Global Solvability of the Free-Boundary Problem for Stellar Models of Self-Gravitating Viscous Radiative and Reactive Gas

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Contents

1 Introduction

For viscous, heat-conductive and isotropic Newtonian fluids, we have long history of study. However, it is mainly in the last fifty years that the mathematical theory for the fundamental system of equations describing the motion of such fluids has been established by many mathematicians. The motion of fluids mentioned above is governed by the following equations in Eulerian coordinate system corresponding to the conservation laws of mass, momentum and energy (see for example, Lamb [35], Landau-Lifshitz [36], Serrin [58] and Imai [16]):

$$\begin{cases} \rho_t + \nabla\cdot(\rho \boldsymbol{v}) = 0, \\ \rho \dfrac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \nabla\cdot \mathbb{T} + \rho \boldsymbol{f}, \\ \rho \dfrac{\mathrm{D}e}{\mathrm{D}t} = \mathbb{T} : \mathbb{D} - \nabla\cdot \boldsymbol{q}_{th} + \rho Q. \end{cases} \tag{1.1}$$

Unknown quantities, functions of time variable $t > 0$ and space variable $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, are the distributions of the density $\rho = \rho(x,t)$, the velocity vector field $\boldsymbol{v} = \boldsymbol{v}(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ and the absolute temperature $\theta = \theta(x,t)$. Here

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla$$

is the material derivative; $\mathbb{T} = (t_{ij})$ $(i,j = 1,2,3)$ is the stress tensor given by

$$\mathbb{T} = (-p + \mu'\nabla\cdot \boldsymbol{v})\,\mathbb{I} + 2\mu\mathbb{D},$$

$p = p(\rho,\theta)$ is the pressure, $\mathbb{D}$ is the velocity deformation tensor with elements

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad (i,j = 1,2,3)$$

$\mathbb{I}$ is the unit tensor of degree 3, $\mu = \mu(\rho,\theta)$ and $\mu' = \mu'(\rho,\theta)$ are coefficients of the shear (or the first) and the dilatational (or the second) viscosity, respectively, which satisfy $\mu > 0$ and $3\mu' + 2\mu \geq 0$; $\boldsymbol{f} = \boldsymbol{f}(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t))$ is the vector field of external forces per unit mass; $e = e(\rho,\theta)$ is the internal energy per unit mass; $\mathbb{T} : \mathbb{D} = \sum_{i,j=1}^{3} t_{ij} d_{ij}$; $\boldsymbol{q}_{th}$ is the thermal flux; Q is the heat supply per unit mass and unit time. In addition to this system, it is necessary to take into account a more phenomenal situation from the physical point of view: the combustion processes which produce the energy of the fluid itself and by which the chemical composition of the medium changes. Introducing the quantity, "the mass fraction of the reactant" $z = z(x,t)$, coupling the equation

$$\rho \frac{\mathrm{D}z}{\mathrm{D}t} = -\nabla\cdot \boldsymbol{q}_{ch} - \rho\phi z^m \tag{1.2}$$

which describes the processes of the unimolecular reactions (see [72]) with (1.1), and taking in $(1.1)^3$ as

$$Q = \lambda \phi z^m, \tag{1.3}$$

we obtain the system of a chemically active fluid model. Here $m \geq 1$ is the kinetic order of the reaction, $\boldsymbol{q}_{ch}$ is the chemical flux, a positive constant λ means the difference in heat between the reactant and the product, and $\phi = \phi(\rho, \theta)$ is the reaction rate function defined by, for example,

$$\phi(\rho, \theta) = \begin{cases} 0 & \text{for } \ 0 \leq \theta \leq \theta_i, \\ K\rho^{m-1}\theta^s \mathrm{e}^{-A/(\theta-\theta_i)} & \text{for } \ \theta > \theta_i \end{cases} \tag{1.4}$$

from the Arrhenius law (see [50]). In (1.4) positive constants A and K are the activation energy and the coefficient of rate of reactant, respectively, $s \in \mathbb{R}$ and non-negative value θ_i is the ignition temperature. Furthermore, according to Newton-Fourier's law, we can take the explicit formulas for the flux

$$\begin{cases} \boldsymbol{q}_{th} = -\kappa \nabla \theta, \\ \boldsymbol{q}_{ch} = -d\rho \nabla z, \end{cases} \tag{1.5}$$

where $\kappa = \kappa(\rho, \theta) > 0$ is the thermal conductivity and a positive constant d is the species diffusion coefficient.

One may consider equations (1.1) or (1.1), (1.2) in $\bigcup_{t>0} \big(\Omega_t \times \{t\}\big)$, where $\Omega_t \subseteq \mathbb{R}^3$ is a domain occupied by the fluid at $t > 0$, together with the initial or the initial-boundary conditions.

1.1 Historical studies of compressible viscous fluid

At first, we mention the history of studies for compressible viscous (and heat-conductive) fluid briefly (see for example, [48, 60]).

1.1.1 Well-posedness of the problems in three-dimensions

In 1959, for the system of equations (1.1) with $(1.5)^1$, Serrin [57] firstly proved the uniqueness theorem for the initial-boundary value problem in a bounded domain. Temporally local existence theorems for the Cauchy problem of (1.1) with $(1.5)^1$ are firstly established by Nash [47] in 1962 (however, it is pointed out in [66] that this work contained several ambiguous aspects), and independently by Itaya [17] in 1971 (uniqueness of the solution was proved in [19]).

As for the initial-boundary value problem of (1.1) with $(1.5)^1$ in both bounded and unbounded domains, the temporally local existence of the unique solution in anisotropic Hölder spaces was proved by Tani when f, p, e, κ are suitably smooth functions of their arguments. More precisely, in 1977 he settled corresponding the first-initial boundary value problem [65]; in 1981 the free-boundary problem [66], in which, since for each $t > 0$ the shape of the domain Ω_t is unknown *a priori*, free-surface represented by the equation $F = F(x,t) = 0$ must be also determined by coupling with (1.1) another equation called the kinematic boundary condition

$$\frac{\mathrm{D}F}{\mathrm{D}t} = 0 \qquad \text{on } S_t, \ \ t > 0. \tag{1.6}$$

Here $S_t := \partial\Omega_t$ for $t > 0$, on which it is imposed the dynamical and the thermal boundary conditions

$$\mathbb{T}\boldsymbol{n} = -p_e\boldsymbol{n}, \qquad \boldsymbol{q}_{th} \cdot \boldsymbol{n} = -\kappa_e(\theta - \theta_e), \tag{1.7}$$

where $\boldsymbol{n} = \boldsymbol{n}(x,t)$ is the unit vector of the outward normal to S_t and $(p_e, \kappa_e, \theta_e) = (p_e, \kappa_e, \theta_e)(x,t)$ are the external pressure, the external thermal conductivity and the external absolute temperature, respectively. For the free-boundary problem Secchi-Valli [56] found a unique solution in Sobolev spaces under the the conditions $\mu = \mu(\rho)$, $\mu' = \mu'(\rho)$, $\kappa = \kappa(\rho, \theta, \boldsymbol{v})$ and $3\mu' + 2\mu > 0$. Secchi also solved, in Sobolev spaces, various initial-boundary value problems of (1.1) with $(1.5)^1$ locally in time: in [52] on the problem in a fixed bounded domain under the conditons $\mu = 0$, $\mu' = \mu'(\rho, \theta, \boldsymbol{v})$, $\kappa = \kappa(\rho, \theta, \boldsymbol{v})$ and $3\mu' + 2\mu > 0$; in [55] on the free-boundary problem for self-gravitating fluids, i.e., the external force field is given by the formula

$$\boldsymbol{f} = -\nabla U_g, \tag{1.8}$$

where $U_g = U_g(x,t)$, the gravitational potential, is defined (with containing the unknown quantity ρ) by

$$U_g(x,t) = -G\int_{\Omega_t} \frac{\rho(s,t)}{|x-s|}\,\mathrm{d}s \tag{1.9}$$

with the Newtonian gravitational constant G. It is also known that U_g satisfies the Poisson equation

$$\triangle U_g = 4\pi G\rho \tag{1.10}$$

in Ω_t for $t > 0$. Other unique local in time existence theorems are found, for example, in [53, 54, 69–71].

1.1.2 Global solvability of the problems

Although local in time well-posedness of the problems for (1.1) with $(1.5)^1$ has been almost established under conditions general enough, as concerns global in time solvability of the problems there exist only partial results. Matsumura-Nishida solved globally in time the Cauchy problem [38] in 1980 and the initial-boundary value problem [39] in 1983 for (1.1) with $(1.5)^1$ under the assumptions that $\boldsymbol{f}$ (a given potential force) is sufficiently small and the initial value $(\rho_0, \boldsymbol{v}_0, \theta_0)$ is sufficiently close to a positive constant state $(\bar{\rho}, 0, \bar{\theta})$. They also showed that the corresponding stationary problem has a unique solution $(\tilde{\rho}, 0, \tilde{\theta})$ near $(\bar{\rho}, 0, \bar{\theta})$ and the global in time solution converges to this stationary one as time tends to infinity. Their methods were applied to various problems by many authors, for example, Kawashima-Nishida [26], Okada-Kawashima [49], Ducomet [6] and so on. It is also noteworthy to pointout another method due to Solonnikov-Tani of obtaining global in time solvability of the problem in a series of papers [61–63]. They considered a free-boundary problem for a barotropic model with a surface tention on the free-boundary, and proved the existence of global in time solution and its convergence to a stationary solution in Sobolev-Slobodetskiĭ spaces under some smallness assumptions of the initial data.

On the other hand, in spacially one-dimensional case, where all the quantities are depending only on x_1 and t, global in time solvability of various problems (mainly under the assumption that coefficients of viscosities and the thermal conductivity are constants) was investigated by many authors without any smallness assumption on the initial data. Firstly, in 1968 the Cauchy problem for a one-dimensional barotropic model was solved globally in time by Kanel' [25]. Itaya [18] and Tani [64] obtained analogous results for the system of generalized Burgers' equations. As for full one-dimensional model of (1.1) with $(1.5)^1$, in 1977 Kazhikhov-Shelukhin [32] firstly proved the global in time solvability of the problem without any external force and with the Dirichlet boundary condition with respect to the velocty, for a polytropic and ideal fluid, which has the equations of state

$$\begin{cases} p(\rho, \theta) = R\rho\theta, \\ e(\rho, \theta) = c\theta \end{cases} \tag{1.11}$$

with the perfect gas constant R and a positive constant c. Moreover, Kazhikhov [29] proved that the solution of this problem converges to the one of the corresponding stationary problem as time tends to infinity. For them it is necessary to get a priori estimates of the solution, among which the most important one is the boundedness of the density form below by a strictly positive constant. To obtain such an estimate they derived a useful representation formula of the density in [32] (an analogous formula of the density for the system of generalized Burgers' equations had been obtained by Itaya [18]). After these pioneering works, many studies have been done including Nagasawa's ones, in which the global existence and the asymptotic behavior in the free-boundary case for the polytropic and ideal gas were investigated under no external force: in [43, 46] with a free-boundary to a surrounding vacuum state, i.e., $p_e \equiv 0$ in (1.7); in [44, 45] with the one pushed inward by surroundings, i.e., $p_e = p(t) > 0$. For other works, see below (§2.4).

2 Formulation of the problems

In this thesis we consider the free-boundary problem decsribing the motion of some typical gaseous stars composed by compressible, viscous, heat-conductive and chemically reactive gas. Such problems are formulated as follows: to determine the domain Ω_t and quantities $\rho, \boldsymbol{v}, \theta, z$ through equations (1.1)-(1.3), (1.5) with the boundary conditions (1.6), (1.7) together with

$$\boldsymbol{q}_{ch} \cdot \boldsymbol{n} = 0 \qquad \text{on } \ S_t, \ \ t > 0 \tag{2.1}$$

and the initial conditions

$$(\Omega_t, \rho, \boldsymbol{v}, \theta, z)|_{t=0} = (\Omega_0, \rho_0, \boldsymbol{v_0}, \theta_0, z_0). \tag{2.2}$$

From the physical point of view, it is natural to take into account the self-gravitation (1.8), (1.9) as an external force driving the motion of gas.

In the stellar interior, the radiation phenomenon is not negligible at the high temperature regime which is relevant to our models. In general, for radiative gas one has to consider the radiative transfer of photons with hydrodynamical movement and the relativistic treatment for the system. However, in the special case that the stellar matter is in local themodynamical equilibrium and the degree of the absorption of emitted radiation is rather high, that is to say, the mean free path of photons is much shorter than the typical length of the gaseous flow, it is known that instead of coupling the radiative transfer one can use the usual hydrodynamic model with the pressure, the internal energy and the conductivity added by the special radiative effects (see for example, [40]). This means that p and e are given by $p = p_G + p_R$ and $e = e_G + e_R$, respectively, where $p_G = p_G(\rho, \theta)$ and $e_G = e_G(\rho, \theta)$ are the *gaseous* (elastic) contributions, whereas $p_R = p_R(\theta)$ and $e_R = e_R(\rho, \theta)$ are the *radiative* ones. As a rule $p_G(\rho, \theta)$ is determined in the complicated way dependent on several factors, mainly the degree of the ionization of gas and the degeneracy of electrons and ions. If stellar matter is *not* in sufficiently low temperature and high density, that is, the degeneracy of both electrons and ions is of sufficiently low degree (including non-degenerate case), the ideal-gas approximation (1.11)[1] is widely accepted for both the electron pressure and the ion pressure. Since almost all parts of the stellar body may be in this situation, we assume $p_G(\rho, \theta) = R\rho\theta$. In this case from the thermodynamical relations, it easily follows that e_G is depending only on θ, i.e., $e_G = C(\theta)$ and $C'(\theta) = c_{\mathrm{v}}(\theta)$, where $c_{\mathrm{v}}(\theta)$ is the specific heat capacity at constant volume. Here for simplicity we assume $c_{\mathrm{v}}(\theta)$ is a positive constant, that is to say, $e_G = c_{\mathrm{v}}\theta$. The gas consisting of normal stars can be regarded as a "black body", so that the radiative pressure p_R and the energy of radiation per unit mass e_R are given by the Stefan-Boltzmann law (see for example, [2])

$$p_R(\theta) = \frac{a}{3}\theta^4, \qquad e_R(\rho, \theta) = \frac{a}{\rho}\theta^4$$

with the radiation-density constant $a > 0$.

We also assume that the thermal conductivity in (1.5)[1] has the form

$$\kappa(\rho, \theta) = \kappa_1 + \kappa_2 \frac{\theta^q}{\rho} \tag{2.3}$$

with positive constants κ_1, κ_2 and q, which is motivated by the fact that in the radiating regime one has to take into account the flux $\boldsymbol{q}_{th}$ from not only the heat-conductive contribution $\boldsymbol{q}_{cd}$, but also the radiative contribution $\boldsymbol{q}_{rad}$ given by

$$\boldsymbol{q}_{cd} = -\kappa_1 \nabla\theta, \qquad \boldsymbol{q}_{rad} = -\frac{1}{3}\frac{c}{\hat{\kappa}\rho}\nabla(\rho e_R)$$

with the speed of light c and the Rossland mean absorption coefficient $\hat{\kappa} = \hat{\kappa}(\rho, \theta)$. Here $\hat{\kappa}$ is defined such that the quantity $1/(\hat{\kappa}\rho)$ is the mean free path of a photon inside the media. Hence

$$\boldsymbol{q}_{th} = \boldsymbol{q}_{cd} + \boldsymbol{q}_{rad} = -\left(\kappa_1 + \frac{4ac}{3}\frac{\theta^3}{\hat{\kappa}\rho}\right)\nabla\theta.$$

If $\hat{\kappa}(\rho, \theta)$ is nearly a constant, then $q \approx 3$ in (2.3). Furthermore we assume that the reaction is first-order and define the reaction rate function as

$$\phi = \phi(\theta) = K\theta^\beta \mathrm{e}^{-A/\theta} \tag{2.4}$$

with a non-negative number β, which corresponds to the case that $m = 1$ in (1.2)-(1.4) and $s \geq 0$, $\theta_i = 0$ in (1.4).

2.1 Several stellar models of self-gravitating viscous gas

We restrict our analysis to the following two models under the assumptions that μ and μ' are constants, and p_e is a non-negative constant, $\kappa_e \equiv 0$ in (1.7).

Problem 1 *A three-dimensional spherically symmetric stellar model*

Until now, many astrophysicists have studied the sytem of equations (1.1) or (1.1)-(1.2) mainly in the spherically symmetric framework (see for example, [2, 33]).

Following them, here we also restrict our analysis to the one in the spherically symmetric case.

From the physical observations it is widely acceptable that almost all mass of a gaseous star is concentrated near its centre despite of the wide distribution of gaseous particles; for example, it is said that only about 10% of the solar mass lies outside the ball of radius $R_\odot/2$, where $R_\odot$ is the radius of the sun. From this, roughly speaking, one can regard that a stellar interior consists of two parts, the central condensation and the *stellar envelope*. Since the motion of gaseous star described by (1.1), (1.2) admits a great variety, it is needless to say that the situation mentioned above is certainly contained in it. However, we also know that stars, in way of their evolution, usually have the core in the centre composed of the heavy chemical elements (for example, helium, carbon, oxygen, etc.) produced by the burning of the light gas, hydrogen. Due to high temperature near the centre of the star, "hydrogen burning" begins first near the centre and the products are gradually accumulated there. From these phenomenal points of view, we may assume that there exists a spherical rigid core in the centre of the star, and focus our interest on the motion of outer gaseous part like the stellar envelope. In this situation it is natural to take into account, as the external forces driving the motion of gas around the core, both the self-gravitation of gas and the potential force of the core, where the latter is usually regarded as the dominant factor of $\boldsymbol{f}$ in $(1.1)^2$.

Let us reduce (1.1), (1.2) to the ones in the polar coordinate system with the spherical symmetricity. Setting with $r := |x|$

$$\rho(x,t) = \hat{\rho}(r,t), \quad \boldsymbol{v}(x,t) = \hat{v}(r,t)\frac{x}{r}, \quad \theta(x,t) = \hat{\theta}(r,t), \quad z(x,t) = \hat{z}(r,t),$$

we have with omitting the hats

$$\begin{cases} \rho_t + \dfrac{(r^2\rho v)_r}{r^2} = 0, \\ \rho(v_t + vv_r) = \left(-p + \zeta\dfrac{(r^2v)_r}{r^2}\right)_r + \rho(f_g + f_c), \\ \rho(e_t + ve_r) = \left(-p + \zeta\dfrac{(r^2v)_r}{r^2}\right)\dfrac{(r^2v)_r}{r^2} - 4\mu\left(\dfrac{(v^2)_r}{r} + \dfrac{v^2}{r^2}\right) \\ \qquad\qquad\qquad\qquad + \dfrac{(r^2\kappa\theta_r)_r}{r^2} + \lambda\rho\phi z, \\ \rho(z_t + vz_r) = \dfrac{(r^2 d\rho z_r)_r}{r^2} - \rho\phi z \end{cases} \tag{2.5}$$

in $\bigcup_{t>0}\left(D_t \times \{t\}\right)$, where $D_t := \left\{r \in \mathbb{R} \mid R_0 < r < R(t)\right\}$ for any $t \geq 0$, and $R_0 > 0$ is a radius of the core. Here fluctuating boundary function $R(t)$ and $\rho = \rho(r,t)$,

$v = v(r,t)$, $\theta = \theta(r,t)$, $z = z(r,t)$ are unknown functions, a positive constant $\zeta := 2\mu + \mu'$ is the bulk viscosity which satisfies the relation $3\zeta - 4\mu \geq 0$. In this spherically symmetric case the self-gravitation of gas per unit mass $f_g = f_g(r,t)$ is directly given by Newton's law

$$f_g(r,t) = -\frac{G}{r^2}\int_{R_0}^{r} 4\pi\rho(s,t)s^2\,\mathrm{d}s, \tag{2.6}$$

whereas the potential force of the core $f_c = f_c(r)$ is given by

$$f_c(r) = -\frac{GM_0}{r^2} \tag{2.7}$$

with the mass of the core M_0.

