H|(p+q+2-|H|) \leq s_1(|H|-1) + r_1 + q|H| \leq |H| - 1 + p + q|H|$ and $(p+q)|H| + 2|H| - |H|^2 \leq |H| - 1 + p + q|H|$. Hence $|H|^2 - |H| - 1 \geq p(|H|-1) \geq r_1(|H|-1) \geq (|H|+1)(|H|-1) = |H|^2 - 1$. This is impossible. □

Claim 2.3.9 $|H| > r_1 - s_1$.

Proof. Suppose that $|H| \leq r_1 - s_1 \leq p - s_1$. Then, $|H|(p+q+2-|H|) \leq s_1(|H| - 1) + r_1 + q|H| \leq (p - |H|)(|H|-1) + p + q|H|$. This shows $2|H| \leq |H|$, but this is a contradiction. □

Claim 2.3.10 $d_G(y) = d_L(y) \geq |L| - s_1 + 1$ *for any* $y \in V(L - U)$.

Proof. For any $x \in V(H)$, $xy \notin E(G)$. Since

$$\sum_{x \in V(H)} d_G(x) \leq |H|(|H|-1) + s_1|H| + r_1 - s_1 + q|H|,$$

we get

$$\begin{aligned} d_G(y) &\geq n + q - (|H|-1) - s_1 - q - \frac{r_1 - s_1}{|H|} \\ &> |L| - s_1 \end{aligned}$$

by Claim 2.3.9. Hence the claim holds. □

Claim 2.3.11 $N(v_1) \cap (U_1 - \{u_1\}) \neq \emptyset$.

Proof. If $N(v_1) \cap (U_1 - \{u_1\}) = \emptyset$, $d_G(v_1) \leq |L| - 1 - (s_1 - 1) = |L| - s_1$. On the other hand, $d_G(v_1) \geq |L| - s_1 + 1$ by Claim 2.3.10. This is a contradiction. □

Without loss of generality, we may assume that $u_2 \in N(v_1) \cap (U_1 - \{u_1\})$. Give orientations to C_1 and C_2 such that $C_1(v_1, u_1) \neq \emptyset$ and $C_2(v_2, u_2) \neq \emptyset$, and take $z = u_1^- \in C_1(v_1, u_1)$ and $y = v_2^+ \in C_2[u_2^+, u_2^-]$. Here and in the following, $C_j[v_j^+, u_j^-]$ will be used as the abbreviation for $V(C_j[v_j^+, u_j^-])$.

Claim 2.3.12 *There exist no disjoint subgraphs $C'_1, C'_2, \ldots, C'_k$ in L satisfying C'_1 is a path connecting u_1 and u_2, $C'_2, \ldots, C'_k$ are cycles, $v_i \in V(C_i)$ for $1 \leq i \leq p$, $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q$ and $|\bigcup_{i=1}^{p+q} V(C'_i) \cap U| \geq r - 1$.*

Proof. Let $L' = \langle \bigcup_{i=1}^{k} V(C'_i) \rangle$ and $M = V(L) - \bigcup_{i=1}^{k} V(C'_i) - U$. For any $x \in M$, $d_G(x) = d_L(x)$ and by Claim 2.3.10,

$$\begin{aligned} d_L(x) &\geq |L| - s_1 + 1 \geq |L| + q - k + 1 \\ &\geq |L'| + |M| + q - k + 1 \\ &\geq \frac{|L'| + q}{2} + (|M| - 1) + \frac{3(k-1) + 2}{2} + \frac{q}{2} - k + 2 \\ &= \frac{|L'| + q}{2} + (|M| - 1) + \frac{k + q + 3}{2} \\ &> \frac{|L'| + q}{2} + (|M| - 1) + \frac{3}{2}. \end{aligned}$$

Then by Lemma 2.5, M can be inserted into L'. Choose any $y \in N_H(u_1)$. Then there exists $y' \in N_H(u_2) - \{y\}$. By adding a path connecting y and y' in H, we get a larger admissible k-cycle-packing. This contradicts the minimality of $|L|$. (We may miss one vertex in U, but they contain two vertices in H.) □

Claim 2.3.13 $d_{C_1}(z) + d_{C_1}(y) + d_{C_1}(v_2) \leq 2|C_1| + 1$.

Proof. $N(y) \cap N(v_2) \cap (V(C_1) - \{u_1, v_1\}) = \emptyset$ (otherwise, we get a disjoint path P connecting u_1 and u_2 through v_1 and a cycle C_2' through v_2 in $\langle V(C_1) \cup V(C_2)\rangle$, contradicting Claim 2.3.12). Then $d_{C_1}(z) + d_{C_1}(y) + d_{C_1}(v_2) \leq |C_1| - 1 + |C_1| + 2 \leq 2|C_1| + 1$. □

Claim 2.3.14 $d_{C_2}(z) + d_{C_2}(y) + d_{C_2}(v_2) \leq 2|C_2| + 1$.

Proof. We may assume that $N(y) \cap C_2(u_2, v_2) = \emptyset$ and $N(v_2) \cap (C_2(y, v_2^-) - \{u_2\}) = \emptyset$, since otherwise we get a disjoint u_1–u_2 path C_1' passing through v_1 and a cycle C_2' passing through v_2 in $\langle V(C_1) \cup V(C_2)\rangle$, contradicting Claim 2.3.12. Therefore, $N_{C_2}(y) \subseteq C_2[v_2, u_2] - \{y\}$ and $N_{C_2}(v_2) \subseteq \{u_2, y, v_2^-\}$. If $N_{C_2}(z) \cap C_2(u_2, v_2] \neq \emptyset$ and $N_{C_2}(z) \cap C_2(v_2, u_2) \neq \emptyset$, we get a disjoint u_1–u_2 path C_1' passing through v_1 and a cycle C_2' passing through v_2. Then $N_{C_2}(z) \subseteq \{u_2, v_2\}$ or $C_2[u_2, v_2)$ or $C_2(v_2, u_2]$. Hence

$$\begin{aligned} d_{C_2}(z) + d_{C_2}(y) + d_{C_2}(v_2) &\leq |C_2| - 1 + |C_2| - 1 + 3 \\ &= 2|C_2| + 1. \end{aligned}$$

□

Claim 2.3.15 $d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) \leq 2|C_i| + 2$ *for* $3 \leq i \leq p + q$.

Proof. Suppose that $d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) > 2|C_i| + 2$ for some i, $3 \leq i \leq p + q$. Then $d_{C_i}(z) \geq 3$. Take $w_1, w_2 \in N_{C_i}(z)$ such that $C_i(w_1, w_2) \cap N(z) = \emptyset$ and $v_i \in C_i[w_1, w_2]$ if $3 \leq i \leq p$ and $e_i \in E(C_i[w_1, w_2])$ if $p + 1 \leq i \leq p + q$. Then $N(v_2) \cap N(y) \cap C_i(w_2, w_1) = \emptyset$ and

$$\begin{aligned} d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) &\leq |C_i[w_2, w_1]| + |C_i(w_2, w_1)| + 2|C_i[w_1, w_2]| \\ &= 2|C_i| + 2. \end{aligned}$$

This is a contradiction. □

Claim 2.3.16 $d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) \leq 2|C_i| + 1$ *for* $p + q + 1 \leq i \leq k$.

Proof. If $d_{C_i}(z) \leq 1$, the claim holds. Suppose that $d_{C_i}(z) = t \geq 2$ and let $w_1, w_2, \ldots, w_t \in N_{C_i}(z) = W$. If $t \geq 3$, only v_2 or y can have neighbors on $C_i(w_j, w_l)$ for $1 \leq j \neq l \leq t$ by Claim 2.3.12. Furthermore, $N_W(v_2) \cap N_W(y) = \emptyset$. Then,

$$d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) \leq 2|C_i|.$$

If $t = 2$, at least one of $N(y) \cap C_i(w_1, w_2)$ and $N(v_2) \cap C_i(w_1, w_2)$ is empty, and also at least one of $N(y) \cap C_i(w_2, w_1)$ and $N(v_2) \cap C_i(w_2, w_1)$ is empty. Hence

$$d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) \leq |C_i| + 4 \leq 2|C_i| + 1.$$

□

Claim 2.3.17 *$L - U$ is not complete.*

Proof. $z \notin N(y) \cap N(v_2)$. □

Claim 2.3.18 $|L| \geq (n + q + 4)/2$.

Proof. By Claim 2.3.17, $2(|L| - 2) \geq \sigma_2(G) = n + q$. Hence $|L| \geq (n + q + 4)/2$. □

By Claim 2.3.10,

$$d_G(z) + d_G(y) + d_G(v_2) \geq 3|L| - 3s_1 + 3. \tag{2.3}$$

On the other hand, by Claims 2.3.13, 14, 15 and 16,

$$\begin{aligned} d_G(z) + d_G(y) + d_G(v_2) &\leq \sum_{i=1}^{2}(2|C_i| + 1) + \sum_{i=3}^{p+q}(2|C_i| + 2) + \sum_{i=p+q+1}^{k}(2|C_i| + 1) \\ &= 2|L| + 2 + 2(p + q - 2) + (k - p - q) \\ &= 2|L| + k + p + q - 2. \end{aligned} \tag{2.4}$$

By (2.3) and (2.4),

$$|L| \leq k + p + q + 3s_1 - 5.$$

By Claim 2.3.18,

$$(n + q + 4)/2 \leq k + p + q + 3s_1 - 5.$$

Then,

$$\begin{aligned} n &\leq 2k + 2p + q + 6s_1 - 14 \\ &\leq 2k + 8p + q - 14 \\ &\leq 10k - 14. \end{aligned}$$

But this is a contradiction. This completes the proof of Theorem 2.6.

Chapter 3

Vertex-Disjoint Short Cycles Containing Specified Edges in a Graph

We say that a cycle is *short* when its length is at most 5. In this chapter, we consider the existence of short cycles containing specified edges in a graph. We obtain a sharp minimum degree condition, which is an improvement of that of the result in [10].

3.1 Introduction

In this chapter, 'disjoint' means 'vertex-disjoint', since we only deal with partitions of the vertex set, and n always denotes the order of a graph.

In [10], Egawa et al. considered the partition of a graph into cycles passing through specified edges and proved the following theorem.

Theorem 3.1 (Egawa et al. [10]) *Suppose that $k \geq 2, n \geq 4k-1$ and $\sigma_2(G) \geq n+2k-2$. Then for any independent edges $e_1, \ldots, e_k \in E(G)$, G contains k disjoint cycles $H_1, \ldots, H_k$ such that $e_i \in E(H_i)$ and $\bigcup_{i=1}^{k} V(H_i) = V(G)$.*

The proof of Theorem 3.1 consists of two steps, solving a packing problem and then extending a packing to a partition. The result of a packing problem is the next theorem.

Theorem 3.2 (Egawa et al. [10]) *Suppose that $k \geq 1$, $n \geq 4k-1$ and $\sigma_2(G) \geq n+2k-2$. Then for any independent edges $e_1, \ldots, e_k \in E(G)$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| \leq 4$.*

The following corollary is immediate from Theorem 3.2.

Corollary 3.3 *Suppose that* $k \geq 1$, $n \geq 4k-1$ *and* $\delta(G) \geq \frac{1}{2}(n+2k-2)$. *Then for any independent edges* $e_1, \ldots, e_k \in E(G)$, G *contains* k *disjoint cycles* $C_1, \ldots, C_k$ *such that* $e_i \in E(C_i)$ *and* $|C_i| \leq 4$.

The result of extending a packing to a partition is the following.

Theorem 3.4 (Egawa et al. [10]) *Suppose that* $k \geq 1$, $n \geq 3k$, $\sigma_2(G) \geq n+k$, *and* $e_1, \ldots, e_k \in E(G)$ *are independent edges. Moreover,* G *contains* k *disjoint cycles* $C_1, \ldots, C_k$ *such that* $e_i \in E(C_i)$. *Then* G *contains* k *disjoint cycles* $H_1, \ldots, H_k$ *such that* $e_i \in E(H_i)$ *and* $\bigcup_{i=1}^{k} V(H_i) = V(G)$.

In [10], the next two examples are shown for Theorem 3.2 and Corollary 3.3.

Example 3.1. Let $G = (K_1 \cup K_{n-2k}) + K_{2k-1}$ and $V(K_1) = \{x\}$. Take any k independent edges $e_1, \ldots, e_k$ in $\langle \{x\} \cup N_G(x) \rangle$, and let x be an endvertex of e_1. Then there is no cycle through e_1 avoiding any endvertices of $e_2, \ldots, e_k$ and $\sigma_2(G) = n+2k-3$.

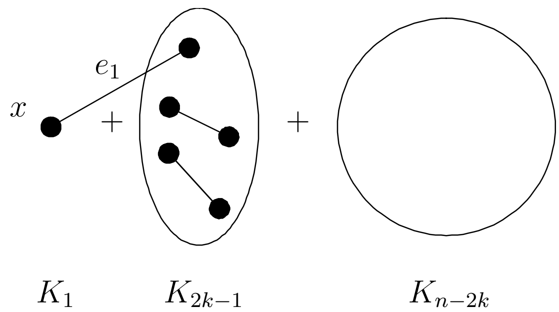

Figure 3.1: The graph G in Example 3.1.

Example 3.2. Let $G = (A \cup B) + K_{2k-2}$ with an edge e_1 joining A and B, where A and B are complete graphs with $|A| = \lceil n/2 \rceil - k + 1$ and $|B| = \lfloor n/2 \rfloor - k + 1$. Take any $k-1$ independent edges $e_2, \ldots, e_k$ in K_{2k-2}. Then $e_1, \ldots, e_k$ are k independent edges, but there is no cycle through e_1 avoiding any vertices in K_{2k-2}, while $\delta(G) = \lfloor n/2 \rfloor + k - 2 = \lfloor \frac{n+2k-4}{2} \rfloor$.

Example 3.2 gives the sharpness of the assumption in Corollary 3.3 only for the case n is even.

In this chapter, we will prove the following theorem.

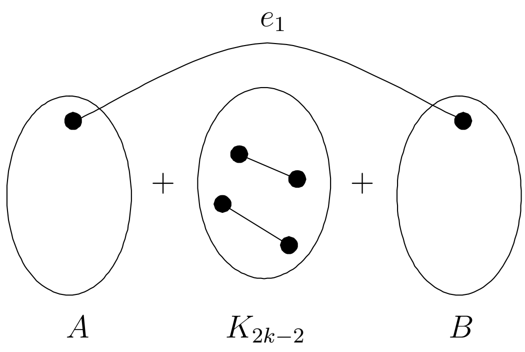

Figure 3.2: The graph G in Example 3.2.

Theorem 3.5 *Suppose that $n \geq \max\{6k, 4k+6\}$, $k \geq 1$ and $\delta(G) \geq (n+2k-3)/2$. Then for any independent edges $e_1, \ldots, e_k$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ for $1 \leq i \leq k$, and '$|C_i| \leq 4$ for $1 \leq i \leq k$' or '$|C_i| = 5$ for some i, $1 \leq i \leq k$ and the rest are all triangles'.*

By Theorem 3.5, the degree condition in Theorem 3.1 can be slightly improved when n is sufficiently large.

