Characterizations of Some Subclasses of Infinitely Divisible Distributions on $\mathbb{R}^d$ by Stochastic Integrals

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Contents

Chapter 1

Introduction.

1.1 Lévy processes and infinitely divisible distributions on $\mathbb{R}^d$

We start with several definitions which are needed throughout the thesis.

Definition 1.1 (Lévy process)**.** A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is a *Lévy process* if the following conditions are satisfied.
(1) For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, random variables $X_{t_0} - X_0, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent. (Independent increment property.)
(2) The distribution of $X_{s+t} - X_s$ does not depend on s. (Stationary increment property.)
(3) $X_0 = 0$ a.s.
(4) X_t is stochastically continuous for any $t \geq 0$.

Definition 1.2 (Infinitely divisible distribution)**.** A probability measure μ on $\mathbb{R}^d$ is infinitely divisible if, for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that

$$\mu = \mu_n^{n*},$$

where μ_n^{n*} is the n-fold convolution of μ_n.

We denote by $I(\mathbb{R}^d)$ (resp. $I_{\mathrm{sym}}(\mathbb{R}^d)$) the class of all infinitely divisible (resp. all symmetric infinitely divisible) distributions on $\mathbb{R}^d$.

Proposition 1.3 (see, e.g. [S99])**.** *If $\{X_t\}$ is a Lévy process on $\mathbb{R}^d$, then, for every t, the distribution of X_t is infinitely divisible.*

Definition 1.4. By the independent and stationary increment property, the distribution of X_t is determined by that of X_1. Therefore, the distribution of X_1 is a characteristic of the Lévy process $\{X_t\}$. We denote by $\{X_t^{(\mu)}\}$ a Lévy process such that its distribution at time 1 is $\mu \in I(\mathbb{R}^d)$. Namely, $\mathcal{L}(X_1^{(\mu)}) = \mu$, where $\mathcal{L}$ means " the law of " throughout this thesis.

Let $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$, be the characteristic function of μ.

Proposition 1.5 (Lévy–Khintchine representation (see, e.g. [S99])). *(i) If $\mu \in \mathbb{R}^d$, then*

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \frac{\mathrm{i}\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right],\ z \in \mathbb{R}^d, \tag{1.1}$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty, \tag{1.2}$$

and $\gamma \in \mathbb{R}^d$.

(ii) The representation of $\widehat{\mu}$ in (i) by A, ν, and γ is unique.

(iii) Conversely, if A is a symmetric nonnegative–definite $d \times d$ matrix, ν is a measure satisfying (1.2), and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function is given by (1.1).

(A, ν, γ) is called the Lévy–Khintchine triplet of $\mu \in I(\mathbb{R}^d)$.

Proposition 1.6 (Polar decomposition of Lévy measures ([R90], [BMS06])). *Let ν be the Lévy measure of some $\mu \in I(\mathbb{R}^d)$ with $\nu(\mathbb{R}^d) > 0$. Then there exists a measure λ on S, the unit sphere on $\mathbb{R}^d$, with $0 < \lambda(S) \leq \infty$ and a family $\{\nu_\xi : \xi \in S\}$ of measures on $(0, \infty)$ such that*

$$\nu_\xi(B) \text{ is measurable in } \xi \text{ for each } B \in \mathcal{B}((0,\infty)), \tag{1.3}$$

$$0 < \nu_\xi((0,\infty)) \leq \infty \text{ for each } \xi \in S, \tag{1.4}$$

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr) \text{ for } B \in \mathcal{B}(\mathbb{R}^d \backslash \{0\}). \tag{1.5}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν in the following sense : If $\lambda, \{\nu_\xi\}$ and $\lambda', \{\nu'_\xi\}$ both have properties (1.3)–(1.5), then there is a measurable function $c(\xi)$ on S such that

$$0 < c(\xi) < \infty, \tag{1.6}$$

$$\lambda'(d\xi) = c(\xi)\lambda(d\xi), \tag{1.7}$$

$$c(\xi)\nu'_\xi(dr) = \nu_\xi(dr) \quad for\ \lambda\text{−}a.e.\ \xi \in S. \tag{1.8}$$

We call the pair (λ, ν_ξ) a polar decomposition of ν and ν_ξ in (1.5) the radial component of ν, respectively.

We also define the cumulant function $C_\mu(z)$ of $\mu \in I(\mathbb{R}^d)$ as follows: $C_\mu(z)$ is the unique complex-valued continuous function on $\mathbb{R}^d$ satisfying $C_\mu(0) = 0$ and $\widehat{\mu}(z) = e^{C_\mu(z)}$. For a random variable X with its distribution μ, we also write $C_X(z)$ for $C_\mu(z)$.

Proposition 1.7 (Stochastic integral with respect to a Lévy process (see, e.g. [RS03], Proposition 29)**.** *Let $\mu \in I(\mathbb{R}^d)$. Let $f(s)$ be a real-valued bounded measurable function on $[a, b], 0 \leq a < b < \infty$, such that there are uniformly bounded step functions $f_n(s)$, $n = 1, 2, \ldots,$ on $[a, b]$ satisfying $f_n \to f$ almost everywhere. Then $\int_a^b f_n(s)dX_s^{(\mu)}$ converges to an $\mathbb{R}^d$-valued random variable X in probability. The limit X does not depend on the choice of $\{f_n\}$ up to probability zero, and we write the limit as $X := \int_a^b f(s)dX_s^{(\mu)}$. Then $\mathcal{L}(X)$ is infinitely divisible and its cumulant function is represented as*

$$C_X(z) = \int_a^b C_\mu(f(s)z)ds.$$

The integral over $[0, \infty)$ is defined as follows when the limit exists:

$$X := \int_0^\infty f(s)dX_s = \lim_{a\to\infty} \int_0^a f(s)dX_s \quad in\ probability.$$

Thus

$$C_X(z) = \int_0^\infty C_\mu(f(s)z)ds.$$

As to the definition of stochastic integrals of nonrandom functions with respect to Lévy processes $\{X_t\}$ on $\mathbb{R}^d$, it is also studied in Sato ([S04], [S06]), whose idea is to define the integrals with respect to $\mathbb{R}^d$-valued independently scattered random measure induced by a Lévy process on $\mathbb{R}^d$. This idea was used in Urbanik and Woyczyński ([UW67]) and Rosinski ([R90]) for the case $d = 1$. See also Barndorff-Nielsen et al. ([BMS06]).

Definition 1.8 (Completely monotone function)**.** A function f on $(0, \infty)$ is said to be completely monotone if it is infinitely many times differentiable and for $n = 0, 1, 2, \cdots$

$$(-1)^n f^{(n)}(s) \geq 0, \quad s \in (0, \infty),$$

where $f^{(n)}(s)$ is the n–th order derivative and $f^{(0)}(s) = f(s)$.

Lemma 1.9. *Let $f(s)$ and $g(s)$ be two completely monotone functions on $(0, \infty)$. Then, $f(s)g(s)$ is again completely monotone on $(0, \infty)$.*

Proof. Let $f(s)$ and $g(s)$ be completely monotone functions on $(0, \infty)$. Then $f(s)g(s) \geq 0$ and, for any $n \in \mathbb{N}$, we have

$$(-1)^n(f(s)g(s))^{(n)} = \sum_{i=0}^{n} {}_nC_i(-1)^i f^{(i)}(s)(-1)^{n-i}g^{(n-i)}(s) \geq 0, \quad s \in (0, \infty),$$

where $(f(s)g(s))^{(n)}$ is the n–th order derivative of $f(s)g(s)$. □

We will to use the following proposition many times in this thesis.

Proposition 1.10 (Bernstein's theorem)**.** *A measurable function f on $(0, \infty)$ is completely monotone if and only if there exists a measure Q on $(0, \infty)$ such that*

$$f(s) = \int_0^\infty e^{-su}Q(du), \quad s \in (0, \infty),$$

holds.

1.2 Some known subclasses of infinitely divisible distributions

In the following, the classification and characterization are given in term of the radial component ν_ξ of the Lévy measure. Classes in $I(\mathbb{R}^d)$ we are going to discuss in this thesis are the following.
(1) Class $U(\mathbb{R}^d)$ (the Jurek class) :

$$\nu_\xi(dr) = \ell_\xi(r)dr, \tag{1.9}$$

where $\ell_\xi(r)$ is measurable in $\xi \in S$ and nonincreasing in $r \in (0, \infty)$.

The class $U(\mathbb{R}^d)$ was introduced by Jurek ([J85]) and $\mu \in U(\mathbb{R}^d)$ is called *s-selfdecomposable.* In his paper ([J85]), he proved the following. (i) $\mu \in U(\mathbb{R}^d)$ if and only if for any $0 < c < 1$ there exists $\mu_c \in I(\mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(cz)^c\widehat{\mu}_c(z)$, and (ii) $\mu \in U(\mathbb{R}^d)$ if and only if there exist probability distributions $\mu_1, \mu_2, ... \in I(\mathbb{R}^d)$ such that

$$\left(\widehat{\mu}_1(n^{-1}z)\widehat{\mu}_2(n^{-1}z)^2 \cdots \widehat{\mu}_n(n^{-1}z)^n\right)^{1/n} \to \widehat{\mu}(z).$$

(2) Class $B(\mathbb{R}^d)$ (the Goldie–Steutel–Bondesson class) :

$$\nu_\xi(dr) = \ell_\xi(r)dr, \tag{1.10}$$

where $\ell_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0,\infty)$ as a function of r.

Bondesson ([B82]) studied generalized convolutions of mixtures of exponential distributions on $\mathbb{R}_+$. (The smallest class that contains all mixtures of exponential distributions and that is closed under convolution and weak convergence on $\mathbb{R}_+$.) $B(\mathbb{R}^d)$ is its generalization to the multidimensional case. (Barndorff-Nielsen et al. [BMS06].) Since completely monotone functions are nonincreasing, ℓ_ξ is nonincreasing. Thus, we have

$$B(\mathbb{R}^d) \subset U(\mathbb{R}^d).$$

(3) Class $L(\mathbb{R}^d)$ (the class of selfdecomposable distributions) :

$$\nu_\xi(dr) = k_\xi(r)r^{-1}dr, \tag{1.11}$$

where $k_\xi(r)$ is measurable in $\xi \in S$ and nonincreasing in $r \in (0,\infty)$.

It is known that $\mu \in L(\mathbb{R}^d)$ if and only if for any $0 < c < 1$, there exists some $\mu_c \in I(\mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(cz)\widehat{\mu}_c(z)$. (This statement is used as the definition of the selfdecomposability usually.) Since $k_\xi(r)r^{-1}$ is nonincreasing, we have

$$L(\mathbb{R}^d) \subset U(\mathbb{R}^d).$$

(4) Class $T(\mathbb{R}^d)$ (the Thorin class) :

$$\nu_\xi(dr) = k_\xi(r)r^{-1}dr, \tag{1.12}$$

where $k_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0,\infty)$ as a function of r.

Thorin ([T77a], [T77b]) studied generalized Γ-convolutions on $\mathbb{R}_+$ and $\mathbb{R}$. (The smallest class that contains all Γ-distributions and that is closed under convolution and weak convergence on $\mathbb{R}_+$ and $\mathbb{R}$.) $T(\mathbb{R}^d)$ is its generalization to the multidimensional case. (Barndorff-Nielsen et al. [BMS06].) r^{-1} is completely monotone and by Lemma 1.9, $k_\xi(r)r^{-1}$ is completely monotone. Furthermore, $k_\xi(r)$ is nonincreasing since completely monotone functions are nonincreasing. Thus, we have

$$T(\mathbb{R}^d) \subset B(\mathbb{R}^d) \cap L(\mathbb{R}^d).$$

(5) Class $G(\mathbb{R}^d)$ (the class of type G distributions) :

$$\mu \in I_{\mathrm{sym}}(\mathbb{R}^d) \text{ and } \nu_\xi(dr) = g_\xi(r^2)dr, \tag{1.13}$$

where $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0,\infty)$ as a function of r.

When $d = 1$, $\mu \in G(\mathbb{R}^1)$ if and only if $\mu = \mathcal{L}(V^{1/2}Z)$, where $\mathcal{L}(V) \in I(\mathbb{R}_+)$, Z is the standard normal random variable, and V and Z are independent. When $d \geq 1$, $\mu \in G(\mathbb{R}^d)$ if and only if $\nu_\mu(B) = E[\nu_0(Z^{-1}B)]$ for some Lévy measure ν_0, where ν_μ is the Lévy measure of μ. (Maejima-Rosiński [MR02].)

1.3 Characterizations of several classes of infinitely divisible distributions by stochastic integrals

In Section 1.2, we explained five known classes of infinitely divisible distributions, which are characterized by their Lévy measures. Here we show some results on characterizations for first four classes by stochastic integrals with respect to Lévy processes.

Proposition 1.11 ([J85]).

$$U(\mathbb{R}^d) = \left\{ \mathcal{L}\left(\int_0^1 t dX_t^{(\mu)} \right), \ \mu \in I(\mathbb{R}^d) \right\}.$$

Proposition 1.12 ([BMS06]).

$$B(\mathbb{R}^d) = \left\{ \mathcal{L}\left(\int_0^1 \log \frac{1}{t} dX_t^{(\mu)} \right), \ \mu \in I(\mathbb{R}^d) \right\}.$$

Proposition 1.13 ([W82] and others).

$$L(\mathbb{R}^d) = \left\{ \mathcal{L}\left(\int_0^\infty e^{-t} dX_t^{(\mu)} \right), \ \mu \in I_{\log}(\mathbb{R}^d) \right\},$$

where $I_{\log}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{|x|>2} \log|x| \mu(dx) < \infty\}$.

Proposition 1.14 ([BMS06]). *Let* $e_1(u) = \int_u^\infty e^{-s}s^{-1}ds$ *and let* $e_1^*(t)$ *be its inverse function, that is,* $t = e_1(u)$ *if and only if* $u = e_1^*(t)$. *Then*

$$T(\mathbb{R}^d) = \left\{ \mathcal{L}\left(\int_0^\infty e_1^*(t) dX_t^{(\mu)} \right), \ \mu \in I_{\log}(\mathbb{R}^d) \right\}.$$

Our first problem in this thesis is to obtain a stochastic integral characterization of $G(\mathbb{R}^d)$, which will be studied in Chapter 2.

1.4 Nested subclasses of selfdecomposable distributions, $L_m(\mathbb{R}^d)$, $m \in \mathbb{N}$

Urbanik ([U73]) and Sato ([S80]) defined and investigated nested subclasses of $L(\mathbb{R}^d)$. They are defined as follows.

Definition 1.15 (Class $L_m(\mathbb{R}^d)$)**.** Let $m = 1, 2, \cdots$ and let $L_0(\mathbb{R}^d) = L(\mathbb{R}^d)$. $\mu \in I(\mathbb{R}^d)$ belongs to $L_m(\mathbb{R}^d)$ if and only if for any $0 < c < 1$, there exists a $\mu_c \in L_{m-1}(\mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(cz)\widehat{\mu}_c(z)$ holds. $L_\infty(\mathbb{R}^d)$ is defined by $\bigcap_{m=0}^{\infty} L_m(\mathbb{R}^d)$.

Then, they showed the following.

Proposition 1.16 ([U73], [S80])**.**

$$L_0(\mathbb{R}^d) \supset L_1(\mathbb{R}^d) \supset L_2(\mathbb{R}^d) \supset \cdots \supset L_m(\mathbb{R}^d) \supset \cdots \supset L_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)},$$

where $\overline{S(\mathbb{R}^d)}$ *is the class of all stable distributions on* $\mathbb{R}^d$ *and the closure is taken by weak convergence and convolution.*

Urbanik [U73] showed this proposition when $d = 1$, and Sato [S80] generalized to the multidimensional case.

This proposition is important, in one sense, in understanding the role of stable distributions in $I(\mathbb{R}^d)$.

The following is also known.

Proposition 1.17 ([J85])**.**

$$L_m(\mathbb{R}^d) = \left\{ \mathcal{L}\left(\int_0^\infty e^{-p_m(t)} dX_t^{(\mu)} \right) : \mu \in I_{\log^m}(\mathbb{R}^d) \right\},$$

where

$$p_m(t) = ((m+1)!t)^{1/(m+1)}$$

and

$$I_{\log^m}(\mathbb{R}^d) = \left\{ \mu \in I(\mathbb{R}^d) : \int_{|x|>2} (\log |x|)^m \mu(dx) < \infty \right\}.$$

1.5 History and motivations

As we mentioned in Definition 1.2, a probability measure μ on $\mathbb{R}^d$ is called infinitely divisible if, for any positive integer n, there exist a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^{n*}$, where μ_n^{n*} is the n-th convolution of μ_n. The class of infinitely divisible distributions is known as the most important class of probability distributions. For instance, normal, exponential, Poisson and stable distributions are in this class. (See e.g. [S99], [SV04].)

Historically, the results on classifying its subclasses were mainly given in terms of Lévy measure ν in the Lévy-Khintchine representation of the characteristic function. The characteristic function is the the Fourier transform of a probability measure. Hence, these results could be said to be analytical ones.

Recently, probabilistic interpretations for such results have been of interest, and, especially, characterizations of subclasses of them by stochastic integrals with respect to Lévy processes have been well studied as we mentioned in Section 1.3. However, only a few classes of infinitely divisible distributions were characterized in this way. Barndorff-Nielsen et al. ([BMS06]) found such characterizations for the Goldie-Steutel-Bondesson class and the Thorin class. (For the details, see Barndorff-Nielsen et al. [BMS06].) As in Section 1.4, nested subclasses of the class of selfdecomposable distributions are studied and that shows the relationship with the class of stable distributions. Our study is on the line of this history.

In Chapter 2, the class of type G distributions on $\mathbb{R}^d$ and its nested subclasses are studied. Type G distribution is a variance mixture of the standard normal distribution. (See e.g. [SV04].) An analytic characterization in terms of Lévy measures for the class of type G distributions is known. In this chapter, probabilistic characterizations by stochastic integral representations for all classes are shown and moreover analytic characterizations for the nested subclasses are also given in terms of Lévy measures. These results correspond to the case of selfdecomposable distributions mentioned in Section 1.3 and 1.4. In Chapter 3, a new class of type G selfdecomposable distributions on $\mathbb{R}^d$ is introduced and characterized in terms of stochastic integrals with respect to Lévy processes. This class is a strict subclass of the intersection of the classes of type G and selfdecomposable distributions, and in dimension one, it is strictly bigger than the class of variance mixtures of normal distributions by selfdecomposable distributions. The relationships with several other known classes of infinitely divisible distributions are established. In Chapter 4, nested subclasses of the new class introduced in Chapter 3 are studied. As in Chapter 2, analytic characterizations for them are given in terms of Lévy measures as well as probabilistic characterizations by stochastic integral representations for all classes are given. A relationship with stable distributions is shown.

Chapter 2

Characterizations of subclasses of type G distributions on $\mathbb{R}^d$.

2.1 The class $G_0(\mathbb{R}^d)$

We have mentioned type G distributions on $\mathbb{R}^d$ in Chapter 1, but we explain them more deeply here.

Summarizing the discussions in Rosinski [R91] and Maejima and Rosiński [MR01], [MR02], we use the following definition of type G distributions on $\mathbb{R}^d$.

Definition 2.1. A probability measure $\mu_0 \in I_{\mathrm{sym}}(\mathbb{R}^d)$ is said to be of type G if its Lévy measure ν_0 is given by

$$\nu_0(B) = E\left[\nu(Z^{-1}B)\right], \quad B \in \mathcal{B}_0(\mathbb{R}^d), \tag{2.1}$$

where ν is another Lévy measure on $\mathbb{R}^d$ and Z is a real valued standard normal random variable. Here $\mathcal{B}_0(\mathbb{R}^d)$ is the class of all Borel sets B in $\mathbb{R}^d$ such that $B \subset \{|x| > \varepsilon\}$ for some $\varepsilon > 0$.

Remark 2.2. ν in (2.1) is not necessarily unique. However, if we let $\bar{\nu}$ be the symmetrization of ν defined by $\bar{\nu}(B) = \frac{1}{2}(\nu(B) + \nu(-B))$, then

$$\nu_0(B) = E\left[\bar{\nu}(Z^{-1}B)\right] = E\left[\bar{\nu}(|Z|^{-1}B)\right]$$

also holds and $\bar{\nu}$ is uniquely determined, (see Maejima and Rosiński [MR02]).

Definition 2.1 is a multidimensional extension of the well-known notion of type G distributions on $\mathbb{R}$. (Another type of multidimensional extension is discussed in Barndorff-Nielsen and Pérez-Abreu [BP02].) In one dimensional case as mentioned in Chapter 1, a type G random variable X can be expressed as $X \stackrel{d}{=} V^{1/2}Z$, where $\stackrel{d}{=}$ means equality in law, V is a nonnegative infinite divisible random variable, independent of Z. Among others, some examples of $\mathbb{R}$-valued type G distributions are symmetric stable distributions, convolution of symmetric stable distributions of different stability indices, symmetric gamma distributions (a special case of which is Laplace distribution), Student t-distributions and normal inverse Gaussian distributions. The first two have multidimensional extensions.