Remark. If we consider a model for the gaseous star without the central rigid core, the external force $\boldsymbol{f}$ in $(1.1)^2$ is given by the self-gravitation (1.8), (1.9) only. In this case it is from this force term in $(1.1)^2$ that a difficulty for temporally global existence problem comes. In fact, multiplying $(1.1)^2$ by $\boldsymbol{v}$, integrating it by part over $\Omega_t \times [0,t]$ and combining the integration of $\rho e + \lambda\rho z$, we have an energy identity

$$E(t) := \int_{\Omega_t}\left(\frac{1}{2}\rho|\boldsymbol{v}|^2 + \rho e + \lambda\rho z + \frac{1}{2}\rho U_g\right)\mathrm{d}x + p_e|\Omega_t| = E(0)$$

with the volume of domain $|\Omega_t|$. Since $U_g < 0$, we cannot obtain *a priori* bounds for other terms in $E(t)$. In addition to this, the spherical symmetricity brings to another serious difficulty, singularity at the origin $r = 0$ even if $\boldsymbol{f} \equiv 0$ in $(1.1)^2$ (see (3.3) with $M_0 = 0$ under $R_0 = 0$ in (2.12) if $\boldsymbol{f} \not\equiv 0$; (3.4) if $\boldsymbol{f} \equiv 0$).

Imposed boundary conditions are on the free-surface for $t > 0$

$$\begin{cases} \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = v(R(t),t), \\ \left(-p + \zeta\dfrac{(r^2 v)_r}{r^2} - 4\mu\dfrac{v}{r},\, \theta_r,\, z_r\right)\Big|_{r=R(t)} = (-p_e,\, 0,\, 0) \end{cases} \tag{2.8}$$

from (1.6), (1.7) and (2.1), on the core for $t > 0$

$$(v,\, \theta_r,\, z_r)\big|_{r=R_0} = (0,\, 0,\, 0).$$

The initial conditons are for $r \in \overline{D_0}$

$$(\rho, v, \theta, z)|_{t=0} = \big(\rho_0(r), v_0(r), \theta_0(r), z_0(r)\big).$$

In order to transform our problem to the one with fixed domain we introduce the Lagrangian transformation. For given smooth velocity field $v(r,t)$ and for arbitrary fixed point $(r,t) \in \bigcup_{t>0} \big(\overline{D_t} \times \{t\}\big)$ we consider the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}R_{r,t}(\tau)}{\mathrm{d}\tau} = v\big(R_{r,t}(\tau),\tau\big) \quad \text{for } \ \tau \in (0,t), \\ R_{r,t}(t) = r \end{cases}$$

and the solution curve $R_{r,t}(\tau)$ uniquely exists as long as v is suitably smooth. Let $R_{r,t}(0) = \xi$. This is uniquely solvable in r as

$$r = R_{\xi,0}(t) = \xi + \int_0^t v\big(R_{\xi,0}(\tau),\tau\big)\,\mathrm{d}\tau,$$

where $R_{\xi,0}(\tau)$ $(0 \le \tau \le t)$ is the solution of the problem

$$\begin{cases} \dfrac{\mathrm{d}R_{\xi,0}(\tau)}{\mathrm{d}\tau} = v\big(R_{\xi,0}(\tau),\tau\big) \quad \text{for } \ \tau \in (0,t), \\ R_{\xi,0}(0) = \xi. \end{cases}$$

Owing to the kinematic boundary condition $(2.8)^1$ this mapping $(r,t) \mapsto (\xi,t)$ is one-to-one from $\overline{D_t} \times \{t\}$ onto $\overline{D_0} \times \{t\}$ for each $t > 0$. Next, we introduce the mass variable

$$\xi \mapsto x = \int_{R_0}^{\xi} \rho_0(s)s^2\,\mathrm{d}s$$

and obtain relations between r and x by $v(R_0,t) = 0$

$$r = \tilde{r}(x,t) = \left({R_0}^3 + 3\int_0^x \frac{\mathrm{d}s}{\tilde{\rho}(s,t)}\right)^{1/3}, \quad \tilde{r}_t = \tilde{v}, \quad \tilde{r}_x = \frac{1}{\tilde{\rho}\,\tilde{r}^2},$$

where tilde "~" represents the transformed functions.

Consequently, by putting the specific volume $v(x,t) := 1/\tilde{\rho}(x,t)$, the velocity $u(x,t) := \tilde{v}(x,t)$ and $(r,\theta,z,p,e,\phi)(x,t) := (\tilde{r},\tilde{\theta},\tilde{z},\tilde{p},\tilde{e},\tilde{\phi})(x,t)$, and normalizing the total mass $\int_{R_0}^{R(0)} \rho_0(s)s^2\,\mathrm{d}s = 1$ our problem becomes in $(0,1)\times(0,\infty)$

$$\begin{cases} v_t = (r^2u)_x \\ u_t = r^2\left(-p + \zeta\dfrac{(r^2u)_x}{v}\right)_x - G\,\dfrac{x+M_0}{r^2}, \\ e_t = \left(-p + \zeta\dfrac{(r^2u)_x}{v}\right)(r^2u)_x - 4\mu(ru^2)_x + \left(\dfrac{r^4\kappa\theta_x}{v}\right)_x + \lambda\phi z, \\ z_t = \left(\dfrac{dr^4z_x}{v^2}\right)_x - \phi z \end{cases} \tag{2.9}$$

with the boundary conditions for $t > 0$

$$\begin{cases} \left(-p + \zeta\dfrac{(r^2u)_x}{v} - 4\mu\dfrac{u}{r}\right)\Big|_{x=1} = -p_e, \\ u|_{x=0} = 0, \\ (\theta_x, z_x)|_{x=0,1} = (0,0), \end{cases} \tag{2.10}$$

the initial conditions for $x \in [0,1]$

$$(v, u, \theta, z)|_{t=0} = \big(v_0(x), u_0(x), \theta_0(x), z_0(x)\big) \tag{2.11}$$

and the relations

$$r = r(x,t) = \left({R_0}^3 + 3\int_0^x v(\xi, t)\,\mathrm{d}\xi\right)^{1/3}, \quad r_t = u, \quad r_x = \frac{v}{r^2}. \tag{2.12}$$

Here we assume the compatibility conditions

$$\begin{cases} \left(-p_0 + \zeta\dfrac{({r_0}^2u_0)'}{v_0} - 4\mu\dfrac{u_0}{r_0}\right)\Big|_{x=1} = -p_e, \\ u_0(0) = {\theta_0}'(0) = {\theta_0}'(1) = {z_0}'(0) = {z_0}'(1) = 0 \end{cases} \tag{2.13}$$

with $p_0 := R\theta_0/v_0 + (a/3){\theta_0}^4$ and $r_0 := ({R_0}^3 + 3\int_0^x v_0(\xi)\,\mathrm{d}\xi)^{1/3}$.

For this problem we shall establish the existence of the unique global in time classical solution to the system (2.9)-(2.11) together with (2.12), (2.4), the equations of state

$$p = R\frac{\theta}{v} + \frac{a}{3}\theta^4, \qquad e = c_{\mathrm{v}}\theta + av\theta^4 \tag{2.14}$$

and the conductivity

$$\kappa = \kappa_1 + \kappa_2 v\theta^q \tag{2.15}$$

under the hypotheses (2.13).

Problem 2 *A one-dimensional stellar model*

Here we consider one-dimensional motion of gaseous star. Denoting x_1 and v_1 by y and v, respectively, for the unknown quantities $(\rho, v, \theta, z) = (\rho, v, \theta, z)(y,t)$ the system of equations to be solved are the following:

$$\begin{cases} \rho_t + (\rho v)_y = 0, \\ \rho(v_t + vv_y) = \Big(-p + (\mu' + 2\mu)v_y\Big)_y + \rho f, \\ \rho(e_t + ve_y) = \Big(-p + (\mu' + 2\mu)v_y\Big)v_y + (\kappa\theta_y)_y + \lambda\rho\phi z, \\ \rho(z_t + vz_y) = (d\rho z_y)_y - \rho\phi z \end{cases}$$

in $\bigcup_{t>0}\big(D'_t\times\{t\}\big)$, where $D'_t := \{y\in\mathbb{R}\,|\,y_1(t)<y<y_2(t)\}$ for any $t\geq 0$, and $y_i(\cdot)$ $(i=1,2)$ are fluctuating unknown boundary functions (we put $y_1(0)=0$, $y_2(0)=L$). Hereafter we denote the bulk viscosity $\mu'+2\mu$, which is a positive constant, by μ. Here we assume that the external force per unit mass $f=f(y,t)$ is given by $f=-U_y$, where $U=U(y,t)$ is the solution of the boundary value problem for each $t>0$

$$\begin{cases} U_{yy}=G\rho \qquad \text{in } \ D'_t, \\ U|_{y=y_1(t)}=U|_{y=y_2(t)}=0 \end{cases} \tag{2.16}$$

with a positive constant G corresponding to the Newtonian gravitational constant. One can regard that this definition of f gives the one-dimensional general self-gravitation similar to the one given by (1.8)-(1.10). Imposed boundary conditions corresponding to (1.6), (1.7) and (2.1) are for $t>0$, $i=1,2$

$$\begin{cases} \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t}=v(y_i(t),t), \\ (-p+\mu v_y,\ \theta_y,\ z_y)|_{y=y_i(t)}=(-p_e,\ 0,\ 0), \end{cases} \tag{2.17}$$

respectively, and the initial conditions are for $y\in\overline{D'_0}$

$$(\rho,v,\theta,z)|_{t=0}=\big(\rho_0(y),v_0(y),\theta_0(y),z_0(y)\big).$$

Similarly to Problem 1, we transform this problem into the one of the Lagrangian coordinate. For given smooth velocity field $v(y,t)$ and for any fixed point $(y,t)\in\bigcup_{t>0}\big(\overline{D'_t}\times\{t\}\big)$, finding the solution $Y_{y,t}(\tau)$ of the problem

$$\begin{cases} \dfrac{\mathrm{d}Y_{y,t}(\tau)}{\mathrm{d}\tau}=v\big(Y_{y,t}(\tau),\tau\big) \quad \text{for } \ 0<\tau<t, \\ Y_{y,t}(t)=y \end{cases}$$

and putting $Y_{y,t}(0)=\xi$, we have

$$y=Y_{\xi,0}(t)=\xi+\int_0^t v\big(Y_{\xi,0}(\tau),\tau\big)\,\mathrm{d}\tau.$$

Then we introduce the mass transformation

$$\xi\mapsto x=\int_0^\xi \rho_0(s)\,\mathrm{d}s.$$

From these changes of variable problem (2.16) is reduced to

$$\begin{cases} (\tilde\rho\,\tilde U_x)_x=G \qquad \text{in } \ (0,M), \\ \tilde U|_{x=0}=\tilde U|_{x=M}=0 \end{cases}$$

for each $t > 0$, where $M = \int_0^L \rho_0(\xi)\,\mathrm{d}\xi$ and tilde "~" represents the transformed functions. Through the relations $\tilde{f} = -\tilde{\rho}\,\tilde{U}_x$ we can get the explicit formula

$$\tilde{f}(x,t) = -G\left(x - \frac{\int_0^M \eta\tilde{\rho}(\eta,t)^{-1}\,\mathrm{d}\eta}{\int_0^M \tilde{\rho}(\eta,t)^{-1}\,\mathrm{d}\eta}\right). \tag{2.18}$$

Consequently, by putting the specific volume $v(x,t) := 1/\tilde{\rho}(x,t)$, the velocity $u(x,t) := \tilde{v}(x,t)$ and $(\theta, z, p, e, \phi)(x,t) := (\tilde{\theta}, \tilde{z}, \tilde{p}, \tilde{e}, \tilde{\phi})(x,t)$, and normalizing $M = 1$ our problem becomes

$$\begin{cases} v_t = u_x, \\ u_t = \left(-p + \mu\dfrac{u_x}{v}\right)_x - G\left(x - \dfrac{\int_0^1 \eta v(\eta,t)\,\mathrm{d}\eta}{\int_0^1 v(\eta,t)\,\mathrm{d}\eta}\right), \\ e_t = \left(-p + \mu\dfrac{u_x}{v}\right)u_x + \left(\kappa\dfrac{\theta_x}{v}\right)_x + \lambda\phi z, \\ z_t = d\left(\dfrac{z_x}{v^2}\right)_x - \phi z \end{cases} \tag{2.19}$$

in $(0,1)\times(0,\infty)$ with the boundary conditions for $t > 0$

$$\left(-p + \mu\frac{u_x}{v},\ \theta_x,\ z_x\right)\Big|_{x=0,\,1} = (-p_e,\ 0,\ 0) \tag{2.20}$$

and the initial conditions for $x \in [0,1]$

$$(v,u,\theta,z)|_{t=0} = \big(v_0(x), u_0(x), \theta_0(x), z_0(x)\big). \tag{2.21}$$

Now, by integration of $(2.19)^2$ with respect to x over $[0,1]$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u\,\mathrm{d}x = -G\left(\frac{1}{2} - \frac{\int_0^1 \eta v(\eta,t)\,\mathrm{d}\eta}{\int_0^1 v(\eta,t)\,\mathrm{d}\eta}\right). \tag{2.22}$$

Denoting $u - \int_0^1 u\,\mathrm{d}x$ by u again, we obtain the final form:

$$\begin{cases} v_t = u_x, \\ u_t = \left(-p + \mu\dfrac{u_x}{v}\right)_x - G\left(x - \dfrac{1}{2}\right), \\ e_t = \left(-p + \mu\dfrac{u_x}{v}\right)u_x + \left(\kappa\dfrac{\theta_x}{v}\right)_x + \lambda\phi z, \\ z_t = d\left(\dfrac{z_x}{v^2}\right)_x - \phi z \end{cases} \tag{2.23}$$

in $(0,1)\times(0,\infty)$ with the same initial-boundary conditions (2.20) and (2.21). For this system it is natural that initial function u_0 (which corresponds to $u_0-\int_0^1 u_0\,\mathrm{d}x$ for the original system (2.19)) satisfies

$$\int_0^1 u_0\,\mathrm{d}x = 0. \tag{2.24}$$

We also assume the compatibility conditions

$$\left(-p_0+\mu\frac{u_0{}'}{v_0}\right)\bigg|_{x=0,1} = -p_e, \quad \theta_0{}'(0)=\theta_0{}'(1)=z_0{}'(0)=z_0{}'(1)=0. \tag{2.25}$$

For this problem we shall establish the existence of the unique global in time classical solution to the system (2.23), (2.20), (2.21) with (2.4), (2.14), (2.15) under the hypotheses (2.24), (2.25). From (2.22) it is easily seen that this solution leads to the one for the original problem (2.19)-(2.21) describing the exact one-dimensional self-gravitating fluid model.

The difficulty of two problems described above is mainly caused by the radiative terms of the equations of state and (v,θ)-dependence of the conductivity. Although our problems can be solved only for some large q (see §2.3), this value of q seems to be physically admissible [75].

2.2 Function spaces

We introduce some function spaces used in this thesis (see for example, [14, 34]). Let $\Omega := (0,1)$ and m a non-negative integer. By $C^{m+\alpha}(\Omega)$ for $0<\alpha<1$ we denote the spaces of functions which are Hölder continuous with exponent α up to order m, with the norm

$$|u|_{m+\alpha} := \sum_{k=0}^{m}\sup_{x\in\Omega}|D^k u(x)| + \sup_{\substack{x,x'\in\Omega\\ x\neq x'}}\frac{|D^m u(x)-D^m u(x')|}{|x-x'|^\alpha},$$

where $D=\partial/\partial x$. Let T be a positive constant and $Q_T := \Omega\times(0,T)$. For a function u defined on Q_T, we denote for $0\le\sigma,\ \sigma'\le 1$

$$|u|^{(0)} := \sup_{(x,t)\in Q_T}|u(x,t)|,$$

$$|u|_x^{(\sigma)} := \sup_{\substack{(x,t),(x',t)\in Q_T\\ x\neq x'}}\frac{|u(x,t)-u(x',t)|}{|x-x'|^\sigma},$$

$$|u|_t^{(\sigma)} := \sup_{\substack{(x,t),(x,t')\in Q_T \\ t\neq t'}} \frac{|u(x,t)-u(x,t')|}{|t-t'|^{\sigma}}.$$

We define $C_{x,\,t}^{\sigma,\,\sigma'}(Q_T)$ as the spaces of continuous functions $u(x,t)$ with the norm

$$|u|_{\sigma,\,\sigma'} := |u|^{(0)} + |u|_x^{(\sigma)} + |u|_t^{(\sigma')}.$$

We also say that $u \in C_{x,\,t}^{2+\alpha,\,1+\alpha/2}(Q_T)$ for $0<\alpha<1$ if u is continuous over $\overline{Q_T}$, has continuous derivatives u_x, u_{xx}, u_t and $(u_{xx}, u_t) \in \left(C_{x,\,t}^{\alpha,\,\alpha/2}(Q_T)\right)^2$. Its norm is defined by

$$|u|_{2+\alpha,\,1+\alpha/2} := |u|^{(0)} + |u_x|^{(0)} + |u_{xx}|_{\alpha,\,\alpha/2} + |u_t|_{\alpha,\,\alpha/2}.$$

2.3 Statements of theorems

Our main result for Problem 1 is

Theorem 1 (Global Solution of Problem 1) *Let $\alpha \in (0,1)$, $3 \le q < 9$ and $0 \le \beta < q+9$. Assume that*

$$(v_0, u_0, \theta_0, z_0) \in C^{1+\alpha}(\Omega) \times \left(C^{2+\alpha}(\Omega)\right)^3$$

satisfies (2.13) *and $v_0(x) > 0$, $\theta_0(x) > 0$, $0 \le z_0(x) \le 1$ for any $x \in \overline{\Omega}$, and $3\zeta - 4\mu > 0$, $p_e > 0$. Then there exists a unique solution (v,u,θ,z) of the initial-boundary value problem* (2.9)-(2.11) *with* (2.12), (2.4), (2.14), (2.15) *such that*

$$(v,\, v_x, v_t, u, \theta, z) \in \left(C_{x,\,t}^{\alpha,\,\alpha/2}(Q_T)\right)^3 \times \left(C_{x,\,t}^{2+\alpha,\,1+\alpha/2}(Q_T)\right)^3$$

for any positive number T. Moreover for any $(x,t) \in \overline{Q_T}$

$$v(x,t) > 0, \quad \theta(x,t) > 0, \quad 0 \le z(x,t) \le 1.$$

This result has been already announced in [68].

