Hamilton Cycles, Paths and Spanning Trees in a Graph

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Kenta Ozeki

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Preface

This thesis is written on the subject "Hamilton cycles, paths and spanning trees in a graph." The basis of this thesis is formed by papers written during these four years.

After an introductory chapter, the reader will find ten chapters. General terminology of Graph Theory is found in Chapter 2. The other chapters can be read independently from one another. The main part of this thesis is divided into two parts; the first one is some relaxed structures of a hamilton cycle, another is spanning trees as a relaxation of a hamilton path.

A cycle in a graph is called a *hamilton cycle* if it passes all vertices of the graph. A problem of determining whether a given graph has a hamilton cycle or not is important in Graph Theory, but it is known as a "difficult" one in a Combinatorial sense. Therefore we do not deal with this problem directly, and consider from the following two aspects; to find better sufficient conditions for the existence of a hamilton cycle, and to study relaxed structures of a hamilton cycle. In this thesis, we focus on the second aspect, in particular, we consider degree sum conditions and independence number conditions for the existence of such structures.

A hamilton cycle must pass all vertices of a graph. Relaxing this property of a hamilton cycle, we consider a cycle passing all specified vertices. We shall give a sufficient condition for the existence of such a cycle in terms of degrees and independent sets of specified vertices. Moreover, a cycle passing not only specified vertices but also specified edges has been studied. We discuss about these cycles in Chapters 3 and 5, respectively.

As another notion of relaxing a hamilton cycle, we consider *a dominating cycle*. A cycle is called dominating if removing all vertices of it results in a graph with no edges. Definitely, a hamilton cycle is dominating, but the converse does not generally hold. We sometimes consider a dominating cycle as "close" to a hamilton cycle, because the outside of the cycle must be small. Moreover, it is known that a dominating cycle has some good properties as a hamilton cycle. Therefore many researchers have studied a dominating cycle. We focus on a dominating cycle in Chapter 4. In Chapter 6, we introduce an invariant "Relative Length," which concerns with a property of a dominating cycle. We mention the relationship between a dominating cycle or “Relative Length” and the length of a longest cycle of a graph in Chapter 7.

A *hamilton path* of a graph is a path passing all vertices. Similarly to a hamilton cycle, it is known that a problem of determining whether a given graph has a hamilton path or not is a “difficult” one. Therefore we are also interested in some relaxed concept of a hamilton path, like a hamilton cycle. The rest five chapters deal with spanning trees with particular properties which are relaxed concepts of a hamilton path.

The most important one is a spanning tree whose maximum degree is at most k, called a *spanning k-tree*. It appears in Chapters 8–10. In particular, we will focus on a k-tree containing specified vertices in Chapter 8, a spanning f-tree, in Chapter 9, and the concept “prism hamiltonian” in Chapter 10, respectively.

In the remaining two chapters, we will study other directions of relaxed structures of a hamilton path; a spanning tree with bounded number of vertices of degrees one, or with bounded number of vertices of degrees at least three. We focus on these spanning trees in Chapters 11 and 12, respectively.

Contents

Chapter 1

Introduction

In this thesis, we deal with cycles, paths and trees with some properties related to a hamilton cycle or a hamilton path. A *hamilton cycle* is a cycle passing through all vertices of a graph. We call a graph having a hamilton cycle *hamiltonian*. Similarly, a *hamilton path* is a path passing through all vertices of a graph. An interest of a hamilton cycle originates in the relationship to *Four Color Problem*, but recently, it concerns with many other topics or areas. So study on a hamilton cycle is one of the most important and basic topics in Graph Theory.

One of the huge targets of this study is to find a necessary and sufficient condition for the existence of a hamilton cycle, except for a trivial one. However, in contrast with an *Eulerian circuit*, it seems to be difficult and no one have succeeded. In fact, the problem of determining whether a given graph has a hamilton cycle or not belongs to the class of NP-complete problems, that is, a difficult problem in a Combinatorial sense. So we have focused on sufficient conditions or some relaxed structures of a hamilton cycle. There are many sufficient conditions for the existence of a hamilton cycle. The following is a classical result on it.

Theorem 1.1 (Dirac [41]) *Let G be a graph of order $n \geq 3$. If the minimum degree is at least $\frac{1}{2}n$, then G has a hamilton cycle.*

To satisfy the assumption of Theorem 1.1, "all vertices" of a graph must have high degrees. However, this condition is too strong in a sense. Ore considered that even if some vertices have low degrees, if all of them are adjacent each other, then the graph might have a hamilton cycle. Actually, he showed the following theorem. When the required number of nonadjacent vertices cannot be taken, we define the value of "the minimum degree sum" as $+\infty$.

Theorem 1.2 (Ore [130]) *Let G be a graph of order $n \geq 3$. If the minimum degree sum of two nonadjacent vertices is at least n, then G has a hamilton cycle.*

As a condition other than the degrees, Chvátal and Erdős considered the one concerning the independence number and the connectivity of a graph. We denote the independence number and the connectivity of a graph G by $\alpha(G)$ and $\kappa(G)$, respectively. Note that it is known that Theorem 1.3 implies Theorem 1.2.

Theorem 1.3 (Chvátal and Erdős [37]) *Let G be a 2-connected graph. If $\alpha(G)$ is at most $\kappa(G)$, then G has a hamilton cycle.*

Starting from Theorems 1.1–1.3, many researchers have considered the conditions concerning with the degrees or the independence number and the connectivity.

A hamilton cycle must pass through "all vertices" of a graph. In this sense, we can consider a relaxation of the concept of a hamilton cycle; a cycle passing through all "specified vertices." A set of specified vertices is called *cyclable* in a graph G if G has a cycle passing through all of them. In order to guarantee the existence of a hamilton cycle, we consider a degree condition or an independence number condition of all vertices as in Theorems 1.1–1.3. However when we would like to show the cyclability of specified vertices, it is not necessary to deal with the degrees or the independence number of all vertices. In fact, it often suffices to consider only that of the specified vertices as follows. For a vertex set S, we denote the independence number and the connectivity of S by $\alpha(S)$ and $\kappa(S)$, respectively. (See Chapter 3).

Theorem 1.4 (Shi [149]) *Let G be a 2-connected graph of order n, and $S \subseteq V(G)$. If the minimum degree sum of pairwise nonadjacent vertices in S is at least n, then S is cyclable in G.*

Theorem 1.5 ([135]) *Let G be a 2-connected graph, and $S \subseteq V(G)$. If $\alpha(S)$ is at most $\kappa(S)$, then S is cyclable in G.*

Motivated by the improvement of Theorem 1.1 to Theorem 1.2, Bauer, Broersma, Li and Veldman [13] considered a condition of degree sum of three vertices for the existence of a hamilton cycle. Recently, this result was extended to the result on cyclability by Broersma, H. Li, J. Li, Tian and Veldman.

Theorem 1.6 (Broersma, H. Li, J. Li, Tian and Veldman [30]) *Let G be a graph of order n, and $S \subseteq V(G)$ with $\kappa(S) \geq 2$. If the minimum degree sum of three pairwise nonadjacent vertices of S is at least $n + \kappa(S)$, then S is cyclable in G.*

Again, motivated by the improvement of the condition of degree sum of two vertices in Theorem 1.4 to three vertices in Theorem 1.6, we show the following result on the degree sum of four vertices. In Chapter 3, we focus on a hamilton cycle and cyclability of specified vertices, and gave the proof of Theorem 1.7.

Theorem 1.7 ([135]) *Let G be a graph of order n, and $S \subseteq V(G)$ with $\kappa(S) \geq 3$. If the minimum degree sum of four pairwise nonadjacent vertices in S is at least $n + \kappa(S) + \alpha(S) - 1$, then S is cyclable in G.*

Recently, Ota gave a result by a condition on the degree sum of more vertices. But Theorems 1.6 and 1.7 cannot be implied by Theorem 1.8. In fact, there exist some examples showing the above fact and we will show them in Chapter 3.

Theorem 1.8 (Ota [132]) *Let G be a graph of order n, and $S \subseteq V(G)$ with $\kappa(S) \geq 2$. If, for any $t \geq \kappa(S)$, the minimum degree sum of $t+1$ pairwise nonadjacent vertices in S is at least $n + t^2 - t$, then S is cyclable in G.*

Along the above stream, many sufficient conditions on a hamilton cycle or cyclability of specified vertices have been found. On the other hand, as other approach to a hamilton property, many relaxations of hamilton cycles are also considered. One of the most important relaxations is the concept of a *dominating cycle*. A cycle C in a graph G is called *dominating* if at least one end-vertex of any edge of G is contained in C. By the definitions, a hamilton cycle is also a dominating cycle but generally the converse does not hold. A dominating cycle has considered as a "pre-hamilton" cycle. This is because, in order to find a hamilton cycle in a given graph, sometimes we first try to find a dominating cycle before it. For example, if some longest cycle in G is dominating and the independence number of G is at most the minimum degree, then G has a hamilton cycle.

Recently, the concept of dominating cycles becomes more important because of not only the meaning of a "pre-hamilton cycle" but also the relationship with other properties, for example, circumference of a graph (the length of a longest cycle). We will mention this relationship in Chapter 7. Therefore we are interested in the concept of dominating cycles itself. In 1980, Bondy showed the following result, which is a generalization of Nash-Williams' result [127]. This is a basic result on dominating cycles using a degree sum condition.

Theorem 1.9 (Bondy [26]) *Let G be a 2-connected graph of order $n \geq 3$. If the minimum degree sum of three pairwise nonadjacent vertices is at least $n+2$, then each longest cycle in G is dominating.*

As well as degree conditions, one might expect an independence number condition on dominating cycles. However, considering the graph $G_1 = K_k + (k+1)K_m$ with $m \geq 2$, in a sense, it fails for general graphs. Since $\alpha(G_1) = k+1 = \kappa(G_1)+1$ and G_1 has no dominating cycle, even if we would like to find a dominating cycle by an independence number condition, the same condition that "$\alpha(G)$ is at most $\kappa(G)$" as Theorem 1.3 is needed. Motivated by the above reason, when we consider an independence number condition for a dominating cycle, it is necessary to restrict ourselves to some particular classes of graphs, at least we must avoid some graphs like G_1. Enomoto, Kaneko, Saito and Wei considered a class of triangle-free graphs and gave an independence number condition for a dominating cycle.

Theorem 1.10 (Enomoto, Kaneko, Saito and Wei [48]) *Let G be a 2-connected triangle-free graph. If $\alpha(G)$ is at most $2\kappa(G) - 2$, then every longest cycle in G is dominating.*

It is unknown whether the condition of Theorem 1.10 is sharp or not. But there exists a triangle-free graph G with $\alpha(G)$ is equal to $2\kappa(G)$ such that any longest cycle of G is not dominating. In Chapter 4, we discuss some results on a dominating cycle and show the following theorem. By the above example, the condition of Theorem 1.11 is best possible.

Theorem 1.11 ([136]) *Let G be a 2-connected triangle-free graph. If $\alpha(G)$ is at most $2\kappa(G) - 1$, then there exists a longest cycle in G which is dominating.*

Similarly to a cycle passing through "specified vertices," sometimes we consider cycles passing through not only "specified vertices," but also "specified edges." Of course there exist some edges that cannot be passed by one cycle. When specified edges induce a graph having a vertex of degree at least three, it is trivially impossible to find a cycle passing through all such edges. So, as those specified vertices and edges, we consider a *linear forest*, that is a forest such that each component of it is a path (possibly it may consist of only one vertex).

In Chapter 5, we consider a cycle passing through specified edges. In particular, we concentrate on a dominating cycle and a hamilton cycle passing through a given linear forest. The following is the one for a dominating cycle. When we take a linear forest F with $E(F) = \emptyset$ and $|V(F)| \leq 3$, the condition of Theorem 1.12 is identical to that of Theorem 1.9. Thus, Theorems 1.12 is a generalization of Theorem 1.9. Note that Theorem 1.12 does not guarantee the existence of a cycle passing through a given linear forest, however, Häggkvist and Thomassen [81] showed that such a cycle exists if the graph is $(m + r)$-connected.

Theorem 1.12 ([137]) *Let G be an $(m + 2)$-connected graph of order n. Let F be a linear forest with $|E(F)| = m$ and let r be the number of the isolated vertices of F. If the minimum degree sum of three pairwise nonadjacent vertices is at least $n+2m+\max\{r-1, 2\}$, then every longest cycle C passing through F is dominating.*

On the other hand, some sufficient conditions for the existence of a hamilton cycle have been also considered. Pósa showed the following theorem as a generalization of Theorem 1.2; for a graph G of order $n \geq 3$ and for a linear forest F with $|E(F)| = m$, if the minimum degree sum of nonadjacent vertices is at least $n + m$, then G has a hamilton cycle passing through F. In Chapter 5, we show some other sufficient conditions for the existence of a hamilton cycle through a given linear forest. One of them is the following. When $m = 0$ in Theorem 1.13, it is identical to Theorem 1.6 for the case where $S = V(G)$. Thus, Theorem 1.13 is a generalization of Theorem 1.6, in a sense. We will survey on a cycle passing through given edges and show Theorems 1.12 and 1.13 in Chapter 5.

Theorem 1.13 ([137]) *Let G be an $(m+2)$-connected graph of order n, and F be a linear forest with $|E(F)| = m$. Suppose that $n \geq 2\kappa(G)+2m+1$. If the minimum degree sum of three pairwise nonadjacent vertices is at least $n + \kappa(G) + m$, then G contains a hamilton cycle passing through F.*

Recently, a new invariant, called *relative length*, is also considered. Now we denote the order (the number of vertices) of a longest path and a longest cycle of a graph G by $p(G)$ and $c(G)$, respectively. The relative length of a graph G, denoted by $\mathrm{diff}(G)$, is defined as the difference between these two invariants, that is, $\mathrm{diff}(G) := p(G) - c(G)$. The relative length looks strange and complex, however, it is useful and has some good applications. It is easy to see that for a connected graph G, $\mathrm{diff}(G) = 0$ if and only if G is hamiltonian and if $\mathrm{diff}(G) \leq 1$, any longest cycle of G is dominating.

In addition to above, more cycle-related properties are implied by the low relative length. For example, it is shown in [111] that for a graph G with $\mathrm{diff}(G) \leq 1$, the circumference of G is at least $\min\{n+\delta-\alpha(G), n\}$, and in [134], for a graph G with $\mathrm{diff}(G) \leq 2$, the circumference of G is at least $\min\{n + 2\delta - 2\alpha(G) - 2, n\}$, where δ be the minimum degree of G. As in Chapter 6, more properties are implied by the low relative length.

As mentioned before, a dominating cycle is regarded as a "pre-hamilton" cycle. Since any longest cycle of G is dominating if $\mathrm{diff}(G) \leq 1$, the property $\mathrm{diff}(G) \leq 1$ can be also regarded as the property "pre-hamiltonian." Moreover, not only $\mathrm{diff}(G) \leq 1$, but also the low relative length, $\mathrm{diff}(G) \leq 2$ or something else, seems to have such property. In fact, by the above result on the circumference, for a graph G with $\mathrm{diff}(G) \leq 2$, if $\alpha(G) + 1$ is at most the minimum degree of G, then G has a hamilton cycle.

Relative length was first posed by Enomoto, van den Heuvel, Kaneko and Saito in 1995. In the same paper, they gave a degree sum condition for a graph G to satisfy $\mathrm{diff}(G) \leq 1$. Note that this result is stronger than Theorem 1.9.

Theorem 1.14 (Enomoto, van den Heuvel, Kaneko and Saito [46]) *Let G be a 2-connected graph of order n. If the minimum degree sum of three pairwise nonadjacent vertices is at least $n + 2$, then* $\mathrm{diff}(G) \leq 1$.

In [111], Li, Saito and Schelp considered the relationship between the property "$\mathrm{diff}(G) \leq 1$" and a minimum degree sum of four vertices condition. They proved that for a 3-connected graph G of order n, if the minimum degree sum of four pairwise nonadjacent vertices is at least $\frac{3}{2}n+1$, then $\operatorname{diff}(G) \le 1$ and also conjectured that the sharp coefficient of n is $\frac{4}{3}$. Lu, Liu and Tian gave a sharp bound on the condition.

Theorem 1.15 (Lu, Liu and Tian [118]) *Let G be a 3-connected graph of order n. If the minimum degree sum of four pairwise nonadjacent vertices is at least $\frac{1}{3}(4n+5)$, then $\operatorname{diff}(G) \le 1$.*

In Chapter 6, we prove the following result on $\operatorname{diff}(G) \le 2$. Moreover, we show some applications of relative length and give some other results on the low relative length.

Theorem 1.16 ([134]) *Let G be a 3-connected graph of order n. If the minimum degree sum of four pairwise nonadjacent vertices is at least $n+6$, then $\operatorname{diff}(G) \le 2$.*

We sometimes regard a graph with a hamilton path as having a good property. So we often try to find sufficient conditions for a graph without a hamilton path to have the low relative length. In [46], Enomoto, van den Heuvel, Kaneko and Saito gave a degree sum of three vertices condition of it.

Theorem 1.17 (Enomoto, van den Heuvel, Kaneko and Saito [46]) *Let G be a connected graph of order n. If the minimum degree sum of three pairwise nonadjacent vertices is at least n, then either $\operatorname{diff}(G) \le 1$ or G has a hamilton path.*

Theorems 1.14 and 1.17 suggest that the connectivity and degree sum condition can be weakened for graphs without a hamilton path. Therefore, one might expect that the conditions of other results on the low relative length can be also weakened for graphs without a hamilton path. By the expectation of Theorem 1.15, we prove the following result.

Theorem 1.18 ([99]) *Let G be a 2-connected graph of order n. If the minimum degree sum of four pairwise nonadjacent vertices is at least $\frac{1}{3}(4n-2)$, then either $\operatorname{diff}(G) \le 1$ or G has a hamilton path.*

On the other hand, in 2002, Schiermeyer and Tewes [148] investigated the relation between a minimum degree sum of three vertices and $\operatorname{diff}(G) \le 2$ in a 2-connected graph G. A path P of a graph G is said to be *dominating* if the removal of all vertices of P from G results in a graph with no edge. They showed that for a 2-connected graph G of order n, if the minimum degree sum of four pairwise nonadjacent vertices is at least $n+3$, then either $\operatorname{diff}(G) \le 2$ or every longest path in G is dominating. However, considering the relations between Theorems 1.14 and 1.17 and between Theorems 1.15 and 1.18, the conclusion of the above result seems to be weak. The following theorem is one improvement of Schiermeyer and Tewes result.

Theorem 1.19 ([99]) *Let G be a 2-connected graph of order n. If the minimum degree sum of four pairwise nonadjacent vertices is at least $n+3$, then either* $\mathrm{diff}(G) \leq 2$ *or G has a hamilton path.*

We also give proofs of Theorems 1.18 and 1.19 in Chapter 6.

If a graph G has a long cycle (for example, comparing the order of G), we can regard G as a graph being "close" to hamiltonian. So we are interested in the circumference $c(G)$, that is the order of a longest cycle. Many researchers have been established the lower bound of the circumference by various invariants. We will survey those of the circumference in Chapter 7. The following results concerns with the circumference and the minimum degree sum of three vertices or $p(G)$.

Theorem 1.20 (Fournier and Fraisse [64]) *Let G be a 2-connected graph of order n. Then $c(G) \geq \min\{2d/3, n\}$, where d is the minimum degree sum of three pairwise nonadjacent vertices.*

Theorem 1.21 (Bondy and Locke [28]) *Let G be a 3-connected graph. Then $c(G) \geq 2(p(G)-1)/5$.*

In Chapter 7, we show the following result, which shows that the bounds of Theorems 1.20 and 1.21 can be improved by considering the minimum degree sum of three vertices and $p(G)$ at the same time.

Theorem 1.22 ([138]) *Let G be a 3-connected graph. Then $c(G) \geq \min\{d-3, p(G)-1\}$, where d is the minimum degree sum of three pairwise nonadjacent vertices.*

In other words, for any 3-connected graph G, $c(G) \geq d-3$ or $\mathrm{diff}(G) \leq 1$. This is an improvement of a result by Fraisse and Jung [66], which says that for any 3-connected graph G, $c(G) \geq d-3$ or any longest cycle in G is dominating. In Chapter 7, we will introduce some results on the circumference and prove Theorem 1.22

As mentioned above, we have considered the relaxation of the concept of a hamilton cycle, for example, a cycle containing specified vertices, dominating cycles, relative length and the circumference. In the rest of introduction of this thesis, we deal with relaxations of the concept of a hamilton path. The most important relaxation of it is a *spanning k-tree*. *A k-tree* is a tree whose maximum degree is at most k. Definitely a spanning 2-tree is equivalent to a hamilton path.

As immediate corollaries of Theorems 1.2 and 1.3, for a connected graph G of order n, we obtain the following result; If the minimum degree sum of nonadjacent vertices is at least $n-1$ or if $\alpha(G)$ is at most $\kappa(G)+1$, then G has a hamilton path. Win, and Neumann-Lara and Rivera-Campo extended those corollaries, and gave a degree sum condition and an independence number condition to have a spanning k-tree, respectively.

Theorem 1.23 (Win [168]) *Let k be an integer with $k \geq 2$, and let G be a connected graph of order n. If the minimum degree sum of k pairwise nonadjacent vertices is at least $n-1$, then G has a spanning k-tree.*

Theorem 1.24 (Neumann-Lara and Rivera-Campo [128]) *Let k be an integer with $k \geq 2$, and let G be an m-connected graph. If $\alpha(G)$ is at most $(k-1)m+1$, then G has a spanning k-tree.*

Similarly to dealing with a cycle containing specified vertices as a relaxation of the concept of a hamilton cycle, it is natural to consider a k-tree containing specified vertices. Motivated by such a consideration, Matsuda and Matsumura proved the following theorem.

Theorem 1.25 (Matsuda and Matsumura [121]) *Let G be a connected graph of order n and let $S \subseteq V(G)$. If the minimum degree sum of k pairwise nonadjacent vertices in S is at least $n-1$, then G has a k-tree containing all vertex in S.*

For a connected graph G and $S \subseteq V(G)$, if $\alpha(S)$ is at most $k-1$, then G and S trivially satisfy the assumption of Theorem 1.25, and hence it has a k-tree containing S. So Theorem 1.25 also gave an independence number condition. However, comparing Theorem 1.24, the condition "$\alpha(S)$ is at most $k-1$" seems too strong for a graph G to have a k-tree containing S. In addition, although the degree sum bound of Theorem 1.25 is best possible, we may be able to decrease it if a graph has high connectivity. Motivated by these consideration, we show the following result, which is a k-tree analogy of Theorem 1.8. In Chapter 8, we will give a proof of Theorem 1.26

Theorem 1.26 ([36]) *Let k be an integer with $k \geq 3$ and let G be a graph of order n. Let $S \subseteq V(G)$ with $\kappa(S) \geq 1$. If for every $l \geq \kappa(S)$, the minimum degree sum of t pairwise nonadjacent vertices in S is at least $n+tl-kl-1$, where $t=(k-1)l+2-\lfloor \frac{l-1}{k} \rfloor$, then G has a k-tree containing S.*

We may consider a more general concept than a spanning k-tree. Let G be a graph and let f be a mapping from $V(G)$ to positive integers. A tree T of G is called an *f-tree* if for any $x \in V(T)$, the degree of x in T is at most $f(x)$. Definitely, when f is a constant mapping taking value k, an f-tree is equivalent to a k-tree.

Matsuda and Matsumura gave a result on the existence of a spanning k-tree with specified leaves, which is an extension of Theorem 1.24.