In Maejima and Rosiński [MR01], they introduced an operator $\mathcal{K} : I_{\mathrm{sym}}(\mathbb{R}^d) \to I_{\mathrm{sym}}(\mathbb{R}^d)$, where $\mathcal{K}(\mu)$ is a symmetric infinitely divisible distribution having the same Gaussian component as μ and the Lévy measure ν_0 in (2.1), where ν is the Lévy measure of $\mu \in I_{\mathrm{sym}}(\mathbb{R}^d)$. Let $G_0(\mathbb{R}^d)$ be the class of all type G distributions on $\mathbb{R}^d$ and define, for $m \in \mathbb{N}$,

$$G_m(\mathbb{R}^d) = \{\mu_0 \in G_0(\mathbb{R}^d) : \nu \text{ in (2.1) is the Lévy measure of} \\ \text{some symmetric infinitely divisible distribution in } G_{m-1}(\mathbb{R}^d)\}.$$

Also, define $G_\infty(\mathbb{R}^d) = \cap_{m \ge 0} G_m(\mathbb{R}^d)$. The classes $G_m(\mathbb{R}^d), 1 \le m \le \infty$, were introduced in Maejima and Rosiński [MR01], and if we use the operator $\mathcal{K}$,

$$G_0(\mathbb{R}^d) = \mathcal{K}(I_{\mathrm{sym}}(\mathbb{R}^d)) \tag{2.2}$$

and $G_m(\mathbb{R}^d) = \mathcal{K}(G_{m-1}(\mathbb{R}^d))$. It was also shown in the paper that

$$I_{\mathrm{sym}}(\mathbb{R}^d) \supset G_0(\mathbb{R}^d) \supset G_1(\mathbb{R}^d) \supset \cdots \supset G_m(\mathbb{R}^d) \supset \cdots \supset G_\infty(\mathbb{R}^d) \supset S_{\mathrm{sym}}(\mathbb{R}^d),$$

where $S_{\mathrm{sym}}(\mathbb{R}^d)$ is the class of all symmetric stable distributions on $\mathbb{R}^d$, and $G_\infty(\mathbb{R}^d)$ is the largest subclass of $I_{\mathrm{sym}}(\mathbb{R}^d)$ which is invariant under the operation $\mathcal{K}$.

One of our purposes in this chapter is to give a characterization of type G distributions by stochastic integrals with respect to Lévy processes.

2.2 Characterization of $G_0(\mathbb{R}^d)$ by stochastic integrals

We start with $G_0(\mathbb{R}^d)$. The following is a known characterization of the Lévy measures of type G distributions.

Proposition 2.3. (Maejima and Rosiński [MR02].) *A probability distribution* $\mu_0 \in I_{\mathrm{sym}}(\mathbb{R}^d)$ *is of type G if and only if its Lévy measure* ν_0 *is either zero or it can be represented as*

$$\nu_0(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) g_\xi(r^2) dr, \quad B \in \mathcal{B}_0(\mathbb{R}^d),$$

where λ *is a symmetric probability measure on the unit sphere* S *in* $\mathbb{R}^d$ *and* $g_\xi(r)$ *is a jointly measurable function such that* $g_\xi = g_{-\xi}, \lambda - a.e.$ *for any fixed* $\xi \in S$, $g_\xi(\cdot)$ *is completely monotone on* $(0,\infty)$ *and satisfies*

$$\int_0^\infty (1 \wedge r^2) g_\xi(r^2) dr = c \in (0,\infty)$$

with c independent of ξ.

The following result for the integrability of stochastic integrals is due to Sato [S06], who studied more general stochastic integrals of matrix valued integrands with respect to additive processes. We state parts of Propositions 2.7 and 3.4 of Sato [S06] as a lemma below for our use.

Lemma 2.4. (Sato [S06].) *Let* $\mu \in I(\mathbb{R}^d)$ *and let* $f(t)$ *be a real-valued measurable function on* $[0,1]$. *If*

$$\int_0^1 f(t)^2 dt < \infty, \tag{2.3}$$

then $Y := \int_0^1 f(t) dX_t^{(\mu)}$ *is integrable,* $C_{\mathcal{L}(Y)}(z) = \int_0^1 C_\mu(f(t)z)dt$ *and* $\int_0^1 |C_\mu(f(t)z)|dt < \infty$. *Furthermore, if we let* (A, ν, γ) *and* (A_Y, ν_Y, γ_Y) *be the generating triplets of* μ *and* $\mathcal{L}(Y)$, *respectively, then*

$$A_Y = A \int_0^1 f(t)^2 dt, \tag{2.4}$$

$$\nu_Y(B) = \int_0^1 dt \int_{\mathbb{R}^d} 1_B(f(t)x)\nu(dx) \tag{2.5}$$

and

$$\gamma_Y = \int_0^1 f(t)\gamma + f(t) \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(t)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) dt. \tag{2.6}$$

Let

$$\phi(u) = (\sqrt{2\pi})^{-1} e^{-u^2/2} \quad \text{(throughout this thesis)}$$

and

$$h(x) = \int_x^\infty \phi(u)du, \quad x \in \mathbb{R}.$$

Define the inverse function of h by h^*, namely, $x = h^*(t)$ if and only if $h(x) = t$. The stochastic integrals we need can be shown to be integrable as follows.

Theorem 2.5. *The stochastic integral*

$$\int_0^1 h^*(t)dX_t^{(\mu)}$$

is integrable for every $\mu \in I(\mathbb{R}^d)$.

Proof of Theorem 2.5. It is enough to show that $f(t) = h^*(t)$ satisfies the conditions in Lemma 2.4 for every $\mu \in I(\mathbb{R}^d)$. Since

$$\int_0^1 h^*(t)^2 dt = \int_{-\infty}^\infty r^2\phi(r)dr = 1,$$

we have (2.3). This completes the proof. □

Definition 2.6. For any $\mu \in I(\mathbb{R}^d)$, define a mapping $\mathcal{G} : I(\mathbb{R}^d) \to I(\mathbb{R}^d)$ by

$$\mathcal{G}(\mu) = \mathcal{L}\left(\int_0^1 h^*(t)dX_t^{(\mu)}\right).$$

Proposition 2.7. (i) *For any $\mu \in I(\mathbb{R}^d)$,*

$$\int_0^1 |C_\mu(zh^*(t))|dt < \infty, \quad z \in \mathbb{R}^d, \tag{2.7}$$

and

$$C_{\mathcal{G}(\mu)}(z) = \int_0^1 C_\mu(zh^*(t))dt, \quad z \in \mathbb{R}^d. \tag{2.8}$$

(ii) *The mapping $\mathcal{G}$ is many-to-one from $I(\mathbb{R}^d)$ into $I_{\mathrm{sym}}(\mathbb{R}^d)$, and one-to-one from $I_{\mathrm{sym}}(\mathbb{R}^d)$ into $I_{\mathrm{sym}}(\mathbb{R}^d)$.*

(iii) *For any $\mu_1, \mu_2 \in I(\mathbb{R}^d)$, $\mathcal{G}(\mu_1 * \mu_2) = \mathcal{G}(\mu_1) * \mathcal{G}(\mu_2)$.*

(iv) *Let $\mu_n \in I(\mathbb{R}^d), n = 1, 2, \cdots$. If $\mu_n \to \mu$, then $\mathcal{G}(\mu_n) \to \mathcal{G}(\mu)$.*

(v) *Let (A, ν, γ) be the triplet of μ and $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ the triplet of $\widetilde{\mu} = \mathcal{G}(\mu)$. Then*

$$\widetilde{A} = A, \tag{2.9}$$

$$\widetilde{\nu}(B) = \int_0^1 dt \int_{\mathbb{R}^d} 1_B(h^*(t)x)\nu(dx) = E\left[\nu(Z^{-1}B)\right], \tag{2.10}$$

$$\widetilde{\gamma} = 0. \tag{2.11}$$

Proof. (i) (2.7) and (2.8) follow from Lemma 2.4. (ii) Since $\widehat{\mathcal{G}(\mu)}(z) = \exp\{C_{\mathcal{G}(\mu)}(z)\}$, in order to show $\mathcal{G}(\mu) \in I_{\rm sym}(\mathbb{R}^d)$, it is enough to show that $C_{\mathcal{G}(\mu)}(z)$ is symmetric in z. Actually, we have

$$\begin{aligned} C_{\mathcal{G}(\mu)}(-z) &= \int_0^1 C_\mu(-zh^*(t))dt = -\int_{-\infty}^{\infty} C_\mu(-zr)dh(r) \\ &= \int_{-\infty}^{\infty} C_\mu(-zr)\phi(r)dr = \int_{-\infty}^{\infty} C_\mu(zs)\phi(s)ds \\ &= -\int_{-\infty}^{\infty} C_\mu(zr)dh(r) = \int_0^1 C_\mu(zh^*(t))dt \\ &= C_{\mathcal{G}(\mu)}(z), \end{aligned}$$

and thus $C_{\mathcal{G}(\mu)}(z)$ is symmetric. This shows that the mapping $\mathcal{G}$ is from $I(\mathbb{R}^d)$ into $I_{\rm sym}(\mathbb{R}^d)$. The fact that $\mathcal{G}$ is one-to-one from $I_{\rm sym}(\mathbb{R}^d)$ into $I_{\rm sym}(\mathbb{R}^d)$ can be shown by Remark 2.2. (iii) and (iv) can be proved by the same idea of Proposition 2.7 (iii) and (iv) of Barndorff-Nielsen et al. ([BMS06]). We show here how to prove them precisely. (iii) is obvious from $\mathcal{L}(X_t^{(\mu_1 * \mu_2)}) = \mathcal{L}(X_t^{(\mu_1)} + X_t^{(\mu_2)})$, where $\{X_t^{(\mu_1)}\}$ and $\{X_t^{(\mu_2)}\}$ are independent. Next we prove (v) before (iv). (v) follows from (2.4)-(2.6) if we notice that $\int_0^1 h^*(t)dt = 0$ and $\int_0^1 h^*(t)^2 dt = 1$. To prove (iv), assume that $\mu_n = \mu_{(A_n,\nu_n,\gamma_n)} \to \mu = \mu_{(A,\nu,\gamma)}$ as $n \to \infty$. Then $C_{\mu_n}(z) \to C_\mu(z)$, and ${\rm tr}A_n$, $\int(|x|^2 \wedge 1)\nu_n(dx)$ and $|\gamma_n|$ are bounded. Since $\mathcal{G}_{\mu_n}$ and $\mathcal{G}_\mu$ have cumulant functions expressed as in (2.8) and since we have already proved (v), we can use the dominated convergence theorem to get $C_{\mathcal{G}(\mu_n)}(z) \to C_{\mathcal{G}(\mu)}(z)$, that is, $\mathcal{G}(\mu_n) \to \mathcal{G}(\mu)$.

Conversely, assume that $\widetilde{\mu}_n = \mathcal{G}(\mu_n) \to \widetilde{\mu}$. Let $(\widetilde{A}_n, \widetilde{\nu}_n, \widetilde{\gamma}_n)$ and (A_n, ν_n, γ_n) be the triplets of $\widetilde{\mu}_n$ and μ_n. We claim that $\{\mu_n\}$ is precompact. The following conditions are necessary and sufficient for precompactness of $\{\mu_n\}$:

$$\sup_n {\rm tr}A_n < \infty, \tag{2.12}$$

$$\sup_n \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu_n(dx) < \infty, \tag{2.13}$$

$$\lim_{l\to\infty} \sup_n \int_{|x|>l} \nu_n(dx) = 0, \tag{2.14}$$

$$\sup_n |\gamma_n| < \infty. \tag{2.15}$$

Since $\{\widetilde{\mu}_n\}$ is precompact, these four facts (2.12)–(2.15) for $\underbrace{(A_n, \nu_n, \gamma_n)}$ also valid with the replacement $\underbrace{(\widetilde{A}_n, \widetilde{\nu}_n, \widetilde{\gamma}_n)}$. We number those facts by $\widetilde{(2.12)}$–$\widetilde{(2.15)}$. Then (2.12) follows from (2.9) and $\widetilde{(2.12)}$; (2.13) follows from $\widetilde{(2.13)}$ since, by (2.10),

$$\begin{aligned}\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\widetilde{\nu}_n(dx) &= \int_0^\infty \phi(s)ds \int_{\mathbb{R}^d}(|sx|^2 \wedge 1)\nu_n(dx)\\ &= \int_{\mathbb{R}^d}|x|^2\nu_n(dx)\int_0^{1/|x|} s^2\phi(s)ds + \int_{\mathbb{R}^d}\nu_n(dx)\int_{1/|x|}^\infty \phi(s)ds\\ &\le \int_{|x|\le 1}|x|^2\nu_n(dx)\int_0^1 s^2\phi(s)ds + \int_{|x|>1}\nu_n(dx)\int_1^\infty \phi(s)ds;\end{aligned}$$

(2.14) is obtained from $\widetilde{(2.14)}$, because

$$\int_{|x|>l}\widetilde{\nu}_n(dx) = \int_0^\infty \phi(s)ds\int_{|x|>l/s}\nu_n(dx) \ge \int_1^\infty \phi(s)ds \int_{|x|>l}\nu_n(dx).$$

To see (2.15), use (2.6) and (2.11). This finishes the proof of precompactness of $\{\mu_n\}$. Now we can choose a convergent sequence of $\{\mu_{n'}\}$ of $\{\mu_n\}$. Thus there is $\mu \in I(\mathbb{R}^d)$ such that $\mu_{n'} \to \mu$. Hence $\mathcal{G}(\mu_{n'}) \to \mathcal{G}(\mu)$ and $\mathcal{G}(\mu) = \widetilde{\mu}$. It follows from (i) that μ does not depend on the choice of subsequence. Hence $\mu_n \to \mu$. □

The following theorem shows that each type G distribution admits the stochastic integral representation defined in Definition 2.1.

Theorem 2.8.

$$G_0(\mathbb{R}^d) = \mathcal{G}(I(\mathbb{R}^d)).$$

Proof. Let $\mu \in I(\mathbb{R}^d)$ and $\widetilde{\mu} = \mathcal{G}(\mu)$. Then by Proposition 2.7 (v), we have (2.1), and thus $\widetilde{\mu} \in G_0(\mathbb{R}^d)$, concluding $\mathcal{G}(I(\mathbb{R}^d)) \subset G_0(\mathbb{R}^d)$.

Conversely, suppose that $\widetilde{\mu} \in G_0(\mathbb{R}^d)$. Then by Definition 2.1 and Proposition 2.7 (v) again, we see that $\widetilde{\mu} = \mathcal{L}\left(\int_0^1 h^*(t)dX_t^{(\mu)}\right)$ for some $\mu \in I(\mathbb{R}^d)$. This means that $\widetilde{\mu} \in \mathcal{G}(I(\mathbb{R}^d))$ and $G_0(\mathbb{R}^d) \subset \mathcal{G}(I(\mathbb{R}^d))$, completing the proof. □

Corollary 2.9. *Let H be a subclass of $I(\mathbb{R}^d)$ and let*

$G_H(\mathbb{R}^d) = \{\mu_0 \in I_{\mathrm{sym}}(\mathbb{R}^d)\ :\ \nu_{\mu_0}(B) = E[\nu(Z^{-1}B)],\ B \in \mathcal{B}_0(\mathbb{R}^d),$ *for some* $\mu_\nu \in H\}$,

where ν_μ s the Lévy measure of $\mu \in I(\mathbb{R}^d)$ and μ_ν is the infinitely divisible distribution with Lévy measure ν. Then we have

$$G_H(\mathbb{R}^d) = \mathcal{G}(H).$$

Remark 2.10. If $H = I(\mathbb{R}^d)$, then the corollary above is nothing but Theorem 2.8. The proof of the corollary can be carried out in the same way as for Theorem 2.8. Also, we see from the discussions above that as mappings from $I_{\mathrm{sym}}(\mathbb{R}^d)$ into $I_{\mathrm{sym}}(\mathbb{R}^d)$, two mappings $\mathcal{K}$ and $\mathcal{G}$ are the same.

2.3 Lévy measures of distributions in $G_m(\mathbb{R}^d)$, $m \in \mathbb{N}$

In this section, we characterize Lévy measures of distributions in $G_m(\mathbb{R}^d), m \in \mathbb{N}$. Write $\phi_0(x) = \phi(x), h_0(x) = h(x)$ and $h_0^*(t) = h^*(t)$.

For $m \in \mathbb{N}$, let $\phi_m(x)$ be the probability density function of the product of $(m+1)$ independent standard normal random variables. Then we have the following.

Lemma 2.11. *For each $m \in \mathbb{N}$,*
(i)

$$\phi_m(x) = \phi_m(-x),$$

(ii)

$$\int_{-\infty}^{\infty} \phi_m(x)dx = 1,$$

(iii)

$$\int_{-\infty}^{\infty} |x|\phi_m(x)dx < \infty \quad \textit{and} \int_{-\infty}^{\infty} x\phi_m(x)dx = 0,$$

(iv)

$$\int_{-\infty}^{\infty} x^2\phi_m(x)dx = 1,$$

(v)

$$\phi_m(x) = \int_{-\infty}^{\infty} \phi_0(u)\phi_{m-1}(x|u|^{-1})|u|^{-1}du. \tag{2.16}$$

Proof. (i)-(iv) are trivial. As to (v), for $B \in \mathcal{B}(\mathbb{R})$, we have

$$P\left(\prod_{i=1}^{m+1} Z_i \in B\right) = \int_{-\infty}^{\infty} 1_B(x)\phi_m(x)dx.$$

On the other hand, we have

$$P\left(\prod_{i=1}^{m+1} Z_i \in B\right) = P\left(|Z_1|\prod_{i=2}^{m+1} Z_i \in B\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1_B(|u|y)\phi_0(u)\phi_{m-1}(y)dudy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1_B(x)\phi_0(u)\phi_{m-1}(x|u|^{-1})|u|^{-1}dudx.$$

This completes the proof of (v). □

For $m \in \mathbb{N}$, let

$$h_m(x) = \int_x^{\infty} \phi_m(u)du, \quad x \in \mathbb{R}$$

and define its inverse $x = h_m^*(t)$ by $t = h_m(x)$. We note that for each $m \in \mathbb{N} \cup \{0\}$,

$$h_m(+\infty) = 0, \quad h_m(-\infty) = 1,$$
$$\int_0^1 h_m^*(t)dt = 0 \quad \text{and} \quad \int_0^1 h_m^*(t)^2 dt = 1,$$

where the last two integrals are given by Lemma 2.11 (iii) and (iv).

Theorem 2.12. *For each $m \in \mathbb{N}$, let $\mu_m \in I_{\mathrm{sym}}(\mathbb{R}^d)$ and denote its Lévy measure by ν_m. Then $\mu_m \in G_m(\mathbb{R}^d)$ if and only if*

$$\nu_m(B) = \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_{m-1}(u)du, \tag{2.17}$$

where ν_0 is the Lévy measure of some $\mu_0 \in G_0(\mathbb{R}^d)$.

Proof. ("Only if" part.) Let $m = 1$. Then, by the definition

$$\nu_1(B) = E\left[\nu_0(Z^{-1}B)\right] = \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_0(u)du$$

for some Lévy measure ν_0 whose distribution is in G_0. Suppose the statement ("only if" part) is true for some $m \in \mathbb{N}$. The Lévy measure ν_{m+1} of $\mu_{m+1} \in G_{m+1}(\mathbb{R}^d)$ is given by

$$\nu_{m+1}(B) = E\left[\nu_m(Z^{-1}B)\right]$$

for some Lévy measure ν_m of a distribution $\mu_m \in G_m(\mathbb{R}^d)$. Then by the induction hypothesis

$$\nu_{m+1}(B) = \int_{-\infty}^{\infty} \phi_0(u)\nu_m(u^{-1}B)du$$
$$= \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} \nu_0(u^{-1}v^{-1}B)\phi_{m-1}(v)dv$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} \nu_0(y^{-1}B)\phi_{m-1}(y|u|^{-1})|u|^{-1}dy \\
&= \int_{-\infty}^{\infty} \nu_0(y^{-1}B)\phi_m(y)dy
\end{aligned}
$$

by (2.16).

("If" part.) Let $m = 1$. Then, by the definition, if a Lévy measure ν_1 is represented as

$$
\nu_1(B) = \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_0(u)du
$$

for some ν_0, the Lévy measure of some $\mu_0 \in G_0(\mathbb{R}^d)$, then $\mu_1 \in G_1(\mathbb{R}^d)$. Suppose that the statement ("if" part) is true for some $m \in \mathbb{N}$. By the same calculation as above,

$$
\begin{aligned}
\int_{-\infty}^{\infty} \nu_0(y^{-1}B)\phi_m(y)dy &= \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} \nu_0(y^{-1}B)\phi_{m-1}(y|u|^{-1})|u|^{-1}dy \\
&= \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} \nu_0(u^{-1}v^{-1}B)\phi_{m-1}(v)dv \\
&= \int_{-\infty}^{\infty} \phi_0(u)\nu_m(u^{-1}B)du \\
&= \nu_{m+1}(B).
\end{aligned}
$$

We have

$$
\begin{aligned}
\nu_{m+1}(B) &= \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_m(u)du \\
&= \int_{-\infty}^{\infty} \phi_0(u)\nu_m(u^{-1}B)du \\
&= E[\nu_m(Z^{-1}B)]
\end{aligned}
$$

for some Lévy measure ν_m having the representation (2.17). Then, by the induction hypothesis, μ_m with the Lévy measure ν_m belong to $G_m(\mathbb{R}^d)$. Thus, $\mu_{m+1} \in G_{m+1}(\mathbb{R}^d)$. This completes the proof. □

The following is a G_m–version of Proposition 2.3, and it characterizes Lévy measures of distributions in $G_m(\mathbb{R}^d)$.