For Problem 2, we obtain the following theorem.

Theorem 2 (Global Solution of Problem 2) *Let $\alpha \in (0,1)$, $q \geq 3$ and $0 \leq \beta < q+9$. Assume that*

$$(v_0, u_0, \theta_0, z_0) \in C^{1+\alpha}(\Omega) \times \left(C^{2+\alpha}(\Omega)\right)^3$$

satisfies (2.24), (2.25) *and $v_0(x) > 0$, $\theta_0(x) > 0$, $0 \leq z_0(x) \leq 1$ for any $x \in \overline{\Omega}$, and $p_e > 0$. Then there exists a unique solution (v, u, θ, z) of the initial-boundary value problem* (2.23), (2.20), (2.21) *with* (2.4), (2.14), (2.15) *such that*

$$(v,\, v_x, v_t, u, \theta, z) \in \left(C^{\alpha,\,\alpha/2}_{x,\,t}(Q_T)\right)^3 \times \left(C^{2+\alpha,\,1+\alpha/2}_{x,\,t}(Q_T)\right)^3$$

for any positive number T. Moreover for any $(x,t) \in \overline{Q_T}$

$$v(x,t) > 0, \quad \theta(x,t) > 0, \quad 0 \leq z(x,t) \leq 1.$$

In [67] for $4 \leq q \leq 16$ and $0 \leq \beta \leq 13/2$ the global in time solvability of Problem 2 was established in the same spaces as in Theorem 2. Theorem 2 is its improvement.

Remark. The range of values of q and β guaranteeing the global in time solvability of Problems 1 and 2 are different from each other. This difference essentially comes from the one of the equations of motion

$$u_t = r^2 \sigma_x + 4\mu r^2 \left(\frac{u}{r}\right)_x - G\,\frac{x+M_0}{r^2}, \quad \sigma = -p + \zeta \frac{(r^2 u)_x}{v} - 4\mu \frac{u}{r} \qquad \text{for Problem 1,}$$

$$u_t = \sigma_x - G\left(x - \frac{1}{2}\right), \qquad \sigma = -p + \mu \frac{u_x}{v} \qquad \text{for Problem 2,}$$

where σ is the stress of gas in each model. The conservation form of the latter allows us to solve Problem 2 for wider range of q and β than that of Problem 1 (see Lemma 4.6, §4.1).

Proof of theorems mentioned above is based on the temporally local existence theorem and a priori estimates. As already mentioned in §1.1.1, the fundamental theorem about the existence and the uniqueness of the local in time classical solution was established by Tani and Secchi; especially in [53, 54] self-gravitating radiative fluid was considered. Since it is easy to see that their argument is applicable without any essential modification to our reacting, three-dimensional spherically symmetric or one-dimensional cases (see for example, [60]), we omit the proof of the following proposition.

Proposition 1 (Local Solutions of Problems 1 and 2) *Let $\alpha \in (0,1)$. Assume that*

$$(v_0, u_0, \theta_0, z_0) \in C^{1+\alpha}(\Omega) \times \Big(C^{2+\alpha}(\Omega)\Big)^3$$

satisfies the compatibility conditions (2.13) *or* (2.25) *and for a positive constant* M

$$|v_0|_{1+\alpha},\ |u_0, \theta_0, z_0|_{2+\alpha} \ \le\ M,$$
$$v_0(x),\ \theta_0(x)\ \ge 1/M, \quad 0 \le z_0(x) \le 1 \quad \textit{for any } x \in \overline{\Omega}.$$

Then there exists a unique solution (v, u, θ, z) of our two initial-boundary value problem such that

$$(v,\ v_x, v_t, u, \theta, z) \in \Big(C^{\alpha,\,\alpha/2}_{x,\,t}(Q_{T^*})\Big)^3 \times \Big(C^{2+\alpha,\,1+\alpha/2}_{x,\,t}(Q_{T^*})\Big)^3$$

for some positive number $T^* = T^*(M)$. Moreover for some positive constant $M^* = M^*(M, T^*)$

$$|v,\ v_x, v_t|_{\alpha,\,\alpha/2},\ |u, \theta, z|_{2+\alpha,\,1+\alpha/2} \ \le\ M^*,$$
$$v(x,t),\ \theta(x,t)\ >\ 1/M^*, \quad 0 \le z(x,t) \le 1 \quad \textit{for any } (x,t) \in \overline{Q_{T^*}}.$$

2.4 Related results

After the pioneering paper [32] due to Kazhikhov and Shelukhin problems with one space variable have been studied under various situations.

Firstly as concerns the one-dimensional problem closely related to Problem 2, we mention the results for models with no external forces. Models for a reacting mixture, in which (1.2) is taken into account and gases are polytropic and ideal, have been studied many authors including Poland-Kassoy [50], Bebernes-Bressan [1], Chen [3], Yanagi [73], Guo-Zhu [15], Chen-Hoff-Trivisa [4] and so on. In [15, 73] the temporal asymptotics for $m \ge 1$, $s = 0$, $\theta_i = 0$ in (1.2)-(1.4) were investigated. The case $\theta_i > 0$ was treated in [3, 4], and especially in [4] the binary mixtures which have different physical parameters in each species of gases were investigated for the particular case $d = 0$ in (1.2). The motion of fluids with some general equations of state and thermal conductivity were investigated by Dafermos-Hsiao [5], Kawohl [27], Jiang [21, 22], Qin [51] and so on. Since most of them considered the situation that the pressure and the internal energy are due to only the gaseous thermal movements, that is, the radiative contribution given by the Stefan-Boltzmann law is not taken into account, the low growth power in θ for p, e, κ are assumed (see [5, 21, 27]). This situation was extended by Qin [51] to the case of any growth power r in θ as follows: for paremeters $r \geq 0, \quad r+1 \leq q < (5r+3)/2$, and positive constants p_1, p_2, c, κ_0 and $p_3(\underline{v}), p_4(\underline{v}), N(\underline{v}), \kappa_1(\underline{v})$ depending on any positive number $\underline{v}$,

(i) $0 < p_1 \leq vp(v,\theta) \leq p_2(1+\theta^{r+1}), \quad |p_\theta(v,\theta)| \leq p_4(\underline{v})(1+\theta^r),$

$\quad - p_3(\underline{v})(1+\theta^{r+1}) \leq p_v(v,\theta) \leq -p_4(\underline{v})(1+\theta^{r+1}),$

(ii) $0 \leq e(v,0), \quad c(1+\theta^r) \leq e_\theta(v,\theta) \leq N(\underline{v})(1+\theta^r),$

(iii) $\kappa_0(1+\theta^q) \leq \kappa(v,\theta) \leq \kappa_1(\underline{v})(1+\theta^q), \quad |\kappa_v(v,\theta)| + |\kappa_{vv}(v,\theta)| \leq \kappa_1(\underline{v})(1+\theta^q)$

for any $v \geq \underline{v}$. However, our radiative case (2.14) is not contained in this assumption (the difference is also seen in the boundary conditions, i.e., he discussed the problem under the Dirichlet condition for u). For radiative (and reactive) gas under the Dirichlet boundary condition for u Ducomet [10] showed the global existence for $q \geq 4$ in (2.15) and for $q \geq 6$ the exponential decay of the solution to a constant steady state determined by initial data. Other explicit forms of state functions were also considered for example, by Lewicka-Mucha [37] for $p(v,\theta) = \theta/v^r$ with any $r \geq 1$, $e(\rho,\theta) = c_{\rm v}\theta$ in the reactive case. Kazhikhov-Nikolaev [30, 31] and Kazhikhov [28] investigated an isothermal model with a non-monotonic state function $p(v)$ satisfying the following:

(i) $p(v) \geq p(v_1)$ for $0 < v < v_1, \quad p(v) \leq p(v_1)$ for $v_1 < v,$

(ii) if $p'(v) < 0$, then $p'(v) \leq kv^{-1}$

for a positive constant k and at least one number $v_1 \in (0,\infty)$. This aimed at the investigation of the model with the well-known van der Waals equations of state

$$p(v,\theta) = \frac{R\theta}{v-b} - \frac{a}{v^2} \tag{2.26}$$

with positive constants a and b. Since the right-hand side of (2.26) is meaningful only for $v > b$, it is necessary to obtain uniform a priori estimate $v(x,t) > b$. However it have not been succeeded until now.

Ducomet [7,8,11] and Ducomet-Zlotnik [12,13] studied one-dimensional stellar models similar to ours, i.e., radiative and reactive gas in the external force field with the free-boundary. In [11] the temporally global existence of the solution was shown for $q = 4$ in (2.15) and $\beta = 0$ in (2.4). However, in a series of papers [7,8,11–13] they adopted as a self-gravitation, a special form characterized by the “pancakes model”, which is relevant to some large-scale structure of the universe (see [59]),

$$\tilde{f}(x,t) = -G\left(x - \frac{1}{2}M\right)$$

with gaseous total mass M, not the exact form (2.18). Although the temporally global existence of the solution for any $q \geq 2$ was established recently in [12, 13], they were discussed not for the pure free-boundary case (2.20) but for the Dirichlet condition of θ.

On the other hand, three-dimensional spherically symmetric motion of a compressible viscous polytropic ideal fluid was also investigated by many authors. Itaya [20] studied the model with no external forces in the annulus domain. Yanagi [74] discussed this problem with a small potential force like (2.7), not the self-gravitation which is described by the unknown quantity ρ. In the exterior domain (outside of a sphere) Jiang [23] considered same equations as in [20] (see also [24]), and by using the method in [23] Nakamura, Nishibata and Yanagi extended Jiang's model to the one with a large potential force (in [42] for the isentropic gas, in [41] for the polytropic and ideal gas). Ducomet [9] also considered a spherically symmetric stellar model of polytropic and ideal gas having central rigid core, however he took f_g only as external force field, but not f_r which is dominant in the present situation.

3 Proof of Theorem 1: Three-dimensional spherically symmetric problem

In this section, we consider Problem 1. In order to prove Theorem 1 it is sufficient to establish the following a priori boundedness since we had the temporally local existence theorem (Proposition 1).

Proposition 2 (A priori Estimates for Problem 1) *Let T be an arbitrary positive number. Assume that α, q, β, μ, ζ, p_e and the initial data satisfy the hypotheses of Theorem* 1, *and that the problem* (2.9)-(2.11) *with* (2.12), (2.4), (2.14), (2.15) *has a solution (v, u, θ, z) such that*

$$(v,\, v_x, v_t, u, \theta, z) \in \left(C^{\alpha,\,\alpha/2}_{x,\,t}(Q_T)\right)^3 \times \left(C^{2+\alpha,\,1+\alpha/2}_{x,\,t}(Q_T)\right)^3.$$

Then there exists a positive constant C depending on the initial data and T such that

$$|v,\, v_x, v_t|_{\alpha,\,\alpha/2},\ |u, \theta, z|_{2+\alpha,\,1+\alpha/2} \ \le\ C,$$

$$v(x,t),\ \theta(x,t)\ \ge\ 1/C, \quad 0 \le z(x,t) \le 1 \quad \textit{for any}\ \ (x,t) \in \overline{Q_T}.$$

In proving Proposition 2, we need several lemmas concerning the estimates of the solution and its derivatives. Our methods are mainly based on the techniques in Dafermos-Hsiao [5], Kawohl [27] and Jiang [21]. We use C_0 and C, C_T as positive constants depending on the initial data and other constants, but the former does not depend on T, and $\|\cdot\|$ denotes the usual $L^2(\Omega)$-norm.

3.1 Estimates in Sobolev spaces

Lemma 3.1 *For any $t \in [0, T]$*

$$\int_0^1 \left(\frac{1}{2}u^2 + e + \lambda z + p_e v\right) \mathrm{d}x \le E_0 \tag{3.1}$$

with

$$E_0 := \int_0^1 \left(\frac{1}{2}{u_0}^2 + e_0 + \lambda z_0 + p_e v_0\right) \mathrm{d}x + \int_0^1 G(x + M_0)\left(\frac{1}{R_0} - \frac{1}{r_0}\right) \mathrm{d}x,$$

$$e_0 := c_{\mathrm{v}}\theta_0 + a v_0 {\theta_0}^4.$$

Proof. Let $\sigma := -p + \zeta\dfrac{(r^2u)_x}{v} - 4\mu\dfrac{u}{r}$. Multiplying $(2.9)^2$ by u and integrating it by part over $[0,1]$ with the help of the boundary condition, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\frac{1}{2}u^2 + p_e v - G\,\frac{x+M_0}{r}\right)\mathrm{d}x + \int_0^1 \sigma(r^2u)_x\,\mathrm{d}x$$

$$= 4\mu\int_0^1 r^2u\left(\frac{u}{r}\right)_x\mathrm{d}x. \tag{3.2}$$

Adding the integration of $e+\lambda z$ over $[0,1]\times[0,t]$ to the integration of (3.2) yields

$$\int_0^1 \left(\frac{1}{2}u^2 + e + \lambda z + p_e v - G\,\frac{x+M_0}{r}\right)\mathrm{d}x$$

$$= \int_0^1 \left(\frac{1}{2}{u_0}^2 + e_0 + \lambda z_0 + p_e v_0 - G\,\frac{x+M_0}{r_0}\right)\mathrm{d}x. \tag{3.3}$$

From $r \geq R_0$ in $\overline{Q_T}$, we have (3.1).

Lemma 3.2 *For any $t \in [0,T]$*

$$U(t) + \int_0^t V(\tau)\,\mathrm{d}\tau \leq C_1, \tag{3.4}$$

where C_1 is a positive constant independent of T and

$$\begin{cases} U(t) := \displaystyle\int_0^1 \Big[\, c_{\mathrm{v}}(\theta - 1 - \log\theta) + R(v - 1 - \log v) \,\Big]\,\mathrm{d}x, \\ V(t) := \displaystyle\int_0^1 \left[\, \eta\frac{{(r^2u)_x}^2}{v\theta} + \eta'\frac{vu^2}{r^2\theta} + \frac{r^4\kappa{\theta_x}^2}{v\theta^2} + \lambda\frac{\phi}{\theta}z \,\right]\mathrm{d}x, \end{cases}$$

$\eta := \dfrac{3\zeta - 4\mu}{6} > 0, \quad \eta' := \dfrac{12(3\zeta - 4\mu)}{3\zeta + 4\mu}\mu > 0.$

Proof. Rewriting $(2.9)^3$ as

$$e_\theta\theta_t + \theta p_\theta(r^2u)_x = \frac{\zeta}{v}{(r^2u)_x}^2 - 8\mu\frac{u}{r}(r^2u)_x + 12\mu\frac{v}{r^2}u^2 + \left(\frac{r^4\kappa}{v}\theta_x\right)_x + \lambda\phi z \tag{3.5}$$

and multiplying this by θ^{-1}, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c_{\mathrm{v}}\log\theta + R\log v + \frac{4}{3}av\theta^3\right)$$

$$= \zeta\frac{{(r^2u)_x}^2}{v\theta} - 8\mu\frac{(r^2u)_xu}{r\theta} + 12\mu\frac{vu^2}{r^2\theta} + \frac{1}{\theta}\left(\frac{r^4\kappa\theta_x}{v}\right)_x + \lambda\frac{\phi}{\theta}z. \tag{3.6}$$

Noting the identity

$$\zeta\frac{{(r^2u)_x}^2}{v\theta}-8\mu\frac{(r^2u)_xu}{r\theta}+12\mu\frac{vu^2}{r^2\theta}=\frac{3\zeta-4\mu}{6}\frac{{(r^2u)_x}^2}{v\theta}$$

$$+\frac{12(3\zeta-4\mu)}{3\zeta+4\mu}\mu\frac{vu^2}{r^2\theta}+\frac{1}{v\theta}\left[\sqrt{\frac{3\zeta+4\mu}{6}}(r^2u)_x-4\mu\sqrt{\frac{6}{3\zeta+4\mu}}\frac{vu}{r}\right]^2$$

and integrating (3.6) over $[0,1]\times[0,t]$, we have

$$U(t)+\int_0^t V(\tau)\,\mathrm{d}\tau\leq C_0\left(1+\int_0^1 v\theta^3\,\mathrm{d}x\right).$$

From Hölder's inequality for $\gamma\in[0,4]$

$$\int_0^1 v\theta^\gamma\,\mathrm{d}x\leq\left(\int_0^1 v\theta^4\,\mathrm{d}x\right)^{\gamma/4}\left(\int_0^1 v\,\mathrm{d}x\right)^{(4-\gamma)/4} \tag{3.7}$$

(3.4) follows.

Lemma 3.3 *For any* $(x,t)\in\overline{Q_T}$

$$\int_0^1 z\,\mathrm{d}x+\int_0^t\int_0^1\phi z\,\mathrm{d}x\,\mathrm{d}\tau=\int_0^1 z_0\,\mathrm{d}x, \tag{3.8}$$

$$\frac{1}{2}\int_0^1 z^2\,\mathrm{d}x+\int_0^t\int_0^1\left(\frac{dr^4}{v^2}{z_x}^2+\phi z^2\right)\mathrm{d}x\,\mathrm{d}\tau=\frac{1}{2}\int_0^1{z_0}^2\,\mathrm{d}x, \tag{3.9}$$

$$0\leq z(x,t)\leq 1. \tag{3.10}$$

Proof. Equalities (3.8)-(3.9) are easily obtained by integrating $(2.9)^4$ over $[0,1]\times[0,t]$. Let b be a positive constant, and define $W:=\mathrm{e}^{-bt}z$. Then W satisfies

$$\begin{cases} W_t+(b+\phi)W=\left(\frac{dr^4}{v^2}W_x\right)_x & \text{in } Q_T,\\ W_x|_{x=0,1}=0 & \text{for } t\in[0,T],\\ W|_{t=0}=z_0\geq 0 & \text{for } x\in[0,1]. \end{cases}$$

Using the comparison theorem (see [3]), we conclude that z is non-negative. Applying the same arguments to the function $1-z$, we finally obtain (3.10).