Theorem 3.6 *Suppose that $n \geq 6k+2$, $k \geq 2$ and either $\sigma_2(G) \geq n+2k-2$ or $\delta(G) \geq (n+2k-3)/2$. Then for any independent edges $e_1, \ldots, e_k \in E(G)$, G contains k disjoint cycles $H_1, \ldots, H_k$ such that $e_i \in E(H_i)$ and $\bigcup_{i=1}^{k} V(H_i)$.*

The following example shows that the conclusion '$|C_i| = 5$ for some i, $1 \leq i \leq k$ and the rest are all triangles' in Theorem 3.5 is necessary.

Example 3.3. Suppose that n is odd. Let G be a graph obtained from $G' = (A \cup B) + K_{2k-2}$, where A and B are complete graphs with $|A| = |B| = (n-2k-1)/2$, by adding new three vertices x, y and z with an edge yz and joining x to A, B and K_{2k-2}, y to A and K_{2k-2}, and z to B and K_{2k-2}. Take any $k-1$ independent edges $e_2, \ldots, e_k$ in K_{2k-2} and let $e_1 = yz$. Then $e_1, \ldots, e_k$ are k independent edges, but e_1 can not be contained in a cycle of length 3 or 4 avoiding the vertices of K_{2k-2}, while $\delta(G) = (n+2k-3)/2$.

For k independent edges $e_1 = x_1y_1, \ldots, e_k = x_ky_k$, a cycle C is called *admissible* if $|E(C) \cap \{e_1, \ldots, e_k\}| = 1$ and $|V(C) \cap \{x_1, \ldots, x_k, y_1, \ldots, y_k\}| = 2$. For $1 \leq r \leq k$, a set of cycles $\{C_1, \ldots, C_r\}$ is *admissible* if each C_i is admissible, mutually disjoint, and $|C_i| \leq 4$ for $1 \leq i \leq r$ or $|C_i| = 5$ for some i, $1 \leq i \leq r$ and the rest are all triangles. If we say 'r admissible cycles', it means that a set of these r cycles is admissible.

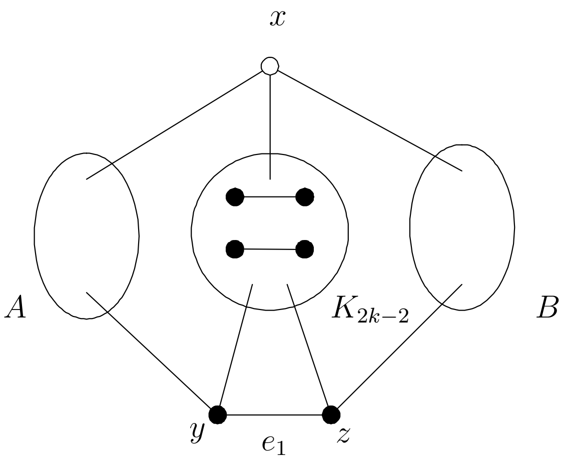

Figure 3.3: The graph G in Example 3.3.

3.2 Proof of Theorem 3.5

We distinguish two cases according to the value of k.

Case 1 $k \geq 2$.

Let G be an edge-maximal counterexample and $e_i = x_i y_i$ for $1 \leq i \leq k$. Since if G is a complete graph, G contains k admissible cycles, G is not complete. Let x and y be nonadjacent vertices of G and define $G' = G + xy$, the graph obtained from G by adding the edge xy. Then G' is not a counterexample by the maximality of G, and so G' has k admissible cycles $C_1, \ldots, C_k$. Without loss of generality, we may assume that $xy \in E(C_k)$. Then G has $k-1$ admissible cycles $C_1, \ldots, C_{k-1}$. We take these cycles such that $|\bigcup_{i=1}^{k-1} V(C_i)|$ is as small as possible. We may assume that $e_i \in E(C_i)$. Let $L = \langle \bigcup_{i=1}^{k-1} V(C_i) \rangle$, $M = G - L$, $D = M - \{x_k, y_k\}$.

Claim 3.2.1 *$d_{C_i}(z) \leq 3$ for any $z \in V(D)$ and $1 \leq i \leq k-1$.*

Proof. Let $z \in V(D)$. If $d_{C_i}(z) \geq 4$ for some i, $1 \leq i \leq k-1$, $\langle V(C_i) \cup \{z\} \rangle$ contains a cycle passing through e_i which is shorter than C_i. $\square$

Claim 3.2.2 *$d_D(x_k) \geq 2$ and $d_D(y_k) \geq 2$.*

Proof. Suppose that $d_D(x_k) \leq 1$. Then

$$\frac{n + 2k - 3}{2} \leq d_G(x_k) \leq |L| + 2 \leq \max\{4k-4, 3k-1\} + 2.$$

Then $n \leq \max\{6k-1, 4k+5\}$. This is a contradiction. $\square$

Take any $z \in N_D(x_k)$ and $z' \in N_D(y_k)$, and let $S = \{x_k, y_k, z, z'\}$. Since M does not contain an admissible cycle passing through e_k length at most 4 (if such cycle exists, it contradicts G does not contain k admissible cycles or the minimality of $|L|$), $zz', x_kz', y_kz \notin E(G)$, and $d_S(w) \leq 2$ for any $w \in V(M) - S$. Then

$$d_M(S) \leq 2(|M| - 4) + 6 = 2|M| - 2.$$

Therefore,

$$\begin{aligned} d_L(S) &\geq 4\delta(G) - (2|M| - 2) = 2n + 4k - 6 - 2(n - |L|) + 2 \\ &= 2|L| + 4k - 4 = \sum_{i=1}^{k-1}(2|C_i| + 4). \end{aligned} \tag{3.1}$$

Claim 3.2.3 $d_{C_i}(S) \leq 2|C_i| + 4$ *for* $1 \leq i \leq k - 1$.

Proof. Suppose that $|C_i| \geq 4$. By Claim 3.2.1, $d_{C_i}(\{z, z'\}) \leq 6$. If $d_{C_i}(\{x_k, y_k\}) \geq |C_i|+3$, there is a triangle $x_ky_kax_k$ for some $a \in V(C_i)-\{x_i, y_i\}$. Hence $d_{C_i}(\{x_k, y_k\}) \leq |C_i| + 2$, and we get $d_{C_i}(S) \leq 2|C_i| + 4$ if $|C_i| = 4$ and $d_{C_i}(S) \leq 2|C_i| + 3$ if $|C_i| = 5$.

Suppose that $|C_i| = 3$, $C_i = x_iy_iax_i$ and $d_{C_i}(S) \geq 2|C_i|+5 = 11$. If $\{zx_i, zy_i, x_ka, z'a\} \subseteq E(G)$, then $x_iy_izx_i$ and $x_ky_kz'ax_k$ are two admissible cycles. Then, since $d_{C_i}(S) \geq 11$, we may assume that $\{za, y_ka, z'x_i, z'y_i\} \subseteq E(G)$. But this means that there are two admissible cycles $x_iy_iz'x_i$ and $x_ky_kazx_k$. □

By Claim 3.2.3, the equality holds for (3.1), that is, $d_{C_i}(S) = 2|C_i| + 4$ for all i, $1 \leq i \leq k - 1$.

Claim 3.2.4 $|C_i| = 3$ *for* $1 \leq i \leq k - 1$.

Proof. By the proof of Claim 3.2.3, we only consider the case $|C_i| = 4$. Let $C_i = x_iy_iabx_i$. Since $d_{C_i}(\{z, z'\}) = 6$, $d_{C_i}(z) = d_{C_i}(z') = 3$ and each of $N_{C_i}(z)$ and $N_{C_i}(z')$ is $\{a, b, x_i\}$ or $\{a, b, y_i\}$. Hence we may assume that $\{za, z'a, zb, z'b, zy_i\} \subseteq E(G)$ by symmetry. Then $x_ka \notin E(G)$ and since $d_{C_i}(\{x_k, y_k\}) = 6$, we may assume that $y_ka \in E(G)$. (Otherwise, we get an admissible triangle $x_ky_kbx_k$.) By Claim 3.2.2, we can take $z'' \in N_D(x_k) - \{z\}$. Since also $d_{C_i}(\{z'', z'\}) = 6$, $z''a \in E(G)$. Then $x_iy_izbx_i$ and $x_ky_kaz''x_k$ are admissible cycles. □

Claim 3.2.5 $d_{C_i}(\{z, z'\}) = 6$ *for some* i, $1 \leq i \leq k - 1$.

Proof. Suppose that $d_{C_i}(\{z, z'\}) \leq 5$ for $1 \leq i \leq k - 1$. Then $d_L(\{z, z'\}) \leq 5k - 5$. Since $N_D(z) \cap N_D(z') = \emptyset$,

$$d_M(\{z, z'\}) \leq |M| - 2 = n - 3(k - 1) - 2 = n - 3k + 1.$$

Hence we get

$$d_G(\{z, z'\}) \leq (5k - 5) + (n - 3k + 1) = n + 2k - 4 < 2\delta(G).$$

This is a contradiction. $\square$

Without loss of generality, we may assume that $d_{C_1}(\{z, z'\}) = 6$. This means that $N_{C_1}(z) = N_{C_1}(z') = V(C_1)$. Let $C_1 = x_1y_1ax_1$ and take any $z'' \in N_D(x_k) - \{z\}$. Let $S' = \{x_k, y_k, z', z''\}$. Then, since $N_{C_1}(S') = 2|C_1| + 4 = 10$ also holds, $d_{C_1}(z'') \geq 2$. Hence $x_1y_1zx_1$ and $x_ky_kz'az''x_k$ or $x_1y_1z''x_1$ and $x_ky_kz'azx_k$ are two admissible cycles, and this gives k admissible cycles which consist of $k-1$ admissible triangles and an admissible cycle of length 5. This completes the proof of Case 1.

Case 2 $k = 1$.

In this case, the assumption is $\delta(G) \geq (n-1)/2$. Let $e_1 = xy$, $x, y \in V(G)$ and $M = V(G) - \{x, y\}$. We may assume that $N(x) \cap N(y) = \emptyset$, since otherwise there is an admissible triangle. If there are $z \in N_M(x)$ and $z' \in N_M(y)$ such that $N(z) \cap N(z') \neq \emptyset$, there is an admissible cycle. Hence we may assume that $N(z) \cap N(z') = \emptyset$ for any $z \in N_M(x)$ and $z' \in N_M(y)$. Let $D = V(G) - (N(x) \cup N(y))$ and take any $z \in N_M(x)$ and $z' \in N_M(y)$. Then

$$\begin{aligned} n &\geq 2 + |N_M(x)| + |N_M(y)| + |N_D(z)| + |N_D(z')| \\ &\geq 2 + |N_M(x)| + |N_M(y)| \\ &\quad + \left(\frac{n-1}{2} - (|N_M(x)| - 1) - 1\right) + \left(\frac{n-1}{2} - (|N_M(y)| - 1) - 1\right) \\ &= n + 1. \end{aligned}$$

This is a contradiction. This completes the proofs of Case 2 and Theorem 3.5.

Chapter 4

Vertex-Disjoint 4-Cycles Containing Specified Edges in a Bipartite Graph

In this chapter, degree conditions are given for a bipartite graph to contain vertex-disjoint 4-cycles each of which contains a previously specified edge.

4.1 Introduction

In this chapter, 'disjoint' means 'vertex-disjoint', since we only deal with partitions of the vertex set. For a bipartite graph G with partite sets V_1 and V_2, we define

$$\sigma_{1,1}(G) = \min\{d_G(x) + d_G(y) | x \in V_1, y \in V_2, xy \notin E(G)\}.$$

(When G is a complete bipartite graph, we define $\sigma_{1,1}(G) = \infty$.)

For a packing of cycles in a graph, Dirac settled the case of triangles.

Theorem 4.1 (Dirac [8]) *Suppose that $|G| = n \geq 3k$ and $\delta(G) \geq (n+k)/2$. Then G contains k disjoint triangles.*

Egawa et al. [10] considered partitions into cycles passing through specified edges and proved the following theorem.

Theorem 4.2 (Egawa et al. [10]) *Suppose that $k \geq 2, |G| = n \geq 3k$ and either*

$$\sigma_2(G) \geq \max\left\{n + 2k - 2, \left\lfloor \frac{n}{2} \right\rfloor + 4k - 2\right\}$$

or

$$\delta(G) \geq \max\left\{\left\lceil \frac{n}{2} \right\rceil + k - 1, \left\lceil \frac{n+5k}{3} \right\rceil - 1\right\}.$$

Then, for any independent edges $e_1, \ldots, e_k$, G can be partitioned into cycles $H_1, \ldots, H_k$ such that $e_i \in E(H_i)$.

Theorem 4.2 is proved by first solving packing and then extending a packing to a partition. Results of packing problems are next two theorems.

Theorem 4.3 (Egawa et al. [10]) *Suppose that $k \geq 1$, $|G| = n \geq 4k - 1$ and $\sigma_2(G) \geq n + 2k - 2$. Then for any independent edges $e_1, \ldots, e_k$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| \leq 4$.*

Theorem 4.4 (Egawa et al. [10]) *Suppose that $k \geq 2$, $3k \leq |G| = n \leq 4k-2$ and either*

$$\sigma_2(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + 4k - 2$$

or

$$\delta(G) \geq \left\lceil \frac{n+5k}{3} \right\rceil - 1.$$

Then for any independent edges $e_1, \ldots, e_k$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| \leq 4$.

In this chapter, we consider the problem of packing in a bipartite graph with specified edges. In the rest of this chapter, G denotes a bipartite graph with partite sets V_1 and V_2 satisfying $|V_1| = |V_2| = n$.

For packing of cycles in a bipartite graph, Wang [24] and Li et al. [16] obtained the following conditions on $\delta(G)$ and $\sigma_{1,1}(G)$, respectively.

Theorem 4.5 (Wang [24]) *Suppose that $n \geq 2k + 1$ and $\delta(G) \geq k + 1$. Then G contains k disjoint cycles.*

Theorem 4.6 (Li et al. [16]) *Suppose that $n \geq 2k+1$ and $\sigma_{1,1}(G) \geq 2k+2$. Then G contains k disjoint cycles.*

The case where edges are specified, Wang [29] and Chen et al.[3] independently obtained the degree conditions. In [3], their proof consists of two steps like that of Theorem 4.2, that is, packing cycles and extending a packing to a partition. The result of a packing problem is the following.

Theorem 4.7 (Chen et al. [3]) *Suppose that $n \geq 2k$, and either*

$$\sigma_{1,1}(G) \geq \max\left\{n+k, \left\lceil \frac{2n-1}{3} \right\rceil + 2k\right\}$$

or

$$\delta(G) \geq \max\left\{\left\lceil \frac{n+k}{2} \right\rceil, \left\lceil \frac{2n+4k}{5} \right\rceil\right\}.$$

Then for any independent edges $e_1, \ldots, e_k$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| \leq 6$.