Theorem 2.13. *Let $m \in \mathbb{N}$. A $\mu_m \in I_{\mathrm{sym}}(\mathbb{R}^d)$ belongs to $G_m(\mathbb{R}^d)$ if and only if its Lévy measure ν_m is either zero or it can be represented as*

$$
\nu_m(B) = \int_S \lambda(d\xi) \int_0^{\infty} 1_B(r\xi)g_{m,\xi}(r^2)dr, \quad B \in \mathcal{B}_0(\mathbb{R}^d),
$$

where λ is a symmetric measure on the unit sphere S on $\mathbb{R}^d$ and $g_{m,\xi}(r)$ is represented as

$$g_{m,\xi}(s) = \int_{-\infty}^{\infty} \phi_{m-1}(\sqrt{s}|r|^{-1})|r|^{-1} g_\xi(r^2)dr,$$

for some function g_ξ on $(0, \infty)$ which has the same properties as in Proposition 2.3.

Proof. We see by Theorem 2.12 and Proposition 2.3, $\mu_m \in G_m(\mathbb{R}^d)$ if and only if ν_m is represented as

$$\begin{aligned}\nu_m(B) &= \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_{m-1}(u)du \\ &= \int_{-\infty}^{\infty} \phi_{m-1}(u)du \int_S \lambda(d\xi) \int_0^{\infty} 1_{u^{-1}B}(r\xi) g_\xi(r^2)dr.\end{aligned}$$

If we use here the facts that $\lambda(d\xi) = \lambda(d(-\xi)), g_\xi = g_{-\xi}$ and $\phi_{m-1}(u) = \phi_{m-1}(-u)$, then we have

$$\begin{aligned}\nu_m(B) &= \int_{-\infty}^{\infty} \phi_{m-1}(y|r|^{-1})|r|^{-1}dy \int_S \lambda(d\xi) \int_0^{\infty} 1_B(y\xi) g_\xi(r^2)dr \\ &= \int_S \lambda(d\xi) \int_{-\infty}^{\infty} 1_B(y\xi) g_{m,\xi}(y^2)dy\end{aligned}$$

where

$$g_{m,\xi}(s) = \int_{-\infty}^{\infty} \phi_{m-1}(\sqrt{s}|r|^{-1})|r|^{-1} g_\xi(r^2)dr.$$

This completes the proof. □

2.4 Characterizations of $G_m(\mathbb{R}^d)$, $m \in \mathbb{N}$, by stochastic integrals

In this section, we characterize distributions in $G_m(\mathbb{R}^d)$ by stochastic integral representations.

Theorem 2.14. *For each $m \in \mathbb{N}$, the stochastic integral*

$$Y_m := \int_0^1 h_m^*(t) dX_t^{(\mu)}$$

is integrable for every $\mu \in I(\mathbb{R}^d)$,

$$\int_0^1 |C_\mu(h_m^*(t)z)|dt < \infty$$

and

$$C_{\mathcal{L}(Y_m)}(z) = \int_0^1 C_\mu(h_m^*(t)z)dt.$$

Proof. Since

$$\int_0^1 h_m^*(t)^2 dt = \int_{-\infty}^{\infty} |x|^2 \phi_m(x)dx < \infty,$$

we have the assertion by Lemma 2.4. □

Let $\mathcal{G}_1 = \mathcal{G}^1 = \mathcal{G}$.

Definition 2.15. Let $m \in \mathbb{N}$. Define a mapping $\mathcal{G}_{m+1}$ by

$$\mathcal{G}_{m+1}(\mu) = \mathcal{L}\left(\int_0^1 h_m^*(t)dX_t^{(\mu)}\right), \quad \mu \in I(\mathbb{R}^d)$$

and

$$\mathcal{G}^{m+1}(\mu) = \mathcal{G}(\mathcal{G}^m((\mu)), \quad \mu \in I(\mathbb{R}^d).$$

Proposition 2.16. *For $m \in \mathbb{N}$,*

$$G_m(\mathbb{R}^d) = \mathcal{G}(G_{m-1}(\mathbb{R}^d)).$$

Proof. The proof is almost the same as that of Theorem 2.8. Let $\mu_{m-1} \in G_{m-1}(\mathbb{R}^d)$ and $\mu_m = \mathcal{G}(\mu_{m-1})$. Also let ν_{m-1} and ν_m be the Lévy measures of μ_{m-1} and μ_m, respectively. Then by Proposition 2.7 (v), we have $\nu_m(B) = E[\nu_{m-1}(Z^{-1}B)]$. Thus $\mu_m \in G_m(\mathbb{R}^d)$, and $\mathcal{G}(G_{m-1}(\mathbb{R}^d)) \subset G_m(\mathbb{R}^d)$.

Conversely, suppose that $\mu_m \in G_m(\mathbb{R}^d)$. Then by the definition of $G_m(\mathbb{R}^d)$ and Proposition 2.7 (v) again, we see that $\mu_m = \mathcal{L}\left(\int_0^1 h^*(t)dX_t^{(\mu)}\right)$ for some $\mu \in G_{m-1}(\mathbb{R}^d)$. This means that $\mu_m \in \mathcal{G}(G_{m-1}(\mathbb{R}^d))$, and $G_m(\mathbb{R}^d) \subset \mathcal{G}(G_{m-1}(\mathbb{R}^d))$, completing the proof. □

Corollary 2.17. *For $m \in \mathbb{N}$,*

$$G_m(\mathbb{R}^d) = \mathcal{G}^{m+1}(I(\mathbb{R}^d)).$$

We next show

Theorem 2.18. *For $m \in \mathbb{N}$*

$$\mathcal{G}_{m+1}(I(\mathbb{R}^d)) = \mathcal{G}^{m+1}(I(\mathbb{R}^d)).$$

Proof. We note that

$$\widetilde{\mu} \in \mathcal{G}_{m+1}(I(\mathbb{R}^d)) \text{ if and only if } \widetilde{\mu} = \mathcal{L}\left(\int_0^1 h_m^*(t) dX_t^{(\mu)}\right), \quad \mu \in I(\mathbb{R}^d)$$

and that

$$\widetilde{\mu} \in \mathcal{G}^{m+1}(I(\mathbb{R}^d)) \text{ if and only if } \widetilde{\mu} = \mathcal{L}\left(\int_0^1 h_0^*(t) dX_t^{(\mu)}\right), \quad \mu \in \mathcal{G}^m(I(\mathbb{R}^d)).$$

We next claim that

$$\int_{-\infty}^{\infty} \phi_0(u) du \int_{-\infty}^{\infty} |C_\mu(uvz)| \phi_{m-1}(v) dv < \infty, \ z \in \mathbb{R}^d. \tag{2.18}$$

If it would be proved, we could exchange the order of the integrals in the calculation of cumulants below.

The proof of (2.18) is as follows. The idea is from Barndorff–Nielsen et al. [BMS06]. If the generating triplet of μ is (A, ν, γ), then

$$|C_\mu(z)| \le 2^{-1}(\mathrm{tr}A)|z|^2 + |\gamma||z| + \int_{\mathbb{R}^d} |g(z,x)| \nu(dx),$$

where

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle(1+|x|^2)^{-1}.$$

Hence

$$\begin{aligned}|C_\mu(uvz)| &\le 2^{-1}(\mathrm{tr}A)u^2v^2|z|^2 + |\gamma||u||v||z| + \int_{\mathbb{R}^d} |g(z,uvx)|\nu(dx) \\ &\quad + \int_{\mathbb{R}^d} |g(uvz,x) - g(z,uvx)|\nu(dx) =: I_1 + I_2 + I_3 + I_4,\end{aligned}$$

say. The finiteness of $\int_{-\infty}^{\infty} \phi_0(u) du \int_{-\infty}^{\infty} (I_1 + I_2)\phi_{m-1}(v) dv$ follows from Lemma 2.11. Noting that $|g(z,x)| \le c_z |x|^2(1+|x|^2)^{-1}$ with a positive constant c_z depending on z, we have

$$\begin{aligned}&\int_{-\infty}^{\infty} \phi_0(u) du \int_{-\infty}^{\infty} I_3 \phi_{m-1}(v) dv \\ &\quad \le c_z \int_{\mathbb{R}^d} \nu(dx) \int_{-\infty}^{\infty} \phi_0(u) du \int_{-\infty}^{\infty} \frac{(uv|x|)^2}{1+(uv|x|)^2} \phi_{m-1}(v) dv \\ &\quad = c_z \left(\int_{|x|\le 1} \nu(dx) + \int_{|x|>1} \nu(dx)\right) \int_{-\infty}^{\infty} \phi_0(u) du \int_{-\infty}^{\infty} \frac{(uv|x|)^2}{1+(uv|x|)^2} \phi_{m-1}(v) dv \\ &\quad =: I_{31} + I_{32},\end{aligned}$$

say, and

$$I_{31} \le c_z \int_{|x|\le 1} |x|^2 \nu(dx) \int_{-\infty}^{\infty} u^2 \phi_0(u)du \int_{-\infty}^{\infty} v^2 \phi_{m-1}(v)dv < \infty,$$
$$I_{32} \le c_z \int_{|x|>1} \nu(dx) \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} \phi_{m-1}(v)dv < \infty.$$

As to I_4, note that for $a \in \mathbb{R}$,

$$\begin{aligned} |g(az,x) - g(z,ax)| &= \frac{|\langle az, x\rangle||x|^2|1-a^2|}{(1+|x|^2)(1+|ax|^2)} \\ &\le \frac{|z||x|^3(|a|+|a|^3)}{(1+|x|^2)(1+|ax|^2)} \\ &\le \frac{|z||x|^2(1+|a|^2)}{2(1+|x|^2)}, \end{aligned}$$

since $|b|(1+b^2)^{-1} \le 2^{-1}$. Then

$$\begin{aligned} &\int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} I_4 \phi_{m-1}(v)dv \\ &\quad \le |z| \int_{\mathbb{R}^d} \frac{|x|^2}{1+|x|^2} \nu(dx) \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} (1+u^2v^2)\phi_{m-1}(v)dv < \infty. \end{aligned}$$

This completes the proof of (2.18).

If we calculate the necessary cumulants, we have

$$\begin{aligned} C_{\mathcal{G}_{m+1}(\mu)}(z) &= \int_0^1 C_\mu(h_m^*(t)z)dt \\ &= -\int_{-\infty}^{\infty} C_\mu(uz)dh_m(u) \\ &= \int_{-\infty}^{\infty} C_\mu(uz)\phi_m(u)du \end{aligned}$$

and

$$\begin{aligned} C_{\mathcal{G}^{m+1}(\mu)}(z) &= \int_0^1 C_{\mathcal{G}^m(\mu)}(h_0^*(t)z)dt \\ &= \int_0^1 dt \int_0^1 C_\mu(h_0^*(t)h_{m-1}^*(s)z)ds \\ &= \int_{-\infty}^{\infty} dh_0(u) \int_{-\infty}^{\infty} C_\mu(uvz)dh_{m-1}(v) \\ &= \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} C_\mu(uvz)\phi_{m-1}(v)dv \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} C_\mu(yz)dy \int_{-\infty}^{\infty} \phi_0(u)\phi_{m-1}(y|u|^{-1})|u|^{-1}du \\
&= \int_{-\infty}^{\infty} C_\mu(yz)\phi_m(y)dy \\
&= C_{\mathcal{G}^{m+1}(\mu)}(z).
\end{aligned}
$$

This completes the proof of Theorem 2.18. □

The following is a goal of this section and a G_m–version of Theorem 2.8. Namely, any $\mu \in G_m(\mathbb{R}^d)$ has the stochastic integral representation defined in Definition 2.15.

Theorem 2.19.

$$G_m(\mathbb{R}^d) = \mathcal{G}_{m+1}(I(\mathbb{R}^d)).$$

Proof. The statement is an immediate consequence of Corollary 2.17 and Theorem 2.18. □

2.5 The class $G_\infty(\mathbb{R}^d)$

We conclude this chapter with two statements for $G_\infty(\mathbb{R}^d)$.

Proposition 2.20.

$$\mathcal{G}(G_\infty(\mathbb{R}^d)) = G_\infty(\mathbb{R}^d).$$

Proposition 2.21. $S_{\mathrm{sym}}(\mathbb{R}^d)$ *is invariant under* $\mathcal{G}$*–mapping and* $G_\infty(\mathbb{R}^d)$ *is the largest class which is invariant under* $\mathcal{G}$*–mapping.*

These two propositions are given in Theorem 2.3 of Maejima and Rosiński [MR01] in terms of operator $\mathcal{K}$. Since we have Remark 2.10 in Section 2.2, we get them above.

Chapter 3

A subclass of type G selfdecomposable distributions on $\mathbb{R}^d$.

3.1 The class $M(\mathbb{R}^d)$

Recall five classes of infinitely divisible distributions introduced in Chapter 1.

(i) The class $U(\mathbb{R}^d)$:

$\nu_\xi(dr) = l_\xi(r)dr$ and $l_\xi(r)$ is nonincreasing.

(ii) The class $B(\mathbb{R}^d)$:

$\nu_\xi(dr) = l_\xi(r)dr$ and $l_\xi(r)$ is completely monotone.

(iii) The class $L(\mathbb{R}^d)$:

$\nu_\xi(dr) = k_\xi(r)r^{-1}dr$ and $k_\xi(r)$ is nonincreasing.

(iv) The class $T(\mathbb{R}^d)$:

$\nu_\xi(dr) = k_\xi(r)r^{-1}dr$ and $k_\xi(r)$ is completely monotone.

(v) The class $G(\mathbb{R}^d)$:

$\nu_\xi(dr) = g_\xi(r^2)dr$ and $g_\xi(r)$ is completely monotone; in this case we also assume that μ is symmetric.

Being motivated by the relations among classes (i)–(v), it is natural to introduce and consider the following new class.

Definition 3.1 (*The class* $M(\mathbb{R}^d)$)**.** $\mu \in M(\mathbb{R}^d)$ if and only if $\mu \in I_{\mathrm{sym}}(\mathbb{R}^d)$ with

$$\nu_\xi(dr) = g_\xi(r^2)r^{-1}dr \text{ and } g_\xi(r) \text{ is completely monotone.} \tag{3.1}$$

It is easy to see that $M(\mathbb{R}^d) \subset L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$, i.e., the elements of $M(\mathbb{R}^d)$ are type G selfdecomposable distributions. For, if we put $k_\xi(x) = g_\xi(x^2)$, then we see that $M(\mathbb{R}^d) \subset L(\mathbb{R}^d)$. Note that $f(x) = x^{-\alpha}$, $\alpha > 0$, is completely monotone and by Lemma 1.9, $h_\xi(x) = g_\xi(x)x^{-1/2}$ is also completely monotone. Hence we see that $M(\mathbb{R}^d) \subset G(\mathbb{R}^d)$. In Theorem 3.6 below we will prove that this inclusion is strict. The purpose of this chapter is to characterize the class $M(\mathbb{R}^d)$ by stochastic integrals with respect to Lévy processes, and compare it with other known classes.

3.2 Characterization of the class $M(\mathbb{R}^d)$ by stochastic integrals

Let $m(x) = \int_x^\infty \phi(s)s^{-1}ds, x > 0$, and denote its inverse by $m^*(t)$, that is, $t = m(x)$ if and only if $x = m^*(t)$.

Theorem 3.2. *Let $\mu \in I(\mathbb{R}^d)$. Then the stochastic integral*

$$\int_0^\infty m^*(t)dX_t^{(\mu)}$$

exists if and only if $\mu \in I_{\log}(\mathbb{R}^d)$.

Proof. ("If" part.) For the proof, we need the following lemma, which is a special case of Proposition 5.5 of [S06].

Lemma 3.3. *Let $\mu \in I(\mathbb{R}^d)$ and $f(t)$ a real-valued measurable function on $[0,\infty)$. Let (A, ν, γ) be the triplet of μ. Then $Y := \int_0^\infty f(t)dX_t^{(\mu)}$ is integrable, if the following conditions are satisfied:*

$$\int_0^\infty f(t)^2 dt < \infty, \tag{3.2}$$

$$\int_0^\infty dt \int_{\mathbb{R}^d} (|f(t)x|^2 \wedge 1)\nu(dx) < \infty, \tag{3.3}$$

$$\int_0^\infty \left| f(t)\gamma + f(t) \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(t)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| dt < \infty. \tag{3.4}$$

Furthermore, $C_{\mathcal{L}(Y)}(z) = \int_0^1 C_\mu(f(t)z)dt$, $\int_0^1 |C_\mu(f(t)z)|dt < \infty$ and if we let (A, ν, γ) and (A_Y, ν_Y, γ_Y) be the generating triplets of μ and $\mathcal{L}(Y)$, respectively, then

$$A_Y = A \int_0^\infty f(t)^2 dt, \tag{3.5}$$

$$\nu_Y(B) = \int_0^\infty dt \int_{\mathbb{R}^d} 1_B(f(t)x)\nu(dx) \tag{3.6}$$

and

$$\gamma_Y = \int_0^\infty f(t)\gamma + f(t)\int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(t)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)dt. \tag{3.7}$$

For the proof of ("if" part), it is enough to show that $f(t) = m^*(t)$ satisfies (3.2) $-$ (3.4) in Lemma 3.3 for every $\mu \in I_{\log}(\mathbb{R}^d)$. Note that $m(+0) = \infty$ and $m(\infty) = 0$. Since

$$\int_0^\infty m^*(t)^2 dt = \int_0^\infty s\phi(s)ds < \infty,$$

we have (3.2).

As to (3.3), we have

$$\begin{aligned}
&\int_0^\infty dt \int_{\mathbb{R}^d} (|m^*(t)x|^2 \wedge 1)\nu(dx) \\
&\quad = -\int_0^\infty dm(s) \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu(dx) \\
&\quad = \int_0^\infty \phi(s)s^{-1}ds \left(\int_{|x|\le 1/s} |sx|^2\nu(dx) + \int_{|x|>1/s} \nu(dx)\right) \\
&\quad =: (J_1 + J_2),
\end{aligned}$$

say. Here

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}^d} |x|^2\nu(dx) \int_0^{1/|x|} s\phi(s)ds \\
&= \left(\int_{|x|\le 1} + \int_{|x|>1}\right) |x|^2\nu(dx) \int_0^{1/|x|} s\phi(s)ds \\
&=: J_{11} + J_{12},
\end{aligned}$$

say, and

$$\begin{aligned}
J_{11} &\le \int_{|x|\le 1} |x|^2\nu(dx) \int_0^\infty s\phi(s)ds < \infty, \\
I_{12} &\le \int_{|x|>1} |x|^2\nu(dx) \int_0^{1/|x|} sds \le 2^{-1} \int_{|x|>1} \nu(dx) < \infty.
\end{aligned}$$

Also,

$$J_2 = \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty \phi(s)s^{-1}ds$$

$$
\begin{aligned}
&= \left(\int_{|x|\le 1} + \int_{|x|>1}\right) \nu(dx) \int_{1/|x|}^{\infty} \phi(s)s^{-1}ds \\
&=: J_{21} + J_{22},
\end{aligned}
$$

say, and

$$
\begin{aligned}
J_{21} &\le C_1 \int_{|x|\le 1} x^2 \nu(dx) < \infty, \\
J_{22} &\le \int_{|x|>1} \nu(dx) \left\{ \int_{1/|x|}^{1} s^{-1}ds + \int_{1}^{\infty} \phi(s)s^{-1}ds \right\} \\
&= \int_{|x|>1} (\log|x| + C_2)\nu(dx) < \infty,
\end{aligned}
$$

since $\mu \in I_{\log}(\mathbb{R}^d)$, where $C_1, C_2 > 0$. This shows (3.3).