Lemma 3.4 *For any* $(x,t) \in \overline{Q_T}$

$$v(x,t) \geq C_T. \tag{3.11}$$

Proof. Since $(2.9)^2$ can be written as

$$\frac{u_t}{r^2} = -p_x + \zeta(\log v)_{xt} - G\,\frac{x+M_0}{r^4},$$

integrating this over $[x,1]\times[0,t]$ with the help of $(2.10)^1$, we have the identity

$$\log\frac{v_0}{v} + \frac{1}{\zeta}\int_0^t p\,\mathrm{d}\tau = \frac{1}{\zeta}\left[\int_x^1\left(\frac{u}{r^2}-\frac{u_0}{{r_0}^2}\right)\mathrm{d}\xi + \int_0^t\!\!\int_x^1 \frac{2u^2}{r^3}\,\mathrm{d}\xi\,\mathrm{d}\tau\right]$$
$$+\frac{p_e}{\zeta}t + \log\left(\frac{r_0(1)}{r(1,t)}\right)^{4\mu/\zeta} + \frac{1}{\zeta}\int_0^t\!\!\int_x^1 \frac{G(\xi+M_0)}{r^4}\,\mathrm{d}\xi\,\mathrm{d}\tau, \tag{3.12}$$

which immediately yields

$$\min_{(x,t)\in\overline{Q_T}} v(x,t) \geq \min_{x\in\overline{\Omega}} v_0(x)\left(\frac{R_0}{r_0(1)}\right)^{4\mu/\zeta}$$
$$\times\exp\left\{-\frac{1}{\zeta}\left[\frac{2\sqrt{2}}{{R_0}^2}{E_0}^{1/2} + \left(p_e + \frac{4E_0}{{R_0}^3} + \frac{G(1+M_0)}{{R_0}^4}\right)T\right]\right\}.$$

Combining this lemma and (3.7), we immediately obtain the next corollary.

Corollary 3.1 *For any* $t \in [0,T]$ *and* $\gamma \in [0,4]$

$$\|\theta(\cdot,t)\|_{L^\gamma(\Omega)} \leq C_T. \tag{3.13}$$

Lemma 3.5 *For any* $t\in[0,T]$ *and* $\gamma \in [0,\, q+4]$, $q \geq 0$

$$\int_0^t \max_{x\in\overline{\Omega}} \theta(x,\tau)^\gamma\,\mathrm{d}\tau \leq C_T. \tag{3.14}$$

Proof. For any $\gamma \geq 0$ and $(x,t) \in Q_T$ we have

$$\begin{aligned}
\theta(x,t)^{\gamma/2} &\leq \left(\int_0^1 \theta \,\mathrm{d}x\right)^{\gamma/2} + \frac{\gamma}{2}\int_0^1 \theta^{\gamma/2-1}|\theta_x| \,\mathrm{d}x \\
&\leq C_0\left(1 + \int_0^1 \frac{v^{1/2}\theta^{\gamma/2}}{r^2\kappa^{1/2}} \cdot \frac{r^2\kappa^{1/2}|\theta_x|}{v^{1/2}\theta}\,\mathrm{d}x\right) \\
&\leq C_0\left[\,1 + \left(\int_0^1 \frac{v\theta^{\gamma}}{r^4\kappa}\,\mathrm{d}x\right)^{1/2} V(t)^{1/2}\,\right]. \qquad (3.15)
\end{aligned}$$

Since $\theta^\gamma \leq C_0(1+\theta^{q+4})$ holds for any $\gamma \in [0,\, q+4]$, we have from (3.1) and (3.13)

$$\int_0^1 \frac{v\theta^\gamma}{r^4\kappa}\,\mathrm{d}x \leq C_0\int_0^1 \frac{v\theta^\gamma}{1+v\theta^q}\,\mathrm{d}x \leq C_0\int_0^1 (v+\theta^4)\,\mathrm{d}x \leq C,$$

which yields (3.14) from (3.15) and (3.4).

In [32], Kazhikhov and Shelukhin firstly derived the useful representation formula of v for the case that the gas is polytropic and ideal. In our radiative case we can derive the similar one.

Lemma 3.6 *The identity*

$$\begin{aligned}
v(x,t) = &\frac{1}{\mathrm{P}(x,t)\mathrm{Q}(x,t)\mathrm{R}(x,t)} \\
&\times\left(v_0(x) + \frac{R}{\zeta}\int_0^t \theta(x,\tau)\mathrm{P}(x,\tau)\mathrm{Q}(x,\tau)\mathrm{R}(x,\tau)\,\mathrm{d}\tau\right) \qquad (3.16)
\end{aligned}$$

holds, where

$$\begin{cases}
\mathrm{P}(x,t) := \left(\dfrac{r_0(1)}{r(1,t)}\right)^{4\mu/\zeta}\exp\left[\,\dfrac{1}{\zeta}\displaystyle\int_x^1\left(\frac{u}{r^2}-\frac{u_0}{{r_0}^2}\right)\mathrm{d}\xi\,\right], \\
\mathrm{Q}(x,t) := \exp\left\{\dfrac{p_e}{\zeta}t + \dfrac{1}{\zeta}\displaystyle\int_0^t\!\!\int_x^1\left[\,\frac{2u^2}{r^3} + \frac{G(\xi+M_0)}{r^4}\,\right]\mathrm{d}\xi\,\mathrm{d}\tau\right\}, \\
\mathrm{R}(x,t) := \exp\left(-\dfrac{a}{3\zeta}\displaystyle\int_0^t \theta(x,\tau)^4\,\mathrm{d}\tau\right).
\end{cases}$$

Proof. Going back to (3.12), taking exponent and dividing the pressure part into $\displaystyle\int_0^t \frac{R}{\zeta}\frac{\theta}{v}\,\mathrm{d}\tau + \frac{a}{3\zeta}\int_0^t \theta^4\,\mathrm{d}\tau$, we have

$$\frac{1}{v}\exp\left(\int_0^t \frac{R}{\zeta}\frac{\theta}{v}\,\mathrm{d}\tau\right) = \frac{1}{v_0}\mathrm{PQR}.$$

Multiplying this by $\dfrac{R}{\zeta}\theta$ and integrating it with respect to t, we obtain

$$\exp\left(\int_0^t \frac{R}{\zeta}\frac{\theta}{v}\,\mathrm{d}\tau\right) = 1 + \frac{1}{v_0}\int_0^t \frac{R}{\zeta}\theta\,\mathrm{PQR}\,\mathrm{d}\tau.$$

Lemma 3.7 *For any* $(x,t) \in \overline{Q_T}$

$$v(x,t) \le C_T. \tag{3.17}$$

Proof. At first, from (3.1) it is easily seen that for any $(x,t) \in \overline{Q_T}$

$$C_0{}^{-1} \le \mathrm{P}(x,t) \le C_0. \tag{3.18}$$

From (3.4), Jensen's inequality and mean value theorem we find a point $x^*(t) \in [0,1]$ for each fixed $t \in [0,T]$ such that

$$\theta(x^*(t),t) - \log\theta(x^*(t),t) - 1 \le \frac{C_1}{c_\mathrm{v}}, \quad \alpha_0 \le \theta(x^*(t),t) \le \beta_0$$

with two positive roots α_0 and β_0 of the equation $y - \log y - 1 = C_1/c_\mathrm{v}$. Since

$$\theta(x,t)^2 = \theta(x^*(t),t)^2 + 2\int_{x^*(t)}^x \theta(\xi,t)\theta_\xi(\xi,t)\,\mathrm{d}\xi,$$

we have

$$\frac{1}{2}\alpha_0{}^4 - C_0V(t) \le \theta(x,t)^4 \le 2\beta_0{}^4 + C_0V(t). \tag{3.19}$$

Let us decompose v into two parts $v_1 + v_2$, where

$$v_1 = v_1(x,t) := \frac{v_0(x)}{(\mathrm{PQR})(x,t)},$$

$$v_2 = v_2(x,t) := \frac{R}{\zeta}\int_0^t \frac{(\mathrm{PQR})(x,\tau)}{(\mathrm{PQR})(x,t)}\theta(x,\tau)\,\mathrm{d}\tau.$$

Using (3.18) and (3.19), we immediately obtain

$$C_0\,\mathrm{e}^{-\frac{t}{\zeta}\left[\left(p_e + \frac{4E_0}{R_0{}^3} + \frac{G(1+M_0)}{R_0{}^4}\right) - \frac{1}{6}a\alpha_0{}^4\right]} \le v_1(x,t) \le C_0\,\mathrm{e}^{-\frac{t}{\zeta}\left(p_e - \frac{2}{3}a\beta_0{}^4\right)}. \tag{3.20}$$

Also (3.15) with $\gamma = 2$, (3.18) and (3.19) yield

$$v_2(x,t) \le C_0 \int_0^t \mathrm{e}^{-\frac{1}{\zeta}\left(p_e - \frac{2}{3}a{\beta_0}^4\right)(t-\tau)} \Big(1 + V(\tau)\Big)\,\mathrm{d}\tau, \tag{3.21}$$

and hence v_2 is bounded from above by (3.4).

Remark. If p_e is sufficiently large, then for any $(x,t) \in \overline{Q_T}$

$${C_0}^{-1} \le v(x,t) \le C_0.$$

Indeed, (3.20) and (3.21) together with the assumption $p_e \ge \dfrac{2}{3}a{\beta_0}^4$ imply that v_1 is decreasing exponentially in t and v_2 is uniformly bounded. Also since we have $\theta(x,t) \ge C_2 - C_3 V(t)$ for some positive constants C_2, C_3 by using (3.15) with $\gamma = 2$, $v_2(x,t)$ is estimated from below by

$$\begin{aligned}
&C_0 \int_0^t \mathrm{e}^{-\frac{1}{\zeta}\left(p_e - \frac{1}{6}a{\alpha_0}^4\right)(t-\tau) - \frac{1}{\zeta}\int_\tau^t\int_x^1 \left[\frac{2u(\xi,\tau')^2}{r(\xi,\tau')^3} + \frac{G(\xi+M_0)}{r(\xi,\tau')^4}\right]\mathrm{d}\xi\,\mathrm{d}\tau'} \times \Big(C_2 - C_3 V(\tau)\Big)\,\mathrm{d}\tau \\
&\ge C_0\left(1 - \mathrm{e}^{-C_0 t}\right) - C_0 \int_0^t \mathrm{e}^{-C_0(t-\tau)} V(\tau)\,\mathrm{d}\tau,
\end{aligned}$$

whose right hand side has a positive lower bound for sufficiently large t.

Lemma 3.8 *For any* $t \in [0,T]$

$$\int_0^t \|(r^2u)_x\|^2\,\mathrm{d}\tau \le C_T. \tag{3.22}$$

Proof. Rewriting (3.2) as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\frac{1}{2}u^2 + p_e v - G\,\frac{x+M_0}{r}\right)\mathrm{d}x + \int_0^1 \left(\frac{\zeta}{v}{(r^2u)_x}^2 + 12\mu\frac{u^2v}{r^2}\right)\mathrm{d}x \\
= \int_0^1 \left(p + 8\mu\frac{u}{r}\right)(r^2u)_x\,\mathrm{d}x
\end{aligned}$$

and integrating this with respect to t, we obtain

$$\begin{aligned}
\int_0^t \|(r^2u)_x\|^2\,\mathrm{d}\tau &\le C\left(1 + \int_0^t \|p\|^2\,\mathrm{d}\tau\right) \\
&\le C\left(1 + \int_0^t \max_{x\in\overline{\Omega}}\theta^4 \cdot \int_0^1 \theta^4\,\mathrm{d}x\,\mathrm{d}\tau\right).
\end{aligned}$$

From this one can easily derive (3.22) by virtue of Lemma 3.5.

Corollary 3.2 *For any* $t \in [0,T]$

$$\int_0^t \|u_x\|^2 \, \mathrm{d}\tau \le C_T. \tag{3.23}$$

Proof. From $u_x = \dfrac{1}{r^2}(r^2u)_x - 2\dfrac{v}{r^3}u$ we have

$$\int_0^t \|u_x\|^2 \, \mathrm{d}\tau \le 2\int_0^t \left(\frac{1}{{R_0}^2}\|(r^2u)_x\|^2 + 4\int_0^1 \frac{v^2}{r^6}u^2 \, \mathrm{d}x \right) \mathrm{d}\tau,$$

and hence (3.23) follows from Lemma 3.8.

Lemma 3.9 *If* $q \ge 2$*, then for any* $t \in [0,T]$

$$\|v_x\|^2 + \int_0^t\int_0^1 \theta {v_x}^2 \, \mathrm{d}x \, \mathrm{d}\tau \le C_T. \tag{3.24}$$

Proof. Since $(2.9)^1$ and $(2.9)^2$ imply

$$\left(\frac{u}{r^2} - \zeta\frac{v_x}{v}\right)_t = -p_x - 2\frac{u^2}{r^3} - G\,\frac{x+M_0}{r^4},$$

multiplying this by $\dfrac{u}{r^2} - \zeta\dfrac{v_x}{v}$ and integrating it with respect to x lead to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{u}{r^2} - \zeta\frac{v_x}{v}\right\|^2 + \int_0^1 \theta {v_x}^2 \, \mathrm{d}x \\
&\le C\left[1 + V(t) + \max_{x\in\overline{\Omega}}(u^2+\theta)\cdot\|u\|^2 + \max_{x\in\overline{\Omega}}\left(1+\theta^2+\frac{\theta^8}{1+\theta^q}\right)\cdot\left\|\frac{u}{r^2} - \zeta\frac{v_x}{v}\right\|^2\right].
\end{aligned}$$

Note that $(2.10)^2$ and Cauchy-Schwarz' inequality imply

$$(r^2u)(x,t) = \int_0^x (r^2u)_x \, \mathrm{d}x \le C_0V(t)^{1/2}\left(\int_0^1 v\theta \, \mathrm{d}x\right)^{1/2},$$

so that for $(x,t) \in \overline{Q_T}$

$$|u(x,t)| \le C_0V(t)^{1/2}, \tag{3.25}$$

and

$$\int_0^t \max_{x\in\overline{\Omega}} \frac{\theta^8}{1+\theta^q}\,\mathrm{d}\tau \le C$$

holds from Lemma 3.5 for $q \ge 2$. Applying Gronwall's inequality to the above inequality gives (3.24).

To obtain higher order estimetes of u and θ we introduce the function

$$\mathrm{K} = \mathrm{K}(v,\theta) := \int_0^\theta \frac{\kappa(v,\xi)}{v}\,\mathrm{d}\xi,$$

which has the estimates

$$|\mathrm{K}| \le C(1+\theta^{q+1}), \qquad |\mathrm{K}_v|,\ |\mathrm{K}_{vv}| \le C\theta \tag{3.26}$$

(see [21, 27]). Multiplying (3.5) by $\left(\frac{1}{r^4}\mathrm{K}\right)_t$, integrating it over $[0,1]\times[0,t]$ and using the boundary condition of θ, we have

$$\begin{aligned}
&\int_0^t\!\!\int_0^1 e_\theta\theta_t\left(\frac{1}{r^4}\mathrm{K}\right)_t \mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_0^1 \frac{r^4\kappa}{v}\theta_x\left(\frac{1}{r^4}\mathrm{K}\right)_{xt}\mathrm{d}x\,\mathrm{d}\tau\\
&\quad= \int_0^t\!\!\int_0^1 \bigg[-\theta p_\theta (r^2u)_x + \frac{\zeta}{v}{(r^2u)_x}^2 - 8\mu\frac{u}{r}(r^2u)_x\\
&\qquad\qquad + 12\mu\frac{v}{r^2}u^2 + \lambda\phi z\bigg]\left(\frac{1}{r^4}\mathrm{K}\right)_t \mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}\tag{3.27}$$

Here

$$\begin{cases}
\left(\frac{1}{r^4}\mathrm{K}\right)_t = \frac{1}{r^4}\frac{\kappa}{v}\theta_t + \frac{1}{r^4}\mathrm{K}_v(r^2u)_x - \frac{4}{r^5}\mathrm{K}u,\\
\left(\frac{1}{r^4}\mathrm{K}\right)_{xt} = \frac{1}{r^4}\left(\frac{\kappa}{v}\theta_x\right)_t + \frac{1}{r^4}\mathrm{K}_v(r^2u)_{xx} + \frac{1}{r^4}\mathrm{K}_{vv}v_x(r^2u)_x + \frac{1}{r^4}\left(\frac{\kappa}{v}\right)_v v_x\theta_t\\
\qquad -\frac{4}{r^7}\kappa\theta_t - \frac{4}{r^5}\frac{\kappa}{v}u\theta_x - \frac{4}{r^5}\mathrm{K}u_x - \frac{4}{r^5}\mathrm{K}_v u v_x + \frac{20v}{r^8}\mathrm{K}u - \frac{4v}{r^7}\mathrm{K}_v(r^2u)_x.
\end{cases}$$

We define the quantities:

$$X := \int_0^t\!\!\int_0^1 \left(1+\theta^{q+3}\right){\theta_t}^2\,\mathrm{d}x\,\mathrm{d}\tau, \quad Y := \max_{t\in[0,T]}\int_0^1 \left(1+\theta^{2q}\right){\theta_x}^2\,\mathrm{d}x,$$

$$Z := \max_{t\in[0,T]} \|(r^2u)_{xx}\|^2.$$

By Cauchy-Schwarz' inequality we have for any $t \in [0,T]$

$$\begin{aligned}\max_{x\in\overline{\Omega}} \theta^{2q+2} &\leq C + C\int_0^1 (1+\theta)^{2q+1}|\theta_x|\,\mathrm{d}x \\ &\leq C + C\max_{x\in\overline{\Omega}}(1+\theta)^{q-1}\cdot\left[\int_0^1 (1+\theta)^4\,\mathrm{d}x\right]^{1/2} Y^{1/2} \\ &\leq C + \frac{1}{2}\max_{x\in\overline{\Omega}}\theta^{2q+2} + CY^{\frac{q+1}{q+3}},\end{aligned}$$

which gives

$$|\theta|^{(0)} \leq C + CY^{\frac{1}{2q+6}}. \tag{3.28}$$

Also by the standard interpolation inequality

$$\begin{aligned}\max_{t\in[0,T]}\|(r^2u)_x\|^2 &\leq C\max_{t\in[0,T]}\Big(\,\|r^2u\|^2 + \|r^2u\|\,\|(r^2u)_{xx}\|\,\Big) \\ &\leq C + CZ^{1/2}\end{aligned} \tag{3.29}$$

we have from Cauchy-Schwarz' inequality

$$\begin{aligned}|(r^2u)_x|^{(0)} &\leq \max_{t\in[0,T]}\Big(\,\|(r^2u)_x\|^2 + 2\|(r^2u)_x\|\,\|(r^2u)_{xx}\|\,\Big)^{1/2} \\ &\leq C\left[\,1 + Z^{1/2} + (1+Z^{1/2})^{1/2}Z^{1/2}\,\right]^{1/2} \\ &\leq C + CZ^{3/8}.\end{aligned} \tag{3.30}$$

Estimating each term in (3.27) by using (3.25), (3.26) and (3.28)-(3.30), we have the following lemma.