Theorem 1.27 (Matsuda and Matsumura [120]) *Let m, k and s be integers with $k \geq 2$, $0 \leq s \leq k$ and $s+1 \leq m$ and let G be an m-connected graph. If $\alpha(G)$ is at most $(m-s)(k-1)+1$, then for any s vertices of G, G has a spanning k-tree T such that the s specified vertices are contained in the set of leaves.*

Extending this result to a spanning f-tree, the following conjecture is proposed.

Conjecture 1.28 ([49]) *Let m be an integer, G be an m-connected graph and f be a mapping from $V(G)$ to positive integers. If $\sum_{x \in V(G)} f(x) \geq 2(|V(G)|-1)$ and $\alpha(G)$ is at most $\min\left\{\sum_{x \in R}(f(x)-1) : R \subseteq V(G), |R| = m\right\}+1$, then there exists a spanning f-tree.*

Suppose that there exists a spanning f-tree T. Then

$$\begin{aligned} \sum_{x \in V(G)} f(x) &\geq \sum_{x \in V(G)} d_T(x) \\ &= 2|E(T)| \\ &= 2(|V(G)|-1). \end{aligned}$$

Therefore for the existence of a spanning f-tree, the condition "$\sum_{x \in V(G)} f(x) \geq 2(|V(G)|-1)$" is a trivial necessary condition. Note that the independence number condition of Conjecture 1.28 is best possible if it is true.

In Chapter 9, we show the following result, which gives a partial solution to Conjecture 1.28. For a mapping f, let $S_i(f) := \{x \in V(G) : f(x) = i\}$ and $s_i(f) := |S_i(f)|$.

Theorem 1.29 ([49]) *Let m be a positive integer, G be an m-connected graph and f be a mapping from $V(G)$ to positive integers. Suppose $s_1(f)+s_2(f) \leq m+1$, $\sum_{x \in V(G)} f(x) \geq 2(|V(G)|-1)$ and $\alpha(G)$ is at most $\min\left\{\sum_{x \in R}(f(x)-1) : R \subseteq V(G), |R| = m\right\}+1$. Then there exists a spanning f-tree in G.*

Let f_1 be a mapping on $V(G)$ which assigns 1 to s given vertices and k to other vertices. Then a spanning f_1-tree is a spanning k-tree satisfying the conclusion of Theorem 1.27. Moreover,

$$\begin{aligned} &\min\left\{\sum_{x \in R}(f_1(x)-1) : R \subseteq V(G), |R| = m\right\}+1 \\ &\quad = s(1-1)+(m-s)(k-1)+1 \\ &\quad = (m-s)(k-1)+1, \end{aligned}$$

and hence Theorem 1.27 is a special case of Conjecture 1.28. If $k \geq 3$, then $s_1(f_1)+s_2(f_1) = s \leq m+1$. This implies that Theorem 1.29 is a generalization of Theorem

1.27 for $k \geq 3$. Note that essential part of the proof of Theorem 1.27 is only the case $k \geq 3$, because the case $k = 2$ is contained in results on a hamilton path like Theorem 1.3.

Again we consider a spanning k-tree as a relaxation of a hamilton path, that is, a spanning tree with maximum degree at most k. Similarly to this consideration for a hamilton cycle, the concept of a spanning *k-walk* has been considered. A *k-walk* is a closed walk that passes through each vertex at most k times. It is clear that a spanning 1-walk is equivalent to a hamilton cycle, so in this sense, a spanning k-walk is a relaxed concept of it. It is known that the existence of a spanning k-tree implies that of a spanning k-walk, and that the existence of a spanning k-walk implies that of a spanning $(k+1)$-tree. Thus, the properties "having a spanning k-tree" and "having a spanning k-walk" provide a hierarchy for measuring how far a graph is from being hamiltonian.

The *prism over G* is defined as the Cartesian product of the graphs G and K_2. Formally, it consists of two copies of G and a matching joining the corresponding vertices. A graph G is called *prism hamiltonian* if the prism over G has a hamilton cycle. The property of "being prism hamiltonian" is between the properties "having a spanning 2-tree" and "having a spanning 2-walk," that is, if G has a spanning 2-tree then G is prism hamiltonian, and if G is prism hamiltonian then G has a spanning 2-walk. Therefore, the property of "being prism hamiltonian" can be added to the above "k-tree and k-walk" hierarchy.

As mentioned in Theorem 1.23, Win gave a sharp degree sum of k vertices condition for a connected graph G to have a spanning k-tree. For a spanning k-walk, Jackson and Wormald prove the following result.

Theorem 1.30 (Jackson and Wormald [92]) *Let G be a connected graph of order n, and let k be an integer with $k \geq 1$. If the minimum degree sum of $k+1$ vertices which are pairwise nonadjacent is at least n, then G has a spanning k-walk.*

By Theorems 1.23 and 1.30 for a connected graph G of order n, if the minimum degree sum of nonadjacent vertices is at least $n-1$, then G has a spanning 2-tree (a hamilton path), and if the minimum degree sum of three pairwise nonadjacent vertices is at least n, then G has a spanning 2-walk. Since the property of "being prism hamiltonian" is between "having a spanning 2-tree" and "having a spanning 2-walk," it is natural to pose the following problem; Determine a sharp degree sum condition for connected graphs to be prism hamiltonian. As an answer to this problem, in Chapter 10, we show the following result.

Theorem 1.31 ([133]) *Let G be a connected graph of order n. If the minimum degree sum of three pairwise nonadjacent vertices is at least n, then G is prism* *hamiltonian.*

Therefore for a connected graph G of order n, if the minimum degree sum of three pairwise nonadjacent vertices is at least n, G has not only the property "having a spanning 2-walk" but also "being prism hamiltonian." Moreover, there exists a graph showing that the degree sum condition of Theorem 1.31 is best possible. In this sense, the property of "being prism hamiltonian" seems closer to the property "having a 2-walk" than "having a spanning 2-tree."

We have considered a spanning tree with bounded degrees as a relaxation of the concept of a hamilton path. But there are some other relaxations of it. In the rest of introduction of this thesis, we concentrate on them, in particular, the following two concepts of spanning trees.

We can regard a hamilton path as a spanning tree such that exactly two vertices have the degree one and others have the degree two. In this sense, a spanning tree with bounded number of vertices of degree one or with bounded number of vertices of degree at least three is a relaxation of the concept of a hamilton path. A vertex in a spanning tree of degree one (at least three) is called a *leaf* (a *branch vertex*, respectively.) Notice that a hamilton path is a spanning tree with exactly two leaves and no branch vertices. In Chapters 11 and 12, we consider a spanning tree with bounded number of leaves and branch vertices, respectively.

This study is also based on the immediate corollaries of Theorems 1.2 and 1.3; for a graph G of order n, if the minimum degree sum of nonadjacent vertices is at least $n-1$ or if $\alpha(G)$ is at most $\kappa(G)+1$, then G has a hamilton path. In other words, such graph has a spanning tree with exactly two leaves and no branch vertices. Broersma and Tuinstra extended the result for a spanning tree with bounded number of leaves.

Theorem 1.32 (Broersma and Tuinstra [31]) *Let $k \geq 2$ be an integer and let G be a connected graph of order $n \geq 2$. If the minimum degree sum of nonadjacent vertices is at least $n - k + 1$, then G has a spanning tree with at most k leaves.*

The case $k = 2$ guarantees the existence of a hamilton path, which is the equivalent to a corollary of Theorem 1.2, so Theorem 1.32 contains it. Note that the graph $K_m + (m + k)K_1$ shows the best possibility of Theorem 1.32. Although the degree condition of Theorem 1.32 is sharp, we may decrease it by restricting ourselves to some special classes of graphs. Of course, such classes have to avoid graphs like $K_m + (m + k)K_1$.

One of the important classes having the above property is a class of *claw-free* graphs. A *claw* is a graph isomorphic to $K_{1,3}$, that is a complete bipartite graph with partite sets of order one and three, respectively. A graph is called *claw-free* if it has no induced claw. Since a claw-free graph has several interesting properties and relationship to some problems of Graph Theory, many researchers have considered about a class of claw-free graphs. In fact, Matthews and Sumner proved that the degree condition of Theorem 1.1 can be decreased if we restrict ourselves to claw-free graphs.

Theorem 1.33 (Matthews and Sumner [123]) *Let G be a 2-connected claw-free graph of order n. If the minimum degree if at least $(n-2)/3$, then G has a hamilton cycle.*

Therefore, in view of Theorem 1.33, for claw-free graphs, a much weaker condition may yield the same conclusion as in results for other structures. Motivated by this observation, we study a degree sum condition for a claw-free graph to have a spanning tree with a bounded number of leaves, and give the following theorem.

Theorem 1.34 ([96]) *Let $k \geq 2$ be an integer and let G be a connected claw-free graph of order n. If the minimum degree sum of $k+1$ nonadjacent vertices is at least $n-k$, then G has a spanning tree with at most k leaves.*

In Chapter 11, we concentrate on a spanning tree with bounded number of leaves.

On the other hand, in Chapter 12, we consider a spanning tree with bounded number of branch vertices. Gargano, Hammar, Hell, Stacho and Vaccaroa gave a degree sum condition for claw-free graphs to have a spanning tree with bounded number of branch vertices.

Theorem 1.35 (Gargano, Hammar, Hell, Stacho and Vaccaroa [72]) *Let $s \geq 0$ be an integer and let G be a connected claw-free graph of order n. If the minimum degree sum of $s+3$ nonadjacent vertices is at least $n-s-2$, then G has a spanning tree with at most s branch vertices.*

Note that it is unknown whether the condition of Theorem 1.35 is sharp or not. Theorem 1.35 also implies an independence number condition; for a connected claw-free graph G, if $\alpha(G)$ is at most $s+2$, then G has a spanning tree with at most s branch vertices. However, this condition is not best possible. In fact, we obtain the following result.

Theorem 1.36 ([122]) *Let $s \geq 0$ be an integer and let G be a connected claw-free graph. If $\alpha(G)$ is at most $2s+2$, then G has a spanning tree with at most s branch vertices.*

By Theorem 1.36, we can find $2s+3$ pairwise nonadjacent vertices in G if we assume that G has no spanning tree with at most s branch vertices. In this sense, we conjecture a weaker condition than Theorem 1.35 can also guarantee the existence of a spanning tree with bounded number of branch vertices as follows;

Conjecture 1.37 ([122]) *Let $s \geq 0$ be an integer and let G be a connected claw-free graph of order n. If the minimum degree sum of $2s+3$ nonadjacent vertices is at least $n-2$, then G has a spanning tree with at most s branch vertices.*

In the last of Introduction, we show the relationship between the relaxed structures of (or invariants concerning with) hamilton cycles or hamilton paths considered in each Chapters 3–12 of this thesis. (See Figure 1.1.) An arrow from A to B means that B is an extended structure (or a generalized invariant) of A.

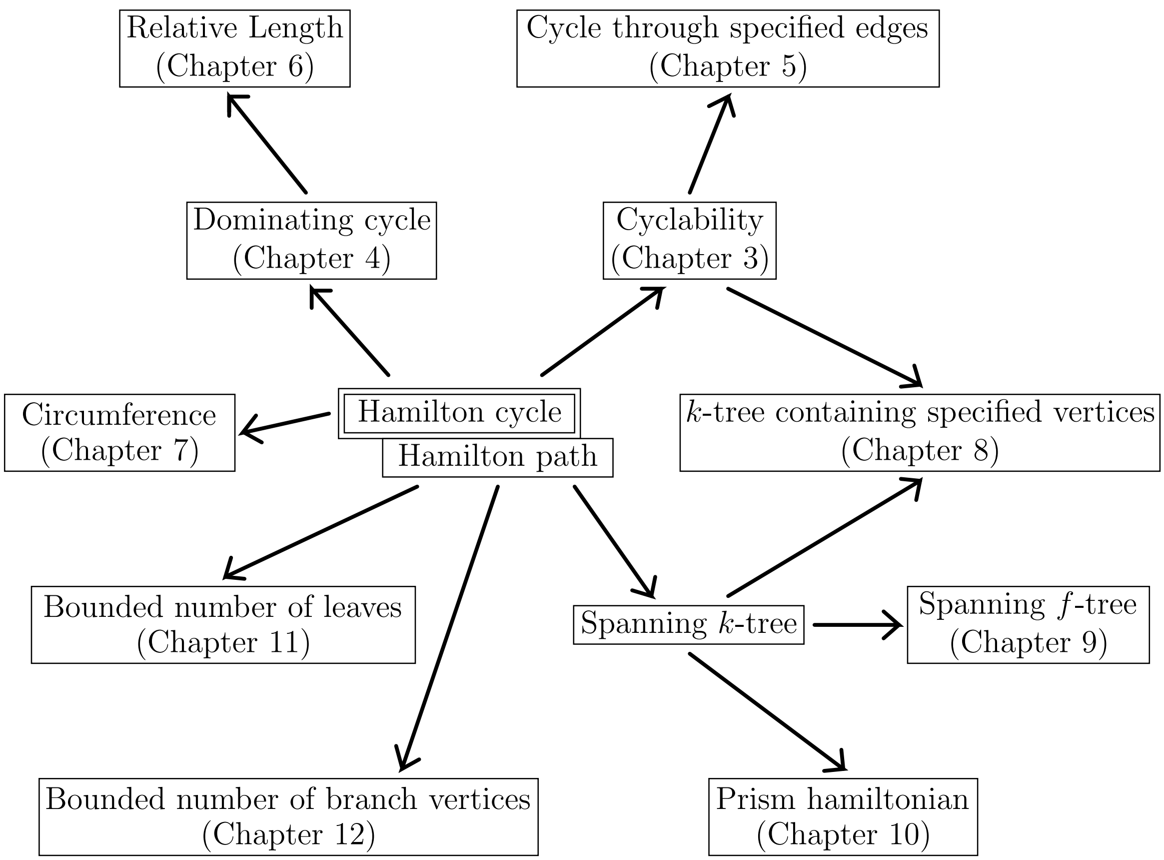


Figure 1.1: Relationship between the structures dealt in this thesis.

Chapter 2

Fundamentals

In this chapter, we define some basic terminology of Graph Theory, which is often used in the following chapters.

2.1 Graphs

A graph G is defined by a pair consisting of a *vertex set* $V(G)$ and an *edge set* $E(G)$ together with a mapping which associates each edge with two unordered vertices (possibly same vertex) called its *end-vertices*. For $u, v \in V(G)$ and for $e \in E(G)$, if u and v are end-vertices of e, we often write $e = uv$ and say that e *joins* u and v. A *loop* is an edge whose end-vertices are equal. *Multiple edges* are the edges which have same pair of end-vertices. We call a graph which has no loops or multiple edges *a simple* graph. If both of $V(G)$ and $E(G)$ are finite sets, a graph G is called *a finite* graph. In this thesis, we consider only simple and finite graphs. For a graph G, the number of vertices is called the *order* of G.

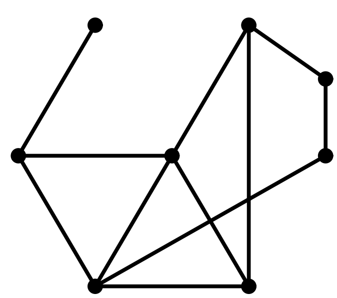


Figure 2.1: A simple and finite graph

2.2 Subgraphs, unions and joins of graphs

Let G and H be two graphs and let $S \subseteq V(G)$. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called a *subgraph of* G. An *induced* subgraph by S, denoted by $G[S]$, is a graph with $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : u, v \in S\}$. We define $G - S := G[V(G) - S]$. When a graph H is a subgraph of G, a new graph $G - H$ is defined by $G - H := G[V(G) - V(H)]$.

A graph $G \cup H$, called the *union* of G and H, is a graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. A graph mG is a graph constructed by the union of m vertex disjoint copies of G. The *join* of G and H, denoted by $G+H$, is a graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G),\ v \in V(H)\}$. For k graphs $G_1, G_2, \ldots, G_k$, the *sequential join* $G_1 + G_2 + \cdots + G_k$ is the union of $k-1$ joins $G_i + G_{i+1}$ for $1 \leq i \leq k-1$. Note that $G_1 + G_2 + G_3 = (G_1 \cup G_3) + G_2$.

2.3 Neighborhoods, degrees and independent sets

For $u, v \in V(G)$, if u and v are end-vertices of an edge, we say that they are *adjacent*. A *neighborhood* of v is the set of all vertices which is adjacent to v, and it is denoted by $N_G(v)$ or simply $N(v)$. The degree of v, denoted by $d_G(v)$ or simply $d(v)$, is the number of the neighborhoods of v. Let $\delta(G) := \min\{d_G(v) : v \in V(G)\}$, called the *minimum degree* in G. For $X \subseteq V(G)$, we define $N_G(X)$ by $N_G(X) := \bigcup_{x \in X} N_G(x)$. In Chapter 3–7, 10 and 11, with a slight abuse of notation, for a subgraph H of G, we write $N_H(x)$, $N_H(X)$ and $d_H(x)$ instead of $N_G(x) \cap V(H)$, $N_G(X) \cap V(H)$ and $|N_H(x)|$, respectively, because we use such notation many times in those chapters. We sometimes write $N_G(H)$ instead of $N_G(V(H))$.

For $X \subseteq V(G)$, X is an *independent* set in G if we have $xy \notin E(G)$ for each $x, y \in X$. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the maximum independent set in G. If $\alpha(G) \geq k$, let

$$\sigma_k(G) := \min\Big\{\sum_{x \in X} d_G(x)\colon X \text{ is an independent set with } |X| = k\Big\};$$

otherwise $\sigma_k(G) := +\infty$. Note that $\sigma_1(G) = \delta(G)$. For $X \subseteq V(G)$ with $|X| \geq k$, we denote

$$\Delta_k(X) := \max\Big\{\sum_{x \in Y} d_G(x) : Y \subseteq X \text{ and } |Y| = k\Big\}.$$

Let $r \geq k$. We define $\sigma_k^r(G)$ as follows; if $\alpha(G) \geq r$, let

$$\sigma_k^r(G) := \min\{\Delta_k(X) : X \text{ is an independent set with } |X| = r\};$$

otherwise $\sigma_k^r(G) := +\infty$. Remark that $\sigma_k^k(G) = \sigma_k(G)$. By the definition of $\sigma_k(G)$ and $\sigma_k^r(G)$, we obtain the following proposition.

Proposition 2.1 *(i) If $k \le l$, then $\lceil \frac{l}{k}\sigma_k(G) \rceil \le \sigma_l(G)$. In particular, $l\delta(G) \le \sigma_l(G)$.*

(ii) If $k \le l \le r$, then $\lceil \frac{k}{l}\sigma_l(G) \rceil \le \sigma_k^r(G)$. In particular, $\sigma_k(G) \le \sigma_k^r(G)$. $\square$

2.4 Particular classes of graphs

2.4.1 Paths

A graph P with $V(P) = \{u_0, u_1, \dots, u_l\}$ and $E(P) = \{u_i u_{i+1} : 0 \le i \le l-1\}$ is called a *path* or particularly $u_0 u_l$*-path*. Also P is called a path *connecting* u_0 and u_l. We say that u_0 (or u_l) is an *end-vertex* of P and u_i ($1 \le i \le l-1$) is an *internal vertex* of P, respectively. We define the *length* of a path P by the number of edges of P. A subgraph of P which forms a path connecting u_i and u_j is called a *subpath* of P and denoted by $u_i P u_j$. A path is often considered as a sequence of vertices along the edges. For example, we write the above path P by $P = u_0 u_1 \cdots u_l$. Sometimes we give an orientation to a path P and write $\overrightarrow{P}$ for the oriented path. For $x \in V(P)$, we denote the h-th successor and the h-th predecessor of x on $\overrightarrow{P}$ (if exist) by x^{+h} and x^{-h}, respectively. For $X \subseteq V(P)$, we define $X^{+h} := \{x^{+h} : x \in X - \{u_{l-h+1}, \dots, u_l\}\}$ and $X^{-h} := \{x^{-h} : x \in X - \{u_0, \dots, u_{h-1}\}\}$. We often write x^+, x^-, X^+ and X^- for x^{+1}, x^{-1}, X^{+1} and X^{-1}, respectively.

Let G be a graph, H be a subgraph of G and $x \in V(G-H)$. A path P is called an H*-path* if both of end-vertices of P are contained in H and all internal vertices and all edges of P are not contained in H.

For two paths P_1 and P_2, we say that P_1 and P_2 are *internally disjoint* if P_1 and P_2 are edge-disjoint and all internal vertices of P_i and all vertices of P_{3-i} are distinct for $i = 1, 2$. (Possibly the end-vertex of P_1 and the one of P_2 are the same vertex.)

Let $S \subseteq V(G)$. A subgraph F of G is called an (x, S)*-fan with width* l if F is a union of $P_1, \dots, P_l$ where every P_i is a path connecting x and a vertex a_i in S with $V(P_i) \cap S = \{a_i\}$ and $V(P_i) \cap V(P_j) = \{x\}$ for $1 \le i < j \le l$. The $a_1, \dots, a_l$ are said to be *end-vertices* of F. Let $\kappa(x, S) := \max\{l :$ there exists an (x, S)-fan with width $l\}$. For a subgraph H of G which does not contain x, we write an (x, H)-fan and $\kappa(x, H)$ instead of an $(x, V(H))$-fan and $\kappa(x, V(H))$, respectively.

For a graph G, let $p(G)$ be the order (the number of vertices) of a longest path in G.

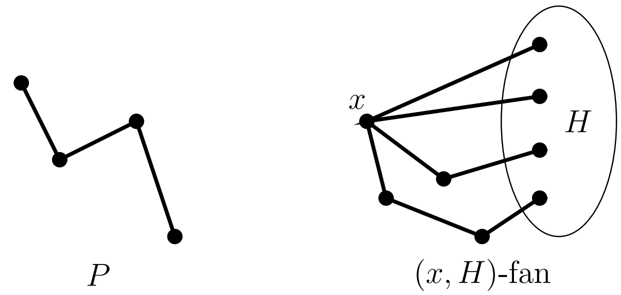


Figure 2.2: A path P and an (x, H)-fan

2.4.2 Cycles

A *cycle* C is a graph with $V(C) = \{u_0, u_1, \dots, u_{l-1}\}$ ($l \geq 3$) and $E(C) = \{u_i u_{i+1} : 0 \leq i \leq l-2\} \cup \{u_{l-1}u_0\}$. The *length* of cycle C is defined as l, that is, the number of edges of C. In particular, we call the cycle with length 3 *a triangle*. For a graph G, let $c(G)$ be the length of a longest cycle in G, called the *circumference* of G.

Like the paths case, we give an orientation to C and write $\overrightarrow{C}$ for the oriented cycle. For $x, y \in V(C)$, we denote the xy-path on $\overrightarrow{C}$ by $x\overrightarrow{C}y$, and write the reverse sequence of $x\overrightarrow{C}y$ by $y\overleftarrow{C}x$. For $x \in V(C)$, we denote the h-th successor and the h-th predecessor of x on $\overrightarrow{C}$ by x^{+h} and x^{-h}, respectively. For $X \subseteq V(C)$, we define $X^{+h} := \{x^{+h} : x \in X\}$ and $X^{-h} := \{x^{-h} : x \in X\}$. We often write x^+, x^-, X^+ and X^- for x^{+1}, x^{-1}, X^{+1} and X^{-1}, respectively.