In this chapter, we get analogous results of Theorem 4.7, that is, we specify the number of 4-cycles. First we consider a condition on $\sigma_{1,1}(G)$.

Theorem 4.8 *Suppose that $k \geq 1$, $1 \leq s \leq k$, $n \geq 2k$, and*

$$\sigma_{1,1}(G) \geq \max\left\{\left\lceil \frac{4n+2s-1}{3} \right\rceil, \left\lceil \frac{2n-1}{3} \right\rceil + 2k\right\}.$$

Then for any independent edges $e_1, \ldots, e_k$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$, $|C_i| \leq 6$, and there are at least s 4-cycles in $\{C_1, \ldots, C_k\}$.

In the case of $\delta(G)$, another conclusion is obtained.

Theorem 4.9 *Suppose that $k \geq 1$, $0 \leq s \leq k$, $n \geq 2k$, and*

$$\delta(G) \geq \max\left\{\left\lceil \frac{2n+2k+s}{4} \right\rceil, \left\lceil \frac{2n+4k}{5} \right\rceil\right\}.$$

Then for any independent edges $e_1, \ldots, e_k$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$, $|C_i| = 4$ for $1 \leq i \leq s$, and $|C_i| \leq 6$ for $s+1 \leq i \leq k$.

Note that a part of Theorem 4.7 is a special case of Theorem 4.9 where $s = 0$.

The next theorem is a corollary of Theorems 4.8 and 4.9.

Theorem 4.10 *Suppose that $k \geq 1$, $n \geq 2k$, and either*

$$\sigma_{1,1}(G) \geq \left\lceil \frac{4n+2k-1}{3} \right\rceil$$

or

$$\delta(G) \geq \left\lceil \frac{2n+3k}{4} \right\rceil.$$

Then for any independent edges $e_1, \ldots, e_k$, G contains k disjoint 4-cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$.

Note that $(4n+2k-1)/3 \geq \lceil (2n-1)/3 \rceil + 2k$ and $(2n+3k)/4 \geq \lceil (2n+4k)/5 \rceil$ always hold.

The degree conditions of Theorem 4.8 and 4.9 are sharp in the following sense. (In the following, $E_{i,j} = \{xy | x \in W_i, y \in W_j\}$.)

Example 4.1. Suppose that $n \geq 2k$, and let $V(G) = \bigcup_{i=1}^{8} W_i$, where $|W_1| = |W_2| = s-1$, $|W_3| = |W_4| = k-s+1$, $|W_5| = |W_8| = (n-s+1)/3$, and $|W_6| = |W_7| = (2n-3k+s-1)/3$ and $E(G) = \bigcup_{i=1}^{4} E_{1,2i} \cup \bigcup_{i=1}^{3} E_{2,2i+1} \cup \bigcup_{i=3}^{7} E_{i,i+1} \cup E_{3,8}$. Let F_1 be any perfect matching in $\langle W_1 \cup W_2 \rangle$ and F_2 be any perfect matching in $\langle W_3 \cup W_4 \rangle$. Then for any edge e of F_2, we cannot take a 4-cycle containing e without using the vertices of $F_1 \cup F_2 - \{e\}$, while $\sigma_{1,1}(G) = (4n+2s-2)/3$.

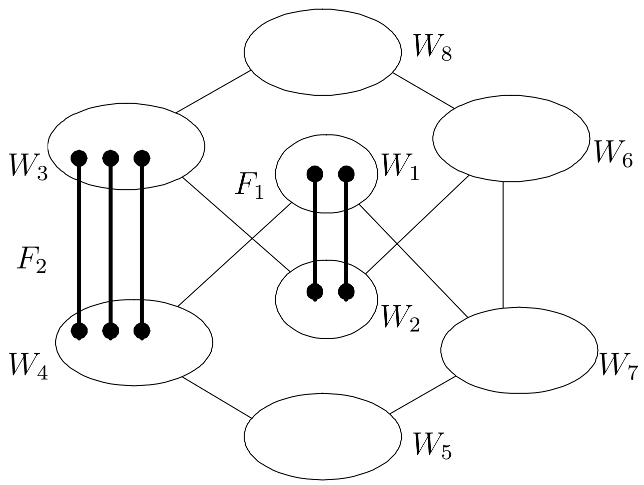

Figure 4.1: The graph G in Example 4.1.

Example 4.2. Suppose that $n \geq 2k$, and let $V(G) = \bigcup_{i=1}^{8} W_i$, where $|W_1| = |W_2| = (s-1)/2$, $|W_3| = |W_4| = k$, $|W_5| = |W_6| = |W_7| = |W_8| = (2n-2k-s+1)/4$ and $E(G) = \bigcup_{i=1}^{4} E_{1,2i} \cup \bigcup_{i=1}^{3} E_{2,2i+1} \cup \bigcup_{i=3}^{7} E_{i,i+1} \cup E_{3,8}$. Let F be any perfect matching in $\langle W_3 \cup W_4 \rangle$. Then since we must use at least one vertex in $V_1 \cup V_2$ to make 4-cycle passing through an edge in F, we cannot make s 4-cycles each of which contains exactly one edge in F, while $\delta(G) = (2n+2k+s-1)/4$.

Other examples are shown in [3].

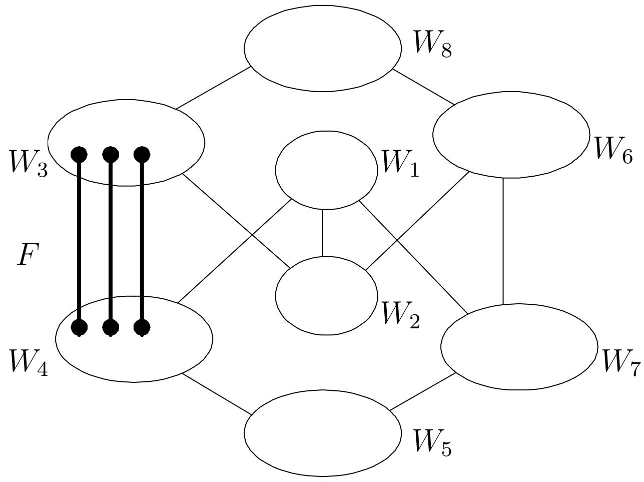

Figure 4.2: The graph G in Example 4.2.

We will use the notation $C[x,y]$ to denote the segment of the cycle C from x to y (including u and v) under some orientation of C, and $C[x,y) = C[x,y] - \{y\}$ and $C(x,y) = C[x,y] - \{x,y\}$.

Let $F = \{e_1, \ldots, e_k\}$ be a set of independent edges, where $e_i = x_iy_i$, $x_i \in V_1$, $y_i \in V_2$, and set $T = \{x_1, y_1, \ldots, x_k, y_k\}$. A cycle C is called *admissible* if $|E(C) \cap F| = 1$, $|V(C) \cap T| = 2$ and $|C| \leq 6$, and a set of disjoint cycles $\{C_1, \ldots, C_r\}$ is *admissible* for $r \leq k$ if each C_i is admissible.

4.2 Proof of Theorem 4.8

The next lemma will be used several times in Sections 4.2 and 4.3.

Lemma 4.1 *Let C be a cycle in G, $e \in E(C)$, $u \in V(G-C) \cap V_1$, $v \in V(G-C) \cap V_2$ and $d_C(u) + d_C(v) \geq |C|/2 + 2$. Then, either $\langle V(C) \cup \{v\} \rangle$ contains a shorter cycle than C passing through e, or there exists $w \in N_C(u)$ such that $\langle V(C) \cup \{v\} - \{w\} \rangle$ contains a cycle passing through e.*

Proof. We may assume $d_C(v) \leq 2$ (otherwise, $\langle V(C) \cup \{v\} \rangle$ contains a shorter cycle than C passing through e). Then $d_C(v) = 2$ and $d_C(u) = |C|/2$. This means that $N_C(u) = V(C) \cap V_2$. Also, we may assume $N_C(v) = \{a, b\}$ with $e \in E(C[b,a])$. Take any $w \in N_C(u) \cap C(b,a)$. Then $\langle V(C) \cup \{v\} - \{w\} \rangle$ contains a cycle passing through e. $\square$

We consider two cases according to the value of k.

Case 1 $k \geq 2$.

Let G be an edge-maximal counterexample to Theorem 4.8. We assume $e_i = x_i y_i$, $x_i \in V_1$ and $y_i \in V_2$ for $1 \leq i \leq k$. Clearly, since G is not a complete bipartite graph, there are nonadjacent vertices $x \in V_1$ and $y \in V_2$. Let G' be the graph obtained from G by adding the new edge xy. Then G' contains k admissible cycles $C_1, \ldots, C_k$ including at least s 4-cycles. Without loss of generality, we may assume $xy \in E(C_k)$. Then G has $k-1$ admissible cycles $C_1, \ldots, C_{k-1}$. We choose admissible cycles $C_1, \ldots, C_{k-1}$ so that $\sum_{i=1}^{k-1} |C_i|$ is as small as possible. Note that there are at least $s-1$ 4-cycles. We may also assume that $e_i \in E(C_i)$ for $1 \leq i \leq k-1$. Let $L = \langle \bigcup_{i=1}^{k-1} V(C_i) \rangle$, $M = G - L$, $|M| = 2m$, and $D = M - \{x_k, y_k\}$.

We consider the following two cases according to the number of 4-cycles.

Case 1.1 There are s or more 4-cycles in $\{C_1, \ldots, C_{k-1}\}$.

Claim 4.2.1 *We may assume $d_D(x_k) > 0$ and $d_D(y_k) > 0$.*

(*Proof.*) Suppose that $d_D(x_k) = 0$ and take any $z \in V(D) \cap V_2$. Then

$$d_M(x_k) + d_M(z) \leq 1 + (m-1) = m.$$

This implies that

$$\begin{aligned} d_L(x_k) + d_L(z) &\geq \frac{2n-1}{3} + 2k - m = \sum_{i=1}^{k-1} \frac{|C_i|}{2} + 2k - \frac{n+1}{3} \\ &> \sum_{i=1}^{k-1} \left(\frac{|C_i|}{2} + 1 \right) \end{aligned}$$

when $n \leq 3k$ and

$$\begin{aligned} d_L(x_k) + d_L(z) &\geq \frac{4n+2s-1}{3} - m = \sum_{i=1}^{k-1} \frac{|C_i|}{2} + \frac{n+2s-1}{3} \\ &> \sum_{i=1}^{k-1} \left(\frac{|C_i|}{2} + 1 \right) \end{aligned}$$

when $n \geq 3k$. Thus

$$d_{C_i}(x_k) + d_{C_i}(z) \geq \frac{|C_i|}{2} + 2$$

for some C_i, $1 \leq i \leq k-1$. By Lemma 4.1, there exists $w \in N_{C_i}(x_k)$ such that $\langle V(C_i) \cup \{z\} - \{w\} \rangle$ contains a cycle passing through e_i.

Similarly, we may assume that $N_D(y_k) \neq \emptyset$. $\square$

Take any $z \in N_D(x_k)$ and $z' \in N_D(y_k)$. Then z and z' are nonadjacent. We consider two cases according to the value $|D|$.

Case 1.1.1 $|D| \geq 4$.

Claim 4.2.2 *We may assume that $d_D(z) > 0$ and $d_D(z') > 0$.*

Proof. Suppose that $N_D(z) = \emptyset$ and take $w \in V(D) \cap V_1 - \{z'\}$. Then

$$d_M(z) + d_M(w) \leq 1 + (m-1) = m.$$

The rest of the proof is similar to that of Claim 4.2.1. $\square$

Take any $w \in N_D(z)$ and $w' \in N_D(w')$. Let

$$D_1 = N_D(y_k) \cap N_D(w') - \{z'\},$$

and

$$D_2 = N_D(x_k) \cap N_D(w) - \{z\}.$$

Note that $|D_i| \leq m-3$ for $i = 1, 2$.

Claim 4.2.3 *We may assume $|D_1| + |D_2| \leq m - 3$.*

(*Proof.*) Suppose that $|D_1| + |D_2| \geq m-2$. Then $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. Take $u \in D_2$ and $u' \in D_1$. Since $N_{D_1}(u) = \emptyset$ and $N_{D_2}(u') = \emptyset$,

$$d_M(u) + d_M(u') \leq (m - |D_1| - 1) + (m - |D_2| - 1) \leq m.$$

By Lemma 4.1, we can replace the cycles to decrease $|D_1| + |D_2|$. $\square$

Let $S = \{x_k, y_k, z, z', w, w'\}$. Since

$$d_M(S) = 10 + |E(S, M-S)| \leq 10 + |M - S| + |D_1| + |D_2| \leq 3m + 1,$$

we get

$$\begin{aligned} d_L(S) &\geq 3\left(\frac{2n-1}{3} + 2k\right) - (3m+1) \\ &= \sum_{i=1}^{k-1} \frac{3}{2}|C_i| + 6k - n - 2 > \sum_{i=1}^{k-1}\left(\frac{3}{2}|C_i| + 3\right) \end{aligned}$$

when $n \leq 3k$ and

$$\begin{aligned} d_L(S) &\geq 3\left(\frac{4n+2s-1}{3}\right)-(3m+1) \\ &= \sum_{i=1}^{k-1}\frac{3}{2}|C_i|+n+2s-2 > \sum_{i=1}^{k-1}\left(\frac{3}{2}|C_i|+3\right) \end{aligned}$$

when $n \geq 3k$. This implies that

$$d_{C_i}(S) \geq \frac{3}{2}|C_i|+4$$

for some C_i, $1 \leq i \leq k-1$.

Suppose that $C_i = x_iy_iaa'x_i$ and $d_{C_i}(S) \geq 10$. Since if $\{wa', y_ka, x_iw', z'y_i\} \subset E(G)$, $\langle S \cup V(C_i)\rangle$ contains two admissible cycles $x_ky_kaa'wzx_k$ and $x_iy_iz'w'x_i$, $|E(G) \cap \{wa', y_ka, x_iw', z'y_i\}| \leq 3$. Similarly, $|E(G) \cap \{w'a, x_ka', y_iw, zx_i\}| \leq 3$. This means $za, z'a' \in E(G)$. Also, if $\{x_ka', x_iz\} \subset E(G)$, there are two admissible cycles $x_ky_kz'a'x_k$ and $x_iy_iazx_i$ in $\langle S \cup V(C_i)\rangle$. Therefore, $|E(G) \cap \{x_ka', x_iz\}| \leq 1$. Similarly, $|E(G) \cap \{y_ka, y_iz'\}| \leq 1$. This means $\{wa', wy_i, w'x_i, w'a\} \subset E(G)$. Then there are two admissible cycles $x_ky_kz'\ a'wzx_k$ and $x_iy_iaw'x_i$ in $\langle S \cup V(C_i)\rangle$.

Next, suppose that $C_i = x_iy_ia'bb'ax_i$ and $d_{C_i}(S) \geq 13$. By the minimality of the number of 4-cycles, $d_{C_i}(s) \leq 2$ for every $s \in S - \{x_k, y_k\}$. By symmetry, we may assume $d_{C_i}(x_k) = 3$ and $d_{C_i}(z') = 2$ since $d_{C_i}(\{x_k, y_k, z, z'\}) \leq 9$. Then x_kb and $z'b$ are edges and there are two admissible cycles $x_ky_kz'bx_k$ which is shorter than C_i. $\square$

Case 1.1.2 $|D| = 2$.