Lemma 3.10 *If* $2 \leq q < 9$ *and* $0 \leq \beta < q+9$*, then then there exists a number* δ*,* $0 < \delta < 1$ *such that*

$$X + Y \leq C_T\left(1 + Z^{\delta}\right). \tag{3.31}$$

Proof. At first, we suppose $q \geq 2$, $\beta \geq 0$. Hereafter we use C_ε as a positive constant depending on ε. One can immediately derive the following inequalities from the definitions of X and Y.

$$\int_0^t\int_0^1 e_\theta\theta_t\cdot\frac{1}{r^4}\frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau \geq CX, \tag{3.32}$$

$$\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x \cdot \frac{1}{r^4}\left(\frac{\kappa}{v}\theta_x\right)_t \mathrm{d}x\,\mathrm{d}\tau = \frac{1}{2}\int_0^1 \left(\frac{\kappa}{v}\theta_x\right)^2 \mathrm{d}x - \frac{1}{2}\int_0^1 \left(\frac{\kappa_0}{v_0}{\theta_0}'\right)^2 \mathrm{d}x$$
$$\geq CY - C \tag{3.33}$$

with $\kappa_0 := \kappa_1 + \kappa_2 v_0 {\theta_0}^q$. In preperation for estimating other terms in (3.27) we have for any $t \in [0,T]$

$$\max_{x\in\overline{\Omega}} \left(\frac{r^4\kappa}{v}\theta_x\right)^2 \leq \int_0^1 \left(\frac{r^4\kappa}{v}\theta_x\right)^2 \mathrm{d}x + 2\int_0^1 \left|\frac{r^4\kappa}{v}\theta_x\right| \left|\left(\frac{r^4\kappa}{v}\theta_x\right)_x\right| \mathrm{d}x$$
$$\leq C|1+\theta^{q+2}|^{(0)}V(t) + CV(t)^{1/2}\left[\int_0^1 (1+\theta^{q+2})\left(\frac{r^4\kappa}{v}\theta_x\right)_x^2 \mathrm{d}x\right]^{1/2}. \tag{3.34}$$

From (3.5) it follows

$$\left(\frac{r^4\kappa}{v}\theta_x\right)_x^2 \leq C\left[\, {e_\theta}^2{\theta_t}^2 + \theta^2{p_\theta}^2{(r^2u)_x}^2 + {(r^2u)_x}^4 + {(r^2u)_x}^2u^2 + u^4 + \phi^2z^2\,\right].$$

Noting the inequalities

$$\int_0^t\int_0^1 (1+\theta^{q+2}){e_\theta}^2{\theta_t}^2 \,\mathrm{d}x\,\mathrm{d}\tau \leq C\,|1+\theta^5|^{(0)}X \leq CX + CXY^{\frac{5}{2q+6}},$$

$$\int_0^t\int_0^1 (1+\theta^{q+2})\theta^2{p_\theta}^2{(r^2u)_x}^2 \,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leq C\left|(1+\theta^2){(r^2u)_x}^2\right|^{(0)}\int_0^t \max_{x\in\overline{\Omega}}(1+\theta^{q+4})\int_0^1 (1+\theta^4)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leq C + CY^{\frac{1}{q+3}} + CY^{\frac{1}{q+3}}Z^{3/4} + CZ^{3/4},$$

$$\int_0^t\int_0^1 (1+\theta^{q+2}){(r^2u)_x}^4 \,\mathrm{d}x\,\mathrm{d}\tau \leq \left|(1+\theta^{q+2}){(r^2u)_x}^2\right|^{(0)}\int_0^t \|(r^2u)_x\|^2\,\mathrm{d}\tau$$
$$\leq C + CY^{\frac{q+2}{2q+6}} + CY^{\frac{q+2}{2q+6}}Z^{3/4} + CZ^{3/4},$$

$$\int_0^t\int_0^1 (1+\theta^{q+2})u^4 \,\mathrm{d}x\,\mathrm{d}\tau \leq C\left|1+\theta^{q+2}\right|^{(0)}\int_0^t V(\tau)\|u\|^2\,\mathrm{d}\tau \leq C + CY^{\frac{q+2}{2q+6}},$$

$$\int_0^t\int_0^1 (1+\theta^{q+2})\phi^2z^2 \,\mathrm{d}x\,\mathrm{d}\tau \leq C\left|1+\theta^{q+2+\beta}\right|^{(0)}\int_0^t\int_0^1 \phi z^2\,\mathrm{d}x\,\mathrm{d}\tau \leq C + CY^{\frac{q+2+\beta}{2q+6}},$$

we have by integrating (3.34)

$$
\begin{aligned}
&\int_0^t \max_{x\in\overline{\Omega}} \left(\frac{r^4\kappa}{v}\theta_x\right)^2 \mathrm{d}\tau \\
&\le C\left(1+X^{1/2}+Y^{\frac{q/2+1+\beta/2}{2q+6}}+Z^{3/8}+X^{1/2}Y^{\frac{5/2}{2q+6}}+Y^{\frac{q/2+1}{2q+6}}Z^{3/8}\right) \qquad (3.35)\\
&\le \varepsilon(X+Y)+C_\varepsilon\left(1+Z^{3/4}\right)
\end{aligned}
$$

for $0\le\beta<3q+10$. Hereafter we assume $0\le\beta<3q+10$. The remaining estimates are as follows.

$$
\begin{aligned}
&\left|\int_0^t\int_0^1 e_\theta\theta_t\cdot\frac{1}{r^4}\mathrm{K}_v(r^2u)_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \\
&\qquad \le \varepsilon X + C_\varepsilon\left|(r^2u)_x{}^2\right|^{(0)}\int_0^t \max_{x\in\overline{\Omega}}(1+\theta)^{1-q}\int_0^1(1+\theta)^4\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad \le \varepsilon X + C_\varepsilon\left(1+Z^{3/4}\right); \qquad (3.36)
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_0^t\int_0^1 e_\theta\theta_t\cdot\frac{4}{r^5}\mathrm{K}u\,\mathrm{d}x\,\mathrm{d}\tau\right| &\le \varepsilon X + C_\varepsilon\left|1+\theta\right|^{(0)}\int_0^t\max_{x\in\overline{\Omega}}(1+\theta^{q+4})\|u\|^2\,\mathrm{d}\tau \\
&\le \varepsilon(X+Y)+C_\varepsilon; \qquad (3.37)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x\cdot\frac{1}{r^4}\mathrm{K}_v(r^2u)_{xx}\,\mathrm{d}x\,\mathrm{d}\tau\right| \\
&\qquad \le C\left|1+\theta^{\frac{q}{2}+2}\right|^{(0)}\max_{t\in[0,T]}\|(r^2u)_{xx}\|\int_0^t\left(1+V(\tau)\right)\mathrm{d}\tau \\
&\qquad \le C+CY^{\frac{q/2+2}{2q+6}}Z^{1/2}\le \varepsilon Y + C_\varepsilon\left(1+Z^{3/4}\right); \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x\cdot\frac{1}{r^4}\mathrm{K}_{vv}v_x(r^2u)_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \\
&\qquad \le C\left|(r^2u)_x\right|^{(0)}Y^{1/2}\left(\int_0^t\max_{x\in\overline{\Omega}}(1+\theta^2)\|v_x\|^2\,\mathrm{d}\tau\right)^{1/2} \\
&\qquad \le \varepsilon Y + C_\varepsilon\left(1+Z^{3/4}\right); \qquad (3.39)
\end{aligned}
$$

$$\left|\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x \cdot \frac{1}{r^4}\left(\frac{\kappa}{v}\right)_v v_x\theta_t \,\mathrm{d}x\,\mathrm{d}\tau\right|$$

$$\le \varepsilon X + C_\varepsilon \int_0^t \max_{x\in\overline{\Omega}}\left(\frac{r^4\kappa}{v}\theta_x\right)^2 \int_0^1 \frac{1}{(1+\theta)^{q+3}}{v_x}^2\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\le \varepsilon(X+Y) + C_\varepsilon\left(1+Z^{3/4}\right); \tag{3.40}$$

$$\left|\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x \cdot \frac{4}{r^7}\kappa\theta_t \,\mathrm{d}x\,\mathrm{d}\tau\right|$$

$$\le \varepsilon X + C_\varepsilon\left|(1+\theta)^{q-7}\right|^{(0)} \int_0^t \max_{x\in\overline{\Omega}}\left(\frac{r^4\kappa}{v}\theta_x\right)^2 \int_0^1 (1+\theta)^4\,\mathrm{d}x\,\mathrm{d}\tau,$$

which is estimated from above by

$$\varepsilon(X+Y) + C_\varepsilon\left(1+Z^{3/4}\right) \tag{3.41}$$

for $2 \le q \le 7$ and from (3.35) by

$$C\left(1 + X^{1/2} + Z^{3/8} + Y^{\frac{(3/2)q-6+\beta/2}{2q+6}} + X^{1/2}Y^{\frac{q-9/2}{2q+6}} + Y^{\frac{(3/2)q-6}{2q+6}}Z^{3/8}\right)$$

$$\le \varepsilon(X+Y) + C_\varepsilon\left(1+Z^{\delta_1}\right) \tag{3.42}$$

with a number δ_1 $(0<\delta_1<1)$ for $7<q<39$ and $0\le\beta<q+24$;

$$\left|\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x \cdot \frac{4}{r^5}\frac{\kappa}{v}u\theta_x \,\mathrm{d}x\,\mathrm{d}\tau\right| \le C\int_0^t \max_{x\in\overline{\Omega}}\left(\frac{r^4\kappa}{v}\theta_x\right)^2 \cdot \|u\|\,\mathrm{d}\tau$$

$$\le \varepsilon(X+Y) + C_\varepsilon\left(1+Z^{3/4}\right); \tag{3.43}$$

$$\left|\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x \cdot \frac{4}{r^5}\mathrm{K}u_x \,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon Y + C_\varepsilon\left|1+\theta^{2q+2}\right|^{(0)}\int_0^t \|u_x\|^2\,\mathrm{d}\tau$$

$$\le \varepsilon Y + C_\varepsilon; \tag{3.44}$$

$$\left|\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x \cdot \frac{4}{r^5}\mathrm{K}_v u v_x \,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon Y + C_\varepsilon\left|1+\theta^2\right|^{(0)}\int_0^t V(\tau)\|v_x\|^2\,\mathrm{d}\tau$$

$$\le \varepsilon Y + C_\varepsilon; \tag{3.45}$$

$$\left|\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x\cdot\frac{20v}{r^8}\mathrm{K}u\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon Y + C_\varepsilon\big|1+\theta^{2q+2}\big|^{(0)}\int_0^t \|u\|^2\,\mathrm{d}\tau$$
$$\le \varepsilon Y + C_\varepsilon; \tag{3.46}$$

$$\left|\int_0^t\int_0^1 \frac{r^4\kappa}{v}\theta_x\cdot\frac{4v}{r^7}\mathrm{K}_v(r^2u)_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon Y + C_\varepsilon\big|1+\theta^{2}\big|^{(0)}\int_0^t \|(r^2u)_x\|^2\,\mathrm{d}\tau$$
$$\le \varepsilon Y + C_\varepsilon; \tag{3.47}$$

$$\left|\int_0^t\int_0^1 \theta p_\theta(r^2u)_x\cdot\frac{1}{r^4}\frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon X + C_\varepsilon\big|1+\theta^{q+5}\big|^{(0)}\int_0^t \|(r^2u)_x\|^2\,\mathrm{d}\tau$$
$$\le \varepsilon(X+Y) + C_\varepsilon; \tag{3.48}$$

$$\left|\int_0^t\int_0^1 \theta p_\theta(r^2u)_x\cdot\frac{1}{r^4}\mathrm{K}_v(r^2u)_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \le C\left|1+\theta^{5}\right|^{(0)}\int_0^t \|(r^2u)_x\|^2\,\mathrm{d}\tau$$
$$\le \varepsilon Y + C_\varepsilon; \tag{3.49}$$

$$\left|\int_0^t\int_0^1 \theta p_\theta(r^2u)_x\cdot\frac{4}{r^5}\mathrm{K}u\,\mathrm{d}x\,\mathrm{d}\tau\right| \le C\left|1+\theta^{q+5}\right|^{(0)}\int_0^t \left(\|(r^2u)_x\|^2+\|u\|^2\right)\mathrm{d}\tau$$
$$\le \varepsilon Y + C_\varepsilon; \tag{3.50}$$

$$\left|\int_0^t\int_0^1 \frac{\zeta}{v}{(r^2u)_x}^2\cdot\frac{1}{r^4}\frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau\right|$$
$$\le \varepsilon X + C_\varepsilon\big|(1+\theta)^{q-3}\big|^{(0)}\big|{(r^2u)_x}^2\big|^{(0)}\int_0^t \|(r^2u)_x\|^2\,\mathrm{d}\tau$$
$$\le \varepsilon(X+Y) + C_\varepsilon\left(1+Z^{\delta_2}\right) \tag{3.51}$$

with a number δ_2 $(0<\delta_2<1)$ for $2\le q<9$;

$$\left|\int_0^t\int_0^1 \frac{\zeta}{v}{(r^2u)_x}^2\cdot\frac{1}{r^4}\mathrm{K}_v(r^2u)_x\,\mathrm{d}x\,\mathrm{d}\tau\right|$$
$$\le C\left|(r^2u)_x\right|^{(0)}\max_{t\in[0,T]}\|(r^2u)_x\|^2\int_0^t\max_{x\in\overline{\Omega}}\theta\,\mathrm{d}\tau \le C\left(1+Z^{7/8}\right); \tag{3.52}$$

$$\left|\int_0^t\int_0^1 \frac{\zeta}{v}(r^2u)_x{}^2\cdot\frac{4}{r^5}\mathrm{K}u\,\mathrm{d}x\,\mathrm{d}\tau\right|$$

$$\le C\left|(1+\theta)^{q-2}\right|^{(0)}\int_0^t \max_{x\in\overline{\Omega}}(1+\theta^{q+4})\|u\|^2\,\mathrm{d}\tau + C\left|(r^2u)_x{}^2\right|^{(0)}\int_0^t \|(r^2u)_x\|^2\,\mathrm{d}\tau$$

$$\le \varepsilon Y + C_\varepsilon\left(1+Z^{3/4}\right); \tag{3.53}$$

$$\left|\int_0^t\int_0^1 \left[-8\mu\frac{u}{r}(r^2u)_x + 12\mu\frac{v}{r^2}u^2\right]\frac{1}{r^4}\frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau\right|$$

$$\le \varepsilon X + C_\varepsilon\left|(1+\theta)^{q-3}\right|^{(0)}\left(\max_{t\in[0,T]}\|(r^2u)_x\|^2\int_0^t V(\tau)\,\mathrm{d}\tau + \int_0^t V(\tau)\|u\|^2\,\mathrm{d}\tau\right)$$

$$\le \varepsilon X + C_\varepsilon\left[(1+Y)^{\frac{q-3}{2q+6}} + Z^{1/2} + (1+Y)^{\frac{q-3}{2q+6}}Z^{1/2}\right]$$

$$\le \varepsilon(X+Y) + C_\varepsilon\left(1+Z^{\frac{q+3}{q+9}}\right); \tag{3.54}$$

$$\left|\int_0^t\int_0^1 \left[-8\mu\frac{u}{r}(r^2u)_x + 12\mu\frac{v}{r^2}u^2\right]\frac{1}{r^4}\mathrm{K}_v(r^2u)_x\,\mathrm{d}x\,\mathrm{d}\tau\right|$$

$$\le C\left|(r^2u)_x{}^2\right|^{(0)}\int_0^t \|\theta\|\,\|u\|\,\mathrm{d}\tau + C\left|(r^2u)_x\right|^{(0)}\int_0^t \max_{x\in\overline{\Omega}}\theta\cdot\|u\|^2\,\mathrm{d}\tau$$

$$\le C\left(1+Z^{3/4}\right); \tag{3.55}$$

$$\left|\int_0^t\int_0^1 \left[-8\mu\frac{u}{r}(r^2u)_x + 12\mu\frac{v}{r^2}u^2\right]\frac{4}{r^5}\mathrm{K}u\,\mathrm{d}x\,\mathrm{d}\tau\right|$$

$$\le C\left|(r^2u)_x\right|^{(0)}\int_0^t \max_{x\in\overline{\Omega}}(1+\theta^{q+1})\cdot\|u\|\,\mathrm{d}\tau + C\left|1+\theta^{q+1}\right|^{(0)}\int_0^t V(\tau)\|u\|\,\mathrm{d}\tau$$

$$\le \varepsilon Y + C_\varepsilon\left(1+Z^{3/8}\right); \tag{3.56}$$

$$\left|\int_0^t\int_0^1 \lambda\phi z\cdot\frac{1}{r^4}\frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon X + C_\varepsilon\left|(1+\theta)^{q-3+\beta}\right|^{(0)}\int_0^t\int_0^1 \phi z^2\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\le \varepsilon(X+Y) + C_\varepsilon \tag{3.57}$$

for $0\le\beta<q+9$;

$$\left|\int_0^t\int_0^1 \lambda\phi z\cdot\frac{1}{r^4}\mathrm{K}_v(r^2u)_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \le C\left|\theta(r^2u)_x\right|^{(0)}\int_0^t\int_0^1 \phi z\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\le \varepsilon Y + C_\varepsilon\left(1+Z^{3/4}\right); \tag{3.58}$$

$$
\left|\int_0^t\int_0^1 \lambda\phi z\cdot\frac{4}{r^5}\mathrm{K}u\,\mathrm{d}x\,\mathrm{d}\tau\right| \le C\left|(1+\theta^{q+1})\,u\right|^{(0)}\int_0^t\int_0^1 \phi z\,\mathrm{d}x\,\mathrm{d}\tau
$$

$$
\le \varepsilon Y + C_\varepsilon\left(1+Z^{1/4}\right) \tag{3.59}
$$

since

$$
|u|^{(0)} \le \max_{t\in[0,T]}\left(\|u\|^2+2\|u\|\,\|u_x\|\right)^{1/2} \le C\left(1+Z^{1/8}\right) \tag{3.60}
$$

results from the standard interpolation inequality and (3.29). Combining (3.32), (3.33), (3.36)-(3.59) and taking ε sufficiently small, we obtain (3.31).

Since the regularity of the solution obtained above is not sufficient, the following arguments are rather formal. However, one can justify them by using the method of difference quotients or mollifiers. In what follows we assume that q and β are real numbers satisfying $3 \le q < 9$ and $0 \le \beta < q+9$.

Lemma 3.11 *For any $t \in [0,T]$*

$$
\|u_t\|^2 + \int_0^t \|(r^2u)_{xt}\|^2\,\mathrm{d}\tau \le C_T\left(1+Z^{\delta}\right) \tag{3.61}
$$

with a number δ, $0<\delta<1$.