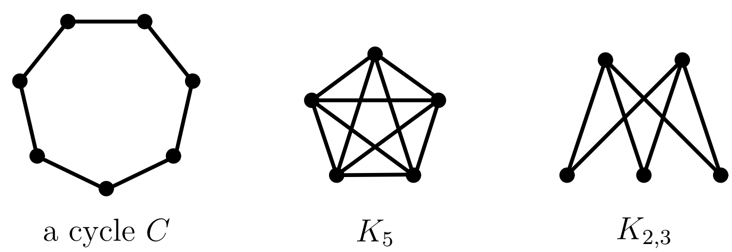


Figure 2.3: A cycle C, the complete graph K_5 and the complete bipartite graph $K_{2,3}$

2.4.3 Complete graphs and bipartite graphs

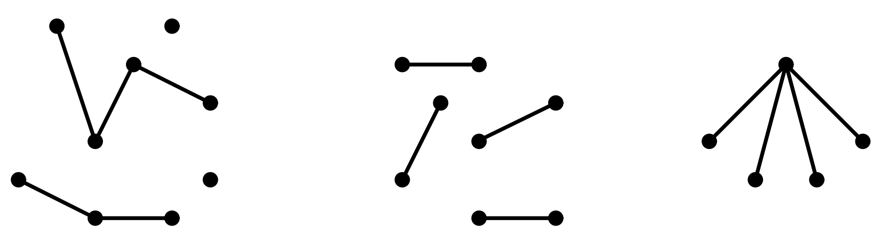
A graph G is *complete* if we have $uv \in E(G)$ for every distinct $u, v \in V(G)$. The complete graph on n vertices is denoted by K_n. A graph G is *bipartite* if we can partition $V(G)$ into two partite sets V_1 and V_2 so that there are no edges joining two vertices of the same partite set. A bipartite graph G is a *complete bipartite* graph if $E(G) = \{uv : u \in V_1,\ v \in V_2\}$. The complete bipartite graph with $|V_1| = l$ and $|V_2| = m$ is denoted by $K_{l,m}$. Clearly a bipartite graph G has no triangles. Like a bipartite graph, a graph which has no triangles is called a *triangle-free* graph.

2.4.4 Trees

A graph T is called a *tree* if it has no cycles and $|E(T)| = |V(T)| - 1$. Let T be a tree. A *leaf* of T is a vertex of degree one in T. We denote by $L(T)$ the set of leaves of T. Let $r \in V(T)$ be a particular vertex of T, called a *root* of T. Then we consider T as an oriented tree from r to leaves, denoted by $\overrightarrow{T}$. We let v^- denote the predecessor of v along $\overrightarrow{T}$. For $u, v \in V(T)$, the unique path in T connecting u and v is denoted by uTv, moreover, if $u \in rTv$, an oriented path $u\overrightarrow{T}v$ is called a path *starting from* u and *reaching* v *along* $\overrightarrow{T}$. In particular, we also regard a path $u\overrightarrow{T}u = u$ consisting of one vertex u as an oriented path starting from u and reaching u along $\overrightarrow{T}$.

A complete bipartite graph $K_{1,m}$ is especially called a *star*. Let V_1, V_2 be partite sets with $|V_1| = 1$ and $|V_2| = m$. The unique vertex in V_1 is called the *center* of the star.

A graph F is called a *forest* if F is a graph having no cycles. A forest F which consists of a union of paths is called a *linear forest*. For a linear forest F, let $\omega_1(F)$ be the number of components of order one in F. Moreover, if all of paths has the length 1, then F is called a *matching*.



a linear forest F a matching M the star $K_{1,4}$

Figure 2.4: A linear forest F, a matching M and the star $K_{1,4}$

2.5 Connectivity, toughness and blocks

A graph G is *connected* if for any $x, y \in V(G)$, there exists a path connecting x to y; otherwise G is *disconnected*. Each maximal connected subgraph of G is called a *component* of G. For $u, v \in V(G)$, we define a *distance* $\mathrm{dist}(u,v)$ *between* u *and* v as $\mathrm{dist}(u,v) := \min\big\{|E(P)| : P$ is a path in G connecting u and $v\big\}$ if u and v are contained in the same component of G; otherwise let $\mathrm{dist}(u,v) := +\infty$.

Let $x, y \in V(G)$. We define the *local connectivity* $\kappa(x,y)$ by the maximum number of internally disjoint paths connecting x and y. The *connectivity* of G, denoted by $\kappa(G)$, is defined by $\kappa(G) := \min\{\kappa(x,y) : x, y \in V(G),\ x \neq y\}$. A graph G is *k-connected* if $k \leq \kappa(G)$. For $T \subseteq V(G) - \{x,y\}$, T *separates* x and y if x and y belong to distinct components of $G - T$. Also T is called a *separating* set if $G - T$ is disconnected. The following theorem is the basic one concerning the connectivity and the separating set, called *Menger's Theorem.*

Theorem 2.2 (Menger [125]) *If $xy \notin E(G)$, then*

$$\kappa(x,y) = \min\{|T| : T \text{ separates } x \text{ and } y\}.$$

In particular, if G is not a complete graph, then

$$\kappa(G) = \min\{|T| : T \text{ is a separating set in } G\}.$$

For the proof of Theorem 2.2, we refer the reader to [167]. By Menger's theorem, a graph G is k-connected if and only if there exists no separating set T such that $|T| < k$ or G is a complete graph on at least $k+1$ vertices.

For a graph G, let $\omega(G)$ be the number of components of G. A graph G is *t-tough* if $t \cdot \omega(G - S) \leq |S|$ for any $S \subset V(G)$ with $\omega(G - S) \geq 2$. The *toughness* of a graph G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough if G is not a complete graph; If $G = K_n$ for some $n \geq 1$, let $\tau(G) := +\infty$. In other word, if $G \neq K_n$,

$$\tau(G) := \min\Big\{\frac{|S|}{\omega(G-S)} : S \subset V(G) \text{ and } \omega(G-S) \geq 2\Big\}.$$

We say that $v \in V(G)$ is a *cut-vertex* if $G - v$ is disconnected. A *block* of G is defined as a maximal subgraph which contains no cut-vertices. An *end block* is a block which has exactly one cut-vertex of G. We remark that each block of a connected graph is 2-connected or isomorphic to K_2.

2.6 Terminology for the specified vertices

In this section, we redefine some invariants for the specified vertices. We define the independence number, the connectivity, the minimum degree and the degree sum of the specified vertices S, as follows.

For $X \subseteq V(G)$, X is called an *independent set of* S, if $X \subseteq S$ and $G[X]$ has no edges. We define

$$\begin{aligned} \alpha(S) &:= \max\{|X| : X \text{ is an independent set of } S\}, \\ \kappa(S) &:= \min\{\kappa(x,y) : x,y \in S,\ x \neq y\} \\ \text{and} \quad \delta(S) &:= \min\{d_G(x) : x \in S\}. \end{aligned}$$

If $\alpha(S) \geq k$, let

$$\sigma_k(S) := \min\Big\{\sum_{x\in X} d_G(x)\colon X \text{ is an independent set of } S \text{ with } |X| = k\Big\};$$

otherwise $\sigma_k(S) := +\infty$. For $r \geq k$, if $\alpha(G) \geq r$, let

$$\sigma_k^r(S) := \min\{\Delta_k(X) : X \text{ is an independent set of } S \text{ with } |X| = r\};$$

otherwise $\sigma_k^r(S) := +\infty$. The following proposition is the same one as Proposition 2.1 if $S = V(G)$.

Proposition 2.3 *(i) If $k \leq l$, then $\lceil \frac{l}{k}\sigma_k(S)\rceil \leq \sigma_l(S)$. In particular, $l\delta(S) \leq \sigma_l(S)$.*

(ii) If $k \leq l \leq r$, then $\lceil \frac{k}{l}\sigma_l(S)\rceil \leq \sigma_k^r(S)$. In particular, $\sigma_k(S) \leq \sigma_k^r(S)$. □

It follows from Theorem 2.2 that the similar result for a fan holds.

Lemma 2.4 *Let G be a graph and let $S \subseteq V(G)$. Then for any $x \in S$, there exists an $\big(x, S - \{x\}\big)$-fan with width at least $\min\big\{|S| - 1, \kappa(S)\big\}$. In particular, $\kappa\big(x, S - \{x\}\big) \geq \min\big\{|S| - 1, \kappa(S)\big\}$.*

Chapter 3

Hamilton cycles and cyclability

A hamilton cycle problem, determining whether a given graph has a hamilton cycle or not, is one of the most important problem in Graph Theory, because of the relationship to some other problems or topics. In this chapter, we introduce some sufficient conditions for the existence of a hamilton cycle. In particular, we concentrate on degree conditions or independence number conditions. In addition to a hamilton cycle problem, we consider a cyclability problem, determining whether a given graph has a cycle passing through given vertices or not. In Sections 3.1 and 3.2, we show some results on hamilton cycles and cyclability, respectively, and in Section 3.3, we show the relationship between these results. In Section 3.4, we prove Theorem 3.23, which is a new sufficient condition for specified vertices to be cyclable.

The contents of this chapter are based on the paper [135] "A degree sum condition concerning the connectivity and the independence number of a graph," joint-work with T. Yamashita.

3.1 Results of Hamilton cycles

A cycle C in a graph G is called a *hamilton cycle* if $V(C) = V(G)$. In particular, a graph which has a hamilton cycle is called *hamiltonian*. The following theorem is the most classical one giving a sufficient condition for a graph to have a hamilton cycle.

Theorem 3.1 (Dirac [41]) *Let G be a graph of order $n \geq 3$. If $\delta(G) \geq \frac{1}{2}n$, then G has a hamilton cycle.*

Ore considered the following result with a $\sigma_2(G)$ condition. By Proposition 2.1 (i), this is a generalization of Theorem 3.1.

Theorem 3.2 (Ore [130]) *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G has a hamilton cycle.*

Chvátal and Erdős considered a condition on concerning with the independence number and the connectivity of a graph. They proved the following theorem.

Theorem 3.3 (Chvátal and Erdős [37]) *Let G be a 2-connected graph. If $\alpha(G) \leq \kappa(G)$, then G has a hamilton cycle.*

By the following proposition, we show that Theorem 3.3 implies Theorem 3.2, which was first shown by Bondy [24] in 1978.

Proposition 3.4 *Let G be a graph of order n. If $\sigma_2(G) \geq n$, then $\alpha(G) \leq \kappa(G)$.*

Proof. Assume that $\alpha(G) > \kappa(G)$. By Menger's Theorem, there exists a separating set $T \subseteq V(G)$ with $|T| = \kappa(G)$. Let X be an independent set with $|X| = \alpha(G)$. Since $|T| = \kappa(G) < \alpha(G)$, we can take $x_1 \in X - T$. Let U_1 be a component of $G - T$ such that $x_1 \in U_1$ and let $U_2 := (G - T) - U_1$.

Suppose that $X \cap U_2 \neq \emptyset$, say $x_2 \in X \cap U_2$. Then since $N(x_i) \subseteq V(G) - X$, we have $|N(x_1) \cup N(x_2)| \leq |V(G) - X| = n - \alpha(G)$, and since $N(x_1) \cap N(x_2) \subseteq T$, we obtain $|N(x_1) \cap N(x_2)| \leq |T| = \kappa(G)$. Therefore

$$\begin{aligned} n &\leq d(x_1) + d(x_2) \\ &= |N(x_1) \cup N(x_2)| + |N(x_1) \cap N(x_2)| \\ &\leq n - \alpha(G) + \kappa(G) \\ &< n, \end{aligned}$$

a contradiction.

Thus, we have $X \cap U_2 = \emptyset$ and hence $X \subseteq U_1 \cup T$. Then since $N(x_1) \subseteq (U_1 \cup T) - X$, we have $d(x_1) \leq |U_1| + |T| - \alpha(G)$. On the other hand, for any $y \in U_2$, $d(y) \leq |U_2| + |T| - 1$. Therefore

$$\begin{aligned} n &\leq d(x_1) + d(y) \\ &\leq |U_1| + |T| - \alpha(G) + |U_2| + |T| - 1 \\ &= n + |T| - \alpha(G) - 1 \\ &< n, \end{aligned}$$

a contradiction, again. $\square$

We remark that all of theorems above are best possible in a sense. Let $m \geq 2$ and $G_1 = K_m + (m+1)K_1$. (See Figure 3.1.) Then $|V(G_1)| = 2m + 1$ and $\delta(G_1) = m = \frac{1}{2}(|V(G_1)| - 1)$. Furthermore G_1 has no hamilton cycles, and hence we cannot replace $\frac{1}{2}n$ by $\frac{1}{2}(n-1)$ without destroying the conclusion of Theorem 3.1. Since $\sigma_2(G_1) = 2m = |V(G_1)| - 1$ and $\alpha(G_1) = m+1 = \kappa(G_1)+1$, we obtain that the lower bound n of Theorem 3.2 and the upper bound $\kappa(G)$ of Theorem 3.3 are sharp, respectively.

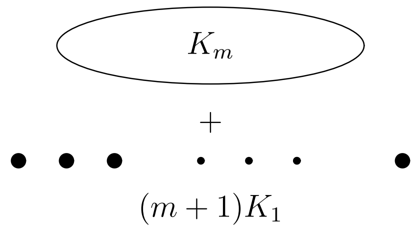


Figure 3.1: The graph G_1

In 1980, Bondy gave the following theorem. If $\alpha(G) \leq \kappa(G)$, then $\sigma_{\kappa(G)+1}(G) = +\infty$, and hence the assumption of Theorem 3.5 holds for $k = \kappa(G)$. Thus, Theorem 3.5 is a generalization of Theorem 3.3.

Theorem 3.5 (Bondy [26]) *Let G be a k-connected graph of order $n \geq 3$. If $\sigma_{k+1}(G) > \frac{1}{2}(k+1)(n-1)$, then G has a hamilton cycle.*

Again, Theorem 3.5 is generalized by Yamashita. By Proposition 2.1 (ii), Theorem 3.6 is a generalization of Theorem 3.5.

Theorem 3.6 (Yamashita [176]) *Let G be a graph of order $n \geq 3$. If $\sigma_2^{\kappa(G)+1}(G) \geq n$, then G has a hamilton cycle.*

In 1991, Flandrin, Jung and Li showed the following theorem with other degree sum condition.

Theorem 3.7 (Flandrin, Jung and Li [59]) *Let G be a 2-connected graph of order $n \geq 3$. If $\sum_{i=1}^{3} d(x_i) \geq n + |\bigcap_{i=1}^{3} N(x_i)|$ for every independent set $\{x_1, x_2, x_3\}$, then G has a hamilton cycle.*

In 1981, Häggkvist and Nicoghossian [80] proved a result on a degree and connectivity condition; A 2-connected graph G of order $n \geq 3$ with $\delta(G) \geq \frac{1}{3}(n+\kappa(G))$ has a hamilton cycle. This result is generalized by Bauer, Broersma, Li and Veldman as follows. In particular, in 1999, Wei [166] gave a short proof of it.

Theorem 3.8 (Bauer, Broersma, Li and Veldman [13]) *Let G be a 2-connected graph of order $n \geq 3$. If $\sigma_3(G) \geq n + \kappa(G)$, then G has a hamilton cycle.*

Suppose that the assumption of Theorem 3.2 holds, that is, $\sigma_2(G) \geq n$. Then for every independent set $\{x_1, x_2, x_3\}$, $d(x_1)+d(x_2) \geq n$ and $d(x_3) \geq |\bigcap_{i=1}^{3} N(x_i)|$, and hence $\sum_{i=1}^{3} d(x_i) \geq n + |\bigcap_{i=1}^{3} N(x_i)|$, that is, the assumption of Theorem 3.7 holds. On the other hand, since $\delta(G) \geq \kappa(G)$, we obtain $\sigma_3(G) \geq \sigma_2(G) + \delta(G) \geq n + \kappa(G)$, that is, the assumption of Theorem 3.8 holds. Therefore each of Theorems 3.7 and 3.8 implies Theorem 3.2, respectively.

Recall $G_1 = K_m + (m+1)K_1$. Since $\sigma_{k+1}(G_1) = (k+1)m = \frac{1}{2}(k+1)(|V(G_1)|-1)$, $\sigma_2^{\kappa(G_1)+1}(G_1) = 2m = |V(G_1)| - 1$, $\sum_{i=1}^{3} d(x_i) = 3m = |V(G_1)| + |\bigcap_{i=1}^{3} N(x_i)| - 1$ for every independent set $\{x_1, x_2, x_3\}$ and $\sigma_3(G_1) = 3m = |V(G_1)| + \kappa(G_1) - 1$, we have that the lower bound of Theorems 3.5–3.8 is best possible.

For other results on a hamilton cycle, we refer surveys [29, 76, 77, 97].

3.2 Cyclability of specified vertices

A hamilton cycle is a cycle passing through all vertices of a graph. Therefore one of the way of generalizing the concept of a hamilton cycle is the concept of a cycle passing through specified vertices. For $S \subseteq V(G)$, S is *cyclable* in G if G contains a cycle passing through S. In fact, in the case $S = V(G)$, S is cyclable in G if and only if G has a hamilton cycle.

Some results on a hamilton cycle are generalized to cyclability. Bollobás and Brightwell, Shi, and Yamashita gave improvements Theorems 3.1, 3.2 and 3.6, respectively.

Theorem 3.9 (Bollobás and Brightwell [23]) *Let G be a graph of order n and $S \subseteq V(G)$ with $|S| \geq 3$. If $\delta(S) \geq \frac{1}{2}n$, then S is cyclable in G.*

Theorem 3.10 (Shi [149]) *Let G be a 2-connected graph of order n and $S \subseteq V(G)$. If $\sigma_2(S) \geq n$, then S is cyclable in G.*

Theorem 3.11 (Yamashita [176]) *Let G be a 2-connected graph of order n and $S \subseteq V(G)$. If $\sigma_2^{\kappa(S)+1}(S) \geq n$, then S is cyclable in G.*

In 1996, Favaron, Flandrin, Li, Liu, Tian and Wu showed the generalization of Theorem 3.7.

Theorem 3.12 (Favaron, Flandrin, Li, Liu, Tian and Wu [56]) *Let G be a 2-connected graph of order n and $S \subseteq V(G)$. If $\sum_{i=1}^{3} d(x_i) \geq n + |\bigcap_{i=1}^{3} N(x_i)|$ for every independent set $\{x_1, x_2, x_3\}$ of S, then S is cyclable in G.*

For a Chvátal-Erdős type condition, in 1985, Fournier gave one of generalizations of Theorem 3.3.

Theorem 3.13 (Fournier [63]) *Let G be a 2-connected graph and $S \subseteq V(G)$. If $\alpha(S) \leq \kappa(G)$, then S is cyclable in G.*

In 1997, Broersma, H. Li, J. Li, Tian and Veldman generalized Theorems of 3.8 and 3.13, respectively. They defined another notion of connectivity of S as follows. If $G[S]$ is not complete, let $\kappa'(S)$ be the minimum cardinality of a set of vertices of G separating two vertices of S. If $G[S]$ is complete, let $\kappa'(S) := |S| - 1$.

Theorem 3.14 (Broersma, H. Li, J. Li, Tian and Veldman [30]) *Let G be a 2-connected graph of order n and $S \subseteq V(G)$. If $\sigma_3(S) \geq n + \min\{\kappa'(S), \delta(S)\}$, then S is cyclable in G.*

Theorem 3.15 (Broersma, H. Li, J. Li, Tian and Veldman [30]) *Let G be a 2-connected graph and $S \subseteq V(G)$. If $\alpha(S) \leq \kappa'(S)$, then S is cyclable in G.*

By the definitions of $\kappa(S)$, $\kappa'(S)$ and $\delta(S)$, the following proposition is obvious.

Proposition 3.16 *If $G[S]$ is not a complete graph, then $\kappa(S) \leq \min\{\kappa'(S), \delta(S)\}$.*

There exist graphs G and $S \subseteq V(G)$ satisfying $\kappa(S) < \min\{\kappa'(S), \delta(S)\}$. For example, let k, m and l be positive integers with $k \leq m$ and $m + 1 \leq k + l$ and we consider the graph $G_2 = K_l + K_1 + kK_1 + mK_1$. Let u be a vertex of K_1, v, w be distinct vertices of kK_1 and $S := \{u, v, w\}$. (See Figure 3.2.) Then $\kappa(S) = k$ since there exist only k internally disjoint paths connecting u and v. On the other hand, we must remove $V(K_1 \cup mK_1)$ to separate v and w, and hence $\kappa'(S) = m+1$. Therefore $\kappa(S) = k < m + 1 = \kappa'(S) = \delta(S)$.

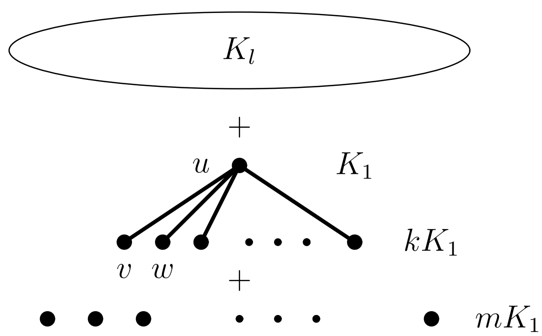


Figure 3.2: The graph G_2

Indeed, the very same proof as Theorem 3.15 yields the following stronger theorem than it.

Theorem 3.17 ([135]) *Let G be a 2-connected graph and $S \subseteq V(G)$. If $\alpha(S) \leq \kappa(S)$, then S is cyclable in G.*

Considering Theorem 3.17, we show that the following lemma, which implies the relationship between $\kappa(S)$ and $\kappa'(S)$ under the condition $\alpha(S) \geq \kappa(S) + 1$.

Lemma 3.18 *Let G be a graph and $S \subseteq V(G)$ with $\alpha(S) \geq \kappa(S) + 1$. Then there exists $T \subseteq V(G)$ such that $|T| = \kappa(S)$ and T separates two vertices of S. In particular, $\kappa(S) = \kappa'(S)$.*

Proof. Let u and v be vertices in S such that $\kappa(u, v) = \kappa(S)$. If $uv \notin E(G)$, then by Menger's theorem, there exists $T \subseteq V(G) - \{u, v\}$ with $|T| = \kappa(S)$ which separates u and v.

Suppose that $uv \in E(G)$. Then $G - uv$ has $T \subseteq V(G) - \{u, v\}$ with $|T| = \kappa(S) - 1$ which separates u and v. If $S - (T \cup \{u, v\}) \neq \emptyset$, then $T \cup \{u\}$ or $T \cup \{v\}$ is a desired separating set. Thus, we may assume that $S \subseteq T \cup \{u, v\}$. Then since $uv \in E(G)$, $\alpha(S) \leq |S| - 1 \leq |T| + 1 = \kappa(S)$, contradicting the assumption. $\square$

By the following proposition, which is the similar one as Proposition 3.4, we show that Theorem 3.17 implies Theorem 3.10.

Proposition 3.19 *Let G be a graph on n vertices and $S \subseteq V(G)$. If $\sigma_2(S) \geq n$, then $\alpha(S) \leq \kappa(S)$.*

Proof. Assume that $\alpha(S) > \kappa(S)$. By Lemma 3.18, there exists $T \subseteq V(G)$ with $|T| = \kappa(S)$ which separates two vertices of S. Let X be an independent set of S with $|X| = \alpha(S)$. Since $|T| = \kappa(S) < \alpha(S)$, we can take $x_1 \in X - T$. Let U_1 be a component of $G - T$ such that $x_1 \in U_1$ and let $U_2 := (G - T) - U_1$.

Suppose that $X \cap U_2 \neq \emptyset$ and let $x_2 \in X \cap U_2$. Then since $N(x_i) \subseteq V(G) - X$, we have $|N(x_1) \cup N(x_2)| \leq |V(G) - X| = n - \alpha(S)$, and since $N(x_1) \cap N(x_2) \subseteq T$, we obtain $|N(x_1) \cap N(x_2)| \leq |T| = \kappa(S)$. Therefore

$$\begin{aligned} n &\leq d(x_1) + d(x_2) \\ &= |N(x_1) \cup N(x_2)| + |N(x_1) \cap N(x_2)| \\ &\leq n - \alpha(S) + \kappa(S) \\ &< n, \end{aligned}$$

a contradiction.