Claim 4.2.4 *For some C_i, $|C_i| = 4$ and $d_{C_i}(z) = d_{C_i}(z') = 2$.*

(*Proof.*) Since $d_M(z) = d_M(z') = 1$,

$$\begin{aligned} d_L(z)+d_L(z') &\geq \frac{2n-1}{3}+2k-2 \\ &= \sum_{i=1}^{k-1}\frac{|C_i|}{2}+2k-\frac{n-1}{3} \geq \sum_{i=1}^{k-1}\left(\frac{|C_i|}{2}+1\right) \end{aligned}$$

when $n \leq 3k$ and

$$\begin{aligned} d_L(x_k)+d_L(z) &\geq \frac{4n+2s-1}{3}-2 \\ &= \sum_{i=1}^{k-1}\frac{|C_i|}{2}+\frac{n+2s-1}{3} > \sum_{i=1}^{k-1}\left(\frac{|C_i|}{2}+1\right). \end{aligned}$$

when $n \geq 3k$. Hence, $d_{C_i}(\{z, z'\}) \geq |C_i|/2 + 2$ for some C_i. On the other hand, by the minimality of L, $d_{C_i}(\{z, z'\}) \leq 4$. Therefore $|C_i| = 4$ and $d_{C_i}(z) = d_{C_i}(z') = 2$. □

We may assume $d_{C_1}(z) = d_{C_1}(z') = 2$ and $C_1 = x_1 y_1 w w' x_1$. Let $L' = L - C_1$, $M' = G - L'$ and $S = \{x_k, y_k, z, z', w, w'\}$.

Since $wy_k, w'x_k, zz' \notin E(G)$,

$$d_G(S) \geq 3\left(\frac{2n-1}{3} + 2k\right) = 2n + 6k - 1.$$

Since $d_{M'}(S) \leq 18$,

$$d_{L'}(S) \geq 2n + 6k - 19 = \sum_{i=2}^{k-1} |C_i| + 6k - 11 > \sum_{i=2}^{k-1} (|C_i| + 6).$$

This implies $d_{C_i}(S) \geq |C_i| + 7$ for some C_i, $2 \leq i \leq k-1$.

Suppose that$C_i = x_i y_i a a' x_i$ and $d_{C_i}(S) \geq 11$. By symmetry, we may assume $d_{C_i}(x_k) = d_{C_i}(z') = d_{C_i}(w') = 2$. Then there are three admissible cycles $x_k y_k z' a' x_k$, $x_1 y_1 w z x_1$, and $x_i y_i a w' x_i$.

Next, suppose that $C_i = x_i y_i a b b' a' x_i$ and $d_{C_i}(S) \geq 13$. By symmetry, we may assume $d_{C_i}(x_k) = 3$ and $d_{C_i}(z') = 2$. Then $x_k b$ and $z' b$ are edges and $x_k y_k b z x_k$ is an admissible cycle shorter than C_i.

This completes the proof of Case 1.1.

Case 1.2 There are exactly $s-1$ 4-cycles in $\{C_1, \ldots, C_{k-1}\}$.

We may assume $|C_i| = 4$ for $1 \leq i \leq s-1$ and $|C_i| = 6$ for $s \leq i \leq k-1$. Note that $|L| = 4(s-1) + 6(k-s) = 6k - 2s - 4$ and $|M| = 2m = 2n - 6k + 2s + 4$.

Claim 4.2.5 *We may assume* $d_M(x_k) \geq (2n - 6k + s + 11)/6$ *and* $d_M(y_k) \geq (2n - 6k + s + 11)/6$.

(*Proof.*) Suppose that$d_M(x_k) \leq (2n - 6k + s + 10)/6$. Since $m - d_M(x_k) \geq (n - 3k + s + 2) - (2n - 6k + s + 11)/6 = (4n - 8k + 5s + 2)/6 > 1$, $V(D) \cap V_2 - N(x_k) \neq \emptyset$. Take any $z \in V(D) \cap V_2 - N(x_k)$. Then,

$$\begin{aligned} d_M(x_k) + d_M(z) &\leq \left(\frac{2n - 6k + s + 10}{6}\right) + (m-1) \\ &= \frac{2n - 6k + s + 4}{6} + m. \end{aligned}$$

Therefore,

$$\begin{aligned} d_L(x_k) + d_L(z) &\geq \frac{4n+2s-1}{3} - \left(\frac{2n-6k+s+4}{6} + m\right) \\ &= \sum_{i=1}^{k-1} \frac{|C_i|}{2} + k + \frac{s}{2} - 1 > \sum_{i=1}^{k-1}\left(\frac{|C_i|}{2} + 1\right). \end{aligned}$$

Hence, for some C_i, $d_{C_i}(\{x_k, z\}) \geq |C_i|/2 + 2$. By Lemma 4.1, we can replace the cycles to increase $d_M(x_k)$.

Similarly, we may assume that $d_M(y_k) \geq (2n - 6k + s + 11)/6$. □

We may assume that $z \in N_M(x_k)$ and $z' \in N_M(y_k)$.

Claim 4.2.6 *For some C_i, $|C_i| = 4$ and $d_{C_i}(\{z, z'\}) = 4$.*

Proof. By Claim 4.2.5,

$$\begin{aligned} d_M(z) + d_M(z') &\leq (m - d_M(y_k) + 1) + (m - d_M(x_k) + 1) \\ &\leq 2m - \left(\frac{2n-6k+s+11}{3}\right) + 2 \\ &= 2m - \frac{2n-6k+s+5}{3}. \end{aligned}$$

Then,

$$\begin{aligned} d_L(z) + d_L(z') &\geq \frac{4n+2s-1}{3} - \left(2m - \frac{2n-6k+s+5}{3}\right) \\ &= \sum_{i=1}^{k-1}|C_i| - 2k + s + \frac{4}{3} > \sum_{i=1}^{s-1}(|C_i| - 1) + \sum_{i=s}^{k-1}(|C_i| - 2) \\ &= \sum_{i=1}^{k-1}\left(\frac{|C_i|}{2} + 1\right). \end{aligned}$$

This implies that for some C_i, $d_{C_i}(z) + d_{C_i}(z') \geq |C_i|/2 + 2$. On the other hand, $d_{C_i}(\{z, z'\}) \leq 4$. Hence $|C_i| = 4$ and $d_{C_i}(z) = d_{C_i}(z') = 2$. □

We may assume that $d_{C_1}(\{z, z'\}) = 4$ and $C_1 = x_1 y_1 w w' x_1$. Let $L' = L - C_1$, $M' = G - L'$, $S = \{x_k, y_k, z, z', w, w'\}$ and $D' = M' - S - \{x_1, y_1\}$.

Claim 4.2.7 *For some C_i, $d_{C_i}(S) \geq |C_i| + 7$, $2 \leq i \leq k-1$.*

(*Proof.*) Since

$$\begin{aligned} d_{M'}(S) &\geq 18 + 2|D'| = 18 + 2(2n - 6k + 2s) \\ &= 4n - 12k + 4s + 18, \end{aligned}$$

we get

$$\begin{aligned} d_{L'}(S) &\geq 3\left(\frac{4n+2s-1}{3}\right) - (4n-12k+4s+18) \\ &= 12k-2s-19. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{i=2}^{k-1}(|C_i|+6) &= |L'|+6(k-2) = (6k-2s-8)+6k-12 \\ &= 12k-2s-20. \end{aligned}$$

Therefore, $d_{L'}(S) > \sum_{i=2}^{k-1}(|C_i|+6)$ and this implies that $d_{C_i}(S) \geq |C_i|+7$ for some C_i, $2 \leq i \leq k-1$. □

The rest of the proof is similar to that of Case 1.1.2 and this completes the proof of Case 1.

Case 2 $k = 1$.

In this case, the assumption is $\sigma_{1,1}(G) \geq (4n+1)/3$. Let $e_1 = xy$, $x \in V_1$ and $y \in V_2$, and $M = V(G) - \{x, y\}$.

Claim 4.2.8 *$d_M(x) \geq (n+1)/3$ and $d_M(y) \geq (n+1)/3$.*

Proof. Suppose that$d_M(x) \leq n/3$. Take any $z \in V_2 \cap M$ such that $xz \notin E(G)$. Then,

$$\frac{4n+1}{3} \leq d_G(x) + d_G(z) \leq \left(\frac{n}{3}+1\right) + (n-1) = \frac{4n}{3}.$$

This is a contradiction. □

If there are adjacent vertices $z \in N_M(x)$ and $z' \in N_M(y)$, we obtain a cycle of length 4 passing through e_1. Hence we may assume that $zz' \notin E(G)$ for any $z \in N_M(x)$ and $z' \in N_M(y)$. Let $D = V(G) - (N(x) \cup N(y))$ and take any $z \in N_M(x)$ and $z' \in N_M(y)$. Then

$$\begin{aligned} 2n &\geq 2 + |N_M(x)| + |N_M(y)| + |N_D(z)| + |N_D(z')| \\ &\geq 2 + \frac{n+1}{3} + \frac{n+1}{3} + \left(\frac{4n+1}{3} - 2\right) \\ &= 2n+1. \end{aligned}$$

This is a contradiction. This completes the proofs of Case 2 and Theorem 4.8.

4.3 Proof of Theorem 4.9

We distinguish three cases according to the value of k and s.

Case 1 $k \geq 2$.

Let G be an edge-maximal counterexample to Theorem 4.9. We assume $e_i = x_i y_i, x_i \in V_1$ and $y_i \in V_2$ for $1 \leq i \leq k$. Let $F' = \{e_1, \ldots, e_s\}$. We define a set of admissible cycles $\mathcal{C} = \{C_1, \ldots, C_r\}$ is *saturated* if $\bigcup_{i=1}^{r} E(C_i) \supset F'$ and $|C_i| = 4$ for all C_i which contains an edge of F' and $\mathcal{C}$ is *nearly-saturated* if $|\bigcup_{i=1}^{r} E(C_i) \cap F'| = s - 1$ and $|C_i| = 4$ for all C_i which contains an edge of F'. Clearly, G is not a complete bipartite graph. Let G' be the graph obtained from G by adding a new edge xy, $x \in V_1$ and $y \in V_2$. Then G' contains admissible and saturated cycles $C_1, \ldots, C_k$. We may assume $xy \in E(C_i)$ for some i, $1 \leq i \leq k$. This means that G has $k-1$ admissible cycles. We distinguish two cases according as these cycles are saturated or nearly-saturated.

Case 1.1 $k-1$ admissible cycles are saturated.

We choose admissible and saturated cycles $C_1, \ldots, C_{k-1}$ so that $\sum_{i=1}^{k-1} |C_i|$ is as small as possible. Without loss of generality, we may also assume that $e_i \in E(C_i)$ for $1 \leq i \leq k-1$.

Let $L = \langle \bigcup_{i=1}^{k-1} V(C_i) \rangle$, $M = G - L$, $|M| = 2m$ and $D = M - \{x_k, y_k\}$.

Claim 4.3.1 *We may assume $d_D(x_k) > 0$ and $d_D(y_k) > 0$.*

Proof. Suppose that$d_D(x_k) = 0$ and take any $z \in V(D) \cap V_2$. Then,

$$d_M(x_k) + d_M(z) \leq 1 + (m-1) = m,$$

and

$$d_L(x_k) + d_L(z) \geq \frac{2n + 2k + s}{2} - m = \sum_{i=1}^{k-1} \frac{|C_i|}{2} + \frac{2k+s}{2} > \sum_{i=1}^{k-1} \left(\frac{|C_i|}{2} + 1 \right).$$

This means that for some C_i, $1 \leq i \leq k-1$,

$$d_{C_i}(x_k) + d_{C_i}(z) \geq \frac{|C_i|}{2} + 2.$$

By Lemma 4.1, there are $w \in N_{C_i}(x_k)$ such that $\langle V(C_i) \cup \{z\} - \{w\} \rangle$ contains a cycle passing through e_i.

Similarly, we may assume that $N_D(y_k) \neq \emptyset$. $\square$

Take any $z \in N_D(x_k)$ and $z' \in N_D(y_k)$. Clearly, z and z' are nonadjacent.
We consider two cases according to the value $|D|$.

Case 1.1.1 $|D| \geq 4$.

Claim 4.3.2 *We may assume $d_D(z) > 0$ and $d_D(z') > 0$.*

(*Proof.*) Suppose that$d_D(z) = 0$ and take any $w \in D \cap V_1$. Then,

$$d_M(z) + d_M(w) \leq 1 + (m - 1) = m.$$

The rest of the proof is similar to that of Claim 4.3.1. □

Take any $w \in N_D(z)$ and $w' \in N_D(z')$. Let

$$D_3 = N_D(y_k) \cap N_D(w') - \{z'\},$$

and

$$D_4 = N_D(x_k) \cap N_D(w) - \{z\}.$$

Claim 4.3.3 *We may assume that $|D_3| + |D_4| \geq m - 3$.*

(*Proof.*) Similar to the proof of Claim 4.2.3. □

Let $S = \{x_k, y_k, z, z', w, w'\}$. Then,

$$d_M(S) = 10 + |E(S, M - S)| \leq 10 + |M - S| + |D_3| + |D_4| \leq 3m + 1.$$

Therefore, we get

$$\begin{aligned} d_L(S) &\geq 6\left(\frac{2n + 2k + s}{4}\right) - (3m + 1) \\ &= \sum_{i=1}^{k-1} \frac{3}{2}|C_i| + 3k + \frac{3}{2}s - 1 > \sum_{i=1}^{k-1} \left(\frac{3}{2}|C_i| + 3\right). \end{aligned}$$

This means that for some C_i, $1 \leq i \leq k - 1$,

$$d_{C_i}(S) \geq \frac{3}{2}|C_i| + 4.$$

The rest of the proof is similar to that of Case 1.1.1 of Theorem 4.8. (Note that every exchange of cycles only produces 4-cycles containing $e_i = x_i y_i$.)

Case 1.1.2 $|D| = 2$.

Claim 4.3.4 *For some C_i, $|C_i| = 4$ and $d_{C_i}(z) = d_{C_i}(z') = 2$.*

Proof. Since $d_M(z) = d_M(z') = 1$,

$$d_L(z) + d_L(z') \geq \left(\frac{2n+2k+s}{2}\right) - 2 = \sum_{i=1}^{k-1} \frac{|C_i|}{2} + \frac{2k+s}{2} > \sum_{i=1}^{k-1} \left(\frac{|C_i|}{2} + 1\right).$$

This implies that $d_{C_i}(\{z, z'\}) \geq |C_i|/2 + 2$ for some C_i. On the other hand, $d_{C_i}(\{z, z'\}) \leq 4$. Hence $|C_i| = 4$ and $d_{C_i}(z) = d_{C_i}(z') = 2$. $\square$

We may assume that $d_{C_j}(z) = d_{C_j}(z') = 2$ and $C_j = x_j y_j w w' x_j$ for some j, $1 \leq j \leq k-1$. Let $L' = L - C_j$, $M' = G - L'$ and $S = \{x_k, y_k, z, z'\}$.