Proof. Differentiating $(2.9)^2$ with respect to t, multiplying it by u_t and integrating it over $[0,1]$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}{u_t}^2\,\mathrm{d}x + \int_0^1 \frac{\zeta}{v}{(r^2u)_{xt}}^2\,\mathrm{d}x = \int_0^1 \Bigg\{ & p_t\left[(r^2u)_{xt} - \frac{4}{r}(r^2u)_x u + \frac{6v}{r^2}u^2\right] \\
& + (r^2u)_{xt}\left[\frac{8\mu}{r}u_t + \frac{\zeta}{v^2}{(r^2u)_x}^2 + \frac{2\zeta}{rv}(r^2u)_x u + \frac{2}{r}pu + \frac{4\mu-6\zeta}{r^2}u^2\right] \\
& - \frac{12\mu v}{r^2}{u_t}^2 + u_t\Bigg[-\frac{2\zeta}{rv}{(r^2u)_x}^2 + \frac{2}{r}(r^2u)_x p + \frac{6\zeta-24\mu}{r^2}(r^2u)_x u \\
& - \frac{6v}{r^2}pu + \frac{24\mu v}{r^3}u^2 + \frac{2G(x+M_0)}{r^3}u\Bigg] - \frac{4\zeta}{rv^2}{(r^2u)_x}^3 u + \frac{14\zeta}{r^2v}{(r^2u)_x}^2u^2 \\
& - \frac{8}{r^2}(r^2u)_x p u^2 - \frac{16\mu+12\zeta}{r^3}(r^2u)_x u^3 + \frac{12v}{r^3}pu^3 + \frac{24\mu v}{r^4}u^4 \Bigg\}\,\mathrm{d}x.
\end{aligned}
$$

Furthermore, integrating this with respect to t and noting the inequality

$$\begin{aligned}&\int_0^t\int_0^1 \left[{p_t}^2+{(r^2u)_x}^4\right] \mathrm{d}x\,\mathrm{d}\tau \\ &\le C\int_0^t\int_0^1 (1+\theta^6){\theta_t}^2\,\mathrm{d}x\,\mathrm{d}\tau + C\left|{(r^2u)_x}^2\right|^{(0)}\int_0^t \left[\|\theta\|^2+\|(r^2u)_x\|^2\right]\mathrm{d}\tau\end{aligned}$$

derived from the equation $p_t=\left(\frac{R}{v}+\frac{4}{3}a\theta^3\right)\theta_t-\frac{R}{v^2}\theta(r^2u)_x$, we have

$$\begin{aligned}&\|u_t\|^2+\int_0^t \|(r^2u)_{xt}\|^2\,\mathrm{d}\tau \\ &\le C+CY^{\frac{4}{2q+6}}+CZ^{3/4}+C\int_0^t\int_0^1\left[{p_t}^2+{(r^2u)_x}^4+{u_t}^2\right]\mathrm{d}x\,\mathrm{d}\tau \\ &\le C\left(1+X+Y+Z^{3/4}+\int_0^t\|u_t\|^2\,\mathrm{d}\tau\right).\end{aligned}$$

This yields (3.61) by using Gronwall's inequality and Lemma 3.10.

Lemma 3.12 *For any $t\in[0,T]$*

$$\begin{aligned}&\|(r^2u)_x\|^2+\|\theta_x\|^2+\|(r^2u)_{xx}\|^2+\|u_t\|^2 \\ &\qquad +\int_0^t\left[\|\theta_t\|^2+\|(r^2u)_{xt}\|^2\right]\mathrm{d}\tau\le C_T, \end{aligned}\tag{3.62}$$

$$|(r^2u)_x|^{(0)}+|u|^{(0)}+|\theta|^{(0)}\le C_T. \tag{3.63}$$

Proof. Squaring the equality

$$\frac{\zeta r^2}{v}(r^2u)_{xx}=u_t+r^2p_x+\frac{\zeta r^2}{v^2}(r^2u)_x v_x+G\,\frac{x+M_0}{r^2},$$

integrating it with respect to x and using Lemmas 3.10 and 3.11 and the relation $p_x=\left(\frac{R}{v}+\frac{4}{3}a\theta^3\right)\theta_x-\frac{R}{v^2}\theta v_x$, we have for any $t\in[0,T]$

$$\begin{aligned}&\|(r^2u)_{xx}\|^2 \\ &\le C\left\{1+\|u_t\|^2+\max_{t\in[0,T]}\int_0^1(1+\theta^6){\theta_x}^2\,\mathrm{d}x+\left[\left|\theta^2\right|^{(0)}+\left|{(r^2u)_x}^2\right|^{(0)}\right]\|v_x\|^2\right\} \\ &\le C\left(1+X+Y+Z^{3/4}\right)\le C\left(1+Z^\delta\right).\end{aligned}$$

This implies

$$Z \le C\left(1 + Z^{\delta}\right)$$

and therefore, Z is bounded. From Lemmas 3.10 and 3.11, (3.60) and (3.28)-(3.30) we conclude that $|u|^{(0)}$, $|(r^2u)_x|^{(0)}$, $\|(r^2u)_x\|$, $\|u_t\|$, $\int_0^t \|(r^2u)_{xt}\|^2 \,\mathrm{d}\tau$, $|\theta|^{(0)}$, $\|\theta_x\|$ and $\int_0^t \|\theta_t\|^2 \,\mathrm{d}\tau$ are also bounded.

Lemma 3.13 *For any* $(x,t) \in \overline{Q_T}$

$$\theta(x,t) \ge C_T. \tag{3.64}$$

Proof. By putting $\Theta := \dfrac{1}{\theta}$, (3.5) becomes

$$e_\theta \Theta_t = \left(\frac{r^4\kappa}{v}\Theta_x\right)_x + \frac{v{p_\theta}^2}{4(\zeta - \frac{4}{3}\mu)} - \left\{\frac{\zeta - \frac{4}{3}\mu}{v}\Theta^2\left[(r^2u)_x - \frac{vp_\theta}{2(\zeta - \frac{4}{3}\mu)\Theta}\right]^2\right.$$
$$\left. + \frac{4\mu}{3v}\Theta^2\left[(r^2u)_x - \frac{3v}{r}u\right]^2 + \frac{2r^4\kappa}{v\Theta}{\Theta_x}^2 + \lambda\phi z\Theta^2\right\}.$$

Since $e_\theta > c_{\mathrm{v}}$, and $p_\theta \le C + C\,|\theta^3|^{(0)} \le C$ from (3.63), there exists a positive constant C_4 such that in Q_T

$$\Theta_t < \frac{1}{e_\theta}\left(\frac{r^4\kappa}{v}\Theta_x\right)_x + C_4.$$

Therefore $\widetilde{\Theta}(x,t) := C_4\,t + \max\limits_{x\in\overline{\Omega}}\left[\theta_0(x)^{-1}\right] - \Theta(x,t)$ satisfies

$$\begin{cases} \mathfrak{L}\widetilde{\Theta} < 0 & \text{in } \ Q_T, \\ \widetilde{\Theta}\big|_{t=0} \ge 0 & \text{for } \ x \in [0,1], \\ \widetilde{\Theta}_x\big|_{x=0,1} = 0 & \text{for } \ t \in [0,T], \end{cases}$$

where $\mathfrak{L}$ is a parabolic operator $\mathfrak{L} := -\dfrac{\partial}{\partial t} + \dfrac{1}{e_\theta}\dfrac{\partial}{\partial x}\left(\dfrac{r^4\kappa}{v}\dfrac{\partial}{\partial x}\right)$. Standard comparison arguments imply $\min\limits_{(x,t)\in\overline{Q_T}} \widetilde{\Theta}(x,t) \ge 0$, which gives for any $(x,t) \in \overline{Q_T}$

$$\theta(x,t) \ge \left\{C_4\,t + \max_{x\in\overline{\Omega}}\left[\theta_0(x)^{-1}\right]\right\}^{-1}.$$

Lemma 3.14 *For any $t \in [0,T]$*

$$\|z_x\|^2 + \|z_{xx}\|^2 + \|z_t\|^2 + \int_0^t \|z_{xt}\|^2 \, \mathrm{d}\tau \le C_T. \tag{3.65}$$

Proof. Multiplying $(2.9)^4$ by z_{xx} and integrating it with respect to x, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}{z_x}^2 \, \mathrm{d}x + \int_0^1 \frac{dr^4}{v^2}{z_{xx}}^2 \, \mathrm{d}x = \int_0^1 \left(\frac{2dr^4}{v^3} v_x z_x - \frac{4dr}{v} z_x + \phi z \right) z_{xx} \, \mathrm{d}x.$$

This yields

$$\|z_x\|^2 + \int_0^t \|z_{xx}\|^2 \, \mathrm{d}\tau \le C + C\int_0^t \|z_x\|^2 \, \mathrm{d}\tau,$$

since $\phi \le C\,|\theta^\beta|^{(0)} \le C$ from (3.63) and

$$\max_{x\in\overline{\Omega}} {z_x}^2 \le \varepsilon \|z_{xx}\|^2 + C_\varepsilon \|z_x\|^2. \tag{3.66}$$

Gronwall's inequality gives bounds of $\|z_x\|$ and $\int_0^t \|z_{xx}\|^2 \, \mathrm{d}\tau$, hence we also obtain the bound of $\int_0^t \|z_t\|^2 \, \mathrm{d}\tau$ by using $(2.9)^4$ again. Next, differentiating $(2.9)^4$ with respect to t, multiplying it by z_t and integrating that over $[0,1]$, we also have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}{z_t}^2 \, \mathrm{d}x + \int_0^1 \frac{dr^4}{v^2}{z_{xt}}^2 \, \mathrm{d}x \\ &\quad = \int_0^1 \left[\frac{2dr^4}{v^3}(r^2u)_x z_x z_{xt} - \frac{4dr^3}{v^2} u z_x z_{xt} - \phi_t z z_t - \phi {z_t}^2 \right] \mathrm{d}x. \end{aligned}$$

Since $|\phi_t| = K\mathrm{e}^{-A/\theta}\left(A\theta^{-2} + \beta\theta^{-1}\right)\theta^\beta |\theta_t| \le C|\theta_t|$ holds from (3.63) and (3.64), we have by Cauchy-Schwarz' inequality

$$\|z_t\|^2 + \int_0^t \|z_{xt}\|^2 \, \mathrm{d}\tau \le C\int_0^t \left(\|z_x\|^2 + \|z_t\|^2 + \|\theta_t\|^2 \right) \mathrm{d}\tau \le C.$$

Therefore, from

$$\frac{dr^4}{v^2} z_{xx} = z_t - \frac{4dr}{v} z_x + \frac{2dr^4}{v^3} v_x z_x + \phi z$$

we obtain

$$\|z_{xx}\|^2 \le C + C \max_{x\in\overline{\Omega}} {z_x}^2 \cdot \|v_x\|^2.$$

This gives a bound of $\|z_{xx}\|$ by using (3.66).

Lemma 3.15 *For any $t \in [0,T]$*

$$\|\theta_{xx}\|^2 + \|\theta_t\|^2 + \int_0^t \|\theta_{xt}\|^2 \, \mathrm{d}\tau \le C_T. \tag{3.67}$$

Proof. Differentiating (3.5) with respect to t, multiplying it by $e_\theta \theta_t$ and integrating that over $[0,1]$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}\left(e_\theta\theta_t\right)^2 \, \mathrm{d}x + \int_0^1 \frac{r^4\kappa}{v} e_\theta {\theta_{xt}}^2 \, \mathrm{d}x \\
&= \int_0^1 \Big\{ \theta_{xt}\big(F_1\theta_t\theta_x + F_2\theta_t v_x + F_3\theta_x\big) + {\theta_t}^2\big(F_4{\theta_x}^2 + F_5\theta_x v_x + F_6\big) + F_7\theta_t{\theta_x}^2 \\
&\qquad + F_8\theta_t\theta_x v_x + \theta_t\big[F_9(r^2u)_{xt} + F_{10}u_t + F_{11}z_t + F_{12}\big] \Big\} \, \mathrm{d}x,
\end{aligned} \tag{3.68}$$

where

$F_1 := -r^4 \left(\dfrac{\kappa}{v}\right)_\theta e_\theta - r^4 \dfrac{\kappa}{v} e_{\theta\theta},$

$F_2 := -r^4 \dfrac{\kappa}{v} e_{\theta v},$

$F_3 := -\dfrac{4r^3\kappa}{v} e_\theta u - r^4 \left(\dfrac{\kappa}{v}\right)_v e_\theta (r^2u)_x,$

$F_4 := -r^4 \left(\dfrac{\kappa}{v}\right)_\theta e_{\theta\theta},$

$F_5 := -r^4 \left(\dfrac{\kappa}{v}\right)_\theta e_{\theta v},$

$F_6 := -\left(p_\theta e_\theta + \theta p_{\theta\theta} e_\theta\right)(r^2u)_x + \lambda K \mathrm{e}^{-A/\theta}\left(\dfrac{A}{\theta^2} + \dfrac{\beta}{\theta}\right)\theta^\beta e_\theta z,$

$F_7 := -\dfrac{4r^3\kappa}{v} e_{\theta\theta} u - r^4 \left(\dfrac{\kappa}{v}\right)_v e_{\theta\theta}(r^2u)_x,$

$F_8 := -\dfrac{4r^3\kappa}{v} e_{\theta v} u - r^4 \left(\dfrac{\kappa}{v}\right)_v e_{\theta v}(r^2u)_x,$

$F_9 := -\theta p_\theta e_\theta + \dfrac{2\zeta}{v} e_\theta (r^2u)_x - \dfrac{8\mu}{r} e_\theta u,$

$F_{10} := -\dfrac{8\mu}{r} e_\theta (r^2u)_x + \dfrac{24\mu v}{r^2} e_\theta u,$

$F_{11} := \lambda\phi e_\theta,$

$$F_{12} := -\theta p_{\theta v} e_\theta (r^2 u)_x{}^2 - \frac{\zeta}{v^2} e_\theta (r^2 u)_x{}^3 + \frac{20\mu}{r^2} e_\theta (r^2 u)_x u^2 - \frac{24\mu v}{r^3} e_\theta u^3.$$

From Lemmas 3.12-3.14 together with the inequalities

$$p_\theta,\ p_{\theta v},\ p_{\theta\theta},\ e_\theta,\ e_{\theta v},\ e_{\theta\theta},\ \frac{\kappa}{v},\ \left(\frac{\kappa}{v}\right)_v,\ \left(\frac{\kappa}{v}\right)_\theta \le C, \quad e_\theta > c_{\mathrm{v}}$$

by integrating (3.68) with respect to t and using Cauchy-Schwarz' inequality one can derive

$$\|\theta_t\|^2 + \int_0^t \|\theta_{xt}\|^2 \,\mathrm{d}\tau \le C + C \int_0^t \max_{x\in\overline{\Omega}} \theta_t{}^2 \cdot \left(\|v_x\|^2 + \|\theta_x\|^2\right) \mathrm{d}\tau.$$

By virtue of

$$\max_{x\in\overline{\Omega}} \theta_t{}^2 \le \varepsilon \|\theta_{xt}\|^2 + C_\varepsilon \|\theta_t\|^2$$

we conclude that $\|\theta_t\|$ and $\int_0^t \|\theta_{xt}\|^2 \,\mathrm{d}\tau$ are bounded. Therefore, squaring the equality

$$\begin{aligned}\frac{r^4\kappa}{v}\theta_{xx} = e_\theta\theta_t + \theta p_\theta (r^2u)_x - \frac{\zeta}{v}(r^2u)_x{}^2 + \frac{8\mu}{r}(r^2u)_x u - \frac{12\mu v}{r^2}u^2 - 4r\kappa\theta_x \\ -r^4\left(\frac{\kappa}{v}\right)_v v_x\theta_x - r^4\left(\frac{\kappa}{v}\right)_\theta \theta_x{}^2 - \lambda\phi z\end{aligned}$$

and integrating it with respect to x, we obtain

$$\|\theta_{xx}\|^2 \le C + C \max_{x\in\overline{\Omega}} \theta_x{}^2 \cdot \left(\|v_x\|^2 + \|\theta_x\|^2\right).$$

From this one can derive (3.67) by using

$$\max_{x\in\overline{\Omega}} \theta_x{}^2 \le \varepsilon \|\theta_{xx}\|^2 + C_\varepsilon \|\theta_x\|^2.$$

3.2 The Hölder estimates

In this section we shall derive the Hölder estimates of the solution following the argument due to Kazhikhov-Shelukhin [32]. First, we easily obtain bounds of $|r_x,\ (r^2u)_x,\ \theta_x,\ z_x|^{(0)}$ from (3.62), (3.65) and (3.67). This implies that r, r^2u, θ and

z are uniformly Lipschitz continuous in x. Applying Cauchy-Schwarz' inequality, we have

$$\begin{aligned}\left|(r^2u)(x,t)-(r^2u)(x,t')\right| &\le \left(\int_{t'}^{t}(r^2u)_t{}^2\,\mathrm{d}\tau\right)^{1/2}|t-t'|^{1/2}\\ &\le \left[\int_{t'}^{t}\Big(\|(r^2u)_t\|^2+2\|(r^2u)_t\|\,\|(r^2u)_{xt}\|\Big)\,\mathrm{d}\tau\right]^{1/2}|t-t'|^{1/2}.\end{aligned}$$

From this together with $(r^2u)_t = r^2u_t + 2ru^2$ and (3.62) it follows that

$$|r^2u\,|_t^{(1/2)} \le C.$$

Similarly we get $|r,\,\theta,\,z|_t^{(1/2)} \le C$. Namely, we have

$$(r,u,r^2u,\theta,z)\in\left(C_{x,\,t}^{1,\,1/2}(Q_T)\right)^5. \tag{3.69}$$

Moreover, we have

$$\left|(r^2u)_x(x,t)-(r^2u)_x(x',t)\right| \le \left(\int_{x'}^{x}(r^2u)_{xx}{}^2\,\mathrm{d}\xi\right)^{1/2}|x-x'|^{1/2},$$

and hence

$$|(r^2u)_x|_x^{(1/2)} \le C$$

by virtue of (3.62). Also $|\theta_x,\,z_x|_x^{(1/2)} \le C$ follows from (3.65) and (3.67) in the same manner. Thus by a standard interpolation lemma (see for example, [34], Chapter II, Lemma 3.1) one can get

$$\big((r^2u)_x,\theta_x,z_x\big)\in\left(C_{x,\,t}^{1/3,\,1/6}(Q_T)\right)^3.$$

Recalling that $(2.9)^1$ and $v|_{t=0}=v_0\in C^{1+\alpha}(\Omega)$, we derive $v\in C_{x,\,t}^{1/3,\,1/6}(Q_T)$. Since it follows from (3.16) that

$$\begin{aligned}v_x(x,t) = \frac{1}{(\mathrm{PQR})(x,t)}\Bigg\{ & v_0{}'(x)-A(x,t)v_0(x)\\ &+\frac{R}{\zeta}\int_0^t\Big[\theta_x(x,\tau)+\theta(x,\tau)\big(A(x,\tau)-A(x,t)\big)\Big](\mathrm{PQR})(x,\tau)\,\mathrm{d}\tau\Bigg\}\end{aligned} \tag{3.70}$$

with

$$A(x,t) := \frac{1}{\zeta}\left\{ \frac{u_0}{{r_0}^2} - \frac{u}{r^2} - \int_0^t \left[\frac{2u^2}{r^3} + \frac{G(x+M_0)}{r^4}\right] \mathrm{d}\tau - \frac{4}{3}a\int_0^t \theta^3\theta_x \,\mathrm{d}\tau \right\},$$

we can easily check $v_x \in C^{\sigma,\sigma/2}_{x,t}(Q_T)$ with $\sigma := \min\{\alpha, 1/3\}$.