For (3.4), we have

$$
\begin{aligned}
&\int_0^\infty \left| m^*(t)\gamma + m^*(t) \int_{\mathbb{R}^d} x \left(\frac{1}{1+|m^*(t)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| dt \\
&\quad \le -|\gamma| \int_0^\infty s dm(s) - \int_0^\infty \left| s \int_{\mathbb{R}^d} x \left(\frac{1}{1+|sx|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| dm(s) \\
&\quad =: J_3 + J_4,
\end{aligned}
$$

say, where

$$
\begin{aligned}
J_3 &\le |\gamma| \int_0^\infty \phi(s)ds < \infty, \\
J_4 &\le \int_0^\infty \phi(s)ds \left| \int_{\mathbb{R}^d} \left(\frac{x|x|^2|s^2-1|}{(1+|sx|^2)(1+|x|^2)} \right) \nu(dx) \right| \\
&\le \int_0^\infty |s^2-1|\phi(s)ds \int_{\mathbb{R}^d} \frac{|x|^3}{(1+|sx|^2)(1+|x|^2)} \nu(dx) \\
&= \int_0^\infty |s^2-1|\phi(s)ds \left(\int_{|x|\le 1} + \int_{|x|>1} \right) \frac{|x|^3}{(1+|sx|^2)(1+|x|^2)} \nu(dx) \\
&=: J_{41} + J_{42},
\end{aligned}
$$

say. Here

$$
J_{41} \le \int_0^\infty |s^2-1|\phi(s)ds \int_{|x|\le 1} \frac{|x|^3}{1+|x|^2} \nu(dx) < \infty,
$$

and

$$
J_{42} \le \int_{|x|>1} \frac{|x|^3}{1+|x|^2} \nu(dx) \int_0^\infty \frac{s^2+1}{1+|sx|^2} \phi(s)ds
$$

$$
\begin{aligned}
&= \int_{|x|>1} \frac{|x|^3}{1+|x|^2}\nu(dx) \left(\int_0^1 + \int_1^\infty\right) \frac{s^2+1}{1+|sx|^2}\phi(s)ds \\
&=: J_{421} + J_{422},
\end{aligned}
$$

say. Furthermore,

$$
\begin{aligned}
J_{421} &\le \int_{|x|>1} \frac{|x|^3}{1+|x|^2}\nu(dx) \int_0^1 \frac{1}{1+|sx|^2}ds \\
&\le \int_{|x|>1} \frac{|x|^2}{1+|x|^2}\nu(dx) \int_0^\infty \frac{1}{1+t^2}dt < \infty,
\end{aligned}
$$

and

$$
J_{422} \le \int_{|x|>1} \frac{|x|^3}{(1+|x|^2)^2}\nu(dx) \int_1^\infty (s^2+1)\phi(s)ds < \infty.
$$

Thus we have (3.4). This completes the proof of ("if" part). □

Proof. ("Only if" part.) Suppose $\int_0^\infty m^*(t)dX_t^{(\mu)}$ exists and let $\widetilde{\nu}$ be its Lévy measure. We have

$$
\begin{aligned}
\int_{|x|>1} \widetilde{\nu}(dx) &= \int_0^\infty dt \int 1_{\{|m^*(t)x|>1\}}(x)\nu(dx) \\
&= -\int_0^\infty dm(s) \int 1_{\{|x|>1/s\}}(x)\nu(dx) \\
&= -\int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty dm(s) \\
&\ge \int_{|x|>1} \nu(dx) \int_{1/|x|}^\infty \phi(s)s^{-1}ds \\
&\ge \int_{|x|>1} \nu(dx)(C_1 \log|x| + C_2),
\end{aligned}
$$

for some $C_1, C_2 > 0$. Thus, $\mu \in I_{\log}(\mathbb{R}^d)$. This competes the proof of ("only if" part). □

Definition 3.4. For any $\mu \in I_{\log}(\mathbb{R}^d)$, define the mapping $\mathcal{M}$ by

$$
\mathcal{M}(\mu) = \mathcal{L}\left(\int_0^\infty m^*(t)dX_t^{(\mu)}\right).
$$

The statement (i) below is one of the main results in this chapter.

Theorem 3.5. *(i)*

$$M(\mathbb{R}^d) = \mathcal{M}(I_{\log}(\mathbb{R}^d)) \cap I_{\rm sym}(\mathbb{R}^d).$$

(ii) Let ν and $\widetilde{\nu}$ be the Lévy measures of $\mu \in I_{\log}(\mathbb{R}^d)$ and $\mathcal{M}(\mu)$, respectively. Then

$$\widetilde{\nu}(B) = \int_0^\infty \nu(s^{-1}B)\phi(s)s^{-1}ds, \; B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{3.8}$$

Proof. We first prove (ii). By (3.6), we have

$$\begin{aligned}\widetilde{\nu}(B) &= \int_0^\infty dt \int_{\mathbb{R}^d} 1_B(xm^*(t))\nu(dx)\\ &= -\int_0^\infty dm(s) \int_{\mathbb{R}^d} 1_B(xs)\nu(dx)\\ &= \int_0^\infty \nu(s^{-1}B)\phi(s)s^{-1}ds.\end{aligned}$$

Now we consider part (i). Let $\mu \in I_{\log}(\mathbb{R}^d)$ and $\widetilde{\mu} = \mathcal{M}(\mu)$. Let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\mu}$, respectively. Then (ii) holds. Thus, if $\nu = 0$, then $\widetilde{\nu} = 0$ and $\widetilde{\mu} \in M(\mathbb{R}^d)$. Assume that $\nu \neq 0$ and ν has a polar decomposition (λ, ν_ξ). Then, for any nonnegative measurable function f,

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\widetilde{\nu}(dx) &= \int_0^\infty \phi(s)s^{-1}ds \int_{\mathbb{R}^d} f(sx)\nu(dx)\\ &= \int_0^\infty \phi(s)s^{-1}ds \int_S \lambda(d\xi) \int_0^\infty f(sr\xi)\nu_\xi(dr)\\ &= \int_S \lambda(d\xi) \int_0^\infty \nu_\xi(dr) \int_0^\infty \phi(s/r)f(s\xi)s^{-1}ds\\ &= \int_S \lambda(d\xi) \int_0^\infty f(s\xi)\widetilde{g}_\xi(s^2)s^{-1}ds,\end{aligned}$$

where

$$\widetilde{g}_\xi(x) = \int_0^\infty \phi(x^{1/2}/r)\nu_\xi(dr) = (2\pi)^{-1/2}\int_0^\infty e^{-x/(2r^2)}\nu_\xi(dr).$$

Define a measure $\widetilde{Q}_\xi$ by

$$\widetilde{Q}_\xi(B) = (2\pi)^{-1/2}\int_0^\infty 1_B(1/(2r^2))\nu_\xi(dr), \qquad B \in \mathcal{B}((0,\infty)).$$

Then $\widetilde{Q}_\xi(B)$ is measurable in ξ and

$$\widetilde{g}_\xi(x) = \int_0^\infty e^{-xu}\widetilde{Q}_\xi(du) \quad \text{for } x > 0.$$

Hence $\widetilde{g}_\xi$ is completely monotone by Proposition 1.10. Letting $\widetilde{\lambda} = \lambda$ and $\widetilde{\nu}_\xi(dr) = \widetilde{g}_\xi(r^2)r^{-1}dr$, we see that $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ is a polar decomposition of $\widetilde{\nu}$ and that $\widetilde{\mu} \in M(\mathbb{R}^d)$. Thus, $\mathcal{M}(I_{\log}(\mathbb{R}^d)) \cap I_{\mathrm{sym}}(\mathbb{R}^d) \subset M(\mathbb{R}^d)$.

Conversely, suppose that $\widetilde{\mu} \in M(\mathbb{R}^d)$ with triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$. If $\widetilde{\nu} = 0$, then $\widetilde{\mu} = \mathcal{M}\mu$ with some $\widetilde{A}$ and $\widetilde{\gamma}$. Suppose that $\widetilde{\nu} \neq 0$. Then, in a polar decomposition $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ of $\widetilde{\nu}$, we have $\widetilde{\nu}_\xi(dr) = \widetilde{g}_\xi(r^2)r^{-1}dr$, where $\widetilde{g}_\xi(x)$ is completely monotone in x and measurable in ξ. Thus, by Proposition 1.10, there are measures $\widetilde{Q}_\xi$ on $(0, \infty)$ satisfying

$$\widetilde{g}_\xi(x) = \int_0^\infty e^{-xu}\widetilde{Q}_\xi(du)$$

such that $\widetilde{Q}_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0, \infty))$. Now define

$$\nu_\xi(B) = (2\pi)^{1/2} \int_0^\infty 1_B((2u)^{-1/2})\widetilde{Q}_\xi(du).$$

Then ν_ξ is a measure on $(0, \infty)$ for each ξ and

$$\int_0^\infty f(r)\nu_\xi(dr) = (2\pi)^{1/2} \int_0^\infty f((2u)^{-1/2})\widetilde{Q}_\xi(du)$$

for all nonnegative measurable functions f on $(0, \infty)$.

Let $\lambda = \widetilde{\lambda}$. Then

$$\begin{aligned}\int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge 1)\nu_\xi(dr) &= (2\pi)^{1/2} \int_S \widetilde{\lambda}(d\xi) \int_{(0,\infty)} ((2u)^{-1} \wedge 1)\widetilde{Q}_\xi(du) \\ &= (2\pi)^{1/2} \int_S \widetilde{\lambda}(d\xi) \left(\int_0^{1/2} \widetilde{Q}_\xi(du) + \int_{1/2}^\infty (2u)^{-1}\widetilde{Q}_\xi(du) \right) < \infty,\end{aligned}$$

where the finiteness of the integral is assured by

$$\int_0^\infty (r^2 \wedge 1)\widetilde{g}_\xi(r^2)r^{-1}dr < \infty,$$

which can be shown by a standard calculation based on the fact that $\widetilde{g}_\xi$ is the Laplace transform of $\widetilde{Q}_\xi$. Define ν by

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Then ν is the Lévy measure of an infinitely divisible distribution and we can check

$$\int_0^\infty \phi(s)s^{-1}ds \int_{\mathbb{R}^d} f(sx)\nu(dx) = \int_{\mathbb{R}^d} f(x)\widetilde{\nu}(dx)$$

for all nonnegative measurable functions f on $\mathbb{R}^d$. This relation can be checked as follows:

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)\widetilde{\nu}(dx) &= \int_S \widetilde{\lambda}(d\xi)\int_0^\infty f(r\xi)\widetilde{\nu}_\xi(dr)\\
&= \int_S \widetilde{\lambda}(d\xi)\int_0^\infty f(r\xi)\widetilde{g}_\xi(r^2)r^{-1}dr\\
&= \int_S \widetilde{\lambda}(d\xi)\int_0^\infty f(r\xi)r^{-1}dr\int_0^\infty e^{-r^2u}\widetilde{Q}_\xi(du)\\
&= (2\pi)^{-1/2}\int_S \widetilde{\lambda}(d\xi)\int_0^\infty f(r\xi)r^{-1}dr\int_0^\infty e^{-r^2/(2u^2)}\nu_\xi(du)\\
&= (2\pi)^{-1/2}\int_S \widetilde{\lambda}(d\xi)\int_0^\infty e^{-r^2/(2u^2)}r^{-1}dr\int_0^\infty f(r\xi)\nu_\xi(du)\\
&= (2\pi)^{-1/2}\int_0^\infty e^{-y^2/2}y^{-1}dy\int_S \widetilde{\lambda}(d\xi)\int_0^\infty f(yu\xi)\nu_\xi(du)\\
&= \int_0^\infty \phi(s)s^{-1}ds\int_{\mathbb{R}^d} f(sx)\nu(dx).
\end{aligned}$$

Define A and γ suitably and let μ be a distribution with the triplet (A,ν,γ). Then $\mathcal{M}\mu = \widetilde{\mu}$, namely $\mathcal{L}\left(\int_0^\infty m^*(t)dX_t^{(\mu)}\right) = \widetilde{\mu}$. Thus by Theorem 3.2, we see that $\mu \in I_{\log}(\mathbb{R}^d)$ and that $\widetilde{\mu} \in \mathcal{M}(I_{\log}(\mathbb{R}^d))$. Since $\widetilde{\mu} \in I_{\mathrm{sym}}(\mathbb{R}^d)$, $\widetilde{\mu} \in \mathcal{M}(I_{\log}(\mathbb{R}^d)) \cap I_{\mathrm{sym}}(\mathbb{R}^d)$. This completes the proof of Theorem 3.5. □

3.3 Relationships of $M(\mathbb{R}^d)$ with other classes (I)

We have the following relations of $M(\mathbb{R}^d)$ with other classes.

Theorem 3.6. *We have*

$$T(\mathbb{R}^d) \cap I_{\mathrm{sym}}(\mathbb{R}^d) \subsetneqq M(\mathbb{R}^d) \subsetneqq L(\mathbb{R}^d) \cap G(\mathbb{R}^d).$$

Proof. (i) We first show that $M(\mathbb{R}^d) \subsetneqq L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$. We have already seen that $M(\mathbb{R}^d) \subset L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$, right after Definition 3.1. To show that $M(\mathbb{R}^d) \neq L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$, it is enough to construct $\mu \in I(\mathbb{R}^d)$ such that $\mu \in L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$ but $\mu \notin M(\mathbb{R}^d)$.

First consider the case $d = 1$. Let

$$\nu(dr) = g(r^2)r^{-1}dr,\ r > 0.$$

For our purpose, it is enough to construct a function $g : (0,\infty) \to (0,\infty)$ such that

(1) $r^{-1/2}g(r)$ is completely monotone on $(0,\infty)$, (meaning that the corresponding μ belongs to $G(\mathbb{R})$),
(2) $g(r^2)$ or, equivalently, $g(r)$ is nonincreasing on $(0,\infty)$, (meaning that the corresponding μ belongs to $L(\mathbb{R})$), and
(3) $g(r)$ is not completely monotone on $(0,\infty)$, (meaning that the corresponding μ does not belong to $M(\mathbb{R})$). Put

$$g(r) = r^{-1/2}\left(e^{-0.9r} - e^{-r} + 0.1e^{-1.1r}\right), \quad r > 0.$$

(1) We have

$$r^{-1/2}g(r) = r^{-1}\left(e^{-0.9r} - e^{-r} + 0.1e^{-1.1r}\right) = \int_{0.9}^{1} e^{-ru}du + 0.1\int_{1.1}^{\infty} e^{-ru}du,$$

which is a sum of two completely monotone functions, and thus, by Proposition 1.10, $r^{-1/2}g(r)$ is completely monotone.
(2) Put

$$k(r) = e^{-0.9r} - e^{-r} + 0.1e^{-1.1r},\ r > 0.$$

If $k(r)$ is nonincreasing, then so is $g(r) = r^{-1/2}k(r)$. To show it, we have

$$\begin{aligned} k'(r) &= -0.9e^{-0.9r} + e^{-r} - 0.11e^{-1.1r} = -0.9e^{-1.1r}\left[\left(e^{0.1r} - \frac{1}{1.8}\right)^2 - \frac{0.604}{3.24}\right] \\ &\le -0.9e^{-1.1r}\left[\left(1 - \frac{1}{1.8}\right)^2 - \frac{0.604}{3.24}\right] = -0.01e^{-1.1r} < 0, \quad r > 0. \end{aligned}$$

(3) To show (3), by Proposition 1.10, we see that

$$k(r) = \int_0^{\infty} e^{-ru}Q(du),$$

where Q is a signed measure such that $Q = Q_1 + Q_2 + Q_3$ and

$$Q_1(\{0.9\}) = 1,\ Q_2(\{1\}) = -1,\ Q_3(\{1.1\}) = 0.1.$$

On the other hand

$$r^{-1/2} = \pi^{-1/2}\int_0^{\infty} e^{-ru}u^{-1/2}du =: \int_0^{\infty} e^{-ru}R(du),$$

where

$$R(du) = (\pi u)^{-1/2}du.$$

Thus

$$g(r)=\int_0^\infty e^{-ru}R(du)\int_0^\infty e^{-rv}Q(dv)=\int_0^\infty e^{-rw}U(dw),$$

where

$$U(B)=\int_0^\infty Q(B-y)R(dy).$$

We are going to show that U is a signed measure, namely, for some interval $(a,b), U\left((a,b)\right)<0$. If so, g is not completely monotone. We have

$$\begin{aligned}
&U\left((a,b)\right)=\pi^{-1/2}\int_0^\infty Q\left((a-y,b-y)\right)y^{-1/2}dy\\
=&\pi^{-1/2}\sum_{i=1}^{3}\int_0^\infty Q_i\left((a-y,b-y)\right)y^{-1/2}dy\\
=&\pi^{-1/2}\left[\int_{a-0.9}^{b-0.9}y^{-1/2}dy-\int_{a-1}^{b-1}y^{-1/2}dy+0.1\int_{a-1.1}^{b-1.1}y^{-1/2}dy\right]\\
=&2\pi^{-1/2}\left[\left(\sqrt{b-0.9}-\sqrt{a-0.9}\right)-\left(\sqrt{b-1}-\sqrt{a-1}\right)+0.1\left(\sqrt{b-1.1}-\sqrt{a-1.1}\right)\right].
\end{aligned}$$

Take $(a,b)=(1.15,1.35)$. Then

$$\begin{aligned}
&U\left((1.15,1.35)\right)\\
&\quad=2\pi^{-1/2}\left[(\sqrt{0.45}-\sqrt{0.25})-(\sqrt{0.35}-\sqrt{0.15})+0.1(\sqrt{0.25}-\sqrt{0.05})\right]\\
&\quad<-0.01\pi^{-1/2}<0.
\end{aligned}$$

This concludes that g is not completely monotone.

A d–dimensional example of $\mu\in I(\mathbb{R}^d)$ such that $\mu\in L(\mathbb{R}^d)\cap G(\mathbb{R}^d)$ but $\mu\notin M(\mathbb{R}^d)$ is given by taking $\nu(dr)$ for the radial component of a Lévy measure. This completes the proof of $M(\mathbb{R}^d)\subsetneqq L(\mathbb{R}^d)\cap G(\mathbb{R}^d)$.

(ii) We next show that $T(\mathbb{R}^d)\cap I_{\mathrm{sym}}(\mathbb{R}^d)\subsetneqq M(\mathbb{R}^d)$. Since $M(\mathbb{R}^d)\subset I_{\mathrm{sym}}(\mathbb{R}^d)$, we consider only $\mu\in I_{\mathrm{sym}}(\mathbb{R}^d)$. We need the following lemma.

Lemma 3.7. (See Feller [F66], p.441, Corollary 2.) *Let ϕ be a completely monotone function on $(0,\infty)$ and let ψ be a nonnegative function on $(0,\infty)$ whose derivative is completely monotone. Then $\phi(\psi)$ is completely monotone.*

If $\mu\in T(\mathbb{R}^d)\cap I_{\mathrm{sym}}(\mathbb{R}^d)$, then the radial component of the Lévy measure of μ has the form $\nu_\xi(dr)=k_\xi(r)r^{-1}dr$, where k_ξ is completely monotone. By the lemma above and the fact that $\psi(r)=r^{1/2}$ has a completely monotone derivative, then

$g_\xi(r) := k_\xi(r^{1/2})$ is completely monotone. Thus $\nu_\xi(dr)$ can be read as $g_\xi(r^2)r^{-1}dr$, where g_ξ is completely monotone, concluding that $\mu \in M(\mathbb{R}^d)$.

To show that $T(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d) \neq M(\mathbb{R}^d)$, it is enough to find a completely monotone function g_ξ such that $k_\xi(r) = g_\xi(r^2)$ is *not* completely monotone. However, the function $g_\xi(r) = e^{-r}$ has such a property. Although e^{-r} is completely monotone, $(-1)^2 \frac{d^2}{dr^2} e^{-r^2} < 0$ for small $r > 0$. This completes the proof of that $T(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d) \subsetneqq M(\mathbb{R}^d)$. ☐

Additional remark. The argument above does not depend on r^{-1} of the radial component $\nu_\xi(dr)$, which gives us the following result between classes $B(\mathbb{R}^d)$ and $G(\mathbb{R}^d)$, namely,

$$B(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d) \subsetneqq G(\mathbb{R}^d).$$

3.4 Relationships of $M(\mathbb{R}^d)$ with other classes (II)

To give more relation of $M(\mathbb{R}^d)$ with other classes, we introduce a mapping.

Definition 3.8.

$$\Phi : I_{\log}(\mathbb{R}^d) \to I(\mathbb{R}^d), \quad \Phi(\mu) = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t^{(\mu)} \right).$$

Remark 3.9 (known)**.** (i) If $\mu \in I_{\log}(\mathbb{R}^d)$, then $\Phi(\mu)$ is a selfdecomposable distribution and $L(\mathbb{R}^d) = \Phi(I_{\log}(\mathbb{R}^d))$. (See, e.g. [J85] and [MR02].)
(ii) $\Phi(B(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)) = T(\mathbb{R}^d)$. (See [BMS06].)