By using the assumption $\delta(G) \geq \frac{2n+4k}{5}$,

$$\begin{aligned} d_{L'}(\{w, w'\}) + 2d_{L'}(S) &\geq 10\delta(G) - 30 \geq 4n - 8k - 30 \\ &= 2\sum_{i=1}^{k-1} |C_i| + 8k - 14 > \sum_{i=1}^{k-1} (2|C_i| + 8). \end{aligned}$$

This implies that

$$d_{C_i}(\{w, w'\}) + 2d_{C_i}(S) \geq 2|C_i| + 9$$

for some C_i, $1 \leq i \leq k-1$.

Suppose that$C_i = x_i y_i a a' x_i$ and $d_{C_i}(\{w, w'\}) + 2d_{C_i}(S) \geq 17$. In particular, $d_{C_i}(S) \geq 7$. By symmetry, we may assume that $d_{C_i}(x_k) = d_{C_i}(z') = 2$. If zx_i and za are edges, $\langle V(M') \cup V(C_i)\rangle$ contains three admissible 4-cycles. Similarly, if $w'x_i$ and $w'a$ are edges, $\langle V(M') \cup V(C_i)\rangle$ contains three admissible 4-cycles. Therefore $|E(G) \cap \{zx_i, za\}| \leq 1$ and $|E(G) \cap \{w'x_i, w'a\}| \leq 1$. This implies that $wa', wy_i, y_k a$ are edges. Furthermore, either za or zx_i is an edge, but in either case $\langle V(M') \cup V(C_i)\rangle$ contains three admissible 4-cycles.

Next, suppose that $C_i = x_i y_i a b b' a' x_i$ and $d_{C_i}(\{w, w'\}) + 2d_{C_i}(S) \geq 21$. By symmetry, we may assume that $d_{C_i}(x_k) = 3$ and $d_{C_i}(z') = 2$. Then $x_k b$ and $z'b$ are edges, and $x_k y_k z' b x_k$ is an admissible cycle shorter than C_i.

This completes the proof of Case 1.1

Case 1.2 $k-1$ admissible cycles are nearly-saturated.

We choose admissible and nearly-saturated cycles $C_2, \ldots, C_k$ so that $\sum_{i=2}^{k} |C_i|$ is as small as possible. Without loss of generality, we may also assume $e_i \in E(C_i)$ for $2 \leq i \leq k$.

Let $L = \langle \bigcup_{i=2}^{k} V(C_i)\rangle$, $M = G - L$, $|M| = 2m$, and $D = M - \{x_1, y_1\}$.

Claim 4.3.5 *We may assume that $d_D(x_1) > 0$ and $d_D(y_1) > 0$.*

(*Proof.*) Suppose that $d_D(x_1) = 0$ and take any $z \in V(D) \cap V_2$. Then,

$$d_M(x_1) + d_M(z) \leq 1 + (m-1) = m.$$

The rest of the proof is similar to that of Claim 4.3.1. $\square$

Take any $z \in N_D(x_1)$ and $z' \in N_D(y_1)$ and let $S = \{x_1, y_1, z, z'\}$. Since $N_D(x_1) \cap N_D(z') = \emptyset$ and $N_D(y_1) \cap N_D(z) = \emptyset$,

$$\begin{aligned} d_L(S) &\geq 4\left(\frac{2n+2k+s}{4}\right) - 2(m+1) = \sum_{i=2}^{k} |C_i| + 2k + s - 2 \\ &= \sum_{i=2}^{k} (|C_i| + 2) + s > \sum_{i=2}^{s} (|C_i| + 3) + \sum_{i=s+1}^{k} (|C_i| + 2). \end{aligned}$$

Since $d_{C_i}(S) \leq 7$ for $2 \leq i \leq s$, $d_{C_i}(S) \geq |C_i| + 3$ for some C_i, $s+1 \leq i \leq k$.

Suppose that $C_i = x_i y_i a b b' a' x_i$ and $d_{C_i}(S) \geq 9$. By symmetry, we may assume that $d_{C_i}(x_1) = 3$ and $d_{C_i}(z') = 2$. Then $x_1 b$ and $z' b$ are edges, and $x_1 y_1 z' b x_1$ is an admissible cycle shorter than C_i.

This completes the proof of Case 1.

Case 2 $k = 1$ and $s = 0$.

In this case, the assumption is $\delta(G) \geq (n+1)/2$ and we must show that for any $e_1 \in E(G)$, G contains a cycle C such that $e \in E(C)$ and $|C| \leq 6$. Let $e_1 = xy$, $x \in V_1$ and $y \in V_2$, and $M = V(G) - \{x, y\}$. If there are adjacent vertices $z \in N_M(x)$ and $z' \in N_M(y)$, we obtain a cycle of length 4 passing through e_1. Hence we may assume that $zz' \notin E(G)$ for any $z \in N_M(x)$ and $z' \in N_M(y)$. Let $D = V(G) - (N(x) \cup N(y))$ and take any $z \in N_M(x)$ and $z' \in N_M(y)$. Again, if there are adjacent vertices $w \in N_D(z)$ and $w' \in N_D(z')$, we obtain a cycle of length 6 passing through e_1. Hence we may assume that $ww' \notin E(G)$ for any $w \in N_D(z)$ and $w' \in N_D(z')$. Let $H = D - (N_M(z) \cup N_M(z'))$ and take any $w \in N_D(z)$ and $w' \in N_D(z')$. Then

$$\begin{aligned} 2n &\geq 2 + |N_M(x)| + |N_M(y)| + |N_D(z)| + |N_D(z')| + |N_H(w)| + |N_H(w')| \\ &\geq 2 + |N_M(x)| + |N_M(y)| + \left(\frac{n+1}{2} - 1\right) + \left(\frac{n+1}{2} - 1\right) \\ &\quad + \left(\frac{n+1}{2} - |N_M(x)|\right) + \left(\frac{n+1}{2} - |N_M(y)|\right) \\ &= 2n + 2. \end{aligned}$$

This is a contradiction. This completes the proof of Case 2.

Case 3 $k = 1$ and $s = 1$.

In this case, the assumption is $\delta(G) \geq (2n+3)/4$ and we must show that for any $e_1 \in E(G)$, G contains a cycle C such that $e \in E(C)$ and $|C| = 4$. Let $e_1 = xy$, $x \in V_1$ and $y \in V_2$, and $M = V(G) - \{x, y\}$. If there are adjacent vertices $z \in N_M(x)$ and $z' \in N_M(y)$, we obtain a cycle of length 4 passing through e_1. Hence we may assume that $zz' \notin E(G)$ for any $z \in N_M(x)$ and $z' \in N_M(y)$. Let $D = V(G) - (N(x) \cup N(y))$ and take any $z \in N_M(x)$ and $z' \in N_M(y)$. Then

$$\begin{aligned} 2n &\geq 2 + |N_M(x)| + |N_M(y)| + |N_D(z)| + |N_D(z')| \\ &\geq 2 + \left(\frac{2n+3}{4} - 1\right) + \left(\frac{2n+3}{4} - 1\right) + \left(\frac{2n+3}{4} - 1\right) + \left(\frac{2n+3}{4} - 1\right) \\ &= 2n + 1. \end{aligned}$$

This is a contradiction. This completes the proofs of Case 3 and Theorem 4.9.

Chapter 5

On a Spanning k-tree Containing Specified Leaves in a Graph

In this section, we give sufficient conditions for a graph G to have a spanning k-tree with specified leaves: Let k, s, and t be integers such that $k \geq 2$, $0 \leq s \leq k$, and $t \geq s+1$. Suppose that (1) G is $(s+1)$-connected and the degree sum of any k independent vertices of G is at least $|G| + (k-1)s - 1$, or (2) G is t-connected and the independence number of G is at most $(t-s)(k-1)+1$. Then for any specified s vertices of G, G has a spanning k-tree containing them as leaves. We also discuss the sharpness of the results.

5.1 Introduction

We first introduce well-known theorems which provide sufficient conditions for graphs to have Hamilton paths or Hamilton cycles.

Theorem 5.1 (Ore [19, 20]) *Let s be an integer with $0 \leq s \leq 2$. Suppose that G is a graph of order $n \geq 3$ satisfying $\sigma_2(G) \geq n+s-1$. Then the following hold:*

(i) if $s=0$, then G has a Hamilton path,

(ii) if $s=1$, then G has a Hamilton cycle, and

(iii) if $s=2$, then G has a Hamilton path connecting any two vertices of G.

Theorem 5.2 (Chvátal and Erdős [6]) *Let t and s be integers with $t \geq 1$ and $0 \leq s \leq 2$. Suppose that G is an t-connected graph satisfying $\alpha(G) \leq t-s+1$. Then*

(i) if $s=0$, then G has a Hamilton path,

(ii) if $s=1$, then G has a Hamilton cycle, and

(iii) if $s=2$, then G has a Hamilton path connecting any two vertices of G.

Theorems 5.1 and 5.2 have lead to many new results and conjectures concerning paths and cycles in graphs. One theme to this research concentrates on Hamilton cycles. Another direction is motivated by the fact a Hamilton path is a spanning tree with the maximum degree at most two. So it is natural to ask for how the preceding theorems might be generalized to guarantee the existence of a spanning tree with maximum degree at most $k \geq 3$. The following results give the answer to this question.

Theorem 5.3 (Win [30]) *Let $k \geq 2$ be an integer and let G be a connected graph of order n. If*

$$\sigma_k(G) \geq n - 1,$$

then G has a spanning k-tree.

Theorem 5.4 (Neumann-Lara and Rivera-Campo [18]) *Let $t \geq 1$ and $k \geq 2$ be integers and let G be an t-connected graph. If*

$$\alpha(G) \leq t(k-1) + 1,$$

then G has a spanning k-tree.

On the other hand, a graph satisfying the conditions of Theorem 5.1 or 5.2 with $s = 2$ has a Hamilton path which contains two specified endvertices. The aim of this paper is to show sufficient conditions for the existence of a spanning k-tree such that the specified s vertices are contained in the set of its leaves.

5.2 Main results and sharpness

We prove the following two results, which are extensions of Theorems 5.1-5.4.

Theorem 5.5 *Let k and s be integers with $k \geq 2$ and $0 \leq s \leq k$. Suppose that a graph G is $(s+1)$-connected of order n and satisfies*

$$\sigma_k(G) \geq n + (k-1)s - 1.$$

Then for any s distinct vertices of G, G has a spanning k-tree such that the specified s vertices are contained in the set of its leaves.

Theorem 5.6 *Let k, s and t be integers with $k \geq 2$, $0 \leq s \leq k$ and $t \geq s + 1$. Suppose that a graph G is t-connected and satisfies*

$$\alpha(G) \leq (t-s)(k-1) + 1.$$

Then for any s distinct vertices of G, G has a spanning k-tree such that the specified s vertices are contained in the set of its leaves.

In Theorems 5.5 and 5.6, the condition 'G is $(s+1)$-connected' is necessary.

Example 5.1 Consider the graph $G = 2K_m + K_s$. Then G is s-connected but not $(s+1)$-connected. Moreover, $\sigma_k(G) = \infty > n + (k-1)s - 1$ hold if $k \geq 3$ and $\alpha(G) = 2 \leq (t-s)(k-1)+1$ hold. However, G has no spanning k-tree such that the s vertices of K_s are contained in the set of its leaves.

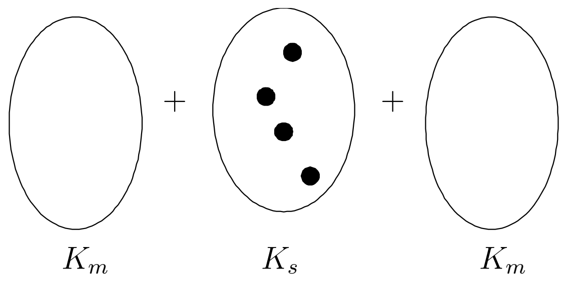

Figure 5.1: The graph G in Example 5.1.

The degree sum condition in Theorem 5.5 is best possible.

Example 5.2 Consider the complete bipartite graph G with partite sets A and B such that $|A| = t+s$ and $|B| = (k-1)t+2$, where t is a sufficiently large integer. Then G is $(s+1)$-connected, $|G| = n = kt+s+2$, and $\sigma_k(G) = k|A| = n+(k-1)s-2$. Suppose that G has a spanning k-tree T such that the s specified vertices in A are contained in the set of leaves of T. Then the number of the edges in T between A and B is at most $kt+s$. However, this is a contradiction since $kt+s < |E(T)| = |G|-1$ hold. Therefore G has no desired spanning k-tree.

Theorem 5.6 is best possible in the following sense.

Example 5.3 Consider the graph $G = (\{(t-s)(k-1)+1\}K_1 \cup K_m) + K_t$, where m is a sufficiently large integer. Then G is t-connected and $\alpha(G) = (t-s)(k-1)+2$. Suppose that G has a spanning k-tree T such that the s specified vertices in K_t are contained in the set of leaves. Then the number of edges in T incident with $V(K_t)$ is at most $s+(t-s)k$. Hence $|E(T)| \leq s+(t-s)k+|E(T)\cap E(K_m)| \leq s+(t-s)k+m-1$. This contradicts $|E(T)| = |G|-1 = (t-s)k+s+m$. Hence G has no desired spanning k-tree.

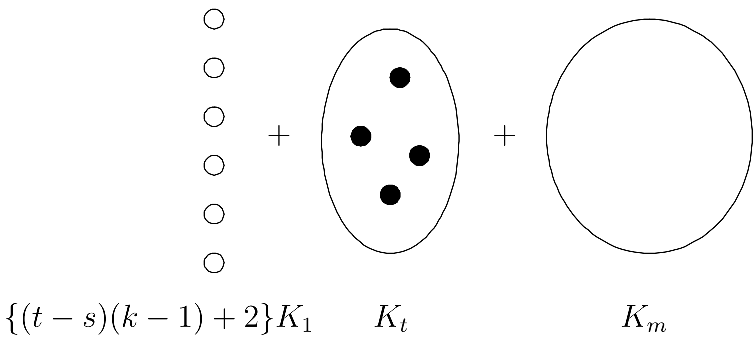

Figure 5.2: The graph G in Example 5.3.

In Theorems 5.5 and 5.6, the condition '$s \leq k$' is natural when $k = 2$, but it might not be sharp for $k \geq 3$.

5.3 Proof of Theorem 5.5

The case $k = 2$ and the case $s = 0$ follow from Theorems 5.1 and 5.3, respectively. Thus we may assume that $k \geq 3$ and $s \geq 1$. Let $U = \{u_1, \ldots, u_s\}$ be the set of specified s vertices in G, and put $H = G - U$. Note that H is connected since $|U| = s$ and G is $(s+1)$-connected.