Next we consider $(2.9)^2$, $(2.9)^3$ and $(2.9)^4$ as the linear equations

$$\begin{cases} (r^2u)_t - \dfrac{\zeta r^4}{v}(r^2u)_{xx} + \dfrac{\zeta r^4 v_x}{v^2}(r^2u)_x - \dfrac{2u}{r}\cdot r^2 u \\ \qquad = -\dfrac{Rr^4\theta_x}{v} + \dfrac{Rr^4\theta v_x}{v^2} - \dfrac{4}{3}ar^4\theta^3\theta_x - G(x+M_0), \\ \theta_t - \dfrac{r^4\kappa}{e_\theta v}\theta_{xx} - \dfrac{1}{e_\theta}\left[4r\kappa + r^4\left(\dfrac{\kappa}{v}\right)_v v_x + r^4\left(\dfrac{\kappa}{v}\right)_\theta \theta_x\right]\theta_x + \dfrac{p_\theta (r^2u)_x}{e_\theta}\theta \\ \qquad = \dfrac{1}{e_\theta}\left[\dfrac{\zeta {(r^2u)_x}^2}{v} - \dfrac{8\mu(r^2u)_x u}{r} + \dfrac{12\mu v u^2}{r^2} + \lambda\phi z\right], \\ z_t - \dfrac{dr^4}{v^2}z_{xx} + \left(\dfrac{2dr^4v_x}{v^3} - \dfrac{4dr}{v}\right)z_x + \phi z = 0, \end{cases} \tag{3.71}$$

whose coefficients and right hand sides are Hölder continuous in x with exponent σ and in t with exponent $\sigma/2$. By the classical Schauder estimates (see for example, [14, 34]) we obtain

$$|r^2u,\, \theta,\, z|_{2+\sigma,\,1+\sigma/2} \le C.$$

This implies

$$\big(v, (r^2u)_x, \theta_x, z_x\big) \in \Big(C^{1,1/2}_{x,t}(Q_T)\Big)^4 \tag{3.72}$$

by the interpolation lemma and $(2.9)^1$. Going back to (3.70) with (3.69) and (3.72), we obtain

$$v_x \in C^{\alpha,\alpha/2}_{x,t}(Q_T). \tag{3.73}$$

Therefore, applying the Schauder estimates to (3.71) again, we have

$$|r^2u,\, \theta,\, z|_{2+\alpha,\,1+\alpha/2} \le C. \tag{3.74}$$

Finally, from (3.69), (3.72)-(3.74) and

$$\begin{cases} u_x = \dfrac{1}{r^2}(r^2u)_x - \dfrac{2v}{r^3}u, \\ u_{xx} = \dfrac{1}{r^2}(r^2u)_{xx} - \dfrac{4v}{r^5}(r^2u)_x + \dfrac{10v^2}{r^6}u - \dfrac{2v_x}{r^3}u, \\ u_t = \dfrac{\zeta r^2}{v}(r^2u)_{xx} - \dfrac{\zeta r^2 v_x}{v^2}(r^2u)_x - \dfrac{Rr^2\theta_x}{v} + \dfrac{Rr^2\theta v_x}{v^2} - \dfrac{4}{3}ar^2\theta^3\theta_x - G\,\dfrac{x+M_0}{r^2} \end{cases}$$

we obtain

$$|u|_{2+\alpha,\,1+\alpha/2} \leq C.$$

4 Proof of Theorem 2: One-dimensional problem

In this section we consider Problem 2. In order to prove Theorem 2, we shall establish the following a priori boundedness.

Proposition 3 (A priori Estimates for Problem 2) *Let T be an arbitrary positive number. Assume that α, q, β, p_e and the initial data satisfy the hypotheses of Theorem 2, and that the problem* (2.23), (2.20), (2.21) *with* (2.4), (2.14), (2.15) *has a solution (v, u, θ, z) such that*

$$(v,\, v_x, v_t, u, \theta, z) \in \left(C^{\alpha,\,\alpha/2}_{x,\,t}(Q_T)\right)^3 \times \left(C^{2+\alpha,\,1+\alpha/2}_{x,\,t}(Q_T)\right)^3.$$

Then there exists a positive constant C depending on the initial data and T such that

$$|v,\, v_x, v_t|_{\alpha,\,\alpha/2},\ |u, \theta, z|_{2+\alpha,\,1+\alpha/2} \ \le\ C,$$

$$v(x,t),\ \theta(x,t)\ \ge\ 1/C, \quad 0 \le z(x,t) \le 1 \quad \textit{for any}\ \ (x,t) \in \overline{Q_T}.$$

We prove this proposition in the following subsections. We use constants C_0, C, C_ε and C_T as the same as in §3.

4.1 Estimates in Sobolev spaces

We first show several lemmas similar to the ones in §3.1.

Lemma 4.1 *For any $t \in [0, T]$*

$$\int_0^1 \left(\frac{1}{2}u^2 + e + \lambda z + f(x)v\right) \mathrm{d}x = E_0, \tag{4.1}$$

$$U(t) + \int_0^t V(\tau)\, \mathrm{d}\tau \le C_0, \tag{4.2}$$

$$\int_0^1 z\, \mathrm{d}x + \int_0^t\!\!\int_0^1 \phi z\, \mathrm{d}x\, \mathrm{d}\tau = \int_0^1 z_0\, \mathrm{d}x, \tag{4.3}$$

$$\frac{1}{2}\int_0^1 z^2\, \mathrm{d}x + \int_0^t\!\!\int_0^1 \left(\frac{d}{v^2}{z_x}^2 + \phi z^2\right) \mathrm{d}x\, \mathrm{d}\tau = \frac{1}{2}\int_0^1 {z_0}^2\, \mathrm{d}x, \tag{4.4}$$

and for any $(x,t) \in \overline{Q_T}$

$$0 \le z(x,t) \le 1. \tag{4.5}$$

Here

$$\begin{cases} E_0 := \displaystyle\int_0^1 \left(\frac{1}{2}{u_0}^2 + e_0 + \lambda z_0 + f(x) v_0 \right) \mathrm{d}x, \\ U(t) := \displaystyle\int_0^1 \Big[c_{\mathrm{v}}(\theta - 1 - \log\theta) + R(v - 1 - \log v) \Big] \mathrm{d}x, \\ V(t) := \displaystyle\int_0^1 \left(\frac{\mu {u_x}^2}{v\theta} + \frac{\kappa {\theta_x}^2}{v\theta^2} + \lambda \frac{\phi}{\theta} z \right) \mathrm{d}x \end{cases}$$

and $f(x) := p_e + \dfrac{1}{2} G x(1-x)$.

Proof. It is easy to see from (2.23) and (2.20) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left(\frac{1}{2} u^2 + f(x) v \right) \mathrm{d}x + \int_0^1 \frac{\mu}{v} {u_x}^2 \,\mathrm{d}x = \int_0^1 p u_x \,\mathrm{d}x \tag{4.6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (e + \lambda z) \,\mathrm{d}x = \int_0^1 \left(-p + \frac{\mu}{v} u_x \right) u_x \,\mathrm{d}x.$$

Adding these equalities and integrating it over $[0,t]$, we obtain (4.1).

Rewriting $(2.23)^3$ as

$$e_\theta \theta_t + \theta p_\theta u_x = \frac{\mu}{v} {u_x}^2 + \left(\frac{\kappa}{v} \theta_x \right)_x + \lambda \phi z \tag{4.7}$$

and multiplying this by θ^{-1}, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(c_{\mathrm{v}} \log\theta + R \log v + \frac{4}{3} a v \theta^3 \right) = \frac{\mu {u_x}^2}{v\theta} + \frac{1}{\theta} \left(\frac{\kappa}{v} \theta_x \right)_x + \lambda \frac{\phi}{\theta} z.$$

Integrating this over $[0,1] \times [0,t]$ yields

$$U(t) + \int_0^t V(\tau) \,\mathrm{d}\tau \le C_0 \left(1 + \int_0^1 v\theta^3 \,\mathrm{d}x \right).$$

From Hölder's inequality

$$\int_0^1 v\theta^\gamma \,\mathrm{d}x \le \left(\int_0^1 v\theta^4 \,\mathrm{d}x \right)^{\gamma/4} \left(\int_0^1 v \,\mathrm{d}x \right)^{(4-\gamma)/4} \quad \text{for } 0 \le \gamma \le 4 \tag{4.8}$$

and (4.1), (4.2) follows.

Equalities (4.3), (4.4) are easily obtained by integrating $(2.23)^4$ over $[0,1]\times[0,t]$ and using (2.20). For the pointwise estimate (4.5) of z is obtained in the same manner as in the proof of Lemma 3.3.

Since $\left(\mu \dfrac{u_x}{v}\right)_x = \mu(\log v)_{xt}$ follows from $(2.23)^1$, integration of $(2.23)^2$ over $[0,x]\times[0,t]$ yields

$$\log\frac{v_0}{v} + \frac{1}{\mu}\int_0^t p \,\mathrm{d}\tau = \frac{1}{\mu}\left[\int_0^x (u_0 - u)\,\mathrm{d}\xi + f(x)\,t\right]. \tag{4.9}$$

Hence, we can obtain a lower bound of v:

$$\min_{(x,t)\in\overline{Q_T}} v(x,t) \geq \min_{x\in\overline{\Omega}} v_0(x)\exp\left\{-\frac{1}{\mu}\left[2\sqrt{2}E_0^{1/2} + \left(p_e + \frac{G}{8}\right)T\right]\right\}. \tag{4.10}$$

This together with (4.8) leads to

$$\int_0^1 \theta^\gamma \,\mathrm{d}x \leq C \quad \text{for } 0 \leq \gamma \leq 4. \tag{4.11}$$

From (4.9) the following representation formula of v holds in the same manner as in the proof of Lemma 3.6.

Lemma 4.2 *The identity*

$$v(x,t) = \frac{1}{\mathrm{P}(x,t)\mathrm{Q}(x,t)\mathrm{R}(x,t)} \times\left(v_0(x) + \frac{R}{\mu}\int_0^t \theta(x,\tau)\mathrm{P}(x,\tau)\mathrm{Q}(x,\tau)\mathrm{R}(x,\tau)\,\mathrm{d}\tau\right) \tag{4.12}$$

holds, where

$$\begin{cases} \mathrm{P}(x,t) := \exp\left[\dfrac{1}{\mu}\displaystyle\int_0^x \big(u_0(\xi) - u(\xi,t)\big)\,\mathrm{d}\xi\right], \\ \mathrm{Q}(x,t) := \exp\left(\dfrac{1}{\mu}f(x)\,t\right), \\ \mathrm{R}(x,t) := \exp\left(-\dfrac{a}{3\mu}\displaystyle\int_0^t \theta(x,\tau)^4\,\mathrm{d}\tau\right). \end{cases}$$

From (4.11) we obtain (see Lemma 3.5)

Lemma 4.3 *For any $t \in [0, T]$ and $\gamma \in [0, q+4]$, $q \geq 0$*

$$\int_0^t \max_{x \in \overline{\Omega}} \theta(x, \tau)^\gamma \, \mathrm{d}\tau \leq C_T. \tag{4.13}$$

Since the pointwise lower bound of v is already obtained in (4.10), here we get the upper one, i.e.,

Lemma 4.4 *For any $(x, t) \in \overline{Q_T}$*

$$v(x, t) \leq C_T. \tag{4.14}$$

Proof. Decomposing v in (4.12) into $v_1 + v_2$, where

$$v_1 = v_1(x, t) := \frac{v_0(x)}{(\mathrm{PQR})(x, t)},$$
$$v_2 = v_2(x, t) := \frac{R}{\zeta} \int_0^t \frac{(\mathrm{PQR})(x, \tau)}{(\mathrm{PQR})(x, t)} \theta(x, \tau) \, \mathrm{d}\tau,$$

we have the following estimates (see the proof of Lemma 3.7):

$$C_0 \, \mathrm{e}^{-\frac{t}{\mu}\left(f(x) - \frac{1}{6} a \alpha_0{}^4\right)} \leq v_1(x, t) \leq C_0 \, \mathrm{e}^{-\frac{t}{\mu}\left(f(x) - \frac{2}{3} a \beta_0{}^4\right)}, \tag{4.15}$$

$$v_2(x, t) \leq C_0 \int_0^t \mathrm{e}^{-\frac{1}{\mu}\left(f(x) - \frac{2}{3} a \beta_0{}^4\right)(t - \tau)} \Big(1 + V(\tau)\Big) \, \mathrm{d}\tau \tag{4.16}$$

with positive roots α_0 and β_0 of the equation $y - \log y - 1 = C_0 / c_{\mathrm{v}}$, where C_0 is the constant appeared in the right-hand side of (4.2). From (4.15) and (4.16) the boundedness of v from above is obtained.

Remark. If p_e is sufficiently large, then for any $(x, t) \in \overline{Q_T}$

$$C_0{}^{-1} \leq v(x, t) \leq C_0.$$

Indeed, (4.15) and the assumption $p_e \geq \dfrac{2}{3} a \beta_0{}^4$ imply that v_1 is decreasing with respect to t exponentially. Therefore, the uniform boundedness of v from above follows from (4.16). Also we have for any $(x,t) \in \overline{Q_T}$

$$v_2(x,t) \geq C_0 \int_0^t \mathrm{e}^{-\frac{1}{\mu}\left(f(x)-\frac{1}{6}a\alpha_0{}^4\right)(t-\tau)}\Big(C_0 - C_0 V(\tau)\Big)\,\mathrm{d}\tau$$

$$\geq C_0\left(1-\mathrm{e}^{-C_0 t}\right) - C_0 \int_0^t \mathrm{e}^{-C_0(t-\tau)} V(\tau)\,\mathrm{d}\tau,$$

whose right-hand side is uniformly bounded from below for sufficiently large t.

Lemma 4.5 *For any* $t \in [0,T]$

$$\text{(i)} \quad \int_0^t \|u_x\|^2\,\mathrm{d}\tau \leq C_T, \tag{4.17}$$

$$\text{(ii)} \quad \|v_x\|^2 + \int_0^t\int_0^1 \theta {v_x}^2\,\mathrm{d}x\,\mathrm{d}\tau \leq C_T \qquad \textit{if } q \geq 2. \tag{4.18}$$

Proof. Integrating (4.6) with respect to t and using Lemma 4.3, we have (4.17) (see the proof of Lemma 3.8).

On the other hand, $(2.23)^1$ and $(2.23)^2$ imply

$$\left(u - \mu\frac{v_x}{v}\right)_t = -p_x - G\left(x - \frac{1}{2}\right).$$

Multiplying this by $u - \mu\frac{v_x}{v}$ and integrating over [0,1], we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}\left(u - \mu\frac{v_x}{v}\right)^2\mathrm{d}x + \int_0^1 \frac{\mu R}{v^3}\theta {v_x}^2\,\mathrm{d}x$$

$$= \int_0^1 \frac{R}{v^2}u\theta v_x\,\mathrm{d}x - \int_0^1\left[\left(\frac{R}{v}+\frac{4}{3}a\theta^3\right)\theta_x + G\left(x-\frac{1}{2}\right)\right]\left(u-\mu\frac{v_x}{v}\right)\mathrm{d}x. \tag{4.19}$$

Firstly, we have for any $\varepsilon > 0$

$$\left|\int_0^1 \frac{R}{v^2}u\theta v_x\,\mathrm{d}x\right| \leq \varepsilon\int_0^1 \theta {v_x}^2\,\mathrm{d}x + C_\varepsilon \max_{x\in\overline{\Omega}}\theta \cdot \int_0^1 u^2\,\mathrm{d}x.$$

The second term of the right-hand side of (4.19) is estimated as follows.

$$\left|\int_0^1\left[\left(\frac{R}{v}+\frac{4}{3}a\theta^3\right)\theta_x + G\left(x-\frac{1}{2}\right)\right]\left(u-\mu\frac{v_x}{v}\right)\mathrm{d}x\right|$$

$$\leq C\left[1+\int_0^1 \kappa\frac{{\theta_x}^2}{\theta^2}\,\mathrm{d}x + \int_0^1 \frac{\theta^2(1+\theta^3)^2}{\kappa}\left(u-\frac{\mu}{v}v_x\right)^2\mathrm{d}x\right]$$

$$\leq C\left[1+V(t)+\max_{x\in\overline{\Omega}}\left(1+\theta^2+\frac{\theta^8}{1+\theta^q}\right)\cdot\int_0^1\left(u-\mu\frac{v_x}{v}\right)^2\mathrm{d}x\right].$$

If $q \geq 2$, then Gronwall's inequality gives (4.18) in virtue of Lemma 4.3.

Lemma 4.6 *If $q \geq 4$, then for any $t \in [0, T]$*

$$\int_0^t \|u_x\|_{L^3(\Omega)}^3 \,\mathrm{d}\tau \leq C_T. \tag{4.20}$$

Proof. We use a method due to Dafermos-Hsiao [5]. Putting $w = \int_0^x u \,\mathrm{d}\xi$ and using (2.24), we get a new system:

$$\begin{cases} w_t = \dfrac{\mu}{v} w_{xx} - p + f(x) & \text{in } Q_T, \\ w|_{t=0} = w_0(x) := \displaystyle\int_0^x u_0(\xi)\,\mathrm{d}\xi & \text{for } x \in [0,1], \\ w|_{x=0,1} = 0 & \text{for } t \in [0,T]. \end{cases}$$

General theory of linear parabolic equations (see for example, [34]) gives

$$\int_0^t \|w_{xx}\|_{L^3(\Omega)}^3 \,\mathrm{d}\tau \leq C\left(\|w_0\|_{W_3^{4/3}(\Omega)} + \int_0^t \|-p + f(x)\|_{L^3(\Omega)}^3 \,\mathrm{d}\tau\right).$$

Therefore, we have

$$\begin{aligned} \int_0^t \|u_x\|_{L^3(\Omega)}^3 \,\mathrm{d}\tau &\leq C\left(1 + \int_0^t \|p\|_{L^3(\Omega)}^3 \,\mathrm{d}\tau\right) \\ &\leq C\left[1 + \int_0^t \left(\int_0^1 \theta^3 \,\mathrm{d}x + \max_{0\leq x\leq 1} \theta^8 \cdot \int_0^1 \theta^4 \,\mathrm{d}x\right) \mathrm{d}\tau\right]. \end{aligned}$$

If $q \geq 4$, then the right-hand side is bounded.

In the same manner as in §3.1 we introduce the function

$$\mathrm{K} = \mathrm{K}(v, \theta) := \int_0^\theta \frac{\kappa(v, \xi)}{v} \,\mathrm{d}\xi.$$

Multiplying (4.7) by K_t and integrating it over $[0,1] \times [0,t]$, we have

$$\begin{aligned} &\int_0^t\int_0^1 e_\theta \theta_t \mathrm{K}_t \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_0^1 \frac{\kappa}{v}\theta_x \mathrm{K}_{xt} \,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad = \int_0^t\int_0^1 \left(-\theta p_\theta u_x + \frac{\mu}{v}{u_x}^2 + \lambda\phi z\right) \mathrm{K}_t \,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{4.21}$$

Here

$$\begin{cases} \mathrm{K}_t = \dfrac{\kappa}{v}\theta_t + \mathrm{K}_v u_x, \\ \mathrm{K}_{xt} = \left(\dfrac{\kappa}{v}\theta_x\right)_t + \mathrm{K}_v u_{xx} + \mathrm{K}_{vv} v_x u_x + \left(\dfrac{\kappa}{v}\right)_v v_x \theta_t, \\ |\mathrm{K}_v|,\ |\mathrm{K}_{vv}| \le C\theta. \end{cases}$$

Let us introduce the quantities:

$$X := \int_0^t\int_0^1 \left(1+\theta^{q+3}\right)\theta_t{}^2\,\mathrm{d}x\,\mathrm{d}\tau, \quad Y := \max_{t\in[0,T]}\int_0^1 \left(1+\theta^{2q}\right)\theta_x{}^2\,\mathrm{d}x,$$
$$Z := \max_{t\in[0,T]}\|u_{xx}\|^2.$$

It is easily seen that the following inequalities hold (see (3.28)-(3.30)):

$$|\theta|^{(0)} \le C + CY^{\frac{1}{2q+6}}, \quad \max_{t\in[0,T]}\|u_x\|^2 \le C + CZ^{1/2}, \quad |u_x|^{(0)} \le C + CZ^{3/8}. \tag{4.22}$$

Estimating each term in (4.21), we can obtain the following lemma.