Thus, we have $X \cap U_2 = \emptyset$ and hence $X \subseteq U_1 \cup T$. Then since $N(x_1) \subseteq (U_1 \cup T) - X$, we have $d(x_1) \leq |U_1| + |T| - \alpha(S)$. On the other hand, for any $y \in S \cap U_2$, $d(y) \leq |U_2| + |T| - 1$. Therefore

$$\begin{aligned} n &\leq d(x_1) + d(y) \\ &\leq |U_1| + |T| - \alpha(S) + |U_2| + |T| - 1 \\ &= n + |T| - \alpha(S) - 1 \\ &< n, \end{aligned}$$

a contradiction, again. $\square$

By Theorems 3.15 and 3.17 and Lemma 3.18, we obtain the following theorem. This is other generalization of Theorem 3.8 and more stronger than Theorem 3.15 by Proposition 3.16.

Theorem 3.20 ([135]) *Let G be a graph of order n and $S \subseteq V(G)$ with $\kappa(S) \geq 2$. If $\sigma_3(S) \geq n + \kappa(S)$, then S is cyclable in G.*

In 2000, Harkat-Benhamadine, Li and Tian gave a $\sigma_4(G)$ condition with the independence number.

Theorem 3.21 (Harkat-Benhamadine, Li and Tian [82]) *Let G be a 3-connected graph of order n and $S \subseteq V(G)$. If $\sigma_4(S) \geq n + 2\alpha(S) - 2$, then S is cyclable in G.*

Since the following proposition holds, Theorem 3.21 is a generalization of Theorem 3.10.

Proposition 3.22 *Let G be a graph of order $n \geq 3$ and $S \subseteq V(G)$. If $\sigma_2(S) \geq n$, then $\sigma_4(S) \geq n + 2\alpha(S) - 2$.*

Proof. First, we shall prove that $\alpha(S) \leq \frac{1}{2}n$. Suppose that $\alpha(S) > \frac{1}{2}n$. Let X be an independent set of S with order $\alpha(S)$. Then for every $x \in X$, $d(x) = |N(x)| \leq |V(G) - X| < n - \frac{1}{2}n = \frac{1}{2}n$. Since $|X| \geq \alpha(S) > \frac{1}{2}n \geq \frac{3}{2}$, we can take two vertices $x_1, x_2 \in X$, and hence $n \leq \sigma_2(S) \leq d(x_1) + d(x_2) < n$, a contradiction. Thus, we have $\alpha(S) \leq \frac{1}{2}n$.

Therefore by Proposition 2.3 (i), we obtain

$$\begin{aligned} \sigma_4(S) &\geq 2\sigma_2(S) \geq 2n \\ &\geq n + 2\alpha(S) \geq n + 2\alpha(S) - 2. \quad \square \end{aligned}$$

On the other hand, we give a $\sigma_4(G)$ condition with the connectivity and the independence number.

Theorem 3.23 ([135]) *Let G be a graph of order n and $S \subseteq V(G)$ with $\kappa(S) \geq 3$. If $\sigma_4(S) \geq n + \kappa(S) + \alpha(S) - 1$, then S is cyclable in G.*

We give an example which shows that both Theorems 3.20 and 3.23 is best possible. Let k, m, l be positive integers with $3 \leq k \leq m - 2$ and $k + l - 1 \leq m$. We consider the graph $G_3 = K_l + K_k + (mK_1 + K_{m-k})$. (See Figure 3.3.) Let $S = K_l \cup mK_1$. Then $|V(G_3)| = 2m + l$, $\kappa(S) = k$ and $\alpha(S) = m + 1$, $\sigma_3(G) = (k + l - 1) + 2m = |V(G_3)| + k - 1$ and $\sigma_4(S) = (k + l - 1) + 3m = |V(G_3)| + k + (m + 1) - 2 = |V(G_3)| + \kappa(S) + \alpha(S) - 2$. Then since G_3 has no cycle which contains all vertices of S, the bounds $n + \kappa(S)$ in Theorem 3.20 and

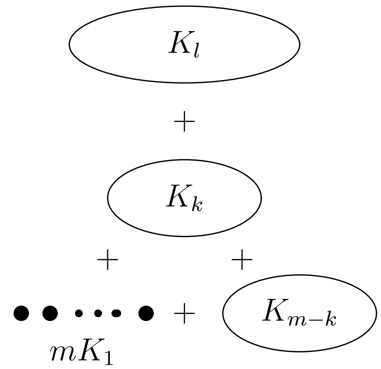


Figure 3.3: The graph G_3

$n + \kappa(S) + \alpha(S) - 1$ in Theorem 3.23 are best possible in a sense.

By combining Theorems 3.23 and 3.17, we obtain Theorem 3.21. Therefore considering Theorem 3.17, Theorem 3.23 is stronger than Theorem 3.21. In Section 3.4, we prove Theorem 3.23.

Moreover, when $S = V(G)$, we obtain the following as a corollary of Theorem 3.23.

Corollary 3.24 *Let G be a 3-connected graph on n vertices. If $\sigma_4(G) \geq n+\kappa(G)+\alpha(G) - 1$, then G has a hamilton cycle.*

On the other hand, in 1995, Ota gave another degree condition concerning cyclability.

Theorem 3.25 (Ota [132]) *Let G be a graph of order n and $S \subseteq V(G)$ with $\kappa(S) \geq 2$. If for any l with $l \geq \kappa(S)$,*

$$\sigma_{l+1}(S) \geq n + l^2 - l,$$

then S is cyclable in G.

By proving the following proposition, we show that the assumption of Theorem 3.25 is weaker than that of Theorem 3.21. Hence, Theorem 3.25 implies Theorem 3.21.

Proposition 3.26 *Let G be a 3-connected graph of order n and $S \subseteq V(G)$. If $\sigma_4(S) \geq n + 2\alpha(S) - 2$, then $\sigma_{l+1}(S) \geq n + l^2 - l$ for any l with $3 \leq l \leq \alpha(S) - 1$.*

Proof. By Proposition 2.3 (i), $\sigma_{l+1}(S) \geq \frac{l+1}{4}\sigma_4(S) \geq \frac{l+1}{4}(n+2\alpha(S)-2)$. Therefore it suffices to show

$$\frac{l+1}{4}(n + 2\alpha(S) - 2) \geq n + l^2 - l.$$

Because the above inequality is a quadratic function on l, it suffices to prove that it holds for $l = 3$ and $l = \alpha(S) - 1$. Since $3 \leq l \leq \alpha(S) - 1$, note that $\alpha(S) \geq 4$.

Case 1. $l = 3$.

In this case, $\frac{l+1}{4}(n + 2\alpha(S) - 2) \geq n + 6 = n + l^2 - l$. Therefore this completes the Case 1.

Case 2. $l = \alpha(S) - 1$.

Suppose that $\frac{l+1}{4}(n + 2\alpha(S) - 2) < n + l^2 - l$. By the assumption of Case 2, this implies $(\alpha(S) - 4)(n - 2\alpha(S) + 2) < 0$. Since $\alpha(S) \geq 4$, we have $n < 2\alpha(S) - 2$.

On the other hand, let X be an independent set of S with $|X| = \alpha(S)$ and choose $x \in X$ so that $d_G(x)$ is as large as possible. Since $|V(G) - X| \geq |N_G(x)| \geq \frac{1}{4}\sigma_4(S)$ by the degree condition, we obtain $n - \alpha(S) \geq \frac{1}{4}(n + 2\alpha(S) - 2)$, and this implies $n \geq 2\alpha(S) - \frac{2}{3}$, a contradiction. This completes the proof. $\square$

Abderrezzak, Flandrin and Amar [1] considered the cyclability of vertices in a bipartite graph. They showed that for a 2-connected bipartite graph G with bipartition (X, Y) and $|X| = |Y|$ and for $S \subset X$, if $d_G(x) + d_G(y) \geq |X| + 1$ for any nonadjacent $x \in S$ and $y \in Y$, then S is cyclable in G. For other results on cycles passing specified vertices, we refer survey of cyclability [78].

3.3 The relationship between the theorems of the cyclability

In Section 3.2, we show several results on cyclability of specified vertices, and compare some of them. In particular, in order to show that the relationship between the assumptions of those results on cyclability with degree conditions, we represent a diagram in Figure 3.4, in which the arrow from Theorem A to Theorem B means that Theorem A implies Theorem B.

In Figure 3.4, there exist five "maximal" theorems, that is, Theorems 3.11, 3.12, 3.20, 3.23 and 3.25. In fact, for each maximal theorem, there exist infinitely many graphs that satisfy the assumption of it and do not satisfy that of other theorems. We shall show such graphs in order of the number of theorems.

First, we show that Theorem 3.11 is not implied by others. Let m be an integer with $m \geq 4$. We consider the graph $G_4 = mK_1 + K_{m+1} + (m+1)K_1 + K_m$. (See Figure 3.5.) Let $S = mK_1 \cup (m+1)K_1$. Then $|V(G_4)| = 4m + 2$, $\kappa(S) = m + 1$, $\alpha(S) = 2m + 1$, and

$$\sigma_2^{\kappa(S)+1}(S) = 2(2m + 1) = |V(G_4)|,$$

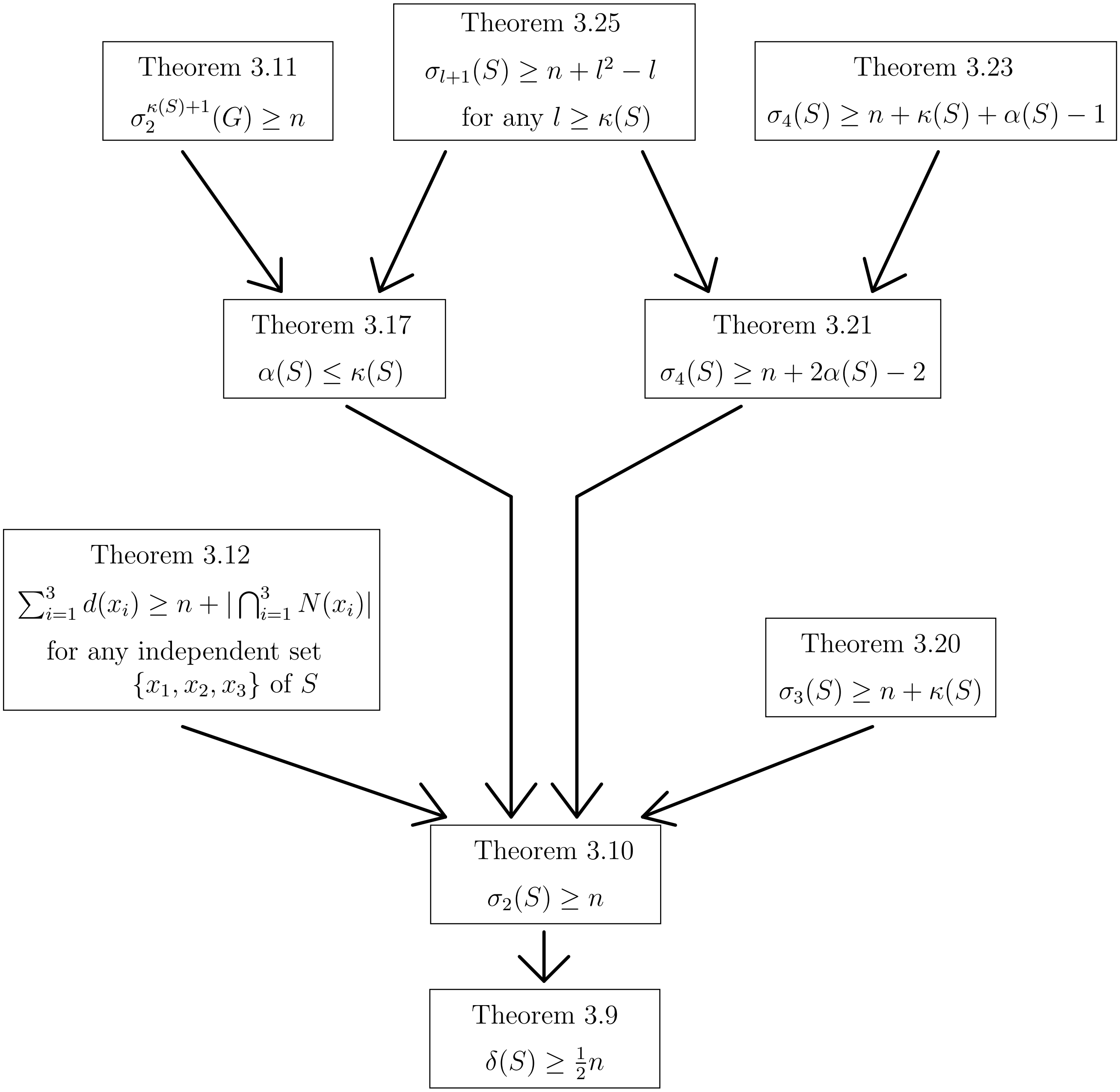


Figure 3.4: The relationship between Theorems on cyclability.

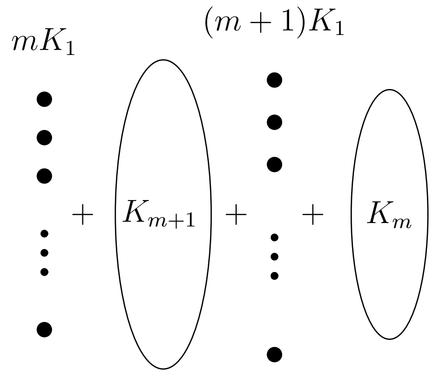


Figure 3.5: The graph G_4

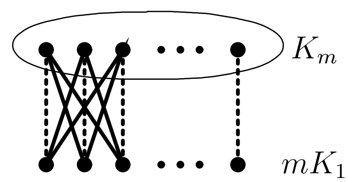


Figure 3.6: The graph G_5

and hence the assumption of Theorem 3.11 holds. However,

$$\sum_{i=1}^{3} d(x_i) \quad = \quad 3(m+1) < |V(G_4)| + |\bigcap_{i=1}^{3} N(x_i)|,$$

for $x_1, x_2, x_3 \in V(mK_1)$,

$$\begin{aligned} \sigma_3(S) &= 3(m+1) < |V(G_4)| + \kappa(S), \\ \sigma_4(S) &= 4(m+1) = |V(G_4)| + 2 < |V(G_4)| + \kappa(S) + \alpha(S) - 1, \end{aligned}$$

and for $l = \alpha(S) - 1 = 2m$,

$$\begin{aligned} \sigma_{l+1}(S) &= m(m+1) + (m+1)(2m+1) \\ &= |V(G_4)| + l^2 - l - (m-1)^2 \\ &< |V(G_4)| + l^2 - l. \end{aligned}$$

Therefore the assumptions of Theorems 3.12, 3.20, 3.23 and 3.25 do not hold.

Next, we show that Theorem 3.12 is not implied by others. Let m be an integer with $m \geq 4$. We consider the graph G_5 is obtained from $K_m + mK_1$ by removing a perfect matching. (See Figure 3.6.) Let $S = mK_1$. Then $|V(G_5)| = 2m$, $\kappa(S) = m - 1$ and $\alpha(S) = m$. For $x_1, x_2, x_3 \in S$, since $|\bigcap_{i=1}^{3} N(x_i)| = m - 3$, we obtain

$$\sum_{i=1}^{3} d(x_i) = 3(m-1) = |V(G_5)| + |\bigcap_{i=1}^{3} N(x_i)|,$$

that is, the assumption of Theorem 3.12 holds. On the other hand,

$$\begin{aligned}
\sigma_2^{\kappa(S)+1}(S) &= 2(m-1) = |V(G_5)| - 2, \\
\sigma_3(S) &= 3(m-1) = |V(G_5)| + \kappa(S) - 2, \\
\sigma_4(S) &= 4(m-1) = |V(G_5)| + \kappa(S) + \alpha(S) - 3,
\end{aligned}$$

and for $l = \alpha(S) - 1 = m - 1$,

$$\begin{aligned}
\sigma_{l+1}(S) &= m(m-1) \\
&= |V(G_5)| + m^2 - 3m \\
&= |V(G_5)| + l^2 - l - 2.
\end{aligned}$$

Therefore the assumptions of Theorems 3.11, 3.20, 3.23 and 3.25 do not hold.

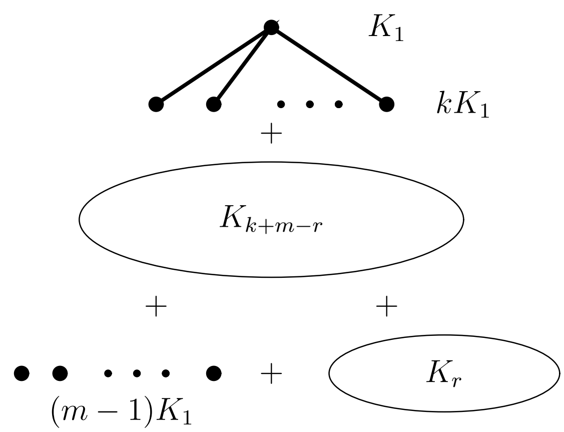


Figure 3.7: The graph G_6

Thirdly, we shall show that Theorem 3.20 is exactly a maximal theorem in Figure 3.4. Let k, r, m be integers such that $k \geq 7$, $r \geq 10$ and $m = 3(r-1)$. We consider the graph $G_6 = K_1 + kK_1 + K_{k+m-r} + ((m-1)K_1 + K_r)$ and let $S = K_1 \cup kK_1 \cup (m-1)K_1$. (See Figure 3.7.) Then $|V(G_6)| = 2k + 2m$, $\kappa(S) = k$, $\alpha(S) = k + m - 1$, and

$$\begin{aligned}
\sigma_3(G) &= \min\{k + 2(k+m), 3(k+m-r+1)\} \\
&= 3k + 2m \\
&= |V(G_6)| + \kappa(S),
\end{aligned}$$

and hence the assumption of Theorem 3.20 holds. On the other hand,

$$\begin{aligned}
\sigma_2^{\kappa(S)+1}(S) &= (k+m) + (k+m-r+1) = |V(G_6)| - r + 1, \\
\sum_{i=1}^{3} d(x_i) &= 3(k+m-r+1) = |V(G_6)| + |\bigcap_{i=1}^{3} N(x_i)| - 2r + 2,
\end{aligned}$$

for $x_1, x_2, x_3 \in V(kK_1)$ and

$$\sigma_4(S) = 4(k+m-r+1) = |V(G_6)| + \kappa(S) + \alpha(S) - r + 2.$$

Therefore the assumptions of Theorems 3.11, 3.12 and 3.23 do not hold. Moreover for $l = \alpha(S) - 1 = k + m - 2$, since $k \geq 7$ and $r \geq 10$, we have

$$\begin{aligned}
&|V(G_6)| + l^2 - l - \sigma_{l+1}(S) \\
&= (2k+2m) + (k+m-2)^2 - (k+m-2) \\
&\quad -\{k(k+m-r+1) + (m-1)(k+m)\} \\
&= kr - 3k - 2m + 6 \\
&= (k-6)(r-3) - 6 \\
&> 0.
\end{aligned}$$

Thus, Theorem 3.25 cannot be applied to the graph G_6.

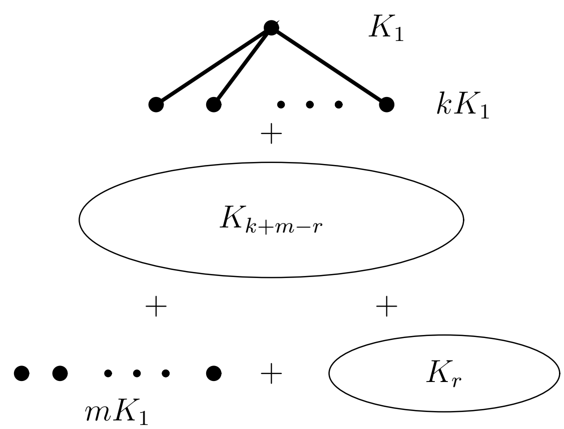


Figure 3.8: The graph G_7

The example which shows that Theorem 3.23 is not weaker than others is constructed by the similar way as G_6. Let k, r, m be integers such that $k \geq 5$, $r \geq 4$ and $m = 4(r-1)$. We consider the graph $G_7 = K_1 + kK_1 + K_{k+m-r} + (mK_1 + K_r)$ and let $S = K_1 \cup kK_1 \cup mK_1$. (See Figure 3.8.) Then $|V(G_7)| = 2k + 2m + 1$, $\kappa(S) = k$ and $\alpha(S) = k + m$. Since

$$\begin{aligned}
\sigma_4(S) &= \min\{k + 3(k+m),\ 4(k+m-r+1)\} \\
&= 4k + 3m \\
&= |V(G_7)| + \kappa(S) + \alpha(S) - 1,
\end{aligned}$$

the assumption of Theorem 3.23 holds. However,

$$\begin{aligned}\sigma_2^{\kappa(S)+1}(S) &= (k+m)+(k+m-r+1) = |V(G_7)|-r,\\ \sum_{i=1}^{3} d(x_i) &= 3(k+m-r+1) = |V(G_7)| + |\bigcap_{i=1}^{3} N(x_i)| - 2r+1,\end{aligned}$$

for $x_1, x_2, x_3 \in V(kK_1)$ and

$$\sigma_3(S) = k+2(k+m) = |V(G_7)|+\kappa(S)-1.$$

Therefore the assumptions of Theorems 3.11, 3.12 and 3.20 do not hold. Moreover for $l = \alpha(S)-1 = k+m-1$, since $k \geq 5$ and $r \geq 4$,

$$\begin{aligned}&|V(G_7)|+l^2-l-\sigma_{l+1}(S)\\ &= (2k+2m+1)+(k+m-1)(k+m-2)\\ &\quad -\{k(k+m-r+1)+m(k+m)\}\\ &= kr-2k-m+3\\ &= (k-4)(r-2)-1\\ &> 0.\end{aligned}$$

Hence the assumption of Theorem 3.25 does not hold.

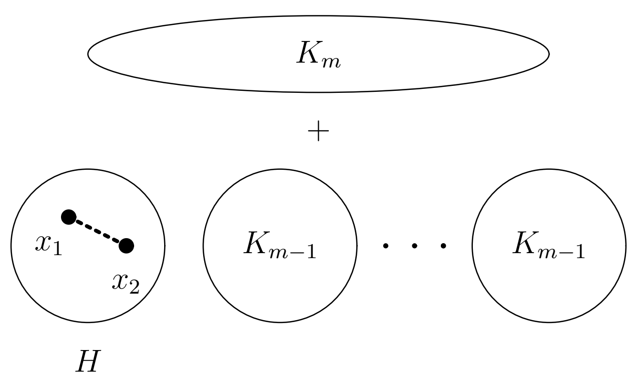


Figure 3.9: The graph G_8

Finally, we show that Theorem 3.25 is not implied by others. Let m be an integer with $m \geq 4$ and H be a graph obtained from K_{m-1} by removing an edge x_1x_2. We consider the graph $G_8 = K_m + (H \cup (m-1)K_{m-1})$. (See Figure 3.9.) Let $S = V(H) \cup (m-1)K_{m-1}$. Then $|V(G_8)| = m^2$, $\kappa(S) = m$ and $\alpha(S) = m+1$. For $l = \kappa(S) = m$, we have

$$\begin{aligned}\sigma_{l+1}(S) &= 2(2m-3)+(m-1)(2m-2)\\ &= |V(G_8)|+l^2-l+m-4\\ &\geq |V(G_8)|+l^2-l\end{aligned}$$

and for $l \geq \kappa(S)+1$, since $l+1 > \alpha(S)$, we have

$$\sigma_{l+1}(S) = +\infty > |V(G_8)| + l^2 - l,$$

and hence the assumption of Theorem 3.25 holds. On the other hand,

$$\begin{aligned}
\sigma_2^{\kappa(S)+1}(S) &= 2(2m-2) < |V(G_8)|, \\
\sum_{i=1}^{3} d(x_i) &= 2(2m-3) + (2m-1) < |V(G_8)| + |\bigcap_{i=1}^{3} N(x_i)|
\end{aligned}$$

for $x_1, x_2 \in V(H)$ and $x_3 \in V((m-1)K_{m-1})$ and

$$\begin{aligned}
\sigma_3(S) &= 2(2m-3) + (2m-2) < |V(G_8)| + \kappa(S), \\
\sigma_4(S) &= 2(2m-3) + 2(2m-2) < |V(G_8)| + \kappa(S) + \alpha(S) - 1.
\end{aligned}$$

Therefore the assumptions of Theorems 3.11, 3.12, 3.20 and 3.23 do not hold.