Claim 5.3.1 *H has a spanning k-tree.*

Proof. If $\alpha(H) < k$, then the claim is true by Theorem 5.4. Hence we may assume that $\alpha(H) \geq k$. Since the number of edges in G joining U to any k vertices in H is at most sk, we obtain $\sigma_k(H) \geq \sigma_k(G) - sk \geq n - s - 1 = |H| - 1$. Hence H has a spanning k-tree by Theorem 5.3. $\square$

We consider the following two cases according to the value of n.

Case 5.3.1 *$n \leq 2s$.*

Take a spanning k-tree T of H and we add the vertices of U to T as many as possible in such a way that the maximum degree of the resulting tree is at most k and each added vertex is a leaf. Let T' be the resulting tree. If $U - V(T') = \emptyset$, then we have nothing to prove. Hence without loss of generality, we may assume that $u_1 \notin V(T')$. Since G is $(s+1)$-connected, u_1 has at least two neighbors v_1, v_2 in T. Note that $d_{T'}(v_i) = k$ for $i = 1, 2$, since otherwise we can add u_1 to T'. Then

$|T'| \geq 2(k-1)+2 = 2k \geq 2s$, which implies $n \geq 2s+1$, a contradiction. This completes the proof of Case 1.

Case 5.3.2 $n \geq 2s+1$.

Claim 5.3.2 *There exists a matching joining U to H, which covers U.*

Proof. Consider the bipartite graph B with partite sets U and $V(H)$, where a vertex in U and one in $V(H)$ are joined by an edge of B if and only if they are adjacent in G. If there exists a subset $U' \subseteq U$ such that $|N_B(U')| < |U'|$, then $(U-U') \cup N_B(U')$ is a separating set of G with cardinality less than s. This contradicts the assumption G is $(s+1)$-connected. Hence we have $|N_B(U')| \geq |U'|$ for all $U' \subseteq U$. By Hall's Marriage Theorem [12], we find the desired matching. □

Let T be a spanning k-tree of H and let $M = \{u_1v_1, \ldots, u_sv_s\}$ be a matching of G which covers U, where $\{v_1, v_2, \ldots, v_s\} = N_M(U) \subseteq V(H)$.

In order to have the desired spanning k-tree of G, we claim that there exists a pair of T and M such that $T \cup M$ is the desired spanning k-tree of G. Note that $T \cup M$ is a spanning $(k+1)$-tree of G. Choose a spanning k-tree T of H and a matching M so that the number of vertices in $N_M(U)$ of degree $k+1$ in $T \cup M$ is as small as possible. Let $T' = T \cup M$. If $d_{T'}(v_i) \leq k$ for each $i = 1, \ldots, s$, then T' is the desired spanning k-tree of G. Thus without loss of generality, we may assume that $d_{T'}(v_1) = k+1$.

We denote by $T_1, T_2, \ldots, T_k$ the components of $T - v_1$ and by $T_1', T_2', \ldots, T_k'$ the components of $T' - v_1$ such that $T_i \subseteq T_i'$ for $1 \leq i \leq k$. For each $i = 1, \ldots, k$, let t_i be the vertex of T_i which is adjacent to v_1 in T and let p_i be a leaf of T contained in $V(T_i)$. Note that $t_i = p_i$ holds for the case $|T_i| = 1$ and that some vertices in $\{p_1, \ldots, p_k\}$ might belong to $N_M(U)$. Put $P = \{p_1, \ldots, p_k\}$.

Claim 5.3.3 *P is an independent set of G.*

Proof. If $p_ip_j \in E(G)$ for some $p_i, p_j \in P$, then $T + p_ip_j - v_1t_i$ is a spanning k-tree of H such that $(T + p_ip_j - v_1t_i) \cup M$ has fewer vertices of degree $k+1$ than $T \cup M$, which contradicts the choice of T. Thus P is an independent set of G. □

Let

$$W_1 = \left(\bigcup_{i=2}^{k} N_G(p_i) \right) \cap V(T_1).$$

Claim 5.3.4 $t_1 \notin W_1$.

Proof. Suppose that $t_1 \in W_1$. Since $t_1 p_i \in E(G)$ for some $p_i \in P - \{p_1\}$, $T - v_1 t_1 + t_1 p_i$ is a spanning k-tree of H such that $(T - v_1 t_1 + t_1 p_i) \cup M$ has fewer vertices of degree $k+1$ than $T \cup M$. This contradicts the choice of T. Hence $t_1 \notin W_1$. $\square$

Claim 5.3.5 *$d_{T'}(w) \geq k$ for all $w \in W_1$.*

Proof. Suppose that there exists a vertex $w \in W_1$ such that $d_{T'}(w) < k$. Since $wp_i \in E(G)$ for some $p_i \in P - \{p_1\}$, $T - v_1 t_1 + wp_i$ is a spanning k-tree of H. Then $(T - v_1 t_1 + wp_i) \cup M$ has fewer vertices of degree $k+1$ than $T \cup M$, which contradicts the choice of T. Therefore $d_{T'}(w) \geq k$ holds for all $w \in W_1$. $\square$

Let $P_T(a, b)$ denote the unique path in T connecting two vertices a and b of T.

Claim 5.3.6 *For each $w \in W_1$, no vertex in $N_T(w) - V(P_T(w, p_1))$ is adjacent to p_1.*

Proof. Suppose that $z \in N_G(p_1) \cap (N_T(w) - V(P_T(w, p_1)))$ for some $w \in W_1$. Since $wp_i \in E(G)$ for some $p_i \in P - \{p_1\}$, $T - wz - v_1 t_1 + p_1 z + wp_i$ is a spanning k-tree of H. This contradicts the choice of T. $\square$

We divide W_1 into three subsets as follows:

$$\begin{aligned} W_{1,1} &:= \{w \in W_1 \mid w \notin N_M(U)\}, \\ W_{1,k} &:= \{w \in W_1 \mid w \in N_M(U) \text{ and } d_{T'}(w) = k\}, \text{ and} \\ W_{1,k+1} &:= \{w \in W_1 \mid w \in N_M(U) \text{ and } d_{T'}(w) = k+1\}. \end{aligned}$$

Claim 5.3.7 $\left|\bigcup_{w \in W_1} N_T(w) - N_G[p_1]\right| \geq (k-1)(|W_{1,1}| + |W_{1,k+1}|) + (k-2)|W_{1,k}|$.

Proof. If $W_1 = \emptyset$, then the above inequality obviously holds. Thus we may assume that $W_1 \neq \emptyset$. Note that $v_1 \notin \bigcup_{w \in W_1} N_T(w)$ since $t_1 \notin W_1$ by Claim 5.3.4.

We consider T_1 as an outdirected tree with root p_1. For any $w_0 \in W_1$ and $z \in N^+_{T_1}(w_0)$, we have $z \notin N_G[p_1]$ by Claim 5.3.6. This implies that $N^+_{T_1}(w_0) \subseteq (\bigcup_{w \in W_1} N_T(w)) - N_G[p_1]$ for any $w_0 \in W_1$. Moreover, for any two distinct vertices w_1 and w_2 of W_1, $N^+_{T_1}(w_1) \cap N^+_{T_1}(w_2) = \emptyset$. Consequently,

$$\begin{aligned} \left|\left(\bigcup_{w \in W_1} N_T(w)\right) - N_G[p_1]\right| &\geq \left|\bigcup_{w \in W_1} N^+_{T_1}(w)\right| = \sum_{w \in W_1} |N^+_{T_1}(w)| \\ &= \sum_{w \in W_{1,1}} |N^+_{T_1}(w)| + \sum_{w \in W_{1,k}} |N^+_{T_1}(w)| + \sum_{w \in W_{1,k+1}} |N^+_{T_1}(w)| \\ &= (k-1)(|W_{1,1}| + |W_{1,k+1}|) + (k-2)|W_{1,k}|. \end{aligned}$$

Hence the claim holds. $\square$

By Claim 5.3.7, we obtain

$$
\begin{aligned}
|V(T_1') \cap N_G(p_1)| &\leq |T_1'| - |\{p_1\}| - \left| \left(\bigcup_{w \in W_1} N_T(w) \right) - N_G[p_1] \right| \\
&\leq |T_1'| - 1 - (k-1)(|W_{1,1}| + |W_{1,k+1}|) - (k-2)|W_{1,k}|. \qquad (5.1)
\end{aligned}
$$

On the other hand, it follows from the definition of W_1 that

$$\sum_{i=2}^{k} |V(T_1') \cap N_G(p_i)| \leq (k-1)|W_1| + (k-1)|V(T_1') \cap U|.$$

This inequality with (5.1) implies that

$$\sum_{i=1}^{k} |V(T_1') \cap N_G(p_i)| \leq |T_1'| - 1 + |W_{1,k}| + (k-1)|V(T_1') \cap U|.$$

Similarly, we define W_j, $W_{j,1}$, $W_{j,k}$ and $W_{j,k+1}$ for each $j = 2, \ldots, k$ as follows:

$$
\begin{aligned}
W_j &= \left(\bigcup_{i=1, i \neq j}^{k} N_G(p_i) \right) \cap V(T_j), \\
W_{j,1} &= \{w \in W_j \mid w \notin N_M(U)\}, \\
W_{j,k} &= \{w \in W_j \mid w \in N_M(U) \text{ and } d_{T'}(w) = k\}, \text{ and} \\
W_{j,k+1} &= \{w \in W_j \mid w \in N_M(U) \text{ and } d_{T'}(w) = k+1\}.
\end{aligned}
$$

By the symmetry, we obtain

$$\sum_{i=1}^{k} |V(T_j') \cap N_G(p_i)| \leq |T_j'| - 1 + |W_{j,k}| + (k-1)|V(T_j') \cap U|$$

for each $j = 2, \ldots, k$. Since $d_G(p_i) \leq |E(\{p_i\}, \{v_1\})| + \sum_{j=1}^{k} |V(T_j') \cap N_G(p_i)| + |E(\{p_i\}, \{u_1\})|$,

$$
\begin{aligned}
\sum_{i=1}^{k} d_G(p_i) &\leq \sum_{i=1}^{k} \Big(|E(\{p_i\}, \{v_1\})| + \sum_{j=1}^{k} |V(T_j') \cap N_G(p_i)| + |E(\{p_i\}, \{u_1\})| \Big) \\
&\leq k + \sum_{j=1}^{k} \Big(|T_j'| - 1 + |W_{j,k}| + (k-1)|V(T_j') \cap U| \Big) + |E(P, \{u_1\})| \\
&= \sum_{j=1}^{k} |T_j'| + \sum_{j=1}^{k} |W_{j,k}| + (k-1) \sum_{j=1}^{k} |V(T_j') \cap U| + |E(P, \{u_1\})|. \qquad (5.2)
\end{aligned}
$$

Since $pu_1 \notin E(G)$ for every $p \in P - N_M(U)$, we have $|E(P, \{u_1\})| \le s - 1 - \sum_{j=1}^{k} |W_{j,k}|$. This inequality together with (5.2) and $s \le k$ implies

$$\begin{aligned}
\sum_{i=1}^{k} d_G(p_i) &\le \sum_{j=1}^{k} |T'_j| + \sum_{j=1}^{k} |W_{j,k}| + (k-1)\sum_{j=1}^{k} |V(T'_j) \cap U| + s - 1 - \sum_{j=1}^{k} |W_{j,k}| \\
&\le n - 2 + \sum_{j=1}^{k} |W_{j,k}| + (k-1)(s-1) + s - 1 - \sum_{j=1}^{k} |W_{j,k}| \\
&= n + (k-1)s - k + s - 2 < n + (k-1)s - 1.
\end{aligned}$$

Since $P = \{p_1, \dots, p_k\}$ is an independent set of G by Claim 5.3.3, this contradicts the assumption of this theorem. This completes the proof of Theorem 5.

5.4 Proof of Theorem 5.6

In order to prove Theorem 5.6, we need the following lemma.

Lemma 5.1 *Let T be a tree and $\{v_1, v_2, \dots, v_l\}$ an independent set of T. Then $T - \{v_1, v_2, \dots, v_l\}$ has exactly $d_T(v_1) + d_T(v_2) + \dots + d_T(v_l) - l + 1$ components.*

Proof. For $l = 1$, clearly $T - v_1$ has exactly $d_T(v_1)$ components. Hence we may assume that $l \ge 2$. By the induction hypothesis, $T - \{v_1, \dots, v_{l-1}\}$ has exactly $d_T(v_1) + \dots + d_T(v_{l-1}) - (l-1) + 1$ components and v_l is contained in some component T'. Note that $d_{T'}(v_l) = d_T(v_l)$ since $\{v_1, \dots, v_l\}$ is independent and $N_T(v_l) \subset V(T')$. By the induction hypothesis, $T' - v_l$ has exactly $d_{T'}(v_l)$ components, and this means that $T - \{v_1, \dots, v_l\}$ has exactly $d_T(v_1) + \dots + d_T(v_l) - l + 1$ components. $\square$

Proof of Theorem 5.6. The case $k = 2$ and the case $s = 0$ follow from Theorems 5.2 and 5.4, respectively. Hence we may assume that $k \ge 3$ and $s \ge 1$. If $|G| = s + 2$, then G is K_{s+2}, and the result follows immediately. Consequently $|G| \ge s + 3$. Let $U = \{u_1, \dots, u_s\}$ be the set of s specified vertices in G.

We define a (k, U)*-tree* of G to be a k-tree T of G satisfying the following conditions;

(i) $U \subseteq V(T)$, and every vertex of U is a leaf of T; and

(ii) $|N_T(w) \cap U| < k - 1$ for any $w \in N_T(U)$.

Claim 5.4.1 *G contains a (k, U)-tree.*

Proof. Since G is $(s+1)$-connected, for any $v \in V(G)-U$ and any edge e, $G-e$ contains s internally-disjoint paths connecting v and U. These paths form a k-tree satisfying (i). If there exists a vertex $v \in V(G)-U$ such that $|N_G(v) \cap U| \le k-1$, then taking an edge e joining v and U if any, we have obtained the desired tree. Thus, we may assume that $k=s$ and $N_G(v) \supseteq U$ for every $v \in V(G)-U$.

Since G is $(s+1)$-connected, $G-U$ is connected. By $|G| \ge s+3$, we can take a path $v_1v_2v_3$ of length two in $G-U$. This path with the edges

$$\{v_1u_i \mid 1 \le i \le s-2\} \cup \{v_2u_{s-1}, v_3u_s\}$$

forms a (k,U)-tree of G. □

We take a (k,U)-tree T of maximum order among all (k,U)-trees of G. If $V(T) = V(G)$, we have nothing to prove. Therefore we may assume that $V(G)-V(T) \neq \emptyset$.

Claim 5.4.2 $|T| \ge t+1$.