Lemma 4.7 *If $q \ge 2$ and $0 \le \beta < q+9$, then there exists a number δ, $0<\delta<1$ such that*

$$X + Y \le C_T\left(1+Z^\delta\right). \tag{4.23}$$

Proof. Let $q \ge 2$ and $\beta \ge 0$ first. Since we already have obtained similar result in Lemma 3.10, and most of terms in (4.21) are estimated in similar ways to that (see for details, [67]), we immediately obtain the following estimates:

$$\int_0^t\int_0^1 e_\theta\theta_t \cdot \frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau \ge CX, \tag{4.24}$$

$$\left|\int_0^t\int_0^1 e_\theta\theta_t \cdot \mathrm{K}_v u_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon X + C_\varepsilon\left(1+Z^{3/4}\right), \tag{4.25}$$

$$\int_0^t\int_0^1 \frac{\kappa}{v}\theta_x\left(\frac{\kappa}{v}\theta_x\right)_t\,\mathrm{d}x\,\mathrm{d}\tau \ge CY - C, \tag{4.26}$$

$$\left|\int_0^t\int_0^1 \frac{\kappa}{v}\theta_x \cdot \mathrm{K}_v u_{xx}\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon Y + C_\varepsilon\left(1+Z^{3/4}\right), \tag{4.27}$$

$$\left|\int_0^t\int_0^1 \frac{\kappa}{v}\theta_x \cdot \mathrm{K}_{vv} v_x u_x \,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon Y + C_\varepsilon \left(1 + Z^{3/4}\right), \tag{4.28}$$

$$\left|\int_0^t\int_0^1 \frac{\kappa}{v}\theta_x \cdot \left(\frac{\kappa}{v}\right)_v v_x \theta_x \,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon\left(X + Y\right) + C_\varepsilon \left(1 + Z^{3/4}\right) \tag{4.29}$$

for $0 \le \beta < 3q + 10$,

$$\left|\int_0^t\int_0^1 \theta p_\theta u_x \cdot \frac{\kappa}{v}\theta_t \,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon\left(X + Y\right) + C_\varepsilon, \tag{4.30}$$

$$\left|\int_0^t\int_0^1 \theta p_\theta u_x \cdot \mathrm{K}_v u_x \,\mathrm{d}x\,\mathrm{d}\tau\right| \le C \left(1 + Z^{3/4}\right), \tag{4.31}$$

$$\left|\int_0^t\int_0^1 \frac{\mu}{v}{u_x}^2 \cdot \mathrm{K}_v u_x \,\mathrm{d}x\,\mathrm{d}\tau\right| \le C \left(1 + Z^{7/8}\right), \tag{4.32}$$

$$\left|\int_0^t\int_0^1 \lambda\phi z \cdot \frac{\kappa}{v}\theta_t \,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon\left(X + Y\right) + C_\varepsilon, \tag{4.33}$$

$$\left|\int_0^t\int_0^1 \lambda\phi z \cdot \mathrm{K}_v u_x \,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon Y + C_\varepsilon \left(1 + Z^{3/4}\right) \tag{4.34}$$

for $0 \le \beta < q + 9$.

An estimate essentially different from the one in Problem 1 is

$$\left|\int_0^t\int_0^1 \frac{\mu}{v}{u_x}^2 \cdot \frac{\kappa}{v}\theta_t \,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon X + C_\varepsilon \int_0^t\int_0^1 (1+\theta)^{q-3}{u_x}^4 \,\mathrm{d}x\,\mathrm{d}\tau, \tag{4.35}$$

whose right-hand side is estimated from above by

$$\varepsilon X + C_\varepsilon \left|{u_x}^2\right|^{(0)} \int_0^t \|u_x\|^2 \,\mathrm{d}\tau \le \varepsilon X + C_\varepsilon \left(1 + Z^{3/4}\right) \tag{4.36}$$

for $2 \le q \le 3$, by

$$\begin{aligned} &\varepsilon X + C_\varepsilon \left|1 + \theta^{q-3}\right|^{(0)} \left|{u_x}^2\right|^{(0)} \int_0^t \|u_x\|^2 \,\mathrm{d}\tau \\ &\quad\le \varepsilon X + C_\varepsilon \left(1 + Y^{\frac{q-3}{2q+6}} + Y^{\frac{q-3}{2q+6}} Z^{3/4} + Z^{3/4}\right) \\ &\quad\le \varepsilon\left(X + Y\right) + C_\varepsilon \left(1 + Z^\delta\right) \end{aligned} \tag{4.37}$$

with a number δ $(0 < \delta < 1)$ for $3 < q < 4$ and by

$$\begin{aligned}
&\varepsilon X + C_\varepsilon \left|1 + \theta^{q-3}\right|^{(0)} \left|u_x\right|^{(0)} \int_0^t \|u_x\|_{L^3(\Omega)}^3 \, \mathrm{d}\tau \\
&\quad \le \varepsilon X + C_\varepsilon \left(1 + Y^{\frac{q-3}{2q+6}} + Y^{\frac{q-3}{2q+6}} Z^{3/8} + Z^{3/8}\right) \\
&\quad \le \varepsilon (X + Y) + C_\varepsilon \left(1 + Z^{3/4}\right) \qquad (4.38)
\end{aligned}$$

for $q \ge 4$ in virtue of Lemma 4.6.

Combining (4.24)-(4.38) and taking ε suitably small, we obtain (4.23).

Lemma 4.8 *If $q \ge 3$ and $0 \le \beta < q + 9$, then for any $t \in [0, T]$*

$$\|u_x\|^2 + \|\theta_x\|^2 + \|u_{xx}\|^2 + \|u_t\|^2 + \int_0^t \left(\|\theta_t\|^2 + \|u_{xt}\|^2\right) \mathrm{d}\tau \le C_T, \quad (4.39)$$

$$|u_x|^{(0)} + |u|^{(0)} + |\theta|^{(0)} \le C_T. \quad (4.40)$$

Proof. The following calculations are formal because the regularity of the solution is not sufficient. However, one can derive the rigorous results by using the arguments of difference quatients and passing to the limit.

Differentiating $(2.23)^2$ with respect to t, multiplying it by u_t and integrating it with respect to x, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{1}{2} {u_t}^2 \, \mathrm{d}x + \int_0^1 \frac{\mu}{v} {u_{xt}}^2 \, \mathrm{d}x = \int_0^1 \left(p_t u_{xt} + \frac{\mu}{v^2} {u_x}^2 u_{xt}\right) \mathrm{d}x.$$

Since $p_t = \left(\frac{R}{v} + \frac{4}{3} a\theta^3\right) \theta_t - \frac{R}{v^2} \theta u_x$, we get for $q \ge 3$

$$\begin{aligned}
&\|u_t\|^2 + \int_0^t \|u_{xt}\|^2 \, \mathrm{d}\tau \\
&\le C \left[1 + \int_0^t \int_0^1 \left({p_t}^2 + {u_x}^4\right) \mathrm{d}x \, \mathrm{d}\tau\right] \\
&\le C \left[\int_0^t \int_0^1 \left(1 + \theta^6\right) {\theta_t}^2 \, \mathrm{d}x \, \mathrm{d}\tau + \left|{u_x}^2\right|^{(0)} \int_0^t \left(\|\theta\|^2 + \|u_x\|^2\right) \mathrm{d}\tau\right] \\
&\le C \left(1 + X + Z^{3/4}\right) \le C \left(1 + Z^\delta\right) \qquad (4.41)
\end{aligned}$$

by Lemma 4.7. By squaring $(2.23)^2$ and noting $p_x = \left(\frac{R}{v} + \frac{4}{3}a\theta^3\right)\theta_x - \frac{R}{v^2}\theta v_x$ it follows from (4.41) that for any $t \in [0, T]$

$$\begin{aligned}\|u_{xx}\|^2 &\leq C\left[1 + \|u_t\|^2 + \int_0^1 \left(1+\theta^6\right)\theta_x{}^2\,\mathrm{d}x + \left(|\theta^2|^{(0)} + \left|u_x{}^2\right|^{(0)}\right)\|v_x\|^2\right] \\ &\leq C\left(1 + Y + Z^\delta\right) \leq C\left(1 + Z^\delta\right).\end{aligned}$$

This implies

$$Z \leq C\left(1 + Z^\delta\right),$$

and hence, we conclude that Z is bounded. Then one can see from (4.22), (4.23) and (4.41) that X, Y, $|\theta|^{(0)}$, $\|u_x\|$, $|u_x|^{(0)}$, $\|u_t\|$ and $\int_0^t \|u_{xt}\|^2\,\mathrm{d}\tau$ are also bounded. The boundedness of u is easily derived from

$$|u|^{(0)} \leq C \max_{t\in[0,T]}\left(\|u\|_{L^1(\Omega)} + \|u_x\|\right).$$

In what follows we assume that q and β are real numbers satisfying $q \geq 3$ and $0 \leq \beta < q + 9$.

Lemma 4.9 *For any* $(x, t) \in \overline{Q_T}$

$$\theta(x, t) \geq C_T, \tag{4.42}$$

and for any $t \in [0, T]$

$$\|z_x\|^2 + \|z_{xx}\|^2 + \|z_t\|^2 + \int_0^t \|z_{xt}\|^2\,\mathrm{d}\tau \leq C_T, \tag{4.43}$$

$$\|\theta_{xx}\|^2 + \|\theta_t\|^2 + \int_0^t \|\theta_{xt}\|^2\,\mathrm{d}\tau \leq C_T. \tag{4.44}$$

Proof. By putting $\Theta := \frac{1}{\theta}$, (4.7) becomes

$$e_\theta\Theta_t = \left(\frac{\kappa}{v}\Theta_x\right)_x + \frac{v{p_\theta}^2}{4\mu} - \left[\frac{2\kappa{\Theta_x}^2}{v\Theta} + \frac{\mu\Theta^2}{v}\left(u_x - \frac{vp_\theta}{2\mu\Theta}\right)^2 + \lambda\phi z\Theta^2\right].$$

Since $e_\theta > c_v$, and $p_\theta \le C + C\,|\theta^3|^{(0)} \le C$ from (4.40), comparison arguments give (4.42) (see the proof of Lemma 3.13).

Multiplying $(2.23)^4$ by z_{xx} and integrating it over $[0,1]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}{z_x}^2\,\mathrm{d}x + \int_0^1 \frac{d}{v^2}{z_{xx}}^2\,\mathrm{d}x = \int_0^1 \left(\frac{2d}{v^2}v_x z_x + \phi z\right) z_{xx}\,\mathrm{d}x.$$

Furthermore, differentiating $(2.23)^4$ with respect to t, multiplying it by z_t and integrating that over $[0,1]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}{z_t}^2\,\mathrm{d}x + \int_0^1 \frac{d}{v^2}{z_{xt}}^2\,\mathrm{d}x = \int_0^1 \left(\frac{2d}{v^3}u_x z_x z_{xt} - \phi_t z z_t - \phi {z_t}^2\right)\mathrm{d}x.$$

Arguments in the proof of Lemma 3.14 give (4.43).

Differentiating (4.7) with respect to t, multiplying it by $e_\theta\theta_t$ and integrating that over $[0,1]$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}\left(e_\theta\theta_t\right)^2\mathrm{d}x + \int_0^1 \frac{\kappa}{v}e_\theta{\theta_{xt}}^2\,\mathrm{d}x \\
&= \int_0^1 \Bigg[- p_\theta e_\theta u_x {\theta_t}^2 - \theta p_{\theta v} e_\theta {u_x}^2\theta_t - \theta p_{\theta\theta} e_\theta u_x {\theta_t}^2 - \theta p_\theta e_\theta u_{xt}\theta_t \\
&\qquad + \frac{2\mu}{v}e_\theta u_x u_{xt}\theta_t - \frac{\mu}{v^2}e_\theta {u_x}^3\theta_t - \left(\frac{\kappa}{v}\right)_v e_{\theta v} v_x u_x \theta_x\theta_t - \left(\frac{\kappa}{v}\right)_v e_{\theta\theta} u_x {\theta_x}^2\theta_t \\
&\qquad - \left(\frac{\kappa}{v}\right)_v e_\theta u_x\theta_x\theta_{xt} - \left(\frac{\kappa}{v}\right)_\theta e_{\theta v} v_x\theta_x{\theta_t}^2 - \left(\frac{\kappa}{v}\right)_\theta e_{\theta\theta}{\theta_x}^2{\theta_t}^2 - \left(\frac{\kappa}{v}\right)_\theta e_\theta\theta_x\theta_t\theta_{xt} \\
&\qquad - \frac{\kappa}{v}e_{\theta v}v_x\theta_t\theta_{xt} - \frac{\kappa}{v}e_{\theta\theta}\theta_x\theta_t\theta_{xt} + \lambda\mathrm{e}^{-A/\theta}\left(\frac{A}{\theta^2}+\frac{\beta}{\theta}\right)\theta^\beta e_\theta z{\theta_t}^2 + \lambda\phi e_\theta z_t\theta_t\Bigg]\mathrm{d}x.
\end{aligned}$$

Calculating each term in a standard manner, we have

$$\|\theta_t\|^2 + \int_0^t \|\theta_{xt}\|^2\,\mathrm{d}\tau \le C\left[1 + \int_0^t \max_{x\in\overline{\Omega}}{\theta_t}^2\cdot\left(\|v_x\|^2 + \|\theta_x\|^2\right)\mathrm{d}\tau\right].$$

Hence one can obtain (4.44) similarly to the proof of Lemma 3.15.

4.2 The Hölder estimates

From (4.43) and (4.44) we see that $|\theta_x|^{(0)}$ and $|z_x|^{(0)}$ are bounded. This and (4.40) yield

$$(u,\theta,z) \in \left(C^{1,0}_{x,t}(Q_T)\right)^3. \tag{4.45}$$

Applying Cauchy-Schwarz' and interpolation inequalities, we have

$$
\begin{aligned}
\big|u(x,t)-u(x,t')\big| &\le \left(\int_{t'}^{t} {u_t}^2 \,\mathrm{d}\tau\right)^{1/2} |t-t'|^{1/2} \\
&\le \left[\int_{t'}^{t} \Big(\|u_t\|^2 + 2\|u_t\|\|u_{xt}\|\Big)\,\mathrm{d}\tau\right]^{1/2} |t-t'|^{1/2}, \\
\big|u_x(x,t)-u_x(x',t)\big| &\le \left(\int_{x'}^{x} {u_{xx}}^2 \,\mathrm{d}\xi\right)^{1/2} |x-x'|^{1/2},
\end{aligned}
$$

from which, together with (4.39), $u \in C^{0,\,1/2}_{x,\,t}(Q_T)$ and $u_x \in C^{1/2,\,0}_{x,\,t}(Q_T)$ follow. Thus by a standard interpolation lemma (see for example, [34], Chapter II, Lemma 3.1) one can get $u_x \in C^{1/3,\,1/6}_{x,\,t}(Q_T)$. Similarly, using Lemmas 4.8, 4.9 and (4.45), we have

$$
(u,\theta,z) \in \Big(C^{1,\,1/2}_{x,\,t}(Q_T)\Big)^3, \quad (u_x,\theta_x,z_x) \in \Big(C^{1/3,\,1/6}_{x,\,t}(Q_T)\Big)^3. \tag{4.46}
$$

Recalling that $v_t = u_x$ and $v|_{t=0} = v_0 \in C^{1+\alpha}(\Omega)$, we deduce $v \in C^{1/3,\,1/6}_{x,\,t}(Q_T)$. Since it follows from (4.12) that

$$
\begin{aligned}
v_x(x,t) = \frac{1}{(\mathrm{PQR})(x,t)}\Bigg\{ & {v_0}'(x) - A(x,t)v_0(x) \\
&+ \frac{R}{\zeta}\int_0^t \Big[\theta_x(x,\tau) + \theta(x,\tau)\big(A(x,\tau)-A(x,t)\big)\Big](\mathrm{PQR})(x,\tau)\,\mathrm{d}\tau\Bigg\}
\end{aligned} \tag{4.47}
$$

with

$$
A(x,t) := \frac{1}{\mu}\left[u_0(x) - u(x,t) - G\left(x-\frac{1}{2}\right)t - \frac{4}{3}a\int_0^t \theta(x,\tau)^3\theta_x(x,\tau)\,\mathrm{d}\tau\right],
$$

one can easily check $v_x \in C^{\sigma,\,\sigma/2}_{x,\,t}(Q_T)$ with $\sigma := \min\{\alpha, 1/3\}$.

Next we consider $(2.23)^2$, $(2.23)^3$ and $(2.23)^4$ as the linear equations

$$
\begin{cases}
u_t - \dfrac{\mu}{v}u_{xx} + \left(\dfrac{\mu}{v^2}v_x\right)u_x = -\dfrac{R}{v}\theta_x + \dfrac{R}{v^2}\theta v_x - \dfrac{4}{3}a\theta^3\theta_x - G\left(x-\dfrac{1}{2}\right), \\
\theta_t - \dfrac{1}{e_\theta}\dfrac{\kappa}{v}\theta_{xx} - \dfrac{1}{e_\theta}\Big[\left(\dfrac{\kappa}{v}\right)_\theta\theta_x + \left(\dfrac{\kappa}{v}\right)_v v_x\Big]\theta_x + \left(\dfrac{p_\theta}{e_\theta}u_x\right)\theta \\
\qquad\qquad = \dfrac{1}{e_\theta}\left(\dfrac{\mu}{v}{u_x}^2 + \lambda\phi z\right), \\
z_t - \dfrac{d}{v^2}z_{xx} + \left(\dfrac{2d}{v^3}v_x\right)z_x + \phi z = 0,
\end{cases} \tag{4.48}
$$

whose coefficients and right-hand sides are Hölder continuous in x with exponent σ and in t with exponent $\sigma/2$. By the classical Schauder estimates (see for example, [14, 34]) we obtain

$$|u, \theta, z|_{2+\sigma,\, 1+\sigma/2} \;\leq\; C.$$

This also implies

$$(v, u_x, \theta_x, z_x) \in \left(C^{1,\, 1/2}_{x,\, t}(Q_T)\right)^4. \tag{4.49}$$

Going back to (4.47) with (4.46) and (4.49), we obtain $v_x \in C^{\alpha,\, \alpha/2}_{x,\, t}(Q_T)$. Hence applying the Schauder estimates to (4.48) again, we finally conclude

$$|u, \theta, z|_{2+\alpha,\, 1+\alpha/2} \;\leq\; C.$$

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