Thus, by these argument, it is proved that assumptions of Theorems 3.11, 3.12, 3.20, 3.23 and 3.25 are not able to compare each other.

3.4 Proof of Theorem 3.23

Let G be a graph and S a subset of $V(G)$ satisfying the assumption of Theorem 3.23. Let C be a cycle in G. If C contains all vertices of S, then there is nothing to prove. By Theorem 3.17, we may assume $\alpha(S) \geq \kappa(S)+1$ and $S \cap V(G-C) \neq \emptyset$, say $x_0 \in S \cap V(G-C)$. By Lemma 3.18, there exists $T \subseteq V(G)$ such that $|T| = \kappa(S)$ and T separates two vertices of S. Choose a cycle C, x_0 and an (x_0, C)-fan F so that

(C1) $|V(C) \cap S|$ is as large as possible;

(C2) $x_0 \notin T$ if possible, subject to (C1);

(C3) $|V(C) \cap V(F)|$ is as large as possible, subject to (C2);

(C4) $|V(F)|$ is as small as possible, subject to (C3).

By (C3), note that $|V(C) \cap V(F)| \geq \kappa(S) \geq 3$. Let P_i be a path of F connecting x_0 and u_i, where $u_i \in V(C) \cap V(F)$ $(1 \leq i \leq m)$. Let $x_i \in S$ be the first vertex from u_i along $\overrightarrow{C}$ for each $i = 1, 2, \ldots, m$. By (C1), $u_j \notin V(u_i \overrightarrow{C} x_i)$ and hence $x_i \neq x_j$ for $i \neq j$. Let $X := \{x_1, x_2, \ldots, x_m\}$ and H be a component of $G - C$ such that $x_0 \in V(H)$.

Claim 3.1 *$N_H(x_i) = \emptyset$ for $1 \leq i \leq m$.*

Proof. Suppose that $N_H(x_i) \neq \emptyset$. Then there exists a $(C \cup F)$-path Q connecting x_i and $v \in V(F)$. If $v \in V(P_j)$ $(j \neq i)$, then $C' = vQx_i\overrightarrow{C}u_iP_ix_0P_jv$ is a cycle containing $(V(C) \cap S) \cup \{x_0\}$, contradicting (C1).

Therefore we may assume that $v \in V(P_i)$. Let $C' = vQx_i\overrightarrow{C}u_iP_iv$ and $F' = \big(F - x_0P_iu_i\big) \cup x_0P_iv$. Then C' is a cycle with $V(C') \cap S = V(C) \cap S$, and F' is an (x_0, C')-fan with $|V(C) \cap V(F)| = |V(C') \cap V(F')|$ and $|V(F')| < |V(F)|$. This contradicts (C4). Hence $N_H(x_i) = \emptyset$ for $1 \leq i \leq m$. $\square$

By (C1), we obtain the following claim.

Claim 3.2 *For $1 \leq i \neq j \leq m$, the following statements hold.*

(i) For any $v \in V(u_j^+\overrightarrow{C}x_j)$, there exists no C-path connecting x_i and v.

(ii) For any $w_1 \in V(x_i^+\overrightarrow{C}u_j)$ and $w_2 \in V(x_i^+\overrightarrow{C}w_1^-)$ with $V(w_2^+\overrightarrow{C}w_1^-) \cap S = \emptyset$, if there exists a C-path connecting x_i and w_1, then there exists no C-path connecting x_j and w_2.

By Claims 3.1 and 3.2 (i), $X \cup \{x_0\}$ is an independent set in $G[S]$, and hence $|X| \leq \alpha(S) - 1$. By (C3), $d_C(x_0) \leq |X|$. Therefore we have

$$d_C(x_0) \leq \alpha(S) - 1. \tag{3.1}$$

Let $x_1, x_2, x_3 \in X$ be three distinct vertices such that x_1, x_2 and x_3 appear in the consecutive order along $\overrightarrow{C}$, where the indices are taken modulo 3. Let $D_i := u_i^+\overrightarrow{C}x_i^-$, $C_i := x_i\overrightarrow{C}u_{i+1}$, $W_i := \{w \in V(C_i) : w^+ \in N_{C_i}(x_i) \text{ and } w^- \in N_{C_i}(x_{i+1})\}$ for each $i = 1, 2, 3$ and let $W := W_1 \cup W_2 \cup W_3$. Note that $x_0, x_1, x_2, x_3 \notin W$.

Claim 3.3 *$W \subseteq S$. Moreover, if $x_0 \in T$, then $W \subseteq T$.*

Proof. Let $w \in W$. Without loss of generality, we may assume that $w \in W_1$. Then $x_1w^+\overrightarrow{C}u_2P_2x_0P_1u_1\overleftarrow{C}x_2w^-\overleftarrow{C}x_1$ is a cycle containing $((V(C) \cap S) \cup \{x_0\}) - \{w\}$. By (C1), we have $w \in S$. Therefore $W \subseteq S$. Moreover, if $x_0 \in T$, then $w \in T$ by (C2). Hence $W \subseteq T$. $\square$

By Claim 3.2 (i), we obtain

$$d_{D_i}(x_j) = 0 \quad \text{for } 1 \leq i \neq j \leq 3 \tag{3.2}$$

and hence

$$\sum_{j=1}^{3} d_{D_i}(x_j) \leq |V(D_i)| \quad \text{for } 1 \leq i \leq 3.$$

By Claim 3.2 (ii), $N_{C_i}(x_i)^- \cap N_{C_i}(x_{i+2}) = \emptyset$ and $N_{C_i}(x_{i+1})^+ \cap N_{C_i}(x_{i+2}) = \emptyset$ for $i = 1, 2, 3$. Clearly, $N_{C_i}(x_i)^- \cap N_{C_i}(x_{i+1})^+ = W_i$ and $N_{C_i}(x_i)^- \cup N_{C_i}(x_{i+1})^+ \cup N_{C_i}(x_{i+2}) \subseteq V(C_i) \cup \{u_{i+1}^+\}$. Therefore for $i = 1, 2, 3$,

$$d_{C_i}(x_1) + d_{C_i}(x_2) + d_{C_i}(x_3) \leq |C_i| + 1 + |W_i|.$$

Thus, we deduce

$$\begin{aligned} d_C(x_1) + d_C(x_2) + d_C(x_3) &= \sum_{i=1}^{3}\sum_{j=1}^{3}(d_{D_i}(x_j) + d_{C_i}(x_j)) \\ &\leq \sum_{i=1}^{3}(|V(D_i)| + |V(C_i)| + 1 + |W_i|) \\ &= |V(C)| + |W| + 3. \end{aligned} \tag{3.3}$$

By Claim 3.2 (i), $N_{G-C-H}(x_i) \cap N_{G-C-H}(x_j) = \emptyset$ for $1 \leq i \neq j \leq 3$. Therefore by Claim 3.1,

$$\begin{aligned} &d_{G-C}(x_0) + d_{G-C}(x_1) + d_{G-C}(x_2) + d_{G-C}(x_3) \\ &\leq |V(H) - \{x_0\}| + |V(G - C - H)| = |V(G - C)| - 1. \end{aligned} \tag{3.4}$$

Claim 3.4 $|X| \geq \kappa(S) + 1$.

Proof. By Claim 3.3, $W \subseteq S$. We prove that W is an independent set. Assume that there exist $w_1 \in W_i$ and $w_2 \in W_j$ with $w_1w_2 \in E(G)$. Suppose first that $i = j$. Without loss of generality, we may assume that $i = j = 1$, and w_1 and w_2 appear in this order along $\overrightarrow{C_1}$. Then $C' = x_1w_1^+\overrightarrow{C}w_2^-x_2\overrightarrow{C}u_1P_1x_0P_2u_2\overleftarrow{C}w_2w_1\overleftarrow{C}x_1$ is a cycle such that $|V(C') \cap S| > |V(C) \cap S|$, contradicting (C1). We may now assume that $i \neq j$. Without loss of generality, we may assume that $i = 1$ and $j = 2$. Then $x_1w_1^+\overrightarrow{C}u_2P_2x_0P_1u_1\overleftarrow{C}w_2^+x_2\overrightarrow{C}w_2w_1\overleftarrow{C}x_1$ is a cycle containing $(V(C) \cap S) \cup \{x_0\}$, a contradiction. Hence W is an independent set in $G[S]$. By Claim 3.2, $W \cup X \cup \{x_0\}$ is an independent set in $G[S]$. Since $x_0, x_1, x_2, x_3 \notin W$, we obtain $\alpha(S) \geq |W \cup X \cup \{x_0\}| \geq |W| + 4$, and hence $|W| \leq \alpha(S) - 4$.

By the inequality (3.3), we deduce

$$\begin{aligned} d_C(x_1) + d_C(x_2) + d_C(x_3) &\leq |V(C)| + |W| + 3 \\ &\leq |V(C)| + (\alpha(S) - 4) + 3 \\ &= |V(C)| + \alpha(S) - 1. \end{aligned}$$

Thus, it follows from the inequality (3.4) that $d_G(x_0)+d_G(x_1)+d_G(x_2)+d_G(x_3) \leq d_C(x_0)+n+\alpha(S)-2$. Since $\sigma_4(S) \geq n+\kappa(S)+\alpha(S)-1$, we have $d_C(x_0) \geq \kappa(S)+1$. Hence $|X| \geq \kappa(S) + 1$. $\square$

Let $U_1, U_2, \ldots, U_p$ be the components of $G - T$. We show that $|\{U_i : X \cap U_i \neq \emptyset\}| \leq 2$. Suppose that $|\{U_i : X \cap U_i \neq \emptyset\}| \geq 3$. Without loss of generality, we may assume that $x_i \in X \cap U_i$ for $i = 1, 2, 3$. By Claims 3.1 and 3.2 (i), we have

$$d_G(x_i) \leq |U_i| + |T| - |(U_i \cup T) \cap (V(H) \cup X)|.$$

Thus, by Claim 3.4, we obtain

$$\begin{aligned}
&d_G(x_1) + d_G(x_2) + d_G(x_3) \\
&\leq \sum_{i=1}^{3} |U_i| + 3|T| - \sum_{i=1}^{3} |(U_i \cup T) \cap (V(H) \cup X)| \\
&= n + 2|T| - \sum_{i=4}^{p} |U_i| - \sum_{i=1}^{3} |(U_i \cup T) \cap (V(H) \cup X)| \\
&\leq n + 2\kappa(S) - (|V(H)| + |X|) \\
&\leq n + \kappa(S) - |V(H)| - 1 \\
&\leq n + \kappa(S) - d_H(x_0) - 2.
\end{aligned}$$

By the inequality (3.1), $d_G(x_0) + d_G(x_1) + d_G(x_2) + d_G(x_3) \leq n + \kappa(S) + \alpha(S) - 3$, a contradiction. Hence, without loss of generality, we may assume that $X \cap \bigcup_{h=3}^{p} U_h = \emptyset$ and $|X \cap U_1| \geq |X \cap U_2|$.

Claim 3.5 $|W| \geq \kappa(S) - 2$.

Proof. Suppose that $|W| \leq \kappa(S) - 3$. By the inequality (3.3), we obtain

$$d_C(x_1) + d_C(x_2) + d_C(x_3) \leq |V(C)| + \kappa(S).$$

Hence the inequalities (3.1) and (3.4) yield

$$\begin{aligned}
&d_G(x_0) + d_G(x_1) + d_G(x_2) + d_G(x_3) \\
&\leq |V(C)| + \kappa(S) + \alpha(S) - 1 + |V(G - C)| - 1 \\
&\leq n + \kappa(S) + \alpha(S) - 2,
\end{aligned}$$

a contradiction. □

Claim 3.6 *$x_0 \notin T$ or $|X \cap T| \leq 1$.*

Proof. Suppose that $x_0 \in T$ and $|X \cap T| \geq 2$. By Claim 3.3, $W \subseteq T$. Since $|X \cap T| \geq 2$, we may assume that x_1, x_2 and x_3 are chosen so that $x_1, x_2 \in X \cap T$. Since $x_0, x_1, x_2 \in T - W$, we obtain $|W| \leq \kappa(S) - 3$, a contradiction. □

Claim 3.7 $|X \cap U_1| \geq 2$.

Proof. First we prove that $|X-T| \geq 2$. Suppose that $|X-T| \leq 1$. Then by Claim 3.4, note that $T \subseteq X$ and $|X-T|=1$. Since $G[V(C) \cup V(H)] - T$ is connected, we have $(C \cup H) - T \subseteq U_1$ and $\bigcup_{h=2}^{p} U_h \subseteq G - (C \cup H)$. Since T is a separating set of S, $U_i \cap S \neq \emptyset$ for some i, $2 \leq i \leq p$. Thus, $\kappa(S) \geq 3$ implies that $|N_C(U_i) \cap T| \geq 3$, that is, $|N_C(U_i) \cap X| \geq 3$. This contradicts Claim 3.2 (i). Therefore $|X-T| \geq 2$.

Suppose that $|X \cap U_1| \leq 1$, that is, $|X \cap U_1| = |X \cap U_2| = 1$ and $|X-T| = 2$. By symmetry, we can assume that x_1 and x_2 are chosen so that $x_1 \in X \cap U_1$ and $x_2 \in X \cap U_2$. By Claim 3.4, we have $|X \cap T| = |X| - |X-T| \geq (\kappa(S)+1) - 2 = \kappa(S) - 1 \geq 2$. Also, we have $|T-X| = |T| - |X \cap T| \leq \kappa(S) - \big(\kappa(S)-1\big) = 1$. Let Q_1 be an x_1x_2-path in $x_1 \overrightarrow{C} u_{\tau(1)} P_{\tau(1)} x_0 P_2 u_2 \overrightarrow{C} x_2$ and Q_2 be an x_2x_1-path in $x_2 \overrightarrow{C} u_{\tau(2)} P_{\tau(2)} x_0 P_1 u_1 \overrightarrow{C} x_1$, where $\tau(i)$ is an integer with $V(x_i^+ \overrightarrow{C} u_{\tau(i)}) \cap X = \emptyset$. Since $x_1 \in U_1$ and $x_2 \in U_2$, we have $V(Q_1) \cap T \neq \emptyset$ and $V(Q_2) \cap T \neq \emptyset$. Moreover, since $V(Q_1) \cap X = V(Q_2) \cap X = \{x_1, x_2\}$, we have $V(Q_1) \cap (T-X) \neq \emptyset$ and $V(Q_2) \cap (T-X) \neq \emptyset$. Since $|T-X| \leq 1$, we have $x_0 \in T$, which contradicts Claim 3.6. $\square$

Without loss of generality, we can assume $x_1, x_2 \in X \cap U_1$ and $x_3 \in X$. Since $x_1, x_2 \in U_1$, we have $N_{D_i}(x_i) \subseteq V(D_i) \cap (U_1 \cup T)$ for $i = 1, 2$. Therefore by the inequality (3.2), we obtain

$$d_{D_i}(x_1) + d_{D_i}(x_2) \leq |V(D_i) \cap U_1| + |V(D_i) \cap T| \quad \text{for } i = 1, 2. \tag{3.5}$$

Let $A_i := \{z \in V(C) \cap U_2 \colon z^+ \in N_C(x_i)\}$ for $i = 1, 2, 3$, and let $B_1 := \{z \in V(C) \cap U_2 \colon z^- \in N_C(x_1)\}$.

Claim 3.8 $X \subseteq U_1 \cup T$.

Proof. Suppose that $X \cap U_2 \neq \emptyset$. We may assume that $x_3 \in X \cap U_2$. By Claim 3.2 (ii), we obtain the following statements.

(I) $N_{C_1}(x_1)^-$ and $N_{C_1}(x_2)$ are disjoint, and $N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \subseteq V(C_1) \cap (U_1 \cup T \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.

(II) $N_{C_2}(x_2)^-$ and $N_{C_2}(x_1)$ are disjoint, and $N_{C_2}(x_2)^- \cup N_{C_2}(x_1) \subseteq V(C_2) \cap (U_1 \cup T \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.

(III) $N_{C_3}(x_1)^+$ and $N_{C_3}(x_2)$ are disjoint, and $N_{C_3}(x_1)^+ \cup N_{C_3}(x_2) \subseteq (V(C_3) \cap (U_1 \cup T \cup B_1 \cup \bigcup_{h=3}^{p} U_h)) \cup \{u_1^+\}$.

Let $A := (V(C_1) \cap A_1) \cup (V(C_2) \cap A_2) \cup (V(C_3) \cap B_1)$. By (I)–(III) and by the inequalities (3.2) and (3.5), we obtain

$$d_C(x_1) + d_C(x_2) \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T| + |A| + 1.$$

On the other hand, by Claim 3.2 (ii), x_3 is not adjacent to any vertex of A. Thus, we have

$$d_C(x_3) \leq |V(C) \cap U_2| + |V(C) \cap T| - |A| - 1,$$

since $x_3 \notin A$. Thus $d_C(x_1)+d_C(x_2)+d_C(x_3) \leq |V(C)|+|V(C)\cap T| \leq |V(C)|+\kappa(S)$. Therefore, by the inequalities (3.1) and (3.4),

$$\begin{aligned} & d_G(x_0) + d_G(x_1) + d_G(x_2) + d_G(x_3) \\ & \leq |V(C)| + \kappa(S) + \alpha(S) - 1 + |V(G-C)| - 1 \\ & = n + \kappa(S) + \alpha(S) - 2, \end{aligned}$$

a contradiction. □

Claim 3.9 $x_0 \in U_1$.

Proof. By Claims 3.4 and 3.8, there exist $|X| \geq \kappa(S)+1$ paths connecting x_0 and each vertex in $X \subseteq U_1 \cup T$, and hence $x_0 \in U_1 \cup T$. Suppose that $x_0 \in T$. Note that $W \subseteq T$ by Claim 3.3. By (C2), $V(G-C) \cap U_2 \cap S = \emptyset$, otherwise we can choose a vertex in $V(G-C) \cap U_2 \cap S$ instead of x_0. Let $y \in V(C) \cap U_2 \cap S$. Then by Claim 3.8, there exist $t_1, t_2 \in V(C) \cap T$ such that $y \in V(t_1^+ \overrightarrow{C} t_2^-)$ and $V(t_1^+ \overrightarrow{C} t_2^-) \subseteq U_2$, because $x_1, x_2 \in X \cap U_1$.

By Claim 3.8, $X \cap U_1 = X - T$. By Claim 3.6, $|X \cap T| \leq 1$, and hence $|X - T| = |X| - |X \cap T| \geq \kappa(S) \geq 3$. Thus we have $|X \cap U_1| \geq 3$. Therefore we may assume that $x_3 \in X \cap U_1$. Since $x_1, x_2 \in X \cap U_1$ and $t_1^+, t_2^- \in U_2$, we have $t_1, t_2 \in T - W$. Thus, $|W| \leq |T - \{x_0, t_1, t_2\}| = \kappa(S) - 3$, contradicting Claim 3.5. □

By Claims 3.1 and 3.2 (i), $N_H(x_i) = \emptyset$ for $i = 1, 2$ and $N_{G-C}(x_1) \cap N_{G-C}(x_2) = \emptyset$. Therefore $d_{G-C}(x_1) + d_{G-C}(x_2) \leq |V(G-C-H) \cap (U_1 \cup T)|$. By Claim 3.9, we have $d_{G-C}(x_0) \leq |V(H) \cap (U_1 \cup T)| - 1$. Thus,

$$\begin{aligned} & d_{G-C}(x_0) + d_{G-C}(x_1) + d_{G-C}(x_2) \\ & \leq |V(G-C) \cap U_1| + |V(G-C) \cap T| - 1. \end{aligned} \tag{3.6}$$

Let $y_0 \in U_2 \cap S$. Then

$$d_G(y_0) \leq |U_2| + |T| - 1 = |U_2| + \kappa(S) - 1, \tag{3.7}$$

and $y_0 \notin N_G(x_0) \cup N_G(x_1) \cup N_G(x_2)$ by Claim 3.9. Let $C_1' := C_1 = x_1 \overrightarrow{C} u_2$ and $C_2' := C_2 \cup D_3 \cup C_3 = x_2 \overrightarrow{C} u_1$.

Based on the results of the previous claims, the proof is completed by considering two cases for the cardinality of $X \cap U_1$: $|X \cap U_1| = 2$ and $|X \cap U_1| = 3$.

Case 1. $|X \cap U_1| = 2$.

By the definition of A_i and Claim 3.2 (i), for $i = 1, 2$, $A_i^+ \subseteq T$ and $A_i^+ \cap X = \emptyset$, and hence $A_i^+ \subseteq T - X$. Moreover, by Claims 3.4 and 3.8 and by the assumption of Case 1, $\kappa(S) - 1 \leq |X \cap T|$. Hence we have $|V(C_1') \cap A_1| + |V(C_2') \cap A_2| \leq |T - X| = |T| - |X \cap T| \leq 1$.

By Claim 3.2 (ii), we obtain the following statements.

(I) $N_{C_1'}(x_1)^-$ and $N_{C_1'}(x_2)$ are disjoint, and $N_{C_1'}(x_1)^- \cup N_{C_1'}(x_2) \subseteq V(C_1') \cap (U_1 \cup T \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.

(II) $N_{C_2'}(x_2)^-$ and $N_{C_2'}(x_1)$ are disjoint, and $N_{C_2'}(x_2)^- \cup N_{C_2'}(x_1) \subseteq V(C_2') \cap (U_1 \cup T \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.

By (I) and (II) and by the inequality (3.5), we have

$$\begin{aligned} & d_C(x_1) + d_C(x_2) \\ & \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T| + |V(C_1') \cap A_1| + |V(C_2') \cap A_2| \\ & \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T| + 1. \end{aligned}$$

Combining with the inequalities (3.1) and (3.6), we obtain $d_G(x_0) + d_G(x_1) + d_G(x_2) \leq \sum_{h \neq 2} |U_h| + |T| + \alpha(S) - 1$. Then by the inequality (3.7), we have $d_G(x_0) + d_G(y_0) + d_G(x_1) + d_G(x_2) \leq |V(G)| + \kappa(S) + \alpha(S) - 2$, a contradiction. □

Case 2. $|X \cap U_1| \geq 3$.

We may assume that $x_3 \in X \cap U_1$. For each $z \in A_i$, we define $\tilde{z}$ to be the vertex satisfying $\tilde{z} \in V(C) \cap T$ and $V(\tilde{z}^+ \overrightarrow{C} z) \subseteq U_2$. Since $x_i \in U_1$, note that $\tilde{z} \in V(x_i^+ \overrightarrow{C} z^-)$ for $i = 1, 2, 3$. Let $\tilde{A}_i = \{\tilde{z} \colon z \in A_i\}$ for $i = 1, 2, 3$.