Theorem 3.10. *(i) Let $\mu \in I(\mathbb{R}^d)$. Then $\mathcal{G}(\mu) \in I_{\log}(\mathbb{R}^d)$ if and only if $\mu \in I_{\log}(\mathbb{R}^d)$.*
(ii) Let

$$a(s) = 2 \int_s^\infty u^{-1} du \int_u^\infty \phi(v) dv, \ s > 0,$$

and define the inverse function $s = a^*(t)$ by $t = a(s)$. Then the stochastic integral

$$\int_0^\infty a^*(t) dX_t^{(\mu)}$$

exists if and only if $\mu \in I_{\log}(\mathbb{R}^d)$.
(iii) If $\mu \in I_{\log}(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d)$, then

$$(\Phi\mathcal{G})(\mu) = (\mathcal{G}\Phi)(\mu) = \mathcal{L}\left(\int_0^\infty a^*(t) dX_t^{(\mu)} \right),$$

where $(\Phi\mathcal{G})(\mu) = \Phi(\mathcal{G}(\mu))$ *and* $(\mathcal{G}\Phi)(\mu) = \mathcal{G}(\Phi(\mu))$, *and the Lévy measure* $\widetilde{\nu}$ *of* $\mathcal{L}\left(\int_0^\infty a^*(t)dX_t^{(\mu)}\right)$ *is*

$$\widetilde{\nu}(B) = \int_0^\infty \nu(s^{-1}B)s^{-1}ds \int_s^\infty \phi(v)dv,$$

where ν *is the Lévy measure of* μ.
(iv)

$$M(\mathbb{R}^d) \supsetneqq \mathcal{G}\left(\Phi\left(I_{\log}(\mathbb{R}^d)\right)\right) = \mathcal{G}\left(L(\mathbb{R}^d)\right) = \Phi(G(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)).$$

Proof of (i). The proof of Theorem C (i) in [BMS06] also works here. Let $\mu \in I(\mathbb{R}^d)$, and $\widetilde{\mu} = \mathcal{G}(\mu)$. Let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\mu}$. We have

$$\begin{aligned}\int_{|x|>2} \log|x|\widetilde{\nu}(dx) &= \int_0^\infty \phi(s)ds \int_{|x|>2/s} \log(s|x|)\nu(dx)\\ &= \int_{\mathbb{R}^d} \nu(dx) \int_{2/|x|}^\infty \phi(s)\log(s|x|)ds = \int_{\mathbb{R}^d} p(x)\nu(dx)\end{aligned}$$

where

$$p(x) = \int_{2/|x|}^\infty \phi(s)\log s ds + \log|x| \int_{2/|x|}^\infty \phi(s)ds.$$

Note that $p(x) = o(|x|^2)$ as $|x| \downarrow 0$ and $j(x) \sim \log|x|$ as $|x| \to \infty$. Thus, $\int_{|x|>2} \log|x|$ $\widetilde{\nu}(dx) < \infty$ if and only if $\int_{|x|>2} \log|x|\nu(dx) < \infty$. □

Proof of (ii). ("If" part.) It is enough to show that $f(t) = a^*(t)$ satisfies (3.2) – (3.4) in Lemma 3.3 for every $\mu \in I_{\log}(\mathbb{R}^d)$. Note that $a(+0) = \infty$ and $a(\infty) = 0$. Since

$$\int_0^\infty a^*(t)^2 dt = -\int_0^\infty s^2 da(s) = 2\int_0^\infty sds \int_s^\infty \phi(v)dv < \infty,$$

we have (3.2).

As to (3.3), we have

$$\begin{aligned}&\int_0^\infty dt \int_{\mathbb{R}^d} (|a^*(t)x|^2 \wedge 1)\nu(dx)\\ &= -\int_0^\infty da(s) \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu(dx)\\ &= 2\int_0^\infty s^{-1}ds \int_s^\infty \phi(v)dv \left(\int_{|x|\le 1/s} |sx|^2\nu(dx) + \int_{|x|>1/s} \nu(dx)\right)\\ &=: 2(I_1 + I_2)\end{aligned}$$

say. Here

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} |x|^2 \nu(dx) \int_0^{1/|x|} s ds \int_s^\infty \phi(v) dv \\
&= \left(\int_{|x|\le 1} + \int_{|x|>1} \right) |x|^2 \nu(dx) \int_0^{1/|x|} s ds \int_s^\infty \phi(v) dv \\
&=: I_{11} + I_{12},
\end{aligned}$$

say, and

$$\begin{aligned}
I_{11} &\le \int_{|x|\le 1} |x|^2 \nu(dx) \int_0^\infty s ds \int_s^\infty \phi(v) dv < \infty, \\
I_{12} &\le \int_{|x|>1} |x|^2 \nu(dx) \int_0^{1/|x|} s ds \le 2^{-1} \int_{|x|>1} \nu(dx) < \infty.
\end{aligned}$$

Also,

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty s^{-1} ds \int_s^\infty \phi(v) dv \\
&= \left(\int_{|x|\le 1} + \int_{|x|>1} \right) \nu(dx) \int_{1/|x|}^\infty s^{-1} ds \int_s^\infty \phi(v) dv \\
&=: I_{21} + I_{22},
\end{aligned}$$

say, and

$$\begin{aligned}
I_{21} &\le \int_{|x|\le 1} \nu(dx) \int_{1/|x|}^\infty s^{-1} ds \int_s^\infty \phi(v) dv, \\
&\le C_1 \int_{|x|\le 1} x^2 \nu(dx) < \infty \\
I_{22} &\le \int_{|x|>1} \nu(dx) \int_{1/|x|}^\infty s^{-1} ds \int_s^\infty \phi(v) dv \\
&\le \int_{|x|>1} \nu(dx) \left\{ \int_{1/|x|}^1 s^{-1} ds \int_0^\infty \phi(v) dv + \int_0^\infty ds \int_s^\infty \phi(v) dv \right\} \\
&= C_2 \int_{|x|>1} (\log |x| + C_3) \nu(dx) < \infty,
\end{aligned}$$

since $\mu \in I_{\log}(\mathbb{R}^d)$, where $C_1, C_2, C_3 > 0$. This shows (3.3).

For (3.4), we have

$$\int_0^\infty \left| a^*(t)\gamma + a^*(t) \int_{\mathbb{R}^d} x \left(\frac{1}{1+|a^*(t)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| dt$$

$$\le -|\gamma|\int_0^\infty s da(s) - \int_0^\infty \left| s\int_{\mathbb{R}^d} x\left(\frac{1}{1+|sx|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right| da(s)$$
$$=: I_3 + I_4,$$

say, where

$$I_3 \le 2|\gamma|\int_0^\infty ds\int_s^\infty \phi(v)dv < \infty,$$
$$I_4 \le 2\int_0^\infty ds\int_s^\infty \phi(v)dv\left|\int_{\mathbb{R}^d}\left(\frac{x|x|^2|s^2-1|}{(1+|sx|^2)(1+|x|^2)}\right)\nu(dx)\right|$$
$$\le 2\int_0^\infty |s^2-1|ds\int_s^\infty \phi(v)dv\int_{\mathbb{R}^d}\frac{|x|^3}{(1+|sx|^2)(1+|x|^2)}\nu(dx)$$
$$= 2\int_0^\infty |s^2-1|ds\int_s^\infty \phi(v)dv\left(\int_{|x|\le 1}+\int_{|x|>1}\right)\frac{|x|^3}{(1+|sx|^2)(1+|x|^2)}\nu(dx)$$
$$=: 2(I_{41}+I_{42}),$$

say. Here

$$I_{41} \le \int_0^\infty |s^2-1|ds\int_s^\infty \phi(v)dv\int_{|x|\le 1}\frac{|x|^3}{(1+|x|^2)}\nu(dx) < \infty,$$

and

$$I_{42} \le \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^\infty \frac{s^2+1}{1+|sx|^2}ds\int_s^\infty \phi(v)dv$$
$$= \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\nu(dx)\left(\int_0^1+\int_1^\infty\right)\frac{s^2+1}{1+|sx|^2}ds\int_s^\infty \phi(v)dv$$
$$=: I_{421}+I_{422},$$

say. Furthermore,

$$I_{421} \le \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^1\frac{s^2+1}{1+|sx|^2}ds\int_0^\infty \phi(v)dv$$
$$\le \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^1\frac{1}{1+|sx|^2}ds$$
$$\le \int_{|x|>1}\frac{|x|^2}{1+|x|^2}\nu(dx)\int_0^\infty\frac{1}{1+t^2}dt < \infty,$$

and

$$I_{422} = \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_1^\infty\frac{s^2+1}{1+|sx|^2}ds\int_s^\infty \phi(v)dv$$

$$\leq \int_{|x|>1} \frac{|x|^3}{(1+|x|^2)^2}\nu(dx)\int_1^\infty (s^2+1)ds\int_s^\infty \phi(v)dv < \infty.$$

Thus we have (3.4). This completes the proof of ("if" part) of (ii).

("Only if" part.) Suppose $\int_0^\infty a^*(t)dX_t^{(\mu)}$ exists and let $\widetilde{\nu}$ be its Lévy measure. We have

$$\begin{aligned}\int_{|x|>1}\widetilde{\nu}(dx) &= \int_0^\infty dt\int 1_{\{|a^*(t)x|>1\}}(x)\nu(dx)\\ &= -\int_0^\infty da(s)\int 1_{\{|x|>1/s\}}(x)\nu(dx)\\ &= -\int_{\mathbb{R}^d}\nu(dx)\int_{1/|x|}^\infty da(s)\\ &\geq 2\int_{|x|>1}\nu(dx)\int_{1/|x|}^\infty s^{-1}ds\int_s^\infty \phi(v)dv\\ &\geq \int_{|x|>1}\nu(dx)(C_1\log|x|+C_2),\end{aligned}$$

for some $C_1, C_2 > 0$. Thus, $\mu \in I_{\log}(\mathbb{R}^d)$. This completes the proof of ("only if" part) of (ii). □

Proof of (iii). Recall that for $\mu \in I_{\log}(\mathbb{R}^d)$

$$C_{\Phi(\mu)}(z) = \int_0^\infty C_\mu(ze^{-t})dt,$$

and for $\mu \in I(\mathbb{R}^d)$,

$$C_{\mathcal{G}(\mu)}(z) = \int_0^1 C_\mu(zh^*(s))ds.$$

Let $\mu \in I_{\log}(\mathbb{R}^d)$. We have, for $z \in \mathbb{R}^d$,

$$\begin{aligned}C_{(\Phi\mathcal{G})(\mu)}(z) &= \int_0^\infty C_{\mathcal{G}(\mu)}(e^{-t}z)dt = \int_0^\infty dt\int_0^1 C_\mu(h^*(s)e^{-t}z)ds\\ C_{(\mathcal{G}\Phi)(\mu)}(z) &= \int_0^1 C_{\Phi(\mu)}(h^*(s)z)ds = \int_0^1 ds\int_0^\infty C_\mu(e^{-t}h^*(s)z)dt.\end{aligned}$$

We claim that

$$\int_0^\infty dt\int_0^1 |C_\mu(h^*(s)e^{-t}z)|ds = \int_0^\infty dt\int_{-\infty}^\infty |C_\mu(ue^{-t}z)|\phi(u)du < \infty. \tag{3.9}$$

Note that $\mathcal{G}(\mu)$ is symmetric and it is unchanged even if we replace μ by $\bar{\mu}(B) = 2^{-1}(\mu(B)+\mu(-B))$. (See [MR02].) Hence, without loss of generality, we assume μ is symmetric. Thus to show (3.9), it is enough to show that

$$\int_0^\infty dt \int_0^\infty |C_\mu(ue^{-t}z)|\phi(u)du < \infty. \tag{3.10}$$

The proof of (3.10) is as follows. The idea is from Barndorff–Nielsen et al. [BMS06]. If the generating triplet of μ is (A, ν, γ), then

$$|C_\mu(z)| \le 2^{-1}(\mathrm{tr}A)|z|^2 + |\gamma||z| + \int_{\mathbb{R}^d} |g(z,x)|\nu(dx),$$

where

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle(1+|x|^2)^{-1}.$$

Hence

$$\begin{aligned}|C_\mu(ue^{-t}z)| \le{}& 2^{-1}(\mathrm{tr}A)u^2e^{-2t}|z|^2 + |\gamma||u|e^{-t}|z| + \int_{\mathbb{R}^d} |g(z, ue^{-t}x)|\nu(dx) \\ &+ \int_{\mathbb{R}^d} |g(ue^{-t}z, x) - g(z, ue^{-t}x)|\nu(dx) =: J_1 + J_2 + J_3 + J_4,\end{aligned}$$

say. The finiteness of $\int_0^\infty dt \int_0^\infty (J_1 + J_2)\phi(u)du$ is trivial.

Noting that $|g(z,x)| \le C_z|x|^2(1+|x|^2)^{-1}$ with a positive constant C_z depending on z, we have

$$\begin{aligned}&\int_0^\infty dt \int_0^\infty J_3\phi(u)du \\ &\le C_z \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty dt \int_0^\infty |ue^{-t}x|^2 \left(1+|ue^{-t}x|^2\right)^{-1} \phi(u)du \\ &= C_z \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty \phi(u)du \int_0^\infty |ue^{-t}x|^2 \left(1+|ue^{-t}x|^2\right)^{-1} dt \\ &= C_z \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty \phi(u)du \int_0^{u|x|} s\left(1+s^2\right)^{-1} ds \\ &= 2^{-1}C_z \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty \phi(u) \log\left(1+u^2|x|^2\right) du \\ &=: K_3\end{aligned}$$

say. Since $\log(1+v) \le C(v1_{(0,2]}(v) + (\log v)1_{(2,\infty)}(v))$ for $v > 0$ for an absolute constant C, we have

$$\begin{aligned}K_3 \le{}& 2^{-1}CC_z \int_{\mathbb{R}^d} |x|^2\nu(dx) \int_0^{\sqrt{2}/|x|} \phi(u)u^2du \\ &+ CC_z \int_{\mathbb{R}^d} \nu(dx) \int_{\sqrt{2}/|x|}^\infty \phi(u)(\log u + \log|x|)du,\end{aligned}$$

which is finite since $\int_{|x|\le 2}|x|^2\nu(dx)<\infty$ and $\int_{|x|>2}\log|x|\nu(dx)<\infty$. say.

As to $\int_0^\infty dt\int_0^\infty J_4\phi(u)du<\infty$, note that for $a>0$,

$$\begin{aligned}|g(az,x)-g(z,ax)| &= |\langle az,x\rangle||x|^2|1-a^2|(1+|x|^2)^{-1}(1+a|x|^2)^{-1}\\ &\le |z||x|^3a(1+a^2)(1+|x|^2)^{-1}(1+a|x|^2)^{-1}.\end{aligned}$$

Then

$$\begin{aligned}\int_0^\infty J_4dt \le&|z|\int_{\mathbb{R}^d}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^\infty\frac{ue^{-t}+u^3e^{-3t}}{1+u^2e^{-2t}|x|^2}dt\\ =&|z|\int_{\mathbb{R}^d}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^{u|x|}\frac{v|x|^{-1}+v^3|x|^{-3}}{(1+v^2)v}dv\\ \le&2^{-1}\pi|z|\int_{\mathbb{R}^d}\frac{|x|^2}{1+|x|^2}\nu(dx)+|z|\int_{\mathbb{R}^d}\frac{1}{1+|x|^2}\nu(dx)\int_0^{u|x|}\frac{v^2}{1+v^2}dv\\ =&:J_{41}+J_{42},\end{aligned}$$

say. Then $\int_0^\infty J_{41}\phi(u)du<\infty$ is evident and

$$\begin{aligned}\int_0^\infty J_{42}\phi(u)du &\le |z|\int_{\mathbb{R}^d}\frac{1}{1+|x|^2}\nu(dx)\int_0^\infty\frac{v^2}{1+v^2}dv\int_{v/|x|}^\infty\phi(u)du\\ &\le |z|\int_{\mathbb{R}^d}\frac{1}{1+|x|^2}\nu(dx)\int_0^\infty 2^{-1}v\phi(v/|x|)dv\\ &\le 2^{-1}|z|\int_{\mathbb{R}^d}\frac{|x|^2}{1+|x|^2}\nu(dx)\int_0^\infty y\phi(y)dy<\infty.\end{aligned}$$

This completes the proof of (3.10). Thus

$$\begin{aligned}C_{(\Phi\mathcal{G})(\mu)}(z) &= \int_0^\infty dt\int_0^1 C_\mu(ze^{-t}h^*(s))ds\\ &= -\int_0^\infty dt\int_{-\infty}^\infty C_\mu(ze^{-t}v)dh(v)\\ &= \int_0^\infty dt\int_0^\infty C_\mu(ze^{-t}v)2\phi(v)dv\\ &= 2\int_0^\infty\phi(v)dv\int_0^\infty C_\mu(ze^{-t}v)dt\\ &= 2\int_0^\infty\phi(v)dv\int_0^v C_\mu(zs)s^{-1}ds\\ &= 2\int_0^\infty C_\mu(zs)s^{-1}ds\int_s^\infty\phi(v)dv,\end{aligned}$$

where the change of the order of integrals is assured by (3.9) and (3.10). Thus we have

$$C_{(\Phi\mathcal{G})(\mu)}(z) = -\int_0^\infty C_\mu(zs)da(s),$$

and hence

$$C_{(\Phi\mathcal{G})(\mu)}(z) = \int_0^\infty C_\mu(za^*(t))dt.$$

We have the form of $\widetilde{\nu}$ as follows by (3.6).

$$\begin{aligned}\widetilde{\nu}(B) &= \int_0^\infty dt \int_{\mathbb{R}^d} 1_B(xa^*(t))\nu(dx)\\ &= -\int_0^\infty da(s) \int_{\mathbb{R}^d} 1_B(xs)\nu(dx)\\ &= \int_0^\infty \nu(s^{-1}B)s^{-1}ds \int_s^\infty \phi(v)dv.\end{aligned}$$

This concludes the proof of (iii). □

Proof of (iv). We first show that the radial component of the Lévy measure of $(\Phi\mathcal{G})(\mu)$ satisfies (3.1). We have

$$\begin{aligned}\widetilde{\nu}(B) = \nu_{(\Phi\mathcal{G})(\mu)}(B) &= \int_0^\infty \nu_{\mathcal{G}(\mu)}(e^tB)dt\\ &= \int_0^\infty dt \int_S \lambda(d\xi) \int_0^\infty 1_{e^tB}(r\xi)g_\xi(r^2)dr,\end{aligned}$$

where λ is a probability measure appearing in the polar decomposition of ν_μ and $g_\xi(r^2)dr$ is the radial component of ν_μ. Then

$$\begin{aligned}\widetilde{\nu}(B) &= \int_S \lambda(d\xi) \int_0^\infty g_\xi(r^2)dr \int_0^\infty 1_B(e^{-t}r\xi)dt\\ &= \int_S \lambda(d\xi) \int_0^\infty g_\xi(r^2)dr \int_0^r 1_B(y\xi)y^{-1}dy\\ &= \int_S \lambda(d\xi) \int_0^\infty 1_B(y\xi)y^{-1}dy \int_y^\infty g_\xi(r^2)dr\\ &=: \int_S \lambda(d\xi) \int_0^\infty 1_B(y\xi)\widetilde{\nu}_\xi(dy),\end{aligned}$$

where

$$\widetilde{\nu}_\xi(dy) = \left(y^{-1}\int_y^\infty g_\xi(r^2)dr\right)dy.$$

This $\widetilde{\nu}_\xi$ satisfies $\int_0^\infty(1\wedge y^2)\widetilde{\nu}_\xi(dy) < \infty$. For

$$\begin{aligned}&\int_0^\infty (1\wedge y^2)\widetilde{\nu}_\xi(dy)\\ &\quad= \int_0^1 ydy \int_y^\infty g_\xi(r^2)dr + \int_1^\infty y^{-1}dy \int_y^\infty g_\xi(r^2)dr\end{aligned}$$

$$= \int_0^1 g_\xi(r^2)dr \int_0^r ydy + \int_1^\infty g_\xi(r^2)dr \int_0^1 ydr + \int_1^\infty g_\xi(r^2)dr \int_1^r y^{-1}dy < \infty,$$

where the last integral is finite because ν is the Lévy measure of a $\mu \in I_{\log}(\mathbb{R}^d)$. Put

$$\widetilde{g}_\xi(x) = \int_{x^{1/2}}^\infty g_\xi(r^2)dr.$$

We then have

$$\frac{d}{dx}\widetilde{g}_\xi(x) = -2^{-1}x^{-1/2}g_\xi(x).$$

Since g_ξ and $x^{-1/2}$ are completely monotone, $x^{-1/2}g_\xi(x)$ is completely monotone. Thus $\widetilde{g}_\xi$ is completely monotone. Hence

$$\widetilde{\nu}_\xi(dy) = \widetilde{g}_\xi(y^2)y^{-1}dy,$$

where $\widetilde{g}_\xi$ is completely monotone. Thus the Lévy measure of $\widetilde{\mu}$ is that of $(\Phi\mathcal{G})(\mu)$ and thus $\widetilde{\mu}$ belongs to the class $M(\mathbb{R}^d)$. Thus $M(\mathbb{R}^d) \supset \mathcal{G}(L(\mathbb{R}^d))$.

The last equality is a consequence of (i) and (iii). Namely, by (i),

$$\mathcal{G}(I_{\log}(\mathbb{R}^d)) = G(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d).$$

Thus by (iii),

$$(\Phi\mathcal{G})(I_{\log}(\mathbb{R}^d)) = \Phi(G(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)) = \left\{ \mathcal{L}\left(\int_0^\infty a^*(t)dX_t^{(\mu)} \right),\ \mu \in I_{\log}(\mathbb{R}^d) \cap I_{\mathrm{sym}}(\mathbb{R}^d) \right\}.$$

It remains to show $M(\mathbb{R}^d) \neq \mathcal{G}(L(\mathbb{R}^d))$. It is enough to show it for $d = 1$.