Proof. Suppose that $|T| \le t$. Since G is t-connected, every vertex in T has at least one neighbor in $G-T$. If there exists $x \in V(T)-U$ with $d_T(x) < k$, we obtain a (k,U)-tree of order more than $|T|$. This contradicts the choice of T. Hence $d_T(x)=k$ for each $x \in V(T)-U$ and

$$2(|T|-1) = \sum_{x \in V(T)} d_T(x) = k(|T|-|U|)+|U| = k|T|-(k-1)s,$$

which implies $(k-2)|T| = (k-1)s-2$. On the other hand, $|T| > s+1$ by the definition and the maximality of T. This inequality together with $(k-2)|T| = (k-1)s-2$ yields $s > k$, which contradicts the assumption $s \le k$. □

Since G is t-connected and $|T| \ge t+1$ by Claim 5.4.2, there exist t internally-disjoint paths in G connecting $v \in V(G)-V(T)$ and t distinct vertices of T. We may assume that each path contains exactly one vertex in $V(T)$. For $i=1,\dots,t$, each path is denoted by $P_G(v,z_i)$, where z_i is the endvertex other than v. Put $Z=\{z_1,\dots,z_t\}$.

Claim 5.4.3 *Z is an independent set of T.*

Proof. Suppose that $z_iz_j \in E(T)$ for some $z_i, z_j \in Z$. Then $(T-z_iz_j) \cup P_G(v,z_i) \cup P_G(v,z_j)$ is a (k,U)-tree of order more than $|T|$, which contradicts the choice of T. □

By Claim 5.4.3, we get $|N_T[u_i] \cap Z| \le 1$ for all $u_i \in U$ with $1 \le i \le s$. Hence we have

$$|Z \cap (V(T)-N_T[U])| \ge t-s.$$

Without loss of generality, we may assume that $z_1, \ldots, z_{t-s} \in Z \cap (V(T) - N_T[U])$. Note that $d_T(z_i) = k$ for any $i = 1, \ldots, t-s$ since otherwise $T \cup P_G(v, z_i)$ is a (k, U)-tree of G larger than T.

By Lemma 5.1, $T - \{z_1, \ldots, z_{t-s}\}$ has $(t-s)(k-1)+1$ components $T_1, \ldots, T_{(t-s)(k-1)+1}$. Note that each T_i contains a vertex not in U by the choice of $z_1, \ldots, z_{t-s}$.

Let $T' = T - U$. We consider T as an outdirected tree $\overrightarrow{T}$ with root z_1. We denote the arc set of $\overrightarrow{T}$ by $A(\overrightarrow{T})$. For every component T_i with $i = 1, \ldots, (t-s)(k-1)+1$, take an arc $x_i z'_i$, if any, such that $x_i \in V(T_i)$ and $z'_i \in \{z_1, \ldots, z_{t-s}\}$, and otherwise take a vertex x_i of T_i such that x_i is a leaf of T'. Moreover, for every T_i, there exists exactly one arc $z''_i y_i \in A(\overrightarrow{T})$ such that $z''_i \in \{z_1, \ldots, z_{t-s}\}$ and $y_i \in V(T_i)$.

If x_i is a leaf of T', then $d_T(x_i) \leq d_{T'}(x_i) + |N_T(x_i) \cap U| \leq 1 + k - 2 = k - 1$ by the condition (iii) for a (k, U)-tree.

Let $P_T(a, b)$ denote the unique path in T connecting two vertices a and b of T.

Claim 5.4.4 *$\{x_i \mid 1 \leq i \leq (t-s)(k-1)+1\} \cup \{v\}$ is an independent set of G.*

Proof. Suppose first that $vx_i \in E(G)$. If x_i is a leaf of T', then $T + vx_i$ is a (k, U)-tree, which is a contradiction. If x_i is not a leaf of T', then $x_i z'_i$ is an arc of $\overrightarrow{T}$, and $(T - x_i z'_i + vx_i) \cup P_G(v, z'_i)$ is a (k, U)-tree, a contradiction.

Next, suppose that $x_i x_j \in E(G)$. Note that either $z''_i \in V(P_T(x_i, x_j))$ or $z''_j \in V(P_T(x_i, x_j))$ holds since T is a tree. We consider three cases.

Case 5.4.1 *Both x_i and x_j are leaves of T'.*

Without loss of generality, we may assume that $z''_j \in V(P_T(x_i, x_j))$. Then $(T - z''_j y_j + x_i x_j) \cup P_G(v, z''_j)$ is a (k, U)-tree larger than T. This contradicts the choice of T.

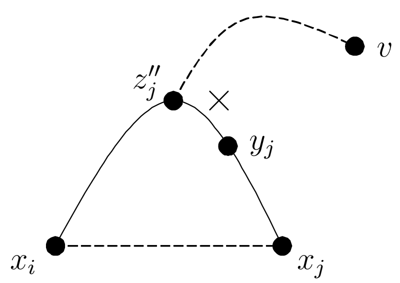

Figure 5.3:

Case 5.4.2 *x_i is a leaf of T' but not x_j.*

In this case, $x_jz'_j \in A(\overrightarrow{T})$. If $z'_j \in V(P_T(z_1, z''_i))$, then $(T - x_jz'_j + x_ix_j) \cup P_G(v, z'_j)$ is a (k, U)-tree, a contradiction.

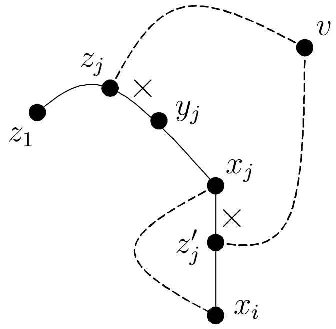

Figure 5.4:

Hence we may assume that $z'_j \notin V(P_T(z_1, z''_i))$. Then

$$T' = \begin{cases} (T - z''_jy_j - x_jz'_j + x_ix_j) \cup P_G(v, z''_j) \cup P_G(v, z'_j) & \text{if } z''_j \in V(P_T(x_i, x_j)), \\ (T - z''_iy_i - x_jz'_j + x_ix_j) \cup P_G(v, z''_i) \cup P_G(v, z'_j) & \text{otherwise}, \end{cases}$$

is a (k, U)-tree. This is a contradiction.

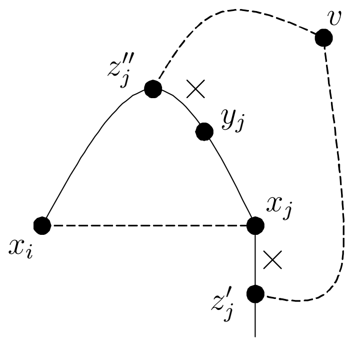

Figure 5.5:

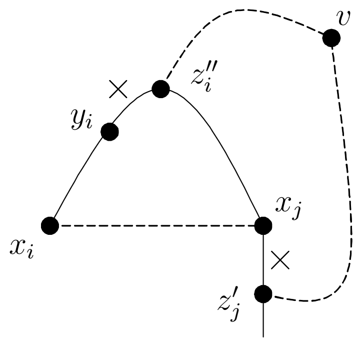

Figure 5.6:

Case 5.4.3 *Neither x_i nor x_j is a leaf of T'.*

In this case, $x_iz'_i, x_jz'_j \in A(\overrightarrow{T})$. By the symmetry, we may assume that $z''_j \in V(P_T(x_i, x_j))$.

If $z'_i \in V(P_T(z_1, z''_j))$, then $(T - x_iz'_i - x_jz'_j + x_ix_j) \cup P_G(v, z'_i) \cup P_G(v, z'_j)$ is a (k, U)-tree, which is a contradiction.

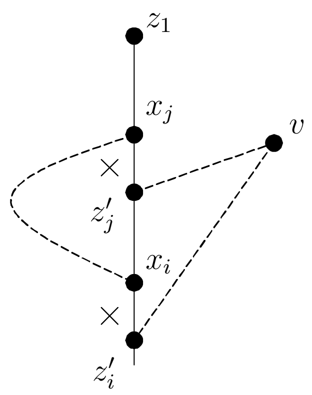

Figure 5.7:

If $z'_i \notin V(P_T(z_1, z''_j))$, then $(T - z''_j y_j - x_i z'_i - x_j z'_j + x_i x_j) \cup P_G(v, z''_j) \cup P_G(v, z'_i) \cup P_G(v, z'_j)$ is a (k, U)-tree. This contradicts the choice of T.

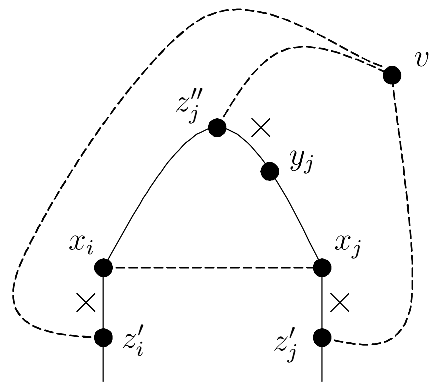

Figure 5.8:

Hence the claim is proved. $\square$

Therefore, by Claim 5.4.4, we obtain $\alpha(G) \geq (t-s)(k-1)+1+1 = (t-s)(k-1)+2$, which contradicts the assumption. This completes the proof.

Chapter 6

Trees with Bounded Degree Covering Specified Vertices

In this chapter, we give sufficient conditions for a graph to have a tree with bounded degree. Let G be a connected graph and A a vertex subset of G. We denote by $\sigma_k(A)$ the minimum value of the degree sum in G of any k independent vertices in A and by $w(G-A)$ the number of components in $G-A$. Our main results are the following: (i) If $\sigma_k(A) \geq |G|-1$, then G contains a tree T with maximum degree at most k and $A \subseteq V(T)$. (ii) If $\sigma_{k-w(G-A)}(A) \geq |A|-1$, then G contains a spanning tree T such that $d_T(x) \leq k$ for every $x \in A$. These are generalizations of a result by Win [30] and degree conditions are sharp.

6.1 Introduction

In this chapter, we use the following notation.

Let G be a graph. For a subset A of $V(G)$, $\alpha(A)$ denotes the independence number of $\langle A \rangle$. For $1 \leq k \leq \alpha(A)$, define

$$\sigma_k(A) = \min \left\{ \sum_{x \in S} d_G(x) \;\middle|\; S \text{ is an independent subset of } A \text{ with } |S| = k. \right\}$$

and $\sigma_k(A) = \infty$ if $\alpha(A) < k$. Note that $\sigma_k(G) = \sigma_k(V(G))$.

We begin with the well-known theorem on the existence of a Hamilton cycle.

Theorem 6.1 (Ore [19]) *Let G be a graph of order $n \geq 3$ and $\sigma_2(G) \geq n$. Then G has a Hamilton cycle.*

This theorem has been generalized in many directions. For example, a cycle containing all the prescribed vertices was considered since a Hamilton cycle is a cycle which contains every vertex of a graph. In particular, the following result was obtained.

Theorem 6.2 (Shi [23], Ota [21]) *Let G be a 2-connected graph of order n and $A \subseteq V(G)$. If $\sigma_2(A) \geq n$, then G has a cycle containing all vertices of A.*

In this paper, we consider analogous extension on degree bounded trees. The starting point is the following result by Win.

Theorem 6.3 (Win [30]) *Let $k \geq 2$ be an integer and G a connected graph of order n. If $\sigma_k(G) \geq n-1$, then G has a spanning k-tree.*

Note that Theorem 6.3 is an extension of the following one since a spanning 2-tree is nothing but a Hamilton path.

Theorem 6.4 (Ore [19]) *Let G be a graph of order n with $\sigma_2(G) \geq n-1$. Then G has a Hamilton path.*

6.2 Main results

We consider two types of extensions of Theorem 6.3. One is on a tree with bounded degree containing all the prescribed vertices.

Theorem 6.5 *Let $k \geq 2$ be an integer, G a connected graph of order n, and $A \subseteq V(G)$. If $\sigma_k(A) \geq n-1$, then G has a k-tree T with $A \subseteq V(T)$.*

The degree condition in Theorem 6.5 is sharp in the sense that we cannot replace the lower bound to $n-2$, which is shown in the following example.

Example 6.1. Consider a complete bipartite graph G with partite sets X and Y such that $|Y| = (k-1)|X| + 2 \geq k+1$. Let $A = V(G) - \{x\}$, where x is any vertex in X. Then $|G| = n = k|X| + 2$ and $\sigma_k(A) = n-2$. Suppose that G has a tree T with the property that $A \subseteq V(T)$ and $d_T(v) \leq k$ for all $v \in V(T)$. If $x \in V(T)$, then $n-1 = |E(T)| \leq k|X| = n-2$, a contradiction. If $x \notin V(T)$, then $n-2 = |E(T)| \leq k(|X|-1) = n-2-k$, which is also a contradiction. Hence G has no desired tree.

The other one is on a spanning tree with bounded degrees on the prescribed vertices. For a graph G and $A \subseteq V(G)$, we denote by $w(G-A)$ the number of components of the subgraph $G-A$. Note that we define $w(G-A) = 0$ if $A = V(G)$.

Theorem 6.6 *Let $k \geq 2$ be an integer, G a connected graph of order n and $A \subseteq V(G)$. Suppose that $w(G-A) \leq k-1$ and $\sigma_{k-w(G-A)}(A) \geq |A|-1$. Then G contains a spanning tree T with $d_T(x) \leq k$ for every $x \in A$.*

The degree condition in Theorem 6.6 is also sharp.

Example 6.2. Let G be a complete bipartite graph with partite sets X and Y, where $X = \{x\}$ and $Y = \{y_1, \ldots, y_{k+1}\}$. Define $A = \{x, y_1, \ldots, y_t\}$ with $2 \leq t \leq k+1$. Then G cannot have a spanning tree T such that $d_T(x) \leq k$, while $w(G-A) = k+1-t$ and $\sigma_{k-w(G-A)}(A) = t-1 = |A|-2$.

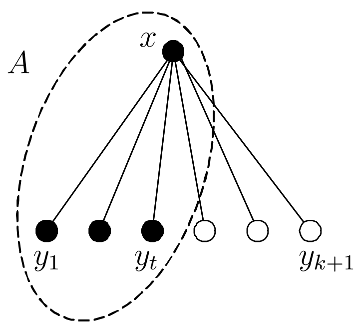

Figure 6.1: The graph G in Example 6.2.

6.3 Proof of Theorem 6.5

Recall that a *k-tree* is a tree T which satisfies $d_T(x) \leq k$ for all $x \in V(T)$. Choose a k-tree T of G such that

(a) $|A \cap V(T)|$ is as large as possible and

(b) subject to (a), $|T|$ is as small as possible.

If $A \subseteq V(T)$, then we have nothing to prove. Hence we may assume that there exists a vertex $x \in A - V(T)$. Since G is connected, there exists a path P which connects x and a vertex of $V(T)$. We may assume that $|V(P) \cap V(T)| = 1$ and let $\{v\} = V(P) \cap V(T)$. By the choice of T, we obtain $d_T(v) = k$.

Let $T_1, \ldots, T_k$ be the components of $T - \{v\}$. For each $i = 1, \ldots, k$, let t_i be the vertex of T_i which is adjacent to v in T and let u_i be a vertex of T_i with $d_T(u_i) = 1$. Note that $u_i \in A$ by the minimality of $|T|$ and that $t_i = u_i$ if $|T_i| = 1$. If $u_i u_j \in E(G)$, then $(T + u_i u_j - v t_i) \cup P$ is a k-tree of G, which contains more vertices of A than T. This contradicts the choice of T. Hence $\{u_1, \ldots, u_k\}$ is an independent set of G.