Claim 3.10 *Let $z \in A_i$. If $|X \cap U_1 \cap V(z^+ \overrightarrow{C} u_i)| \geq 2$, then $V(\tilde{z}^+ \overrightarrow{C} z) \cap S = \emptyset$.*

Proof. By symmetry, we may assume that there exists $z_3 \in A_3$ such that $|X \cap U_1 \cap V(z_3^+ \overrightarrow{C} u_3)| \geq 2$ and $V(\tilde{z}_3^+ \overrightarrow{C} z_3) \cap S \neq \emptyset$. Let $y_3 \in V(\tilde{z}_3^+ \overrightarrow{C} z_3) \cap S$. Choose y_3 so that $|V(y_3 \overrightarrow{C} z_3)|$ is as small as possible. Then note that $y_3 \in U_2$. Since $|X \cap U_1 \cap V(z_3^+ \overrightarrow{C} u_3)| \geq 2$, we may assume that $x_1, x_2 \in X \cap U_1 \cap V(z_3^+ \overrightarrow{C} u_3)$. We partition C_3 into F_1, F_2, F_3 so that $F_1 := x_3 \overrightarrow{C} \tilde{z}_3$ $F_2 := \tilde{z}_3^+ \overrightarrow{C} z_3$ and $F_3 := z_3^+ \overrightarrow{C} u_1$. Note that $V(F_2) \subseteq U_2$ and x_i has no neighbors in U_2 for $i = 1, 2$.

By Claim 3.2 (ii), we obtain the following statements.

(I) $N_{C_1}(x_1)^-$ and $N_{C_1}(x_2)$ are disjoint, and $N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \subseteq V(C_1) \cap (U_1 \cup T \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.

(II) $N_{C_2}(x_2)^-$ and $N_{C_2}(x_1)$ are disjoint, and $N_{C_2}(x_2)^- \cup N_{C_2}(x_1) \subseteq V(C_2) \cap (U_1 \cup T \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.

(III) $N_{F_1}(x_2)^-$ and $N_{F_1}(x_1)$ are disjoint, and $N_{F_1}(x_2)^- \cup N_{F_1}(x_1) \subseteq V(F_1) \cap (U_1 \cup T \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.

(IV) $N_{F_2}(x_i) = \emptyset$ for $i = 1, 2$.

(V) $N_{F_3}(x_1)^+$ and $N_{F_3}(x_2)$ are disjoint, and $N_{F_3}(x_1)^+ \cup N_{F_3}(x_2) \subseteq (V(F_3) \cap (U_1 \cup T \cup B_1 \cup \bigcup_{h=3}^{p} U_h)) \cup \{u_1^+\}$.

Let $A' := (V(C_1) \cap A_1) \cup (V(C_2) \cap A_2) \cup (V(F_1) \cap A_2) \cup (V(F_3) \cap B_1)$. By (I)–(V) and by the inequalities (3.2) and (3.5), we obtain

$$d_C(x_1) + d_C(x_2) \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap V(T)| + |A'| + 1.$$

Suppose that $A' \cap N_C(y_3) \neq \emptyset$, say $z \in A' \cap N_C(y_3)$. Let

$$C' = \begin{cases} x_1 z^+ \overrightarrow{C} u_3 P_3 x_0 P_1 u_1 \overleftarrow{C} z_3^+ x_3 \overrightarrow{C} y_3 z \overleftarrow{C} x_1 & \text{if } z \in V(C_1) \cap A_1, \\ x_2 z^+ \overrightarrow{C} u_3 P_3 x_0 P_1 u_2 \overleftarrow{C} z_3^+ x_3 \overrightarrow{C} y_3 z \overleftarrow{C} x_2 & \text{if } z \in V(C_2) \cap A_2, \\ x_2 \overrightarrow{C} u_3 P_3 x_0 P_2 u_2 \overleftarrow{C} z_3^+ x_3 \overrightarrow{C} z y_3 \overleftarrow{C} z^+ x_2 & \text{if } z \in V(F_1) \cap A_2, \\ x_1 \overrightarrow{C} u_3 P_3 x_0 P_1 u_1 \overleftarrow{C} z y_3 \overleftarrow{C} x_3 z_3^+ \overrightarrow{C} z^- x_1 & \text{if } z \in V(F_3) \cap B_1. \end{cases}$$

Note that by the choice of y_3, there are no vertices of S between y_3 and z_3. Then C' is a cycle containing $(V(C) \cap S) \cup \{x_0\}$, a contradiction. Hence $A' \cap N_C(y_3) = \emptyset$. Moreover, by the definition of A', we have $y_3 \notin A'$. Therefore we obtain

$$d_C(y_3) \leq |V(C) \cap U_2| + |V(C) \cap T| - |A'| - 1,$$

which implies

$$\begin{aligned} d_C(x_1) + d_C(x_2) + d_C(y_3) &\leq |V(C)| + |V(C) \cap T| \\ &\leq |V(C)| + \kappa(S). \end{aligned} \tag{3.8}$$

By Claim 3.2 (ii), $N_{G-C}(x_i) \cap N_{G-C}(y_3) = \emptyset$ for $i = 1, 2$. On the other hand, by a similar argument as in the proof of Claim 3.1, we obtain $N_H(y_3) = \emptyset$. Hence

$$d_{G-C}(x_0) + d_{G-C}(x_1) + d_{G-C}(x_2) + d_{G-C}(y_3) \leq |V(G-C)| - 1. \tag{3.9}$$

Therefore, by the inequalities (3.1), (3.8) and (3.9),

$$\begin{aligned} & d_G(x_0) + d_G(x_1) + d_G(x_2) + d_G(y_3) \\ \leq \;& |V(C)| + \kappa(S) + \alpha(S) - 1 + |V(G-C)| - 1 \\ \leq \;& n + \kappa(S) + \alpha(S) - 2, \end{aligned}$$

a contradiction. □

Claim 3.11 *Let $z \in A_i$. If $|X \cap U_1 \cap V(z^+ \overrightarrow{C} u_i)| \geq 2$, then $\tilde{z} \notin N_C(x_i)^- \cup N_C(x_j)$ for any $x_j \in X \cap U_1 \cap V(z^+ \overrightarrow{C} u_i)$.*

Proof. By Claim 3.10, $V(\tilde{z}^+ \overrightarrow{C} z) \cap S = \emptyset$. Hence, by Claim 3.2 (ii), we have $\tilde{z} \notin N_C(x_j)$. On the other hand, since $x_i \in U_1$ and $\tilde{z}^+ \in U_2$, we have $\tilde{z} \notin N_C(x_i)^-$. Thus, we obtain $\tilde{z} \notin N_C(x_i)^- \cup N_C(x_j)$. □

Case 2.1. $|X \cap U_1| = 3$.

By Claims 3.4 and 3.8, we have $|T - X| = |T| - |T \cap X| = |T| - (|X| - |X \cap U_1|) \leq \kappa(S) - (\kappa(S) + 1 - 3) = 2$. Therefore there exists an index i such that $V(C_i) \cap (T - X) = \emptyset$. By symmetry, we may assume that $i = 3$. Then by the definition of A_2, $V(C_3) \cap A_2 = \emptyset$. Recall that $\tilde{A}_i \subseteq T$. By Claims 3.2 (ii) and 3.11, we obtain

(I) $N_{C_1}(x_1)^-$ and $N_{C_1}(x_2)$ are disjoint, and $N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \subseteq V(C_1) \cap (U_1 \cup (T - \tilde{A}_1) \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.

(II) $N_{C_2}(x_2)^-$ and $N_{C_2}(x_1)$ are disjoint, and $N_{C_2}(x_2)^- \cup N_{C_2}(x_1) \subseteq V(C_2) \cap (U_1 \cup (T - \tilde{A}_2) \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.

(III) $N_{C_3}(x_2)^-$ and $N_{C_3}(x_1)$ are disjoint, and $N_{C_3}(x_2)^- \cup N_{C_3}(x_1) \subseteq V(C_3) \cap (U_1 \cup T)$.

By (I)–(III) and by the inequalities (3.2) and (3.5), we have

$$d_C(x_1) + d_C(x_2) \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T|.$$

By the inequalities (3.1), (3.6) and (3.7), $d_G(x_0) + d_G(y_0) + d_G(x_1) + d_G(x_2) \leq |V(G)| + \kappa(S) + \alpha(S) - 3$, a contradiction. □

Case 2.2. $|X \cap U_1| \geq 4$.

Since $|X \cap U_1| \geq 4$, we can choose $x_1, x_2 \in X \cap U_1$ so that $V(x_1^+ \overrightarrow{C} x_2^-) \cap X \cap U_1 \neq \emptyset$ and $V(x_2^+ \overrightarrow{C} x_1^-) \cap X \cap U_1 \neq \emptyset$. By Claims 3.2 (ii) and 3.11, we obtain

(I) $N_{C_1'}(x_1)^-$ and $N_{C_1'}(x_2)$ are disjoint, and $N_{C_1'}(x_1)^- \cup N_{C_1'}(x_2) \subseteq V(C_1') \cap (U_1 \cup (T - \tilde{A}_1) \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.

(II) $N_{C_2'}(x_2)^-$ and $N_{C_2'}(x_1)$ are disjoint, and $N_{C_2'}(x_2)^- \cup N_{C_2'}(x_1) \subseteq V(C_2') \cap (U_1 \cup (T - \tilde{A}_2) \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.

By (I) and (II) and by the inequality (3.5), we obtain

$$d_C(x_1) + d_C(x_2) \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T|.$$

By the inequalities (3.1), (3.6) and (3.7), $d_G(x_0) + d_G(y_0) + d_G(x_1) + d_G(x_2) \leq |V(G)| + \kappa(S) + \alpha(S) - 3$, a contradiction. $\square$

Chapter 4

Dominating cycles

A dominating cycle has some good properties like a hamilton cycle, so we often deal with it as a "pre-hamilton" cycle. For example, we sometimes use the existence of a dominating cycle in order to find a hamilton cycle. In this sense, a topic on a dominating cycle is one of the most important relaxations of a hamilton cycle. Like results on a hamilton cycle, we consider a degree condition or an independence number condition for the existence of a dominating cycle. In Sections 4.1 and 4.2, we concentrate on such sufficient conditions. In particular, in Section 4.2, we consider a triangle-free graph, that is a graph having no triangles. Since any bipartite graph is trivially triangle-free, we are interested in a class of triangle-free graphs. As one of the results on a dominating cycle of a triangle-free graph, we prove Theorem 4.14 in Section 4.2.2.

The contents of this chapter are based on the paper [136] "Dominating cycles in triangle-free graphs," jointwork with T. Yamashita.

4.1 Results on dominating cycles

A cycle C is *dominating* if for every edge uv, $u \in V(C)$ or $v \in V(C)$. Clearly a hamilton cycle is a dominating cycle but the converse does not hold. In this section, we study on a dominating cycle, in particular a degree condition of it. Bondy showed the following theorem, which is a generalization of a result on a minimum degree condition by Nash-Williams [127].

Theorem 4.1 (Bondy [26]) *Let G be a 2-connected graph of order n. If $\sigma_3(G) \geq n+2$, then each longest cycle in G is dominating.*

Theorem 4.2 (Nash-Williams [127]) *Let G be a 2-connected graph of order n. If $\delta(G) \geq \frac{1}{3}(n+2)$, then each longest cycle in G is dominating.*

Let $m \geq 2$ and $G_1 = mK_1 + (m+1)K_2$. Then $|V(G_1)| = 3m+2$ and $\sigma_3(G_1) = 3(m+1) = |V(G_1)| + 1$. Since any longest cycle in G_1 is not dominating, the lower bound $n+2$ of Theorem 4.1 is best possible. Including this graph G_9, Bauer, Schmeichel and Veldman [18] characterized all 2-connected graph of order n with $\sigma_3(G) \geq n$ that has a longest cycle in G which is not dominating.

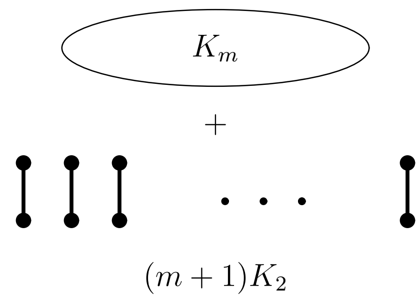


Figure 4.1: The graph G_1

Lu, Liu and Tian considered a $\sigma_4(G)$ condition and they proved the following theorem on a 3-connected graph.

Theorem 4.3 (Lu, Liu and Tian [116]) *Let G be a 3-connected graph of order n. If $\sigma_4(G) \geq \frac{4}{3}n + \frac{5}{3}$, then each longest cycle in G is dominating.*

The graph G_1 with $m \geq 3$ also gives an example that the condition of Theorem 4.3 is sharp, again. Since $\sigma_4(G_1) = 4(m+1) = \frac{4}{3}|V(G_1)| + \frac{4}{3}$, we cannot relax the bound $\frac{4}{3}n + \frac{5}{3}$.

If $\sigma_3(G) \geq n+2$, then by Proposition 2.1 (i) we obtain $\sigma_4(G) \geq \frac{4}{3}\sigma_3(G) \geq \frac{4}{3}n+\frac{8}{3}$. Therefore the graph satisfying the condition in Theorem 4.1 also satisfies the one in Theorem 4.3 (except for the connectivity condition.) Hence Theorem 4.3 is a generalization of Theorem 4.1, in a sense.

Tsugaki and Yamashita showed the following. By Proposition 2.1 (ii), Theorem 4.4 is stronger than Theorem 4.3.

Theorem 4.4 (Tsugaki and Yamashita [159]) *Let G be a 2-connected graph of order n. If $\sigma_3^{\kappa(G)+1}(G) \geq n+2$, then each longest cycle in G is dominating.*

On the other hand, similarly to the improvement of a $\sigma_2(G)$ condition to a $\sigma_3(G)$ condition for a hamilton cycle, we consider a degree condition with the connectivity. Sun, Tian and Wei [150] showed that for a 3-connected graph G of order n, if $\sigma_4(G) \geq n + 2\kappa(G)$, then there exists a longest cycle in G which is dominating. Lu, Liu and Tian [116] improved this conclusion to "each longest cycles in G is dominating." When $\kappa(G) = 3$, again the graph G_1 with $m = 3$ showed that the condition of these results is best possible. But if $\kappa(G) \geq 4$, it was unknown whether the lower bound is sharp or not. Motivated by this fact, Yamashita [175] improved of the result by Sun, Tian and Wei; if $\sigma_4(G) \geq n+\kappa(G)+3$ for a 3-connected graph of order n, then there exists a longest cycle in G which is dominating. Recently, Yamashita [177] showed the following result, which is a common generalization of above three theorems.

Theorem 4.5 (Yamashita [177]) *Let G be a 3-connected graph of order n. If $\sigma_4(G) \geq n + \kappa(G) + 3$, then each longest cycles in G is dominating.*

Since $\sigma_4(G_1) = 4(m+1) = (3m+2) + m + 2 = |V(G_1)| + \kappa(G_1) + 2$, the lower bound of Theorem 4.5 is best possible.

On the other hand, Jackson, Li and Zhu [90] considered the relationship between dominating cycles and regular graphs. They showed that for a 3-connected d-regular graph of order n, if $d \geq \frac{n}{4}$, then each longest cycle in G is dominating.

4.2 Dominating cycles in triangle-free graphs

4.2.1 Results

There exists another approach for dominating cycles. The degree condition of Theorems 4.1 or 4.2 is best possible in a sense. But if we restrict ourselves to a particular class of graphs, then the lower bound of a degree condition may be able to decrease. As one of the class of graphs, we consider triangle-free graphs in this section. A graph G is said to be *triangle-free*, if G has no triangles. Note that any bipartite graph is triangle-free, so a class of triangle-free graphs is important. Aung proved the following result.

Theorem 4.6 (Aung [10]) *Let G be a 2-connected triangle-free graph of order n. If $\delta(G) \geq \frac{1}{6}(n+6)$, then there exists a longest cycle in G which is dominating.*

Comparing Theorems 4.2 and 4.6, we obtain that if we restrict ourselves to triangle-free graphs, we can decrease the bound of the minimum degree which guarantees the existence of a dominating cycle.

On the other hand, Veldman considered an *edge degree* condition in general graphs, instead of a (vertex) degree condition. For $e = uv \in E(G)$, the edge degree $d(e)$ is defined as the number of neighborhoods of e, that is, $d(e) := |N_G(u) \cup N_G(v) - \{u,v\}| = |N_G(u) \cup N_G(v)| - 2$. Note that for a triangle-free graph G and an edge $e = uv$ of G, $d(e) = d_G(u) + d_G(v) - 2$ because $N_G(u) \cap N_G(v) = \emptyset$. Two edges are called *remote* if there are no edges joining an end-vertex of one of the edges and an end-vertex of the other edge. (See Figure 4.2.) Veldman proved the following theorem.

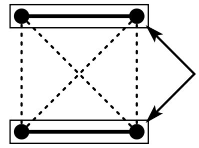


Figure 4.2: Remote edges

Theorem 4.7 (Veldman [160]) *Let G be a k-connected graph of order n. If for any $k+1$ pairwise remote edges $e_0, e_1, \cdots, e_k$, $\sum_{l=0}^{k} d(e_l) \geq \frac{1}{2}k(n-k)+1$, then G has a dominating cycle.*

Theorem 4.7 does not guarantee the existence of a longest cycle which is dominating. But Broersma, Yoshimoto and Zhang improved this theorem for the case where $k = 2$.

Theorem 4.8 (Broersma, Yoshimoto and Zhang [32]) *Let G be a 2-connected graph of order n. If $d(e_0)+d(e_1)+d(e_2) \geq n-1$ for any pairwise remote edges e_0, e_1, e_2, then there exists a longest cycle in G which is dominating.*

By proving the following proposition, we refer to concerning between Theorems 4.6 and 4.8.

Proposition 4.9 *Let G be a triangle-free graph. If $\sigma_3(G) \geq \frac{1}{2}(n+5)$, then $d(e_0)+d(e_1)+d(e_2) \geq n-1$ for any pairwise remote edges e_0, e_1, e_2.*

Proof. Suppose that G is triangle-free. Then since $N(u) \cap N(v) = \emptyset$ for every $e = uv \in E(G)$, we have $d(e) = d(u)+d(v)-2$. On the other hand, for any pairwise remote edges e_0, e_1, e_2, say $e_i = u_i v_i$ $(i = 0,1,2)$, $\{u_i : i = 0,1,2\}$ and $\{v_i : i = 0,1,2\}$ are independent sets, respectively. Therefore if $\sigma_3(G) \geq \frac{1}{2}(n+5)$, then we have

$$\begin{aligned} d(e_0)+d(e_1)+d(e_2) &= \sum_{i=0}^{2} d(u_i) + \sum_{i=0}^{2} d(v_i) - 6 \\ &\geq 2\sigma_3(G) - 6 \\ &\geq n-1. \qquad \square \end{aligned}$$

Therefore we have the following result as a corollary of Theorem 4.8, which is an improvement of Theorem 4.6.

Corollary 4.10 *Let G be a 2-connected triangle-free graph of order n. If $\sigma_3(G) \geq \frac{1}{2}(n+5)$, then there exists a longest cycle in G which is dominating.*

Wang [163] constructed the following triangle-free graph, which shows the lower bounds of the degree sum of three pairwise remote edges condition in Theorem 4.8

and of a $\sigma_3(G)$ condition in Corollary 4.10 are best possible. Let m be an integer with $m \geq 1$ and H be a graph with $V(H) = \{u_1, u_2\}$ and $E(H) = \emptyset$. We consider the graph G_2 obtained from $3K_{m,m} \cup H$ by joining u_1 and one of the partite sets of $3K_{m,m}$, u_2 and another partite set, respectively, like a figure below. (See Figure 4.3.) Then $|V(G_2)| = 6m + 2$. If we choose three edges which are pairwise remote, then we must take an edge from each $K_{m,m}$. Thus, for any pairwise remote edges e_0, e_1, e_2, we have $d(e_0) + d(e_1) + d(e_2) = 3(2(m-1)+2) = 6m = |V(G_2)| - 2$. Furthermore there is no longest cycle which is dominating, hence the lower bound $n-1$ of Theorem 4.8 and then $\frac{1}{2}(n+5)$ of Corollary 4.10 are best possible.

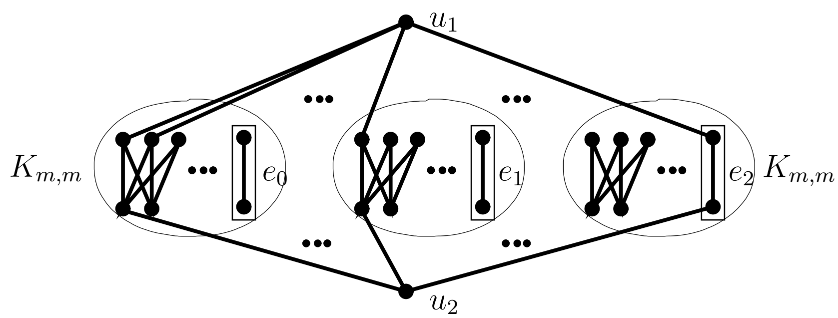


Figure 4.3: The graph G_2

Furthermore, the following graph G_3, which was constructed by Ash and Jackson [8], shows that in Theorem 4.8, we cannot replace the conclusion "there exists a longest cycle in G which is dominating," with "any longest cycle in G is dominating." Let $m \geq 2$ and H_1 and H_2 be vertex-disjoint $K_{m,m+2}$'s. Let X_i and Y_i be the partite sets of H_i with $|X_i| = m$ and $|Y_i| = m+2$, respectively. We choose $\{y_0^i, y_1^i, y_2^i\} \subseteq Y_i$ $(i = 1, 2)$ and consider the graph G_3 obtained from $H_1 \cup H_2$ by joining y_j^1 and y_j^2 $(j = 0, 1, 2)$, respectively. (See Figure 4.4.)

If we choose three edges e_0, e_1, e_2 which are pairwise remote, then we must take $e_j = y_j^1 y_j^2$ $(j = 0, 1, 2)$. Since $|V(G_3)| = 4m + 4$ and $d(e_0) + d(e_1) + d(e_2) = 6m \geq |V(G_3)| - 1$, by Theorem 4.8, there exists a longest cycle in G_3 which is dominating. However, there also exists a longest cycle which is not dominating.

Yoshimoto considered that if we decrease a degree condition of Theorem 4.8, then we can replace the conclusion to "any longest cycle is dominating."

Theorem 4.11 (Yoshimoto [178]) *Let G be a 2-connected graph of order n. If $d(e_1) + d(e_2) \geq n - 3$ for any remote edges e_1, e_2, then each longest cycle in G is dominating.*

For any remote edges e_1, e_2 of G_3, $d(e_1) + d(e_2) = 4m = |V(G_3)| - 4$. Therefore the lower bound of Theorem 4.11 is best possible. The following corollary on a

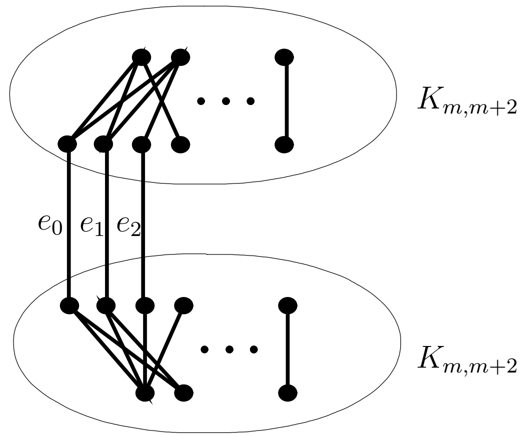


Figure 4.4: The graph G_3

$\sigma_2(G)$ condition is obtained from Theorem 4.11 using the proof of Proposition 4.9.