Consider a Lévy measure $\nu(dr) = \phi(r)|r|^{-1}dr$. Then the corresponding infinitely divisible distribution μ belongs to $M(\mathbb{R})$. Suppose $\mu \in \mathcal{G}(L(\mathbb{R}))$. Then, by (iii), ν also satisfies

$$\nu(B) = \int_0^\infty \nu_0(s^{-1}B)h(s)s^{-1}ds,$$

where $h(s) = \int_s^\infty \phi(x)dx$ and ν_0 is a symmetric Lévy measure. Consider $B \in \mathcal{B}((0,\infty))$. Then we have

$$\begin{aligned}\nu(B) &= \int_0^\infty \int_{\mathbb{R}} 1_B(sx)\nu_0(dx)h(s)s^{-1}\,ds \\ &= \int_0^\infty \int_0^\infty 1_B(r)h(rx^{-1})r^{-1}\,dr\nu_0(dx).\end{aligned}$$

Thus

$$\nu(dr) = \left(\int_0^\infty h(rx^{-1})\nu_0(dx) \right) r^{-1}dr, \quad r > 0.$$

By our assumption, for any $r > 0$,

$$\phi(r) = \int_0^\infty h(rx^{-1})\,\nu_0(dx).$$

Let $h > 0$ and consider

$$\frac{1}{h}(\phi(r+h) - \phi(r)) = \int_0^\infty \frac{1}{h}\left(h((r+h)x^{-1}) - h(rx^{-1})\right)\,\nu_0(dx). \tag{3.11}$$

We have

$$|h((r+h)x^{-1}) - h(rx^{-1})| = \phi((r+\theta h)x^{-1})hx^{-1} \le \phi(rx^{-1})hx^{-1},$$

where $0 < \theta < 1$. Thus we can interchange the limit as $h \to 0$ and the integral in (3.11), and we get

$$-r\phi(r) = -\int_0^\infty \phi(rx^{-1})x^{-1}\,\nu_0(dx), \quad \text{for any } r > 0.$$

Changing variable from r to $r^{1/2}$, we get

$$r^{1/2}\phi(r^{1/2}) = \int_0^\infty \phi(r^{1/2}x^{-1})x^{-1}\,\nu_0(dx).$$

The right hand side is completely monotone, but the left had side is not. This contradicts our assumption that $\mu \in \mathcal{G}(L(\mathbb{R}))$. The proof of (iv) is now completed. □

3.5 More about the classes $M(\mathbb{R})$ and $\mathcal{G}(L(\mathbb{R}))$ when $d = 1$

We first note that

$$\begin{aligned}\mathcal{G}(L(\mathbb{R}^d)) = \{\mu \in I_{\mathrm{sym}}(\mathbb{R}^d)\ : \nu_\mu(B) = E[\nu_0(Z^{-1}B)],\ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \\ \text{for the Lévy measure } \nu_0 \text{ of } \mu_0 \in L(\mathbb{R}^d)\}.\end{aligned} \tag{3.12}$$

This follows from Proposition 2.7 (v). When $d = 1$, we again mention that μ is of type G if and only if $\mu = \mathcal{L}(V^{1/2}Z)$ for some infinitely divisible nonnegative random variable V independent of the standard normal random variable Z. That is, μ is a variance mixture of normal distributions. The goal here is to characterize the distribution of the random variance V in the case of $\mu \in M(\mathbb{R})$. We begin with the following.

Proposition 3.11. *$\mu \in \mathcal{G}(L(\mathbb{R}))$ if and only if $\mu = \mathcal{L}(V^{1/2}Z)$ with $\mathcal{L}(V) \in L(\mathbb{R}_+)$.*

Proof. ("Only if" part.) Suppose $\mu \in \mathcal{G}(L(\mathbb{R}))$. Since $\mu \in G(\mathbb{R})$, there exists V such that $\mu = \mathcal{L}(V^{1/2}Z)$ and $\mathcal{L}(V) \in I(\mathbb{R}_+)$. Also from (3.12), there exists a Lévy measure ν_0 of an element in $L(\mathbb{R})$ such that $\nu_\mu(B) = E[\nu_0(Z^{-1}B)]$. It is known ([MR01]) that for every $x > 0$,

$$\nu_0([x,\infty)) = 2^{-1}\nu_V([x^2,\infty)). \tag{3.13}$$

Since ν_0 is the Lévy measure of some $\mu_0 \in L(\mathbb{R})$,

$$\nu_0(dx) = k_0(x)x^{-1}dx,\ x > 0, \tag{3.14}$$

for some nonincreasing function k_0. It follows from (3.13) and (3.14) that

$$\int_x^\infty k_0(y)y^{-1}dy = 2^{-1}\int_{x^2}^\infty \nu_V(dy),\ x > 0.$$

By the change of variables $u = y^2$ on the left hand side above, we have

$$2^{-1}\int_{x^2}^\infty k_0(u^{1/2})u^{-1}du = 2^{-1}\int_{x^2}^\infty \nu_V(dy),\ x > 0.$$

Thus, we have

$$\nu_V(dy) = k_1(y)y^{-1}dy,$$

where $k_1(y) = k_0(y^{1/2})$ is nonincreasing. Hence $\mathcal{L}(V) \in L(\mathbb{R}_+)$.

("If" part.) Suppose $\mu = \mathcal{L}(V^{1/2}Z)$ and $\mathcal{L}(V) \in L(\mathbb{R}_+)$. Then there exits a nonincreasing function $k_1(y)$ such that

$$\nu_V(dy) = k_1(y)y^{-1}dy.$$

Then by (3.13),

$$\begin{aligned}\int_x^\infty \nu_0(dy) &= 2^{-1}\int_{x^2}^\infty k_1(y)y^{-1}dy \\ &= \int_x^\infty k_1(u^2)u^{-1}du,\ x > 0.\end{aligned}$$

Thus, $\nu_0(dy) = k_0(y)y^{-1}dy$, where $k_0(y) = k_1(y^2)$ is nonincreasing. Hence ν_0 is the Lévy measure of some $\mu_0 \in L(\mathbb{R})$. Since $\nu_\mu(B) = E[\nu_0(Z^{-1}B)]$, where ν_0 is defined by (3.13) from ν_V, we have $\mu \in \mathcal{G}(L(\mathbb{R}))$. This completes the proof. □

We have the following.

Theorem 3.12. *$\mu \in M(\mathbb{R})$ if and only if $\mu = \mathcal{L}(V^{1/2}Z)$, where $\mathcal{L}(V) \in I(\mathbb{R}_+)$ has an absolutely continuous Lévy measure ν_V of the form*

$$\nu_V(dr) = \ell(r)r^{-1}\,dr, \quad r > 0. \tag{3.15}$$

The function ℓ is given by

$$\ell(r) = \int_r^\infty (x-r)^{-1/2}\,\rho(dx), \tag{3.16}$$

where ρ is a measure on $(0,\infty)$ satisfying the integrability condition

$$\int_0^1 x^{1/2}\,\rho(dx) + \int_1^\infty (1+\log x)x^{-1/2}\,\rho(dx) < \infty. \tag{3.17}$$

Proof. (i) ("Only if" part.) Suppose $\mu \in M(\mathbb{R})$. Since $M(\mathbb{R}) \subset G(\mathbb{R})$, we have $\mu = \mathcal{L}(V^{1/2}Z)$ for some $V \in I(\mathbb{R}_+)$. Thus, we get for $z \in \mathbb{R}$,

$$\begin{aligned}
E\left[e^{izV^{1/2}Z}\right] &= E\left[e^{-Vz^2/2}\right] \\
&= \exp\left\{-2^{-1}Az^2 + \int_{0+}^\infty (e^{-vz^2/2}-1)\,\nu_V(dv)\right\} \\
&= \exp\left\{-2^{-1}Az^2 + \int_{0+}^\infty \nu_V(dv)\int_{-\infty}^\infty (e^{izv^{1/2}u}-1)\phi(u)\,du\right\} \\
&= \exp\left\{-2^{-1}Az^2 + \int_{-\infty}^\infty (e^{izx}-1)dx\int_{0+}^\infty \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv)\right\},
\end{aligned}$$

where $A \geq 0$. Therefore, the Lévy measure ν_μ of μ is of the form

$$\nu_\mu(dx) = \left(\int_{0+}^\infty \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv)\right)dx. \tag{3.18}$$

By the definition, $\mu \in M(\mathbb{R})$ if and only if $\nu_\mu(dx) = |x|^{-1}g(x^2)dx$, where g is completely monotone. Thus, by Proposition 1.10, g can be written as

$$g(r) = \int_0^\infty e^{-ry/2}\,Q(dy), \quad r > 0,$$

for some measure Q on $(0,\infty)$. By (3.18), we get

$$\int_{0+}^\infty \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv) = |x|^{-1}g(x^2). \tag{3.19}$$

Since

$$r^{-1/2} = (2\pi)^{-1/2}\int_0^\infty e^{-rw/2}w^{-1/2}\,dw, \quad r > 0,$$

we obtain

$$
\begin{aligned}
r^{-1/2}g(r) &= (2\pi)^{-1/2}\int_0^\infty\int_0^\infty e^{-r(w+y)/2}w^{-1/2}\,dwQ(dy)\\
&= (2\pi)^{-1/2}\int_0^\infty Q(dy)\int_y^\infty e^{-ru/2}(u-y)^{-1/2}\,du\\
&= (2\pi)^{-1/2}\int_0^\infty e^{-ru/2}du\int_0^u (u-y)^{-1/2}\,Q(dy).
\end{aligned}
$$

Taking $x = r^{1/2} > 0$ in (3.19), we get

$$
(2\pi)^{-1/2}\int_{0+}^\infty e^{-r/2v}v^{-1/2}\,\nu_V(dv) = (2\pi)^{-1/2}\int_0^\infty e^{-ru/2}du\int_0^u (u-y)^{-1/2}\,Q(dy). \quad (3.20)
$$

Let

$$
\rho(dx) = -x^{1/2}Q(d(x^{-1})).
$$

Then $\ell(r)$ in (3.16) becomes

$$
\begin{aligned}
\ell(r) &= -\int_r^\infty (x-r)^{-1/2}x^{1/2}Q(d(x^{-1}))\\
&= \int_0^{r^{-1}} (y^{-1}-r)^{-1/2}y^{-1/2}Q(dy)\\
&= \int_0^{r^{-1}} (1-yr)^{-1/2}Q(dy)\\
&= r^{-1/2}\int_0^{r^{-1}} (r^{-1}-y)^{-1/2}Q(dy).
\end{aligned}
$$

Thus by (3.20),

$$
\int_{0+}^\infty e^{-r/2v}v^{-1/2}\,\nu_V(dv) = \int_0^\infty e^{-ru/2}u^{-1/2}\ell(u^{-1})\,du
$$

or

$$
\int_{0+}^\infty e^{-r/2v}v^{-1/2}\,\nu_V(dv) = \int_0^\infty e^{-r/2v}v^{-3/2}\ell(v)\,dv, \quad r > 0.
$$

Therefore

$$
v^{-1/2}\,\nu_V(dv) = v^{-3/2}\ell(v)\,dv, \quad v > 0,
$$

which yields (3.15).

The integrability condition (3.17) for ρ is obtained from the fact that

$$
\infty > \int_{\mathbb{R}}(x^2\wedge 1)\,\nu_\mu(dx) = \int_{\mathbb{R}}(|x|\wedge|x|^{-1})g(x^2)dx.
$$

For, this yields that

$$\int_0^1 x dx \int_0^\infty e^{-x^2y/2}\, Q(dy) < \infty \quad \text{and} \quad \int_1^\infty x^{-1} dx \int_0^\infty e^{-x^2y/2}\, Q(dy) < \infty,$$

and hence

$$\int_0^\infty \left[y^{-1}(1-e^{-y/2}) + 2^{-1}\int_y^\infty u^{-1}e^{-u/2}\, du \right] Q(dy) < \infty.$$

It is obvious that the above condition is equivalent to

$$\int_0^1 (1+\log y^{-1})Q(dy) + \int_1^\infty y^{-1}Q(dy) < \infty. \tag{3.21}$$

On the other hand,

$$\int_0^1 x^{1/2}\rho(dx) = -\int_0^1 xQ(d(x^{-1})) = \int_1^\infty y^{-1}Q(dy)$$

and

$$\int_1^\infty (1+\log x)x^{-1/2}\rho(dx) = -\int_1^\infty (1+\log x)Q(d(x^{-1})) = \int_0^1 (1+\log y^{-1})Q(dy).$$

Thus, we get (3.17) from (3.21). The ("only if" part) is thus proved.

(ii) ("If" part.) Suppose $\mu = \mathcal{L}(V^{1/2}Z)$ and the Lévy measure ν_V of V satisfies (3.15)–(3.17).

We first claim that the integrability condition (3.17) implies that ν_V is really a Lévy measure on $(0,\infty)$ of a positive infinitely divisible random variable, namely it satisfies

$$\int_0^\infty (r\wedge 1)\nu_V(dr) < \infty. \tag{3.22}$$

We have

$$\int_0^\infty (r\wedge 1)\nu_V(dr) = \int_0^1 r\nu_V(dr) + \int_1^\infty \nu_V(dr).$$

As to the first integral, we have

$$\begin{aligned}\int_0^1 r\nu_V(dr) &= \int_0^1 \ell(r)dr = \int_0^1 dr\int_r^\infty (x-r)^{-1/2}\rho(dx)\\ &= \int_0^1 \rho(dx)\int_0^x (x-r)^{-1/2}dr + \int_1^\infty \rho(dx)\int_0^1 (x-r)^{-1/2}dr\\ &= 2\int_0^1 x^{1/2}\rho(dx) + 2\int_1^\infty \left(x^{1/2}-(x-1)^{1/2}\right)\rho(dx)\end{aligned}$$

$$\le 2\int_0^1 x^{1/2}\rho(dx) + C\int_1^\infty x^{-1/2}\rho(dx),$$

where $C > 0$ is a constant. Next, as to the second integral,

$$\begin{aligned}\int_1^\infty \nu_V(dr) &= \int_1^\infty r^{-1}\ell(r)dr \\ &= \int_1^\infty r^{-1}dr\int_r^\infty (x-r)^{-1/2}\rho(dx) \\ &= \int_1^\infty \rho(dx)\int_1^x r^{-1}(x-r)^{-1/2}dr \\ &= \int_1^\infty 2x^{-1/2}\log(x^{1/2}+(x-1)^{1/2})\rho(dx).\end{aligned}$$

Therefore, (3.17) implies (3.22). Furthermore, as we have already seen, ν_μ is expressed as in (3.18). So, to complete the proof, it is enough to show that when we put

$$g(x^2) = |x|\int_0^\infty \phi(v^{-1/2}x)v^{-1/2}\nu_V(dv),$$

then $g(r)$ is completely monotone on $(0,\infty)$. However, for that, it is enough to follow the proof of the ("only if" part) from the bottom to the top. This concludes the proof. □

Example 3.13. Suppose that the measure ρ in Theorem 3.12 has the density and for some $0 < \alpha < 1$,

$$\rho(dx) = x^{-\alpha-1/2}dx.$$

This ρ satisfies the integrability condition (3.17). Then $\ell(r)$ in (3.16) turns out to be

$$\ell(r) = Kr^{-\alpha}, \quad \text{where} \quad K = \int_1^\infty (u-1)^{-1/2}u^{-\alpha-1/2}du < \infty.$$

Thus, ν_V in (3.15) is the Lévy measure of a positive α-stable distribution, and thus $\mu \in \mathcal{G}(L(\mathbb{R})) \subsetneqq M(\mathbb{R})$.

Example 3.14. (Another example of μ such that $\mu \in M(\mathbb{R})$ but $\mu \notin \mathcal{G}(L(\mathbb{R}))$.) Let ρ in (3.16) satisfy (3.17) and that

$$\rho([r_1, r_2]) = 0 \quad \text{for some } 0 < r_1 < r_2 < \infty$$

and $\rho((r_2,\infty)) > 0$. Then the resulting μ belongs to $M(\mathbb{R})$. However,

$$\ell(r_1) = \int_{r_1}^\infty (x-r_1)^{-1/2}\rho(dx) = \int_{r_2}^\infty (x-r_1)^{-1/2}\rho(dx)$$

$$< \int_{r_2}^{\infty} (x - r_2)^{-1/2} \rho(dx) = \ell(r_2).$$

Thus $\ell(r)$ is not a nonincreasing function so that $\mathcal{L}(V) \notin L((0,\infty))$. It follows from Proposition 3.11 that $\mu = \mathcal{L}(V^{1/2}Z) \notin \mathcal{G}(L(\mathbb{R}))$.

Chapter 4

Nested subclasses of some subclass of the class of type G selfdecomposable distributions on $\mathbb{R}^d$.

4.1 Nested subclasses of $M(\mathbb{R}^d)$ and their Lévy measures

In this section, we construct nested subclasses of $M(\mathbb{R}^d)$ as follows. Write $M_0(\mathbb{R}^d) = M(\mathbb{R}^d)$ and we call $g_\xi(r)$ in (3.1) the g–function of ν (or μ).

We define nested subclasses of $M(\mathbb{R}^d)$ in terms of their Lévy measures.

Definition 4.1 (The class $M_k(\mathbb{R}^d)$)**.** For any $k \in \mathbb{N}$, define

$$M_k(\mathbb{R}^d) = \{\widetilde{\mu} \in M_0(\mathbb{R}^d) : \\ \nu \text{ in (3.8) is the Lévy measure of some distribution in } M_{k-1}(\mathbb{R}^d)\}.$$

$M_\infty(\mathbb{R}^d)$ is defined by $\cap_{k=0}^{\infty} M_k(\mathbb{R}^d)$.

For characterizations, we need the following functions. Let $\eta_0(x) = \phi(x)$ and for $k = 1, 2, \ldots,$

$$\eta_k(x) = \int_0^\infty \phi(xu^{-1})\eta_{k-1}(u)u^{-1}du. \tag{4.1}$$

Remark 4.2. (1) $\lim_{x\to+0} \eta_k(x)x^{-1} = \infty$ and $\lim_{x\to\infty} \eta_k(x)x^{-1} = 0$.
(2) $\eta_k(x)$ can be written as follows;

$$\eta_k(x) \;\; = \;\; \int_0^\infty \phi(u_1)u_1^{-1}du_1$$

$$\cdots \int_0^\infty \phi(u_{k-1})u_{k-1}^{-1}du_{k-1}\int_0^\infty \phi\left(x\left(\prod_{i=1}^k u_i\right)^{-1}\right)\phi(u_k)u_k^{-1}du_k.$$

Then we have following.

Theorem 4.3 (A representation of the Lévy measures of $\mu_k \in M_k(\mathbb{R}^d)$)**.** *Let $\mu_k \in I_{\mathrm{sym}}(\mathbb{R}^d)$, $k = 1, 2, \ldots$, and denote its Lévy measure by ν_k. Then, $\mu_k \in M_k(\mathbb{R}^d)$ if and only if*

$$\nu_k(B) = \int_0^\infty \nu_0(u^{-1}B)\eta_{k-1}(u)u^{-1}du,\ B \in \mathcal{B}_0(\mathbb{R}^d) \tag{4.2}$$

where ν_0 is the Lévy measure of some $\mu_0 \in M_0(\mathbb{R}^d)$.

Proof. (i) ("Only if" part.) Let $k = 1$ and suppose $\mu_1 \in M_1(\mathbb{R}^d)$. Then, by the definition

$$\nu_1(B) = \int_0^\infty \nu_0(u^{-1}B)\phi(u)u^{-1}du$$

for some Lévy measure ν_0 whose distribution is in $M_0(\mathbb{R}^d)$. We are going to show the assertion by the induction. Suppose that the assertion is true for some $k \in \mathbb{N}$. Namely, suppose the Lévy measure ν_k of $\mu_k \in M_k(\mathbb{R}^d)$ is given by

$$\nu_k(B) = \int_0^\infty \nu_0(u^{-1}B)\eta_{k-1}(u)u^{-1}du.$$

Suppose $\mu_{k+1} \in M_{k+1}(\mathbb{R}^d)$ and denote its Lévy measure by ν_{k+1}. Then,

$$\begin{aligned}
\nu_{k+1}(B) &= \int_0^\infty \nu_k(u^{-1}B)\phi(u)u^{-1}du \quad (\text{by the definition of } M_{k+1}(\mathbb{R}^d)) &(4.3)\\
&= \int_0^\infty \phi(u)u^{-1}du\int_0^\infty \nu_0(u^{-1}v^{-1}B)\eta_{k-1}(v)v^{-1}dv\\
&= \int_0^\infty \eta_{k-1}(v)v^{-1}dv\int_0^\infty \nu_0(y^{-1}B)\phi(yv^{-1})y^{-1}dy\\
&= \int_0^\infty \nu_0(y^{-1}B)y^{-1}dy\int_0^\infty \eta_{k-1}(v)\phi(yv^{-1})v^{-1}dv\\
&= \int_0^\infty \nu_0(y^{-1}B)\eta_k(y)y^{-1}dy \quad (\text{by (4.1)}). &(4.4)
\end{aligned}$$

This shows the assertion is also true for $k+1$.