Let

$$Y_1 = \bigcup_{i=2}^{k} N_G(u_i) \cap V(T_1).$$

Note that $t_1 \notin Y_1$ since otherwise $t_1 u_i \in E(G)$ for some u_i with $2 \leq i \leq k$ and thus $(T - vt_1 + t_1 u_i) \cup P$ contradicts the choice of T. If $d_T(y) < k$ for some $y \in Y_1$, then $y u_i \in E(G)$ for some u_i with $2 \leq i \leq k$ and thus $(T - t_1 v + u_i y) \cup P$ is a k-tree of G, a contradiction. Hence $d_T(y) = k$ for all $y \in Y_1$.

For $x, y \in V(T)$, we denote by $P_T(x, y)$ the unique path in T connecting x and y.

Claim 6.3.1 *For each $y \in Y_1$, $N_G[u_1] \cap (N_T(y) - V(P_T(y, u_1))) = \emptyset$.*

Proof. Suppose that there exists $z \in N_G[u_1] \cap (N_T(y) - V(P_T(y, u_1)))$ for some $y \in Y_1$. Since $y u_i \in E(G)$ for some u_i with $2 \leq i \leq k$, a k-tree $(T - yz - vt_1 + u_1 z + y u_i) \cup P$ contains more vertices of A than T. This contradicts the choice of T. $\square$

Claim 6.3.2 $|N_T(Y_1) - N_G[u_1]| \geq (k-1)|Y_1|$.

Proof. We may assume that $Y_1 \neq \emptyset$ since otherwise the above inequality obviously holds. Furthermore, $v \notin N_T(Y_1)$ since $t_1 \notin Y_1$.

We consider T_1 as an outdirected tree with root u_1. For any $y_0 \in Y_1$ and $z \in N^+_{T_1}(y_0)$, $z \notin N_G[u_1]$ holds by Claim 6.3.1. This implies that $N^+_{T_1}(y_0) \subseteq N_T(Y_1) - N_G[u_1]$ for any $y_0 \in Y_1$. Since $N^+_{T_1}(y_1) \cap N^+_{T_1}(y_2) = \emptyset$ holds for any two distinct vertices y_1 and y_2 of Y_1, we obtain $|N_T(Y_1) - N_G[u_1]| \geq \left|N^+_{T_1}(Y_1)\right| = \sum_{y \in Y_1} |N^+_{T_1}(y)| = (k-1)|Y_1|$. $\square$

Claim 6.3.3 $\sum_{i=1}^{k} |V(T_j) \cap N_G(u_i)| \leq |T_j| - 1$ *for each $j = 1, 2, \ldots, k$.*

Proof. By Claim 6.3.1, we obtain

$$\begin{aligned} |V(T_1) \cap N_G(u_1)| &\leq |T_1| - 1 - |N_T(Y_1) - N_G[u_1]| \\ &\leq |T_1| - 1 - (k-1)|Y_1|. \end{aligned}$$

Since $\sum_{i=2}^{k} |V(T_1) \cap N_G(u_i)| \leq (k-1)|Y_1|$ by the definition of Y_1, we have

$$\sum_{i=1}^{k} |V(T_1) \cap N_G(u_i)| \leq |T_1| - 1.$$

Similarly, $\sum_{i=1}^{k} |V(T_j) \cap N_G(u_i)| \leq |T_j| - 1$ holds for each $j = 2, \ldots, k$. □

Claim 6.3.4 $\sum_{i=1}^{k} |V(G-T) \cap N_G(u_i)| \leq |G-T| - 1$.

Proof. It is easy to see that $u_i x \notin E(G)$ for all u_i, since otherwise $T + u_i x$ contradicts the choice of T. Suppose that $\sum_{i=1}^{k} |V(G-T) \cap N_G(u_i)| \geq |G-T|$. Then there exists $w \in N_G(u_i) \cap N_G(u_j) \cap (V(G-T) - \{x\})$ for some $1 \leq i < j \leq k$. If $w \in V(P)$, then $T + u_i x \cup P'$ is a k-tree containing more vertices of A than T, where P' is a subpath of P from w to x. Hence $w \notin V(P)$. However, $T' = (T + wu_i + wu_j - vt_i) \cup P$ is a k-tree such that $|V(T') \cap A| > |V(T) \cap A|$, a contradiction. □

Since $d_G(u_i) \leq |\{v\}| + \sum_{j=1}^{k} |V(T_j) \cap N_G(u_i)| + |V(G-T) \cap N_G(u_i)|$,

$$\begin{aligned}
\sum_{i=1}^{k} d_G(u_i) &\leq k + \sum_{i=1}^{k} \sum_{j=1}^{k} |V(T_j) \cap N_G(u_i)| + \sum_{i=1}^{k} |V(G-T) \cap N_G(u_i)| \\
&\leq k + \sum_{j=1}^{k} (|T_j| - 1) + |G-T| - 1 \\
&= \sum_{j=1}^{k} |T_j| + |G-T| - 1 \\
&= |T| - 1 + |G-T| - 1 = n - 2,
\end{aligned}$$

a contradiction. This completes the proof of Theorem 6.5. □

6.4 Proof of Theorem 6.6

To prove this theorem, we consider into two cases.

Case 1 $k = 2$.

If $w(G-A) = 0$, then $A = V(G)$ and the theorem holds by Theorem 6.3. Hence we may assume that $w(G-A) = 1$. We divide A into two subsets such that $A_1 = \{x \in A \mid N_G(x) \cap V(G-A) \neq \emptyset\}$ and $A_2 = A - A_1$. Note that $A_1 \neq \emptyset$ since G is connected. Since $\sigma_1(A) \geq |A| - 1$ and $N_G(A_2) \subseteq A$, $\langle A_2 \rangle$ is complete and $xy \in E(G)$ for any $x \in A_1$ and $y \in A_2$. By $w(G-A) = 1$, $G - A$ has a spanning tree T. Then we get the desired spanning tree by joining each vertex of A_1 to some vertex in T

and adding a Hamilton path of $\langle A_2 \rangle$ to some vertex of A_1. This completes the proof of this theorem for the case $k = 2$.

Case 2 $k \geq 3$.

In the following, a tree T is called a (k, A)-*tree* if $d_T(x) \leq k$ for any $x \in V(T) \cap A$. We construct a new graph H from G by contracting each component of $G - A$ to a single vertex. Take a (k, A)-tree T of H such that $|T|$ is as large as possible. We may assume that $V(H) - V(T) \neq \emptyset$ since otherwise we obtain the desired spanning tree by replacing each contracted vertex with a spanning tree of the corresponding component. Take $x \in V(H) - V(T)$ such that $N_H(x) \cap V(T) \neq \emptyset$ and let $v \in N_H(x) \cap V(T)$. Note that $v \in A$ and $d_T(v) = k$ by the choice of T.

Let $T_1, \dots, T_k$ be the components of $T - \{v\}$ and let t_i be the vertex of T_i which is adjacent to v in T, where $i = 1, \dots, k$. Since $|V(H) - A| = w(G - A)$, we may assume that $V(T_i) \subset A$ for $1 \leq i \leq k - w(G - A)$. Put $k' = k - w(G - A)$ and let u_i be a vertex of T_i such that $d_T(u_i) = 1$ for each $i = 1, \dots, k'$. If $u_i u_j \in E(G)$ for some $1 \leq i < j \leq k'$, then $T + u_i u_j - vt_i + vx$ is a (k, A)-tree larger than T, a contradiction. Hence $\{u_1, \dots, u_{k'}\}$ is an independent set of H, also of G.

If $k' \neq 1$, then we define

$$Y_1 = \bigcup_{i=2}^{k'} N_H(u_i) \cap V(T_1).$$

For the case of $k' = 1$, let $Y_1 = \emptyset$. If $d_T(y) < k$ for some $y \in Y_1$, then $yu_i \in E(H)$ for some $i = 2, \dots, k'$ and thus $T - vt_1 + yu_i + vx$ contradicts the choice of T. Consequently we obtain

$$Y_1 \subset A \text{ and } d_T(y) = k \text{ for all } y \in Y_1. \tag{6.1}$$

For $y, z \in V(T)$, we denote by $P_T(y, z)$ the unique path in T connecting y and z.

Claim 6.4.1 *For each $y \in Y_1$, $(N_T(y) - V(P_T(y, u_1))) \cap N_H[u_1] = \emptyset$.*

Proof. Suppose that $z \in N_H[u_1]$ for some $z \in N_T(y) - V(P_T(y, u_1))$. Since $yu_i \in E(G)$ for some $i = 2, \dots, k'$, a (k, A)-tree $T - yz - vt_1 + u_1 z + yu_i + vx$ contradicts the maximality of T. □

Claim 6.4.2 $|N_T(Y_1) - N_H[u_1]| \geq (k - 1)|Y_1|$.

Proof. If $Y_1 = \emptyset$, then the above inequality holds, and so we may assume that $Y_1 \neq \emptyset$. We obtain $t_1 \notin Y_1$, in particular, $v \notin N_T(y)$ for every $y \in Y_1$, since if $t_1 \in Y_1$, then $t_1 u_i \in E(H)$ for some u_i and $T - vt_1 + t_1 u_i + vx$ contradicts the maximality of T.

We regard T_1 as an outdirected tree with root u_1. For any $y_0 \in Y_1$ and $z \in N^+_{T_1}(y_0)$, it follows from Claim 6.4.1 that $z \notin N_H[u_1]$. This implies that $N^+_{T_1}(y_0) \subseteq N_T(Y_1) - N_H[u_1]$ for any $y_0 \in Y_1$. Since $N^+_{T_1}(y_1) \cap N^+_{T_1}(y_2) = \emptyset$ holds for any two distinct vertices $y_1, y_2 \in Y_1$, we obtain $|N_T(Y_1) - N_H[u_1]| \geq |N^+_{T_1}(Y_1)| = \sum_{y \in Y_1} |N^+_{T_1}(y)| = (k-1)|Y_1|$.
$\square$

For $1 \leq i \leq k'$, let

$$W_{i,k} = N_H(u_i) \cap V(T_k).$$

Note that $t_k \notin W_{i,k}$ since otherwise $t_k u_i \in E(H)$ and $T - vt_k + t_k u_i + vx$ contradicts the choice of T. If $w \notin A$ or $d_T(w) < k$ for some $w \in W_{i,k}$, then $T - vt_k + wu_i + vx$ also contradicts the maximality of T. Hence

$$W_{i,k} \subset A \text{ and } d_T(w) = k \text{ for any } w \in W_{i,k}. \tag{6.2}$$

Suppose that $ww' \in E(T)$ for some $w, w' \in W_{i,k}$. Then $T - ww' + wu_i + w'u_i - vt_k + vx$ is a contradiction. Thus $W_{i,k}$ is an independent set in T for each $i = 1, \ldots, k'$.

Claim 6.4.3 $|W_{i,k}| \leq \frac{1}{k}(|T_k| - 1)$.

Proof. We consider T_k as an outdirected tree with root u_k. Since $W_{i,k}$ is independent in T, we have $N^+_{T_k}[w] \cap N^+_{T_k}[w'] = \emptyset$ for any $w, w' \in W_{i,k}$. This together with $t_k \notin W_{i,k}$ implies $|T_k| \geq 1 + \sum_{w \in W_{i,k}} |N^+_{T_k}[w]| = 1 + k|W_{i,k}|$. $\square$

By Claim 6.4.2, we obtain

$$\begin{aligned} |N_H(u_1) \cap V(T_1)| &\leq |T_1| - 1 - |N_T(Y_1) - N_H[u_1]| \\ &\leq |T_1| - 1 - (k-1)|Y_1|. \end{aligned}$$

By the definition of Y_1, we have $\sum_{i=2}^{k'} |N_H(u_i) \cap V(T_1)| \leq (k'-1)|Y_1|$. Hence

$$\begin{aligned} \sum_{i=1}^{k'} |N_H(u_i) \cap V(T_1)| &\leq |T_1| - 1 - (k-1)|Y_1| + (k'-1)|Y_1| \\ &= |T_1| - 1 - (k-k')|Y_1| \leq |T_1| - 1. \end{aligned}$$

By symmetry, we have $\sum_{i=1}^{k'} |N_H(u_i) \cap V(T_j)| \leq |T_j| - 1$ for each j, $1 \leq j \leq k'$. On the other hand, by Claim 6.4.3,

$$\begin{aligned} \sum_{i=1}^{k'} |N_H(u_i) \cap V(T_k)| = \sum_{i=1}^{k'} |W_{i,k}| &\leq \sum_{i=1}^{k'} \frac{1}{k}(|T_k| - 1) \\ &= \frac{k'}{k}(|T_k| - 1) \leq |T_k| - 1. \end{aligned}$$

By the same argument above, $\sum_{i=1}^{k'} |N_H(u_i) \cap V(T_j)| \le |T_j| - 1$ holds for each $j = k'+1, \ldots, k$.

By the maximality of T, $N_H(u_i) \cap V(G-T) = \emptyset$ for all $i = 1, \ldots, k'$. By (6.1) and (6.2), we have $N_H(u_1) \subseteq A$. This means that $d_G(u_1) = d_H(u_1)$. Analogously, $d_G(u_i) = d_H(u_i)$ holds for each $i = 2, \ldots, k'$. Since $d_G(u_i) \le |\{v\}| + \sum_{j=1}^{k} |V(T_j) \cap N_H(u_i)|$,

$$\begin{aligned}\sum_{i=1}^{k'} d_G(u_i) &\le k' + \sum_{i=1}^{k'} \sum_{j=1}^{k} |V(T_j) \cap N_H(u_i)| \le k' + \sum_{j=1}^{k} (|T_j| - 1) \\ &= \sum_{j=1}^{k} |T_j| - w(G-A). \end{aligned} \tag{6.3}$$

Recall that $x \in V(H) - V(T)$ is a vertex such that $N_H(x) \cap V(T) \neq \emptyset$ and $N_H(x) \cap V(T) \subset A$.

If $x \notin A$, then $\sum_{j=1}^{k} |T_j| \le \sum_{j=1}^{k} |V(T_j) \cap A| + w(G-A) - 1$. By (6.3),

$$\sum_{i=1}^{k} d_G(u_i) \le \sum_{j=1}^{k} |V(T_j) \cap A| - 1 = (|V(T) \cap A| - |\{v\}|) - 1 \le |A| - 2,$$

which is a contradiction. Hence we may assume that $x \in A$. In this case, $\sum_{j=1}^{k} |T_j| \le \sum_{j=1}^{k} |V(T_j) \cap A| + w(G-A)$ holds. This inequality together with (6.3) yields

$$\sum_{i=1}^{k} d_G(u_i) \le \sum_{j=1}^{k} |V(T_j) \cap A| = |V(T) \cap A| - 1 \le |A| - 2.$$

This also contradicts the assumption. This completes the proof of Case 2 and Theorem 6.6.

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