Corollary 4.12 *Let G be a 2-connected triangle-free graph of order n. If $\sigma_2(G) \geq \frac{1}{2}(n+1)$, then each longest cycle in G is dominating.*

Like Theorems 4.1–4.5, there are several results concerning a dominating cycle and the degree conditions. On the other hand, Chvátal and Erdős [37] gave an independence number condition for the existence of a hamilton cycle; any graph G with $\alpha(G) \leq \kappa(G)$ has a hamilton cycle. So one might expect that we can decrease the upper bound of $\alpha(G)$ condition for a dominating cycle like degree conditions. But it is impossible. In order to show that, we construct infinite many graphs as follows; Let k, m be nonnegative integers with $m \geq 2$ and we consider the graph $G_4 = K_k + (k+1)K_m$. (See Figure 4.5.) Then $\alpha(G_4) = k+1 = \kappa(G_4) + 1$ and G_4 has no dominating cycles.

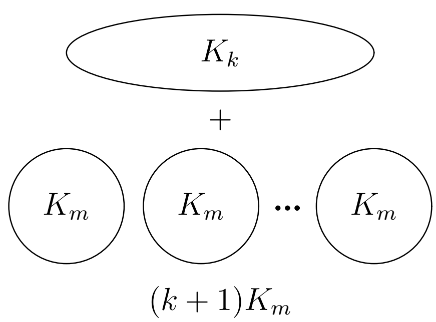


Figure 4.5: The graph G_4

Motivated by the above reason, when we consider an independence number condition for a dominating cycle, it is necessary to restrict ourselves to some particular classes of graphs, at least we must avoid some graphs like G_4. Enomoto, Kaneko, Saito and Wei consider a class of triangle-free graphs and gave an independence number condition for a dominating cycle.

Theorem 4.13 (Enomoto, Kaneko, Saito and Wei [48]) *Let G be a 2-connected triangle-free graph. If $\alpha(G) \le 2\kappa(G) - 2$, then every longest cycle in G is dominating.*

They also showed that the condition of Theorem 4.13 cannot be replaced with "$\alpha(G) \le 2\kappa(G)$" by giving a triangle-free graph G with $\alpha(G) = 2\kappa(G)$ that does not have a dominating cycle. But for a graph G with $\alpha(G) = 2\kappa(G) - 1$, we do not know whether we can guarantee any longest cycle is a dominating cycle. Motivated by the reason, we show the existence of a dominating cycle in a set of longest cycles of a graph satisfying $\alpha(G) \le 2\kappa(G) - 1$.

Theorem 4.14 ([136]) *Let G be a 2-connected triangle-free graph. If $\alpha(G) \le 2\kappa(G) - 1$, then there exists a longest cycle in G which is dominating.*

In the next section, we show Theorem 4.14.

4.2.2 Proof of Theorem 4.14

A graph G is said to be *k-path-connected* if for every pair of distinct vertices u and v, there exists a uv-path of length at least k. The following lemma is used in the proof of Claim 4.4.

Lemma 4.15 ([48, Lemma 3]) *Let G be a 2-connected triangle-free graph and let $x_0 \in V(G)$. If $d_G(x) \ge 3$ for each $x \in V(G) - \{x_0\}$, then G is 4-path-connected.*

Proof of Theorem 4.14.

Take a longest cycle C so that $|E(G-C)|$ is as small as possible. If $E(G-C) = \emptyset$, then there is nothing to prove. Therefore we may assume $E(G-C) \ne \emptyset$, and let H be a component of $G - C$ with $|V(H)| \ge 2$.

Since C is a longest cycle, the following fact holds.

Fact 4.1 (i) *$N_C(H) \cap N_C(H)^+ = \emptyset$.*

(ii) *There exists no C-path joining two vertices of $N_C(H)^+$ (or $N_C(H)^-$).*

(iii) *Let $u_i \in V(H)$ and $x_i \in N_C(u_i)$ for $i = 1, 2$. If $x_1 \ne x_2$ and there exists a u_1u_2-path in H of length at least k, then $x_1^{+k} \ne x_2$ and $x_1^{+l}x_2^{+m} \notin E(G)$ for any positive integers l, m with $l + m = k + 2$.*

By Fact 4.1, we obtain the following fact.

Fact 4.2 *Let S be an independent set in H. Then $S \cup N_C(H)^+$ is independent. In particular, for every $v \in V(H)$, $N_H(v) \cup N_C(H)^+$ is an independent set.*

We will show several claims concerning the structure of H.

Claim 4.3 *For every $u \in V(H)$, $N_C(u) \neq \emptyset$.*

Proof. By Fact 4.2, for every $u \in V(H)$, $N_H(u) \cup N_C(H)^+$ is an independent set. Since $|N_C(H)^+| = |N_C(H)| \geq \kappa(G)$, we have

$$\begin{aligned} 2\kappa(G) - 1 \ \geq \ \alpha(G) \ &\geq \ |N_H(u) \cup N_C(H)^+| \\ &\geq \ d_H(u) + \kappa(G). \end{aligned}$$

This implies $d_H(u) \leq \kappa(G) - 1$. Therefore, we obtain $d_C(u) = d_G(u) - d_H(u) \geq \kappa(G) - (\kappa(G) - 1) = 1$. $\square$

Claim 4.4 $\delta(H) \leq 2$.

Proof. We use a similar argument as the proof of Theorem 1 in [48]. Suppose $\delta(H) \geq 3$.

Case 1. H is not 2-connected.

Let B be an end block of H, $c_B \in V(B)$ be the cut-vertex of H and B' be an end block of H other than B. Since $d_B(x) \geq 3$ for every $x \in V(B) - \{c_B\}$, by Lemma 4.15, B is 4-path-connected. Let $T := N_C(B - \{c_B\})$,

$$\begin{aligned} T_0 := \big\{x \in T \ : \ &N_{B-\{c_B\}}(x) = N_{B-\{c_B\}}(x_0) = \{u\} \\ &\text{for some } x_0 \in T - \{x\} \text{ and } u \in V(B) - \{c_B\}\big\}, \end{aligned}$$

and $T_1 := T - T_0$. Let S_B ($S_{B'}$) be a maximum independent set in B (B', respectively). Since $\delta(H) \geq 3$ and G is triangle-free, we have $|S_B|, |S_{B'}| \geq 3$. Moreover $|V(B)| \geq 4$. Let $S := ((S_B \cup S_{B'}) - \{c_B\}) \cup T^+ \cup T_1^{+3}$.

Subclaim 4.4.1 *S is an independent set of order at least* $2|T_1| + |T_0| + 4$.

Proof. By Fact 4.2, $((S_B \cup S_{B'}) - \{c_B\}) \cup T^+$ is an independent set.

For every $x_1, x_2 \in T_1$ ($x_1 \neq x_2$), by the definition of T_1, there exist vertices $u \in N_{B-\{c_B\}}(x_1)$ and $v \in N_{B-\{c_B\}}(x_2)$ such that $u \neq v$. Since B is 4-path-connected, there exists a uv-path P in B with length at least 4. Therefore by Fact 4.1 (iii), we have $x_1^{+3}x_2^{+3} \notin E(G)$, and hence T_1^{+3} is independent.

By the similar argument, we can show that any vertex in T_1^{+3} is not adjacent to any vertex in $((S_B \cup S_{B'}) - \{c_B\}) \cup T^+$, and hence S is an independent set.

Moreover we have

$$\begin{aligned}|S| &\geq (|S_B|-1)+(|S_{B'}|-1)+|T^+|+|T_1^{+3}| \\ &\geq 2+2+|T|+|T_1| \\ &= 2|T_1|+|T_0|+4. \qquad \square\end{aligned}$$

Case 1.1. $N_{B-\{c_B\}}(T_0) \neq V(B)-\{c_B\}$.

Then $T_1 \cup N_{B-\{c_B\}}(T_0) \cup \{c_B\}$ is a separating set, and hence

$$|T_1 \cup N_{B-\{c_B\}}(T_0) \cup \{c_B\}| = |T_1| + |N_{B-\{c_B\}}(T_0)| + 1 \geq \kappa(G).$$

By the definition of T_0, $|N_{B-\{c_B\}}(T_0)| \leq \frac{1}{2}|T_0|$, and therefore $|T_1|+\frac{1}{2}|T_0|+1 \geq \kappa(G)$. This implies $2|T_1|+|T_0|+2 \geq 2\kappa(G)$. Thus, by Subclaim 4.4.1 we obtain

$$2\kappa(G)-1 \geq \alpha(G) \geq |S| \geq 2|T_1|+|T_0|+4 \geq 2\kappa(G)+2,$$

a contradiction.

Case 1.2. $N_{B-\{c_B\}}(T_0) = V(B)-\{c_B\}$.

Choose $u \in V(B)-\{c_B\}$ so that $|N_C(u) \cap T_0|$ is as small as possible. Let $d_0 := |N_C(u) \cap T_0|$, $d_1 := |N_C(u) \cap T_1|$ and $b := |V(B)|$. Then $|T_1| \geq d_1$ and $b \geq 4$. By the definition of T_0, we have $d_0 \geq 2$. Since $N_C(u) \cup (V(B)-\{u\})$ is a separating set,

$$|N_C(u) \cup (V(B)-\{u\})| = d_0+d_1+b-1 \geq \kappa(G). \tag{4.1}$$

By the definition of T_0 and the choice of u, $|T_0| \geq d_0(b-1)$. Hence by Subclaim 4.4.1, we obtain

$$|S| \geq 2|T_1|+|T_0|+4 \geq 2d_1+d_0(b-1)+4. \tag{4.2}$$

By (4.1) and (4.2), we have

$2(d_0+d_1+b-1)-1 \geq 2\kappa(G)-1 \geq \alpha(G) \geq 2d_1+d_0(b-1)+4$. This implies that $(d_0-2)(b-3)+1 \leq 0$, contradicting that $d_0 \geq 2$ and $b \geq 4$.

Case 2. H is 2-connected.

In this case, we use a similar argument as in Case 1. By Lemma 4.15, H is 4-path-connected. Let $T := N_C(H)$, $T_0 := \{x \in T : N_H(x) = N_H(x_0) = \{u\}$ for some $x_0 \in T-\{x\}$, and $u \in V(H)\}$, $T_1 := T - T_0$, and let S_H be a maximum independent set in H. Then $|S_H| \geq 3$ and $|V(H)| \geq 4$.

Let $S := S_H \cup T^+ \cup T_1^{+3}$. Then by the same argument in the proof of Subclaim 4.4.1, S is an independent set with $|S| = |S_H|+|T^+|+|T_1^{+3}| \geq 3+|T|+|T_1| =$

$2|T_1| + |T_0| + 3$.

Case 2.1. $N_H(T_0) \neq V(H)$.

Since $T_1 \cup N_H(T_0)$ is a separating set and $|N_H(T_0)| \leq \frac{1}{2}|T_0|$, we have $|T_1| + \frac{1}{2}|T_0| \geq \kappa(G)$. Therefore we have

$$2\kappa(G) - 1 \geq \alpha(G) \geq |S| \geq 2|T_1| + |T_0| + 3 \geq 2\kappa(G) + 3,$$

a contradiction.

Case 2.2. $N_H(T_0) = V(H)$.

By the same way as in Case 1.2, choose $u \in V(H)$ so that $|N_C(u) \cap T_0|$ is as small as possible, and let $d_0 := |N_C(u) \cap T_0|$ and $d_1 := |N_C(u) \cap T_1|$. Clearly $|V(H)| \geq 4$. Because $N_C(u) \cup (V(H) - \{u\})$ is a separating set, we have $d_0 + d_1 + |V(H)| - 1 \geq \kappa(G)$.

On the other hand, since $|T_0| \geq d_0|V(H)|$, $|S| \geq 2d_1 + d_0|V(H)| + 3$. Therefore $2(d_0 + d_1 + |V(H)| - 1) - 1 \geq 2d_1 + d_0|V(H)| + 3$, and then $(d_0 - 2)(|V(H)| - 2) + 2 \leq 0$. This contradicts that $d_0 \geq 2$ and $|V(H)| \geq 4$. This complete the proof of Claim 4.4. $\square$

Claim 4.5 $\delta(H) = 1$.

Proof. Assume $\delta(H) = 2$. Let u be a vertex in H with $d_H(u) = 2$, and let $N_H(u) = \{v_1, v_2\}$. Without loss of generality, we may assume that $|N_C(v_1)| \leq |N_C(v_2)|$. Let $S := N_C(\{u, v_1\})^+ \cup N_H(v_1)$. By Fact 4.2, S is an independent set. Since $N_C(u) \cap N_C(v_1) = \emptyset$, we have

$$\begin{aligned} |S| &= |N_C(u)^+| + |N_C(v_1)^+| + |N_H(v_1)| \\ &= d_C(u) + d_G(v_1) \\ &\geq (\kappa(G) - 2) + \kappa(G) = 2\kappa(G) - 2. \end{aligned} \tag{4.3}$$

Since $\delta(H) = 2$, there exists $w_2 \in N_H(v_2) - \{u\}$. By Claim 4.3, there exists $x_2 \in N_C(w_2)$.

Subclaim 4.5.1 (i) *$S \cup \{x_2^{+3}\}$ is an independent set of order $2\kappa(G) - 1$.*

(ii) $d_C(u) = \kappa(G) - 2$.

Proof. There exist a uw_2-path and a v_1w_2-path in H of length at least 2. By Fact 4.1 (iii), $x_2^{+2} \notin N_C(\{u, v_1\})$, and hence $x_2^{+3} \notin S$. Again, by Fact 4.1 (iii), $x^+x_2^{+3} \notin E(G)$ for any $x \in N_C(\{u, v_1\}) - \{x_2\}$. Since G is triangle-free, $x_2^+x_2^{+3} \notin E(G)$. Hence $N_C(\{u, v_1\})^+ \cup \{x_2^{+3}\}$ is independent.

For any $w \in N_H(v_1) - \{w_2\}$, there exists a ww_2-path in H of length at least 2. By Fact 4.1 (iii), $x_2^{+3} \notin N_C(w)$. Therefore $(N_H(v_1) - \{w_2\}) \cup \{x_2^{+3}\}$ is independent. Suppose $x_2^{+3} \in N_C(w_2)$. Then by Fact 4.1 (iii), $x_2, x_2^{+3} \notin N_C(\{u, v_1\})$. Therefore by Fact 4.2, $S \cup \{x_2^+, x_2^{+4}\}$ is an independent set of order $2\kappa(G)$, a contradiction. Thus, we have $x_2^{+3} \notin N_C(w_2)$. Therefore we have $S \cup \{x_2^{+3}\}$ is an independent set. Since $\alpha(G) \leq 2\kappa(G) - 1$, inequality (4.3) implies that $|S \cup \{x_2^{+3}\}| = 2\kappa(G) - 1$ and $d_C(u) = \kappa(G) - 2$. □

Subclaim 4.5.2 $x_2 \in N_C(\{u, v_1\})$.

Proof. Suppose not. Then $x_2^+ \notin S$. Since G is triangle-free, $x_2^+ x_2^{+3} \notin E(G)$. By Subclaim 4.5.1 and Facts 4.1 (i) and (ii), $S \cup \{x_2^+, x_2^{+3}\}$ is an independent set of order $2\kappa(G)$, a contradiction. □

Subclaim 4.5.3 *$w_2 \in N_H(v_1)$ or $N_H(w_2) \cap N_H(v_1) \neq \emptyset$.*

Proof. Suppose that $w_2 \notin N_H(v_1)$ and $N_H(w_2) \cap N_H(v_1) = \emptyset$. Then $N_H(v_1) \cup \{w_2\}$ is independent. Since there exist a w_2u-path and a w_2v_1-path of length at least 2, by Subclaim 4.5.2, $S \cup \{w_2, x_2^{+3}\}$ is an independent set. This contradicts that $\alpha(G) \leq 2\kappa(G) - 1$. □

Subclaim 4.5.4 $N_C(v_1) = N_C(v_2)$.

Proof. By Subclaim 4.5.3, there exist a v_2u-path and a v_2v_1-path in H of length at least 2. Hence $S \cup \{x_2^{+3}\} \cup N_C(v_2)^+$ is an independent set. Since $\alpha(G) \leq 2\kappa(G) - 1$ and G is triangle-free, $N_C(v_2) \subseteq N_C(v_1)$. Thus, since $|N_C(v_1)| \leq |N_C(v_2)|$, we have $N_C(v_1) = N_C(v_2)$. □

Subclaim 4.5.5 $N_C(H) = N_C(\{u, v_1\})$.

Proof. Suppose that $N_C(H) - N_C(\{u, v_1\}) \neq \emptyset$. Let $w \in V(H)$ such that $N_C(w) - N_C(\{u, v_1\}) \neq \emptyset$. By Subclaim 4.5.4, we have $w \notin \{u, v_1, v_2\}$, and hence there exist a wu-path and a wv_1-path or a wv_2-path in H of length at least 2. Then $|S \cup \{x_2^{+3}\} \cup N_C(w)^+| \geq |S| + 2 = 2\kappa(G)$, a contradiction. □

Subclaim 4.5.6 $|N_C(v_1)| = 1$.

Proof. Assume that $|N_C(v_1)| \geq 2$. Let $x_0, x_1 \in N_C(v_1)$ with $x_0 \neq x_1$. By Subclaim 4.5.4, $x_0, x_1 \in N_C(v_2)$. Since G is triangle-free, we have $x_2 \neq x_0, x_1$. By Subclaim

4.5.3, there exists a uv_1-path in H of length at least 2. Hence $S \cup \{x_i^{+3}\}$ is an independent set for $i = 0, 1$. Since G is triangle-free, there exist $i, j \in \{0, 1, 2\}$ such that $x_i^{+3} x_j^{+3} \notin E(G)$. Then $S \cup \{x_i^{+3}, x_j^{+3}\}$ is an independent set of order $2\kappa(G)$, a contradiction. $\square$

By Subclaims 4.5.1 (ii), 4.5.5 and 4.5.6, $|N_C(H)| = |N_C(u)| + |N_C(v_1)| = (\kappa(G) - 2) + 1 = \kappa(G) - 1$. This contradicts the connectivity of G, and completes the proof of Claim 4.5. $\square$

Claim 4.6 *H is a star.*

Proof. Suppose that H is not a star. By Claim 4.5, there exists a vertex $u \in V(H)$ with $d_H(u) = 1$. Since H is not a star, there exists a path uvw_1w_2 in H of length 3. Note that $w_2 \notin N_H(v)$ since G is triangle-free. Let $S := N_C(\{u, v\})^+ \cup N_H(v)$. Since

$$\begin{aligned} |S| &= d_C(u) + d_C(v) + d_H(v) \\ &\geq \kappa(G) - 1 + \kappa(G) \\ &= 2\kappa(G) - 1, \end{aligned}$$

it follows from Fact 4.2 that S is a maximum independent set. By Claim 4.3, there exists $x_2 \in N_C(w_2)$.

We show that $(S - \{w_1\}) \cup \{x_2^{+3}\}$ is also a maximum independent set. There exist a w_2u-path and a w_2v-path in H of length at least 2. By Fact 4.1 (iii), $x_2^{+2} \notin N_C(\{u, v\})$ and $x^+ x_2^{+3} \notin E(G)$ for any $x \in N_C(\{u, v\}) - \{x_2\}$. Since G is triangle-free, $x_2^+ x_2^{+3} \notin E(G)$. Thus, $x_2^{+3} \notin N_C(\{u, v\})^+$ and $N_C(\{u, v\})^+ \cup \{x_2^{+3}\}$ is independent. For any $w \in N_H(v) - \{w_1\}$, there exist a ww_2-path in H of length at least 2. By Fact 4.1, $(N_H(v) - \{w_1\}) \cup \{x_2^{+3}\}$ is independent. Therefore $(S - \{w_1\}) \cup \{x_2^{+3}\}$ is a maximum independent set.

Suppose that $N_H(w_2) \cap N_H(v) \neq \{w_1\}$. Then there exists a w_1w_2-path in H of length at least 2. Thus, $S \cup \{x_2^{+3}\}$ is a independent set of order $2\kappa(G)$, a contradiction.

Therefore we may assume that $N_H(w_2) \cap N_H(v) = \{w_1\}$. By Fact 4.2, $S \cup \{x_2^+\}$ is independent. Since S is a maximum independent set, $x_2 \in N_C(\{u, v\})$. Since there exist a w_2u-path and a w_2v-path in H of length at least 2, by Fact 4.1 (iii) we have $x_2^{+3} \notin N_C(w_2)$. Therefore $(S - \{w_1\}) \cup \{w_2, x_2^{+3}\}$ is an independent set of order $2\kappa(G)$, a contradiction. This completes the proof of Claim 4.6. $\square$

Let v be the center vertex of H and $X := N_C(H)^+$. By Fact 4.2, $N_H(v) \cup X$ is independent. For every $u \in N_H(v)$, we obtain $|X| = |N_C(H)| \geq d_C(v) + d_C(u) \geq d_C(v) + (d_G(u) - 1)$. Therefore $|N_H(v)| + |X| \geq d_G(v) + \kappa(G) - 1 \geq 2\kappa(G) - 1$.

Let $X_0 := \{x \in X : N_{G-C}(x) = \emptyset\}$ and $X_1 := X - X_0 = \{x \in X : N_{G-C}(x) \neq \emptyset\}$. By the minimality of $|E(G-C)|$, we obtain the following fact.

Fact 4.7 (i) $N_C(H) \cap X_0^+ = \emptyset$.

(ii) *There exists no C-path joining a vertex of X and a vertex of X_0^+.*

For each $x \in X_1$, we choose an arbitrary vertex x^* of $N_{G-C}(x)$, and let $Y^* := \{x^* : x \in Y\}$ for $Y \subseteq X_1$. By Fact 4.1 (ii), for any $x_1, x_2 \in X_1$ with $x_1 \neq x_2$, we have $x_1^* \neq x_2^*$ and $x_1^* x_2^* \notin E(G)$. Moreover, for any $x_1 \in X_1$ and $x_2 \in X$ with $x_1 \neq x_2$, we have $x_1^* x_2 \notin E(G)$. Therefore for every $Y_1 \subseteq X_1$, $Y_1^* \cup (X - Y_1)$ is independent and $|Y_1^*| = |Y_1|$. By Fact 4.1 (i), $N_H(v) \cup X_1^*$ is an independent set.

By Fact 4.7, $N_G(x_0^+) \cap (N_H(v) \cup (X - \{x_0\})) = \emptyset$ for every $x_0 \in X_0$. Moreover we have $N_G(x_0^+) \cap (X_1^* \cup (X_0 - \{x_0\})) = \emptyset$. We divide the proof into two cases.

Case 1. There exists $x \in X$ such that $x^+, x^{+2} \notin N_C(H)$.

Let $x \in X$ such that $x^+, x^{+2} \notin N_C(H)$. We partition X_i into Y_i and Z_i for $i = 0, 1$ so that $Y_i := X_i \cap N_G(x^{+2})$, and $Z_i := X_i - N_G(x^{+2})$. By the triangle-free condition, $(Y_0^+ \cup Y_1^*) \cap N_G(x^{+2}) = \emptyset$. Since $|X_i| = |Y_i| + |Z_i|$ for $i = 0, 1$, we have

$$\begin{aligned}
&|N_H(v) \cup Y_0^+ \cup Z_0 \cup Y_1^* \cup Z_1 \cup \{x^{+2}\}| \\
&= |N_H(v)| + |X_0| + |X_1| + 1 \\
&= |N_H(v)| + |X| + 1 \\
&\geq 2\kappa(G).
\end{aligned}$$

Therefore $N_H(v) \cup Y_0^+ \cup Z_0 \cup Y_1^* \cup Z_1 \cup \{x^{+2}\}$ is not an independent set, and hence there exist $x_1, x_2 \in Y_0$ such that $x_1^+ x_2^+ \in E(G)$.

Claim 4.8 *$N_H(v) \cup X \cup \{x^{+3}\}$ is an independent set.*

Proof. Without loss of generality, we may assume that $x^{+3} \in V(x_2^{+3} \overrightarrow{C} x_1^-)$.