(ii) ("If" part.) The assertion is true for $k = 1$. Namely, by the definition of $M_1(\mathbb{R}^d)$, if

$$\nu_1(B) = \int_0^\infty \nu_0(u^{-1}B)\phi(u)u^{-1}du$$

for some ν_0, the Lévy measure of some $\mu_0 \in M_0(\mathbb{R}^d)$, then μ_1 whose Lévy measure is ν_1 belongs to $M_1(\mathbb{R}^d)$. Suppose that the assertion is true for some $k \in \mathbb{N}$ and suppose that $\mu_{k+1} \in I_{\text{sym}}(\mathbb{R}^d)$ have the Lévy measure $\nu_{k+1}(B) = \int_0^\infty \nu_0(u^{-1}B)\eta_k(u)u^{-1}du$. Then from the calculation from (4.3) to (4.4), we have

$$
\begin{aligned}
\nu_{k+1}(B) &= \int_0^\infty \phi(u)u^{-1}du \int_0^\infty \nu_0(v^{-1}B)\eta_{k-1}(v)v^{-1}dv \\
&= \int_0^\infty \phi(u)u^{-1}\nu_k(u^{-1}B)du
\end{aligned}
$$

and μ_k with the Lévy measure ν_k belongs to $M_k(\mathbb{R}^d)$ by the induction hypothesis. Thus $\mu_{k+1} \in M_{k+1}(\mathbb{R}^d)$ follows from Definition 4.1. This completes the proof. □

The following is a characterization of the Lévy measures of distributions in $M_k(\mathbb{R}^d)$ in terms of the g–function of the Lévy measure.

Theorem 4.4. *Let $k \in \mathbb{N}$. A $\mu_k \in I_{\text{sym}}(\mathbb{R}^d)$ belongs to $M_k(\mathbb{R}^d)$ if and only if its Lévy measure ν_k is either zero or it can be represented as*

$$
\nu_k(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)g_{k,\xi}(r^2)r^{-1}dr,\ B \in \mathcal{B}_0(\mathbb{R}^d),
$$

where $g_{k,\xi}(r)$ is represented as

$$
g_{k,\xi}(s) = \int_0^\infty \eta_{k-1}(s^{1/2}y^{-1})g_\xi(y^2)y^{-1}dy. \tag{4.5}
$$

Here $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in r for λ– a.e. ξ.

Proof. Recall from (1.5) and (3.1) that

$$
\nu_0(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)g_\xi(r^2)r^{-1}dr.
$$

We see by Theorem 4.3 that $\mu_k \in M_k(\mathbb{R}^d)$ if and only if ν_k is represented as

$$
\begin{aligned}
\nu_k(B) &= \int_0^\infty \nu_0(u^{-1}B)\eta_{k-1}(u)u^{-1}du \\
&= \int_0^\infty \eta_{k-1}(u)u^{-1}du \int_S \lambda(d\xi) \int_0^\infty 1_{u^{-1}B}(y\xi)g_\xi(y^2)y^{-1}dy. \\
&= \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)r^{-1}dr \int_0^\infty \eta_{k-1}(ry^{-1})g_\xi(y^2)y^{-1}dy \\
&= \int_S \lambda(d\xi) \int_{-\infty}^\infty 1_B(r\xi)g_{k,\xi}(r^2)r^{-1}dr.
\end{aligned}
$$

This completes the proof. □

4.2 Characterizations of $M_k(\mathbb{R}^d)$, $k \in \mathbb{N}$, by stochastic integrals

In this section, we characterize distributions in $M_k(\mathbb{R}^d)$ by stochastic integral representations. Let $m_k(x) = \int_x^\infty \eta_k(u)u^{-1}du, x > 0$. Since $m_k(x)$ is strictly monotone, we can define its inverse by $m_k^*(t)$, that is, $t = m_k(x)$ if and only if $x = m_k^*(t)$.

Lemma 4.5. *For each $k \in \mathbb{N}$ and $|x| > 1$, there exists $C > 0$ such that*

$$\int_0^\infty \phi(u_1)u_1^{-1}du_1 \cdots \int_0^\infty \phi(u_k)u_k^{-1}\phi\left(\left(\prod_{i=1}^k u_i|x|\right)^{-1}\right) du_k \leq C(\log|x|)^k. \tag{4.6}$$

Proof. For the proof, we use induction argument. Let $k = 1$ and for $|x| > 1$, we have

$$\begin{aligned}
\int_0^\infty \phi(u)u^{-1}\phi\left(u^{-1}|x|^{-1}\right) du &= \left(\int_0^1 + \int_1^\infty\right) \phi(u)u^{-1}\phi(u^{-1}|x|^{-1})du \\
&\leq \int_0^{|x|} \phi(w/|x|)w^{-1}\phi(w^{-1})dw + C \\
&\leq \left(\int_0^1 + \int_1^{|x|}\right) w^{-1}\phi(w^{-1})dw + C \\
&\leq C + \int_1^{|x|} w^{-1}dw + C \\
&\leq C\log|x| + C,
\end{aligned}$$

where and in what follows C will denote an absolute positive constant which may be different from one to another.

Next suppose the statement is true for some $k \geq 1$ and show it is also true for $k+1$. We have, for $|x| > 1$,

$$\begin{aligned}
&\int_0^\infty \phi(u_1)u_1^{-1}du_1 \cdots \int_0^\infty \phi(u_{k+1})u_{k+1}^{-1}\phi\left(\left(\prod_{i=1}^{k+1} u_i|x|\right)^{-1}\right) du_{k+1} \\
&= \int_0^\infty \phi(u_1)u_1^{-1}du_1 \cdots \int_0^\infty \phi(u_k)u_k^{-1}du_k \\
&\qquad \left(\int_0^1 + \int_1^\infty\right) \phi(u_{k+1})u_{k+1}^{-1}\phi\left(\left(\prod_{i=1}^{k+1} u_i|x|\right)^{-1}\right) du_{k+1} \\
&=: H_1 + H_2,
\end{aligned}$$

say. Here

$$
\begin{aligned}
H_1 =& \int_0^\infty \phi(u_1)u_1^{-1}du_1 \\
&\cdots \int_0^\infty \phi(u_k)u_k^{-1}du_k \int_{1/\prod_{i=1}^k u_i|x|}^\infty \phi\left(\left(\prod_{i=1}^k u_i|x|y\right)^{-1}\right)\phi(y)y^{-1}dy \\
\le& \int_0^\infty \phi(u_1)u_1^{-1}du_1 \cdots \int_0^\infty \phi(u_k)u_k^{-1}du_k \left(\int_{1/\prod_{i=1}^k u_i|x|}^{1/\prod_{i=1}^k u_i} + \int_{1/\prod_{i=1}^k u_i}^\infty\right)\phi(y)y^{-1}dy \\
\le& \int_0^\infty \phi(u_1)u_1^{-1}du_1 \\
&\cdots \int_0^\infty \phi(u_k)u_k^{-1}du_k\phi\left(\left(\prod_{i=1}^k u_i|x|\right)^{-1}\right)\log y\Big|_{y=1/\prod_{i=1}^k u_i|x|}^{1/\prod_{i=1}^k u_i} + C \\
\le& \log|x|\int_0^\infty \phi(u_1)u_1^{-1}du_1 \cdots \int_0^\infty \phi(u_k)u_k^{-1}\phi\left(\left(\prod_{i=1}^k u_i|x|\right)^{-1}\right)du_{k+1} + C \\
\le& C(\log|x|)^{k+1} + C,
\end{aligned}
$$

since (4.6) is supposed. And

$$
\begin{aligned}
H_2 =& \int_0^\infty \phi(u_1)u_1^{-1}du_1 \cdots \int_0^\infty \phi(u_k)u_k^{-1}du_k \int_0^1 \phi\left(y\left(\prod_{i=1}^k u_i|x|\right)^{-1}\right)y^{-1}\phi(y^{-1})dy \\
\le& \int_0^1 C(\log y^{-1}|x|)^k y^{-1}\phi(y^{-1})dy \quad \text{(since (4.6) is supposed)} \\
=& \int_1^\infty C(\log|x|w)^k w^{-1}\phi(w)dw \le C\sum_{i=0}^k (\log|x|)^i.
\end{aligned}
$$

This shows the statement is true for $k+1$. This completes the proof. □

Theorem 4.6. *For each $k \in \mathbb{N}$, the stochastic integral*

$$\int_0^\infty m_k^*(t)dX_t^{(\mu)}$$

exists for every $\mu \in I_{\log^{k+1}}(\mathbb{R}^d)$.

Proof. For the proof, we use Lemma 3.3 again. It is enough to show that $f(t) = m_k^*(t)$ satisfies (3.2) – (3.4) in Lemma 3.3 for every $\mu \in I_{\log^{k+1}}(\mathbb{R}^d)$. Note that $m_k(+0) = \infty$ and $m_k(\infty) = 0$. Since

$$\int_0^\infty m_k^*(t)^2dt = \int_0^\infty s^2\eta_k(s)s^{-1}ds$$

$$= \int_0^\infty \phi(u_1)u_1^{-1}du_1 \cdots \int_0^\infty \phi(u_k)u_k^{-1}du_k \int_0^\infty s\phi\left(s\left(\prod_{i=1}^k u_i\right)^{-1}\right) ds$$

$$= \int_0^\infty \phi(u_1)u_1 du_1 \cdots \int_0^\infty \phi(u_k)u_k du_k \int_0^\infty y\phi(y)dy = (2\pi)^{-(k+1)/2} < \infty,$$

we have (3.2).

As to (3.3), we have

$$\int_0^\infty dt \int_{\mathbb{R}^d} (|m_k^*(t)x|^2 \wedge 1)\nu(dx) = -\int_0^\infty dm_k(s) \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu(dx)$$
$$= \int_0^\infty \eta_k(s)s^{-1}ds \left(\int_{|x|\le 1/s} |sx|^2\nu(dx) + \int_{|x|>1/s} \nu(dx)\right) =: I_1 + I_2,$$

say. Here

$$I_1 = \int_{\mathbb{R}^d} |x|^2\nu(dx) \int_0^{1/|x|} s\eta_k(s)ds$$
$$= \left(\int_{|x|\le 1} + \int_{|x|>1}\right) |x|^2\nu(dx) \int_0^{1/|x|} s\eta_k(s)ds =: I_{11} + I_{12},$$

say, and

$$I_{11} \le \int_{|x|\le 1} |x|^2\nu(dx) \int_0^\infty s\eta_k(s)ds < \infty,$$
$$I_{12} = \int_{|x|>1} |x|^2\nu(dx) \int_0^\infty \phi(u_1)u_1du_1 \cdots \int_0^\infty \phi(u_k)u_kdu_k \int_0^{1/|x|} \phi(y)ydy$$
$$= (2\pi)^{-k/2} \int_{|x|>1} |x|^2\nu(dx) \int_0^{1/|x|} \phi(y)ydy$$
$$= (2\pi)^{-k/2} \int_{|x|>1} |x|^2\nu(dx)|x|^{-2} \int_0^1 \phi(w|x|^{-1})wdw$$
$$\le (2\pi)^{-k/2} \int_{|x|>1} \nu(dx) < \infty.$$

Also,

$$I_2 = \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty \eta_k(s)ds = \left(\int_{|x|\le 1} + \int_{|x|>1}\right) \nu(dx) \int_{1/|x|}^\infty \eta_k(s)s^{-1}ds =: I_{21} + I_{22},$$

say. As to I_{21}, we have

$$I_{21} \le \int_{|x|\le 1} \nu(dx) \int_0^\infty \phi(u_1)u_1^{-1}du_1 \cdots \int_0^\infty \phi(u_k)u_k^{-1}du_k \left(\prod_{i=1}^k u_i\right)^2 \int_{1/|x|}^\infty 2s^{-3}ds$$

$$\leq C \int_{|x|\leq 1} |x|^2 \nu(dx) < \infty.$$

And

$$\begin{aligned}
I_{22} = &\int_{|x|>1} \nu(dx) \int_0^\infty \phi(u_1) u_1^{-1} du \\
&\cdots \int_0^\infty \phi(u_k) u_k^{-1} du_k \int_{1/|x|}^\infty \phi\left(s \left(\prod_{i=1}^k u_i |x| \right)^{-1} \right) s^{-1} ds \\
= &\int_{|x|>1} \nu(dx) \int_0^\infty \phi(u_1) u_1^{-1} du \cdots \int_0^\infty \phi(u_k) u_k^{-1} du_k \int_{1/\prod_{i=1}^k u_i |x|^2}^\infty \phi(y) y^{-1} dy \\
= &\int_{|x|>1} \nu(dx) \int_0^\infty \phi(u_1) u_1^{-1} du \\
&\cdots \int_0^\infty \phi(u_k) u_k^{-1} du_k \left(\int_{1/\prod_{i=1}^k u_i |x|^2}^{1/\prod_{i=1}^k u_i} + \int_{1/\prod_{i=1}^k u_i}^\infty \right) \phi(y) y^{-1} dy \\
\leq &\int_{|x|>1} \nu(dx) \int_0^\infty \phi(u_1) u_1^{-1} du \\
&\cdots \int_0^\infty \phi(u_k) u_k^{-1} \left(\phi\left(\left(\prod_{i=1}^k u_i |x|^2 \right)^{-1} \right) \log y \Big|_{y=1/\prod_{i=1}^k u_i |x|^2}^{1/\prod_{i=1}^k u_i} + \prod_{i=1}^k u_i \right) du_k \\
= &\int_{|x|>1} \nu(dx) \\
&\left(\log |x|^2 \int_0^\infty \phi(u_1) u_1^{-1} du_1 \cdots \int_0^\infty \phi(u_k) u_k^{-1} \phi\left(\left(\prod_{i=1}^k u_i |x|^2 \right)^{-1} \right) du_k + C \right) \\
\leq &\int_{|x|>1} \left(C (\log |x|^2)^{k+1} + C \right) \nu(dx) \quad \text{(by Lemma 4.5)} \\
\leq &\int_{|x|>1} \left(C (\log |x|)^{k+1} + C \right) \nu(dx) < \infty.
\end{aligned}$$

For (3.4), we have

$$\begin{aligned}
&\int_0^\infty \left| m_k^*(t) \gamma + m_k^*(t) \int_{\mathbb{R}^d} x \left(\left(1 + |m_k^*(t) x|^2\right)^{-1} - \left(1 + |x|^2\right)^{-1} \right) \nu(dx) \right| dt \\
&\quad \leq - |\gamma| \int_0^\infty s dm_k(s) \\
&\qquad - \int_0^\infty \left| s \int_{\mathbb{R}^d} x \left(\left(1 + |sx|^2\right)^{-1} - \left(1 + |x|^2\right)^{-1} \right) \nu(dx) \right| dm_k(s) =: I_3 + I_4,
\end{aligned}$$

say, where

$$
\begin{aligned}
I_3 \leq& |\gamma| \int_0^\infty \eta_k(s)ds < \infty, \\
I_4 \leq& \int_0^\infty \eta_k(s)ds \left| \int_{\mathbb{R}^d} \left(\left(x|x|^2|s^2-1|\right) \left((1+|sx|^2)(1+|x|^2)\right)^{-1} \right) \nu(dx) \right| \\
\leq& \int_0^\infty |s^2-1|\eta_k(s)ds \int_{\mathbb{R}^d} |x|^3 \left((1+|sx|^2)(1+|x|^2)\right)^{-1} \nu(dx) \\
=& \int_0^\infty |s^2-1|\eta_k(s)ds \\
& \left(\int_{|x|\leq 1} + \int_{|x|>1} \right) |x|^3 \left((1+|sx|^2)(1+|x|^2)\right)^{-1} \nu(dx) =: I_{41} + I_{42},
\end{aligned}
$$

say. Here

$$
I_{41} \leq \int_0^\infty |s^2-1|\eta_k(s)ds \int_{|x|\leq 1} |x|^3 \left(1+|x|^2\right)^{-1} \nu(dx) < \infty,
$$

and

$$
\begin{aligned}
I_{42} \leq& \int_{|x|>1} |x|^3 \left(1+|x|^2\right)^{-1} \nu(dx) \int_0^\infty \left(s^2+1\right) \left(1+|sx|^2\right)^{-1} \eta_k(s)ds \\
=& \int_{|x|>1} |x|^3 \left(1+|x|^2\right)^{-1} \nu(dx) \\
& \left(\int_0^1 + \int_1^\infty \right) \left(s^2+1\right) \left(1+|sx|^2\right)^{-1} \eta_k(s)ds =: I_{421} + I_{422},
\end{aligned}
$$

say. Furthermore,

$$
\begin{aligned}
I_{421} =& \int_{|x|>1} |x|^3 \left(1+|x|^2\right)^{-1} \nu(dx) \int_0^1 \left(s^2+1\right) \left(1+|sx|^2\right)^{-1} \eta_k(s)ds \\
=& \int_{|x|>1} |x|^3 \left(1+|x|^2\right)^{-1} \nu(dx) \\
& \left(\int_0^{1/|x|} + \int_{1/|x|}^1 \right) \left(s^2+1\right) \left(1+|sx|^2\right)^{-1} \eta_k(s)ds =: I_{4211} + I_{4212},
\end{aligned}
$$

say. We have

$$
\begin{aligned}
I_{4211} \leq& \int_{|x|>1} |x|\nu(dx) \int_0^{1/|x|} \eta_k(s)ds \\
=& \int_{|x|>1} |x|\nu(dx) \int_0^\infty \phi(u_1)u_1du_1 \cdots \int_0^\infty \phi(u_k)u_kdu_k \int_0^{1/|x|} \phi(y)dy
\end{aligned}
$$

$$= (2\pi)^{-2/k} \int_{|x|>1} |x|\nu(dx) \int_0^{1/|x|} \phi(y)dy$$
$$= (2\pi)^{-2/k} \int_{|x|>1} |x|\nu(dx)|x|^{-1} \int_0^1 \phi(w|x|)dw \le (2\pi)^{-2/k} \int_{|x|>1} \nu(dx) < \infty,$$

and

$$\begin{aligned} I_{4212} &\le \int_{|x|>1} \nu(dx) \int_{1/|x|}^1 \left(|sx|(s^2+1)\right)\left(1+|sx|^2\right)^{-1} \eta_k(s)s^{-1}ds \\ &\le \int_{|x|>1} \nu(dx) \int_{1/|x|}^1 \eta_k(s)s^{-1}ds \\ &\le \int_{|x|>1} \nu(dx) \int_{1/|x|}^\infty \eta_k(s)s^{-1}ds = I_{22} < \infty. \end{aligned}$$

Also

$$\begin{aligned} I_{422} &= \int_{|x|>1} |x|^3 \left(1+|x|^2\right)^{-1} \nu(dx) \int_1^\infty \left(s^2+1\right)\left(1+|sx|^2\right)^{-1} \eta_k(s)ds \\ &\le \int_{|x|>1} |x|^3 \left(1+|x|^2\right)^{-2} \nu(dx) \int_1^\infty (s^2+1)\eta_k(s)ds < \infty. \end{aligned}$$

Thus we have (3.4). This completes the proof. □

Let $\mathcal{M}_1 = \mathcal{M}^1 = \mathcal{M}$.

Definition 4.7. Let $k \in \mathbb{N}$. Define the mapping $\mathcal{M}_{k+1}$ by

$$\mathcal{M}_{k+1}(\mu) = \mathcal{L}\left(\int_0^\infty m_k^*(t)dX_t^{(\mu)}\right), \quad \mu \in I_{\log^{k+1}}(\mathbb{R}^d)$$

and $\mathcal{M}^{k+1}$ be the $(k+1)$ times iteration of $\mathcal{M}$. That is, $\mathcal{M}^{k+1}(\mu)$ can be defined with $\mathcal{M}^{k+1}(\mu) = \mathcal{M}(\mathcal{M}^k(\mu))$ if and only if $\mathcal{M}^k(\mu)$ is defined and belongs to $I_{\log}(\mathbb{R}^d)$.

Theorem 4.8. *For $k \in \mathbb{N}$,*

$$M_k(\mathbb{R}^d) = \mathcal{M}(M_{k-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)).$$

Proof. The proof is almost the same as that of Theorem 3.5 (i) in Chapter 3. Let $\mu_{k-1} \in M_{k-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)$ and $\mu_k = \mathcal{M}(\mu_{k-1})$. Also let ν_{k-1} and ν_k be the Lévy measures of μ_{k-1} and μ_k, respectively. Then by Theorem 3.5, we have $\nu_k(B) = \int_0^\infty \nu_{k-1}(s^{-1}B)\phi(s)s^{-1}ds$. Thus $\mu_k \in M_k(\mathbb{R}^d)$ by the definition 4.1, and $\mathcal{M}(M_{k-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)) \subset M_k(\mathbb{R}^d)$.

Conversely, suppose that $\mu_k \in M_k(\mathbb{R}^d)$. Then by the definition of $M_k(\mathbb{R}^d)$ and Theorem 3.5 again, we see that $\mu_k = \mathcal{L}\left(\int_0^\infty m^*(t)dX_t^{(\mu)}\right)$ for some $\mu \in M_{k-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)$. This means that $\mu_k \in \mathcal{M}(M_{k-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d))$, and $M_k(\mathbb{R}^d) \subset \mathcal{M}(M_{k-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d))$, completing the proof. □

Corollary 4.9. *For $k \in \mathbb{N}$,*

$$M_k(\mathbb{R}^d) = \mathcal{M}^{k+1}(I_{\log^{k+1}}(\mathbb{R}^d)) \cap I_{\mathrm{sym}}(\mathbb{R}^d).$$

We next show

Theorem 4.10. *For $k \in \mathbb{N}$*

$$\mathcal{M}_{k+1}(I_{\log}(\mathbb{R}^d)) = \mathcal{M}^{k+1}(I_{\log^{k+1}}(\mathbb{R}^d)).$$

Proof. We note that $\widetilde{\mu} \in \mathcal{M}_{k+1}(I_{\log^{k+1}}(\mathbb{R}^d))$ if and only if

$$\widetilde{\mu} = \mathcal{L}\left(\int_0^\infty m_k^*(t)dX_t^{(\mu)}\right), \quad \mu \in I_{\log^{k+1}}(\mathbb{R}^d)$$

and that $\widetilde{\mu} \in \mathcal{M}^{k+1}(I_{\log^{k+1}}(\mathbb{R}^d))$ if and only if

$$\widetilde{\mu} = \mathcal{L}\left(\int_0^\infty m^*(t)dX_t^{(\mu)}\right),$$

where $\mu \in I_{\log}(\mathbb{R}^d)$ and has the Lévy measure ν_{k-1} of the form in (4.2).

We next claim that, for any $\mu \in I_{\log^{k+1}}(\mathbb{R}^d)$,

$$\int_0^\infty \phi(u)u^{-1}du \int_0^\infty |C_\mu(uvz)|\eta_{k-1}(v)v^{-1}dv < \infty, \ z \in \mathbb{R}^d. \tag{4.7}$$

If it would be proved, we could exchange the order of the integrals in the calculation of cumulants below.

The proof of (4.7) is as follows. The idea is from Barndorff–Nielsen et al. [BMS06]. If the generating triplet of μ is (A, ν, γ), then

$$|C_\mu(z)| \le 2^{-1}(\mathrm{tr}A)|z|^2 + |\gamma||z| + \int_{\mathbb{R}^d} |g(z,x)|\nu(dx),$$

where

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle(1+|x|^2)^{-1}.$$

Hence

$$|C_\mu(uvz)| \le 2^{-1}(\mathrm{tr}A)u^2v^2|z|^2 + |\gamma||u||v||z| + \int_{\mathbb{R}^d} |g(z,uvx)|\nu(dx)$$

$$+\int_{\mathbb{R}^d}|g(uvz,x)-g(z,uvx)|\nu(dx)=: J_1+J_2+J_3+J_4,$$

say. The finiteness of $\int_0^\infty \phi(u)u^{-1}du\int_0^\infty (J_1+J_2)\eta_{k-1}(v)v^{-1}dv$ is easily to be shown by the same calculation as in the proof of Theorem 4.6.

Noting that $|g(z,x)|\le C_z|x|^2(1+|x|^2)^{-1}$ with a positive constant C_z depending on z, we have

$$\begin{aligned}
&\int_0^\infty \phi(u)u^{-1}du\int_0^\infty J_3\eta_{k-1}(v)v^{-1}dv\\
&\quad\le C_z\int_{\mathbb{R}^d}\nu(dx)\int_0^\infty \phi(u)u^{-1}du\int_0^\infty |uvx|^2\left(1+|uvx|^2\right)^{-1}\eta_{k-1}(v)v^{-1}dv\\
&\quad= C_z\int_{\mathbb{R}^d}\nu(dx)\int_0^\infty |sx|^2\left(1+|sx|^2\right)^{-1}\eta_k(s)s^{-1}ds\\
&\quad= C_z\left(\int_{|x|\le 1}\nu(dx)+\int_{|x|>1}\nu(dx)\right)\int_0^\infty |sx|^2\left(1+|sx|^2\right)^{-1}\eta_k(s)s^{-1}ds\\
&\quad=: J_{31}+J_{32},
\end{aligned}$$

say, and

$$\begin{aligned}
J_{31}&\le C_z\int_{|x|\le 1}|x|^2\nu(dx)\int_0^\infty s\eta_k(s)ds<\infty,\\
J_{32}&= C_z\int_{|x|>1}\nu(dx)\left(\int_0^{1/|x|}+\int_{1/|x|}^\infty\right)|sx|^2\left(1+|sx|^2\right)^{-1}\eta_k(s)s^{-1}ds\\
&=: J_{321}+J_{322},
\end{aligned}$$

say. We have

$$J_{321}\le 2^{-1}\int_{|x|>1}|x|\nu(dx)\int_0^{1/|x|}\eta_k(s)ds<\infty,$$

by the finiteness of I_{4211} in the proof of Theorem 4.6.

Also, we have the finiteness of J_{322} by Lemma 4.5.

As to J_4, note that for $a>0$,

$$\begin{aligned}
|g(az,x)-g(z,ax)|&=|\langle az,x\rangle||x|^2|1-a^2|(1+|x|^2)^{-1}(1+a|x|^2)^{-1}\\
&\le |z||x|^3a(1+a^2)(1+|x|^2)^{-1}(1+a|x|^2)^{-1}.
\end{aligned}$$

Then

$$\int_0^\infty \phi(u)u^{-1}du\int_0^\infty J_4\eta_{k-1}(v)v^{-1}dv$$

$$
\begin{aligned}
\le & |z|\int_{\mathbb{R}^d}\nu(dx)\int_0^\infty \phi(u)u^{-1}du \\
& \int_0^\infty |x|^3uv(1+u^2v^2)(1+|x|^2)^{-1}(1+u^2v^2|x|^2)^{-1}\eta_{k-1}(v)v^{-1}dv \\
= & |z|\int_{\mathbb{R}^d}\nu(dx)\int_0^\infty |x|^3s(1+s^2)(1+|x|^2)^{-1}(1+|sx|^2)^{-1}\eta_k(s)s^{-1}ds \\
= & |z|\left(\int_{|x|\le 1}+\int_{|x|>1}\right)\nu(dx)\int_0^\infty |x|^3(1+s^2)(1+|x|^2)^{-1}(1+|sx|^2)^{-1}\eta_k(s)ds \\
= & : J_{41}+J_{42},
\end{aligned}
$$

say. Here

$$
\begin{aligned}
J_{41} &\le |z|\int_{|x|\le 1}|x|^2\nu(dx)\int_0^\infty |x|(1+s^2)(1+|x|^2)^{-1}(1+|sx|^2)^{-1}\eta_k(s)ds \\
&\le 2^{-1}|z|\int_{|x|\le 1}|x|^2\nu(dx)\int_0^\infty (1+s^2)\left(1+|sx|^2\right)^{-1}\eta_k(s)ds \\
&\le 2^{-1}|z|\int_{|x|\le 1}|x|^2\nu(dx)\int_0^\infty (1+s^2)\eta_k(s)ds<\infty,
\end{aligned}
$$

and

$$
J_{42}=|z|\int_{|x|>1}|x|^3\left(1+|x|^2\right)^{-1}\nu(dx)\int_0^\infty \left(1+s^2\right)\left(1+|sx|^2\right)^{-1}\eta_k(s)ds<\infty.
$$

The finiteness of J_{42} follows from that of I_{42} in the proof of Theorem 4.6.

This completes the proof of (4.7).

If we calculate the necessary cumulants, we have

$$
\begin{aligned}
C_{\mathcal{M}_{k+1}(\mu)}(z) &= \int_0^\infty C_\mu(m_k^*(t)z)dt \\
&= -\int_0^\infty C_\mu(uz)dm_k(u)=\int_0^\infty C_\mu(uz)\eta_k(u)u^{-1}du \\
C_{\mathcal{M}^{k+1}(\mu)}(z) &= \int_0^\infty C_{\mathcal{M}^k(\mu)}(m^*(t)z)dt=\int_0^\infty dt\int_0^\infty C_\mu(m^*(t)m_{k-1}^*(s)z)ds \\
&= \int_0^\infty dm(u)\int_0^\infty C_\mu(uvz)dm_{k-1}(v) \\
&= \int_0^\infty \phi(u)u^{-1}du\int_0^\infty C_\mu(uvz)\eta_{k-1}(v)v^{-1}dv \\
&= \int_0^\infty C_\mu(yz)y^{-1}dy\int_0^\infty \phi(yv^{-1})\eta_{k-1}(v)v^{-1}dv \\
&= \int_0^\infty C_\mu(yz)\eta_k(y)y^{-1}dy=C_{\mathcal{M}^{k+1}(\mu)}(z).
\end{aligned}
$$

This completes the proof of Theorem 4.10. □

The following is a goal of this section and a M_k–version of Theorem 3.5 (i). Namely, any $\mu \in M_k(\mathbb{R}^d)$ has the stochastic integral representation defined in Definition 4.7.

Theorem 4.11.

$$M_k(\mathbb{R}^d) = \mathcal{M}_{k+1}(I_{\log^{k+1}}(\mathbb{R}^d)) \cap I_{\mathrm{sym}}(\mathbb{R}^d).$$

Proof. The statement is an immediate consequence of Corollary 4.9 and Theorem 4.10. □

4.3 The classes $M_\infty(\mathbb{R}^d)$

Theorem 4.12.

$$M_\infty(\mathbb{R}^d) \supset S_{\mathrm{sym}}(\mathbb{R}^d),$$

where $S_{\mathrm{sym}}(\mathbb{R}^d)$ is the class of all symmetric stable distributions on $\mathbb{R}^d$.

Proof. Let $k \geq 1$. When μ_A is Gaussian with zero mean and covariance matrix A, suppose $\{X_t\}$ is a Gaussian Lévy process such that the covariance matrix of X_1 is $c_k^{-1}A$, where $c_k = \left(\int_0^\infty m_k^*(t)^2 dt\right)$. Then we have

$$\mu_A = \mathcal{L}\left(\int_0^\infty m_k^*(t)dX_t\right) \in M_k(\mathbb{R}^d)$$

for any $k \geq 1$. Hence $\mu \in M_\infty(\mathbb{R}^d)$.

When μ is non–Gaussian α–stable with the Lévy measure ν, we have

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)r^{-(1+\alpha)}dr = \int_S \lambda_k(d\xi) \int_0^\infty 1_B(r\xi)c_k r^{-(1+\alpha)}dr,$$

where

$$c_k = \int_0^\infty m_{k-1}^*(t)^\alpha dt \quad \text{and} \quad \lambda_k(d\xi) = c_k^{-1}\lambda(d\xi).$$

We also have

$$\begin{aligned} c_k r^{-(1+\alpha)} &= -r^{-(1+\alpha)} \int_0^\infty u^\alpha dm_{k-1}(u) = r^{-1} \int_0^\infty (ur^{-1})^\alpha \eta_{k-1}(u)u^{-1}dt \\ &= r^{-1} \int_0^\infty \eta_{k-1}(ry^{-1})y^{-(1+\alpha)}dy = r^{-1} \int_0^\infty \eta_{k-1}(ry^{-1})g(y^2)y^{-1}dy, \end{aligned}$$

where

$$g(s) = s^{-\alpha/2},$$

which is completely monotone. Thus, by Theorem 4.4, $c_k r^{-(1+\alpha)}$ can be regarded as $g_{k,\xi}(r)r^{-1}$, implying that ν is the Lévy measure of a distribution in $M_k(\mathbb{R}^d)$. This is true for all k, and thus $\mu \in M_\infty(\mathbb{R}^d)$. $\square$

4.4 More about the classes $M_k(\mathbb{R})$ when $d = 1$

In Chapter 3, we have characterized class $M_0(\mathbb{R})$ in terms of V in $V^{1/2}Z$ as a variance mixture of normal distribution. We characterize the distribution of the random variance V in the case of $\mu \in M_k(\mathbb{R})$.

Theorem 4.13. *Let $k = 1, 2, \cdots$. A necessary and sufficient condition for that $\mu \in M_0(\mathbb{R})$ belongs to a smaller class $M_k(\mathbb{R})$ is that*

$$\rho(dx) = 2^{-1}(2\pi x)^{-1/2}\left\{\int_0^\infty \phi(u_1)u_1^{-1}du_1 \cdots \int_0^\infty \phi(u_{k-2})u_{k-2}^{-1}du_{k-2}\right. \tag{4.8}$$
$$\left.\int_0^\infty \phi(u_{k-1})u_{k-1}^{-1} g\left(x\left(\prod_{i=1}^{k-1} u_i\right)^{-2}\right) du_{k-1}\right\}dx,$$

where $g(\cdot)$ is completely monotone.

The proof is almost the same as that of Theorem 3.12 in Chapter 3.

Proof. (i) ("Only if" part.) Suppose $\mu \in M_k(\mathbb{R})$. Since $M_k(\mathbb{R}) \subset G(\mathbb{R})$, we have $\mu = \mathcal{L}(V^{1/2}Z)$ for some $V \in I(\mathbb{R}_+)$. Thus, we get for $z \in \mathbb{R}$,

$$\begin{aligned}
E\left[e^{izV^{1/2}Z}\right] &= E\left[e^{-Vz^2/2}\right] = \exp\left\{-2^{-1}Az^2 + \int_{0+}^\infty (e^{-vz^2/2} - 1)\,\nu_V(dv)\right\}\\
&= \exp\left\{-2^{-1}Az^2 + \int_{0+}^\infty \nu_V(dv)\int_{-\infty}^\infty (e^{izv^{1/2}u} - 1)\phi(u)\,du\right\}\\
&= \exp\left\{-2^{-1}Az^2 + \int_{-\infty}^\infty (e^{izx} - 1)dx\int_{0+}^\infty \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv)\right\},
\end{aligned}$$

where $A \geq 0$. Therefore, the Lévy measure ν of μ is of the form

$$\nu(dx) = \left(\int_{0+}^\infty \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv)\right)dx. \tag{4.9}$$

By Theorem 4.4, $\mu \in M_k(\mathbb{R})$ if and only if $\nu(dx) = |x|^{-1} g_k(x^2)dx$, where g_k is given by (4.5). Since $\mu \in M_0(\mathbb{R})$, g_k is completely monotone. By Proposition 1.10, it can be written as

$$g_k(r) = \int_0^\infty e^{-ry/2}\, Q(dy), \quad r > 0,$$

for a measure Q on $(0, \infty)$ given by,

$$Q(dy) = (2\pi)^{-1/2}\, (2y)^{-1} \left\{ \int_0^\infty \phi(u_1) u_1^{-1} du_1 \cdots \int_0^\infty \phi(u_{k-2}) u_{k-2}^{-1} du_{k-2} \right.$$
$$\left. \int_0^\infty \phi(u_{k-1}) u_{k-1}^{-1} g\left(y^{-1} \left(\prod_{i=1}^{k-1} u_i \right)^{-2} \right) du_{k-1} \right\} dy,$$

where $g(\cdot)$ is completely monotone.

By (4.9), we get

$$\int_{0+}^\infty \phi(v^{-1/2}x) v^{-1/2}\, \nu_V(dv) = |x|^{-1} g_k(x^2). \tag{4.10}$$

Since

$$r^{-1/2} = (2\pi)^{-1/2} \int_0^\infty e^{-rw/2} w^{-1/2}\, dw, \quad r > 0,$$

we obtain

$$\begin{aligned} r^{-1/2} g(r) &= (2\pi)^{-1/2} \int_0^\infty \int_0^\infty e^{-r(w+y)/2} w^{-1/2}\, dw Q(dy) \\ &= (2\pi)^{-1/2} \int_0^\infty Q(dy) \int_y^\infty e^{-ru/2} (u-y)^{-1/2}\, du \\ &= (2\pi)^{-1/2} \int_0^\infty e^{-ru/2} du \int_0^u (u-y)^{-1/2}\, Q(dy). \end{aligned}$$

Taking $x = r^{1/2} > 0$ in (4.10), we get

$$\begin{aligned} &(2\pi)^{-1/2} \int_{0+}^\infty e^{-r/2v} v^{-1/2}\, \nu_V(dv) \\ &\qquad = (2\pi)^{-1/2} \int_0^\infty e^{-ru/2} du \int_0^u (u-y)^{-1/2}\, Q(dy). \end{aligned} \tag{4.11}$$

Let

$$\rho(dx) = -\, x^{1/2} Q(d(x^{-1}))$$

$$\left(= -\, 2^{-1}\,(2\pi x)^{-1/2} \left\{ \int_0^\infty \phi(u_1) u_1^{-1} du_1 \cdots \int_0^\infty \phi(u_{k-2}) u_{k-2}^{-1} du_{k-2} \right.\right. \tag{4.12}$$

$$\left.\left. \int_0^\infty \phi(u_{k-1}) u_{k-1}^{-1} g \left(x \left(\prod_{i=1}^{k-1} u_i \right)^{-2} \right) du_{k-1} \right\} dx \right).$$

Then $\ell(r)$ in (3.16) becomes

$$\begin{aligned} \ell(r) &= -\int_r^\infty (x-r)^{-1/2} x^{1/2} Q(d(x^{-1})) = \int_0^{r^{-1}} (y^{-1}-r)^{-1/2} y^{-1/2} Q(dy) \\ &= \int_0^{r^{-1}} (1-yr)^{-1/2} Q(dy) = r^{-1/2} \int_0^{r^{-1}} (r^{-1}-y)^{-1/2} Q(dy). \end{aligned}$$

Thus by (4.11),

$$\int_{0+}^\infty e^{-r/2v} v^{-1/2}\, \nu_V(dv) = \int_0^\infty e^{-ru/2} u^{-1/2} \ell(u^{-1})\, du$$

or

$$\int_{0+}^\infty e^{-r/2v} v^{-1/2}\, \nu_V(dv) = \int_0^\infty e^{-r/2v} v^{-3/2} \ell(v)\, dv, \quad r > 0.$$

Therefore

$$v^{-1/2}\, \nu_V(dv) = v^{-3/2} \ell(v)\, dv, \quad v > 0,$$

which yields (3.15).

The integrability condition (3.17) for Q is obtained from the fact that

$$\infty > \int_{\mathbb{R}} (x^2 \wedge 1)\, \nu(dx) = \int_{\mathbb{R}} (|x| \wedge |x|^{-1}) g_k(x^2) dx.$$

For, this yields that

$$\int_0^1 x dx \int_0^\infty e^{-x^2 y/2}\, Q(dy) < \infty \quad \text{and} \quad \int_1^\infty x^{-1} dx \int_0^\infty e^{-x^2 y/2}\, Q(dy) < \infty,$$

and hence

$$\int_0^\infty \left[y^{-1}(1-e^{-y/2}) + 2^{-1} \int_y^\infty u^{-1} e^{-u/2}\, du \right] Q(dy) < \infty.$$

It is obvious that the above condition is equivalent to

$$\int_0^1 (1+\log y^{-1}) Q(dy) + \int_1^\infty y^{-1} Q(dy) < \infty. \tag{4.13}$$

On the other hand,

$$\int_0^1 x^{1/2}\rho(dx) = -\int_0^1 xQ(d(x^{-1})) = \int_1^\infty y^{-1}Q(dy)$$

and

$$\int_1^\infty (1+\log x)x^{1/2}\rho(dx) = -\int_1^\infty (1+\log x)Q(d(x^{-1})) = \int_0^1 (1+\log y^{-1})Q(dy)).$$

Thus, we get (3.17) from (4.13) and (4.8) by (4.12). The ("only if" part) is thus proved.

(ii) ("If" part.) Suppose $\mu = \mathcal{L}(V^{1/2}Z)$ and the Lévy measure ν_V of V satisfies (3.15)–(3.17).

We first claim that the integrability condition (3.17) implies that ν_V is really a Lévy measure on $(0,\infty)$ of a positive infinitely divisible random variable, namely it satisfies

$$\int_0^\infty (r\wedge 1)\nu_V(dr) < \infty. \tag{4.14}$$

We have

$$\int_0^\infty (r\wedge 1)\nu_V(dr) = \int_0^1 r\nu_V(dr) + \int_1^\infty \nu_V(dr).$$

As to the first integral, we have

$$\begin{aligned}
\int_0^1 r\nu_V(dr) &= \int_0^1 \ell(r)dr = \int_0^1 dr\int_r^\infty (x-r)^{-1/2}\rho(dx)\\
&= \int_0^1 \rho(dx)\int_0^x (x-r)^{-1/2}dr + \int_1^\infty \rho(dx)\int_0^1 (x-r)^{-1/2}dr\\
&= 2\int_0^1 x^{1/2}\rho(dx) + 2\int_1^\infty \left(x^{1/2}-(x-1)^{1/2}\right)\rho(dx)\\
&\le 2\int_0^1 x^{1/2}\rho(dx) + const.\times\int_1^\infty x^{-1/2}\rho(dx)\\
&= -2\int_0^1 xQ(d(x^{-1})) - const.\times\int_1^\infty Q(d(x^{-1}))\\
&= 2\int_1^\infty x^{-1}Q(dx) + const.\times\int_0^1 Q(dx).
\end{aligned}$$

Next, as to the second integral,

$$\int_1^\infty \nu_V(dr) = \int_1^\infty r^{-1}\ell(r)dr = \int_1^\infty r^{-1}dr\int_r^\infty (x-r)^{-1/2}\rho(dx)$$

$$= \int_1^\infty \rho(dx) \int_1^x r^{-1}(x-r)^{-1/2} dr = \int_1^\infty (\log x + const.) x^{-1/2} \rho(dx)$$
$$= - \int_1^\infty (\log x + const.) Q(d(x^{-1})) = \int_0^1 (\log x^{-1} + const.) Q(dx).$$

Therefore, (3.17) implies (4.14). Furthermore, as we have already seen, ν_μ is expressed as in (4.9). So, to complete the proof, it is enough to show that when we put

$$g_k(x^2) = |x| \int_0^\infty \phi(v^{-1/2}x) v^{-1/2} \nu_V(dv),$$

then $g_k(r)$ is as (4.5) in Theorem 4.4. However, for that, it is enough to follow the proof of the ("only if" part) from the bottom to the top. This completes the proof. □

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