First, we show that $N_G(x^{+3}) \cap X = \emptyset$. Suppose that there exists $x_3 \in N_G(x^{+3}) \cap X$. Since G is triangle-free, we have $x_3 \neq x_1, x_2$. Let Q be a C-path joining x_3^- and x_2^-. We define a cycle C' as follows:

$$C' = \begin{cases}
x^{+3} \overrightarrow{C} x_3^- Q x_2^- \overleftarrow{C} x_1^+ x_2^+ \overrightarrow{C} x^{+2} x_1 \overleftarrow{C} x_3 x^{+3} & \text{if } x_3 \in V(x^{+3} \overrightarrow{C} x_1^-), \\
x^{+3} \overrightarrow{C} x_1 x^{+2} \overleftarrow{C} x_2^+ x_1^+ \overrightarrow{C} x_3^- Q x_2^- \overleftarrow{C} x_3 x^{+3} & \text{if } x_3 \in V(x_1^+ \overrightarrow{C} x_2^-), \\
x^{+3} \overrightarrow{C} x_2^- Q x_3^- \overleftarrow{C} x_2 x^{+2} \overleftarrow{C} x_3 x^{+3} & \text{if } x_3 \in V(x_2^+ \overrightarrow{C} x^{+2}).
\end{cases}$$

Note that in each case $V(C') \supseteq V(C) - \{x_2\}$ and C' passes a vertex in H. Since $x_2 \in Y_0 \subseteq X_0$, $N_{G-C}(x_2) = \emptyset$. Therefore $|V(C')| > |V(C)|$, or $|V(C')| = |V(C)|$ and $|E(G - C')| < |E(G - C)|$, which contradicts the maximality of $|V(C)|$ or the minimality of $|E(G - C)|$.

Next, we show that $N_G(x^{+3}) \cap N_H(v) = \emptyset$. Let R be a C-path joining x^{+3} and x_2^-. Then $C' = x^{+3}\overrightarrow{C}x_1x^{+2}\overleftarrow{C}x_2^+x_1^+\overrightarrow{C}x_2^-Rx^{+3}$ is a cycle such that $|V(C')| > |V(C)|$ or $|V(C')| = |V(C)|$ and $|E(G-C')| < |E(G-C)|$. Again this contradicts (C1) or (C2). $\square$

Since $x^{+2} \notin N_C(H)$, we have $x^{+3} \notin X$. Therefore $|N_H(v) \cup X \cup \{x^{+3}\}| \geq 2\kappa(G)$, a contradiction.

Case 2. For every $x \in X$, $x^+ \in N_C(H)$ or $x^{+2} \in N_C(H)$.

By Claim 4.3, we can choose $w \in N_C(v)$. Note that $N_G(w) \cap N_H(v) = \emptyset$, since G is triangle-free. In this case, we partition X_i into Y_i and Z_i for $i = 0, 1$ so that $Y_i := X_i \cap N_G(w)$ and $Z_i := X_i - N_G(w)$.

By the similar argument as in Case 1, we have $|N_H(v) \cup Y_0^+ \cup Z_0 \cup Y_1^* \cup Z_1 \cup \{w\}| \geq 2\kappa(G)$, and hence there exist $x_1, x_2 \in Y_0$ such that $x_1^+x_2^+ \in E(G)$.

On the other hand, by Fact 4.7 (i) $x_1^+, x_2^+ \notin N_C(H)$. Therefore $x_1^{+2}, x_2^{+2} \in N_C(H)$, which implies $x_1^+, x_2^+ \in N_C(H)^-$. Then by Fact 4.1 (ii), $x_1^+x_2^+ \notin E(G)$, a contradiction. $\square$

Chapter 5

Cycles passing through edges

As an extension of a hamilton cycle, we considered a cycle passing through specified vertices in Chapter 3. Extending this concept, it is natural to deal with a cycle passing through not only specified vertices but also specified edges. So we are interested in such a cycle and many researchers have studied it. In this chapter, we concentrate on a cycle passing through given edges.

We show sufficient conditions for the existence of a cycle passing through a given matching in Section 5.1. In the rest section of this chapter, we discuss about a cycle through a given linear forest. In particular, we deal with a long cycle, a dominating cycle and a hamilton cycle, in Sections 5.2, 5.3 and 5.4, respectively.

The contents of this chapter are based on the paper [137] "Hamilton cycles and dominating cycles passing through a linear forest," jointwork with T. Yamashita.

5.1 Cycles passing through given matching

5.1.1 Connectivity conditions

Dirac [41] showed that for any k vertices in a k-connected graph, the graph has a cycle containing all of them. Analogously, several authors have considered a cycle passing through given edges. In particular, it is natural to consider the following problem from Dirac's result; for any matching with k edges in a k-connected graph G, does there exist a cycle passing through all of them? But unfortunately, this answer is "NO." When k is odd and M is a matching with k edges such that $G-M$ is disconnected, clearly it is impossible to find the desired cycle.

Considering this situation, Lovàsz [115] and Woodall [171] independently proposed the following conjecture; for any matching M with k edges in a k-connected graph G, if k is even or $G-M$ is connected, then there exists a cycle passing through M in G. For $k=2$, the conjecture follows from Menger's Theorem (Theorem 2.2). The case $k=3$, $k=4$, and $k=5$ were proved by Lovàsz [115], Erdős and Győri

[50], and Sanders [147], respectively. On the other hand, Woodall [171] showed that $(2k-2)$-connected is enough for the conjecture, and Thomassen [152] improved the connectivity condition to $\frac{3k-1}{2}$. Moreover, Häggkvist and Thomassen gave the following famous partial solution of it.

Theorem 5.1 (Häggkvist and Thomassen [81]) *For any matching with k edges in a $(k+1)$-connected graph G, there exists a cycle passing through M in G.*

By this result and Menger's Theorem, we can easily obtain the following result.

Theorem 5.2 *Let k, m be integers with $k \geq 2$ and $m \geq 0$. Let G be an $(m+k)$-connected graph, and let M be a matching with m edges and $S \subset V(G)$ with $|S| \leq k$. Then there exists a cycle passing through M and S.*

Finally, Kawarabayashi gave a positive answer to the conjecture.

Theorem 5.3 (Kawarabayashi [98]) *For any matching M with k edges in a k-connected graph G, if k is even or $G - M$ is connected, then there exists a cycle passing through M in G.*

5.1.2 Cyclically edge-connectivity conditions for cubic graphs

Some researchers consider a cycle passing through given edges in a cubic graph. A graph G is called *cyclically k-edge-connected* if the resultant graph removing any k edges does not have two components containing at least one cycle. The first result on this field is due to Thomassen [153]; for any matching M with k edges in a cyclically 2^{k+1}-connected cubic graph G, then there exists a cycle passing through M in G. Later Holton and Thomassen conjectured the following stronger statement.

Conjecture 5.4 (Holton and Thomassen [85]) *For any matching M with k edges in a cyclically $(k+1)$-connected cubic graph G, then there exists a cycle passing through M in G.*

Conjecture 5.4 was verified for the case $k = 3$ by Lovász [115], the case $k = 4$ by Aldred, Holton and Thomassen [3], and the case $k = 5$ by Aldred and Holton [2]. McCuaig [124] showed that Conjecture 5.4 is true if the girth (the length of a shortest cycle) is at least $\lceil \frac{k^2}{4} \rceil + 1$. For general case, Conjecture 5.4 is still open.

5.1.3 Degree conditions

Berman proved the following result conjectured by Häggkvist [79].

Theorem 5.5 (Berman [20]) *For any matching M in a graph G of order $n \geq 3$, if $\sigma_2(G) \geq n + 1$, then there exists a cycle passing through M in G.*

Moreover, Jackson and Wormald [94] characterized graphs of order n with $\sigma_2(G) = n$ and having no cycle passing through a given matching. Amar, Flandrin, Gancarzewicz and Wojda [7] considered analogous result for a bipartite graph; for any matching M in a balanced bipartite graph G of order $2n \geq 10$ with bipartition X and Y, if $d_G(x) + d_G(y) \geq \frac{5n}{4}$ for any $x \in X$ and $y \in Y$ with $xy \notin E(G)$, then there exists a cycle passing through M in G.

5.1.4 Number of edges

Instead of the degree conditions, Benhocine and Wojda [19] considered the condition of the number of edges for the existence of a cycle passing through a given matching. For $n \geq 3$ and for $1 \leq d \leq n-1$, let

$$f(n,d) := \frac{d-1}{2} + \frac{(n-d)(n-d+1)}{2} + (d-1)^2.$$

Moreover, let $F(n,d) := \max\left\{f(n,d), f(n, \frac{n}{2})\right\}$. They showed that for any matching M in a graph G of order n with $\delta(G) \geq d$, if $|E(G)| \geq F(n,k)$, then there exists a cycle passing through M in G. They also determined all extremal graphs.

5.2 Long cycles through given edges

In this section, we consider a long cycle passing through given edges. For the existence of a long cycle, we also refer the reader to Chapter 7.

The oldest result on it was due to Enomoto [45]; for an $(m+2)$-connected graph G of order n and for any path P with m edges, G has a cycle of length at least $\min\{\sigma_2(G) - m, n\}$ containing P. Later, Hirohata [83] improved the lower bound of the length to $\min\left\{\frac{2}{k+1}\sigma_{k+1}(G) - m, n\right\}$ for an $(m+k)$-connected graph.

Enomoto, Hirohata and Ota [47] considered other lower bound of the length of a cycle, but passing through only one edge. They proved that for a 3-connected graph G of order n, G has a cycle of length at least $\min\{c-1, n\}$ containing any given egde, where $c := \min\{d_G(x) + d_G(y) : \mathrm{dist}(x,y) = 2\}$. Sun, Tian and Wei [151] characterized graphs that have an edge contained in no cycle of length at least $\min\{c, n\}$. Lu, Liu and Tian [117] improved the lower bound in the result by Enomoto et al. to $\min\{c'-1, n\}$, where $c' := 2\min\{\max\{d_G(x), d_G(y)\} : \mathrm{dist}(x,y) = 2\}$. They also characterized graphs that have an edge contained in no cycle of length at least $\min\{c', n\}$.

For the case not path, Egawa, Glas and Locke [42] showed that every k-connected graph of order n has a cycle containing any k vertices of length at least $\min\{2\delta(G), n\}$. On the other hand, Hu, Tian and Wei [87] considered a long cycle passing through not only given edges, but also vertices. They proved that for an $(m+k)$-connected graph of order n and for any linear forest F with $|E(F)| = m$ and $\omega_1(F) \le k-2$, G has a cycle of length at least $\min\left\{\frac{2}{k+1}\sigma_{k+1}(G) - m, n\right\}$ passing through F. Recently, Fujisawa and Yamashita [68] improved the lower bound of the length to $\min\left\{\sigma_2^{k+1}(G) - m, n\right\}$. But, Hu, Tian and Wei [87] also proved that the assumption "$\omega_1(F) \le k-2$" cannot replaced to "$\omega_1(F) \le k-1$" for $k = 2$. Motivated by the fact, Hirohata and Zhang [84] showed the existence of a cycle of length at least $\min\left\{2\delta(G) - m, n\right\}$ passing through F for any linear forest F with $|E(F)| = m$ edges and $\omega_1(F) \le k$.

5.3 Dominating cycles through given edges

5.3.1 Results

In 1980, Bondy gave a $\sigma_3(G)$ condition for a dominating cycle.

Theorem 5.6 (Bondy [26]) *Let G be a 2-connected graph of order n. If $\sigma_3(G) \ge n+2$, then any longest cycle is dominating.*

Now we show the following result concerning a dominating cycle passing through a linear forest.

Theorem 5.7 ([137]) *Let m, s be nonnegative integers. Let G be an $(m+2)$-connected graph of order n, and let F be a linear forest with $|E(F)| = m$ and $\omega_1(F) = s$. Suppose $\sigma_3(G) \ge n + 2m + 2 + \max\{s-3, 0\}$. Then every longest cycle passing through F is dominating.*

The conditions of Theorem 5.7 are sharp. First, we show that Theorem 5.7 does not hold for an $(m+1)$-connected graph. Let s, t, m be positive integers and let $G_1 := K_s + K_{m+1} + K_t$. If we take a path of length m from K_{m+1} as a linear forest F_1, then G_1 contains a cycle passing through F_1, but no dominating cycle passing through F_1, while $\sigma_3(G_1) = +\infty$. Thus, "$(m+2)$-connected" is necessary. Secondly, the degree condition of Theorem 5.7 is sharp. By considering the following graph G_2 and linear forest F_2. Let s, t, m be positive integers, let $G_2 = sK_1 + K_{m+s+t} + (t+1)K_2$ and let $F_2 := P \cup sK_1$ be a linear forest, where P is a path of length m in K_{m+s+t}. Then $|V(G_2)| = m + 2s + 3t + 2$, $|E(F_2)| = m$, $\omega_1(F_2) = s$, and $\sigma_3(G_2) = 3(m+s+t) + \max\{0, 3-s\} = |V(G_2)| + 2m + 2 + \max\{s-3, 0\} - 1$, but any longest cycle passing through F_2 is not dominating. The proof of Theorem 5.7 is given in Section 5.3.3.

In Theorem 5.7, if a graph G does not have a cycle passing through F, the conclusion holds in a vacuous way. Therefore we consider a condition for a cycle to pass through a linear forest. By using the following result which is easily obtain from Theorem 5.2, we obtain a corollary of Theorem 5.7.

Theorem 5.8 ([137]) *Let k, m be integers with $k \geq 2$ and $m \geq 0$. Let G be an $(m+k)$-connected graph, and let F be a linear forest m edges and $\omega_1(F) = s \leq k$. Then there exists a cycle passing through M and S.*

Corollary 5.9 *Let k, m, s be integers with $k \geq 2$, $m \geq 0$ and $s \geq 0$. Let G be an $(m+k)$-connected graph of order n, and let F be a linear forest m edges and $\omega_1(F) = s \leq k$. Let C be a longest cycle passing through F. If $\sigma_3(G) \geq n + 2m + 2 + \max\{s-3, 0\}$, then C is dominating.*

5.3.2 The matching case

First, we shall prove Theorem 5.7 for the case M is a matching. This result will be used in the proof of Theorem 5.7 for the general case in the next section.

Theorem 5.10 ([137]) *Let $m, s \geq 0$ and G be an $(m+2)$-connected graph of order n. Let M be a matching with $|M| = m$ and $S \subset V(G)$ with $|S| = s$ and $S \cap V(M) = \emptyset$. Suppose $\sigma_3(G) \geq n + 2m + 2 + \max\{s-3, 0\}$. Then every longest cycle passing through M and S is dominating.*

Proof of Theorem 5.10.

If there exists no cycle passing through M and S, then trivially the statement holds. Therefore we may assume that there exists a cycle passing through M and S. Let C be a longest cycle passing through M and S. If $E(G-C) = \emptyset$, then there is nothing to prove. Hence we may assume that $E(G-C) \neq \emptyset$. Let H be a component of $G - C$ with $|V(H)| \geq 2$ and let $v_1, v_2 \in V(H)$ such that $v_1 \neq v_2$. Let $T = N_C(H) = \{u_1, u_2, \ldots, u_t\}$. Let $W := \{w \in V(C) : ww^+ \in M\}$. Since $M \subset E(C)$, note that $|W| = |M| = m$. Moreover, since M is a matching, we have:

Claim 5.1 $W \cap W^+ = \emptyset$.

Let $X = T^+ - W^+$. Since C is a longest cycle passing through M and S, we obtain the following claim.

Claim 5.2 *Let $x_a, x_b \in X$ with $x_a = u_a^+$ and $x_b = u_b^+$ $(a \neq b)$. Let $C_a := x_a \overrightarrow{C} u_b$. Then the following statements hold.*

(i) $x_a \notin T$.

(ii) *There exists no C-path connecting x_a and x_b.*

(iii) $N_{C_a}(x_a)^- \cap N_{C_a}(x_b) \subset W$.

(iv) *If* $N_H(u_a) - \{v_i\} \neq \emptyset$*, then* $N_{C_a}(x_a)^- \cap N_{C_a}(v_i)^+ \subset W \cup W^+ \cup S$ *for* $i = 1, 2$.

(v) *If* $|N_H(\{u_a, u_b\})| \geq 2$*, then* $N_{C_a}(x_a)^- \cap N_{C_a}(x_b)^+ \subset W \cup W^+ \cup S$.

Since G is $(m+2)$-connected, we have $|T| \geq m+2$. Therefore there exist two vertices $x_1, x_2 \in X$. By Claim 5.2 (i) and (ii), $N_H(x_1) = N_H(x_2) = \emptyset$ and $N_{G-C}(x_1) \cap N_{G-C}(x_2) = \emptyset$. Therefore, for $i = 1, 2$,

$$\begin{aligned} & d_{G-C}(x_1) + d_{G-C}(x_2) + d_{G-C}(v_i) \\ & \leq \ |V(G-C)| - |V(H)| + |V(H)| - 1 \\ & = \ |V(G-C)| - 1. \end{aligned} \tag{5.1}$$

We define $C_1 := x_1 \overrightarrow{C} u_2$, $C_2 := x_2 \overrightarrow{C} u_1$ and $W_i := W \cap V(C_i)$ for $i = 1, 2$.

Claim 5.3 $d_C(v_i) \geq m+3$ *for* $i = 1, 2$.

Proof. By Claim 5.2 (i) and (ii), $\{x_1, x_2, v_i\}$ is independent. Clearly, $N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \subset V(C_1)$. By Claim 5.2 (iii), $N_{C_1}(x_1)^- \cap N_{C_1}(x_2) \subset W_1$. Therefore we obtain

$$\begin{aligned} d_{C_1}(x_1) + d_{C_1}(x_2) &= |N_{C_1}(x_1)^-| + |N_{C_1}(x_2)| \\ &= |N_{C_1}(x_1)^- \cup N_{C_1}(x_2)| + |N_{C_1}(x_1)^- \cap N_{C_1}(x_2)| \\ &\leq |V(C_1)| + |W_1|. \end{aligned}$$

By the symmetry, $d_{C_2}(x_1) + d_{C_2}(x_2) \leq |V(C_2)| + |W_2|$. Thus, by the inequality (5.1), we obtain

$$\begin{aligned} & n + 2m + 2 \\ & \leq \ d_G(x_1) + d_G(x_2) + d_G(v_i) \\ & \leq \ |V(C_1)| + |W_1| + |V(C_2)| + |W_2| + d_C(v_i) + |V(G-C)| - 1 \\ & \leq \ n + m + d_C(v_i) - 1. \end{aligned}$$

This implies $d_C(v_i) \geq m+3$. $\square$

Let $X_i = N_C(v_i)^+ - W^+$. Let $S_i := N_C(v_i)^+ \cap S$ and $s_i := |S_i|$. For $R \subset V(G)$ and $u \in V(G)$, we define

$$\varepsilon_R(u) := \begin{cases} 1 & \text{if } u \in R, \\ 0 & \text{otherwise.} \end{cases}$$

Claim 5.4 *Suppose that* $x_1, x_2 \in X_{3-i}$ *for* $i = 1, 2$*. Then* $d_G(x_1) + d_G(x_2) + d_G(v_i) \leq n + 2m + s_i + 1 - \varepsilon_S(x_1) - \varepsilon_S(x_2)$.

Proof. Clearly, we have $N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \cup N_{C_1}(v_i)^+ \subset V(C_1) \cup \{x_2\}$. If $x_2 \in S - S_i$, then $x_2 \notin N_{C_1}(v_i)^+$, which implies $N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \cup N_{C_1}(v_i)^+ \subset V(C_1)$. Therefore we have

$$|N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \cup N_{C_1}(v_i)^+| \leq |V(C_1)| + 1 - \varepsilon_{S-S_i}(x_2).$$

By Claim 5.2 (iv), $N_{C_1}(x_1)^- \cap N_{C_1}(v_i)^+ \subset W_1 \cup W_1^+ \cup (S_i \cap V(C_1))$. Let $A := N_{C_1}(x_1)^- \cap N_{C_1}(x_2)$ and $B := N_{C_1}(x_2) \cap N_{C_1}(v_i)^+$. By Claim 5.2 (ii) and (iii), we have $A \subset W_1$ and $B \subset W_1^+$. Suppose that $N_{C_1}(x_1)^- \cap N_{C_1}(x_2) \cap N_{C_1}(v_i)^+ = N_{C_1}(x_1)^- \cap B = A \cap N_{C_1}(v_i)^+ \neq \emptyset$, say $y \in N_{C_1}(x_1)^- \cap N_{C_1}(x_2) \cap N_{C_1}(v_i)^+$. Then $y \in A \cap B$ and so $y \in W_1 \cap W_1^+$, contradicting Claim 5.1. Therefore $N_{C_1}(x_1)^- \cap N_{C_1}(v_i)^+ \subset (W_1 - A) \cup (W_1^+ - B) \cup (S_i \cap V(C_1))$. On the other hand, note that $x_1 \notin N_{C_1}(v_i)^+$ since $x_1^- \notin V(C_1)$. Thus, if $x_1 \in S_i$ then $N_{C_1}(x_1)^- \cap N_{C_1}(v_i)^+ \subset (W_1 - A) \cup (W_1^+ - B) \cup (S_i \cap V(C_1) - \{x_1\})$, and hence we have

$$|N_{C_1}(x_1)^- \cap N_{C_1}(v_i)^+| \leq |W_1 - A| + |W_1^+ - B| + |S_i \cap V(C_1)| - \varepsilon_{S_i}(x_1).$$

Therefore we have

$$\begin{aligned}
&d_{C_1}(x_1) + d_{C_1}(x_2) + d_{C_1}(v_i) \\
&\quad = \ |N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \cup N_{C_1}(v_i)^+| + |N_{C_1}(x_1)^- \cap N_{C_1}(x_2)| \\
&\qquad + |N_{C_1}(x_2) \cap N_{C_1}(v_i)^+| + |N_{C_1}(x_1)^- \cap N_{C_1}(v_i)^+| \\
&\quad \leq \ |V(C_1)| + 1 - \varepsilon_{S-S_i}(x_2) + |A| + |B| \\
&\qquad + |W_1 - A| + |W_1^+ - B| + |S_i \cap V(C_1)| - \varepsilon_{S_i}(x_1) \\
&\quad = \ |V(C_1)| + 2|W_1| + |S_i \cap V(C_1)| + 1 - \varepsilon_{S-S_i}(x_2) - \varepsilon_{S_i}(x_1).
\end{aligned}$$

By the symmetry, we have $d_{C_2}(x_1) + d_{C_2}(x_2) + d_{C_2}(v_i) \leq |V(C_2)| + 2|W_2| + |S_i \cap V(C_2)| + 1 - \varepsilon_{S-S_i}(x_1) - \varepsilon_{S_i}(x_2)$. Therefore we deduce

$$\begin{aligned}
&d_C(x_1) + d_C(x_2) + d_C(v_i) \\
&\quad \leq \ |V(C_1)| + 2|W_1| + |S_i \cap V(C_1)| + 1 - \varepsilon_{S-S_i}(x_2) - \varepsilon_{S_i}(x_1) \\
&\qquad + |V(C_2)| + 2|W_2| + |S_i \cap V(C_2)| + 1 - \varepsilon_{S-S_i}(x_1) - \varepsilon_{S_i}(x_2) \\
&\quad = \ |V(C)| + 2m + s_i + 2 - \varepsilon_S(x_1) - \varepsilon_S(x_2).
\end{aligned}$$

Thus, it follows from the inequality (5.1) that

$$d_G(x_1) + d_G(x_2) + d_G(v_i) \leq n + 2m + s_i + 1 - \varepsilon_S(x_1) - \varepsilon_S(x_2). \qquad \square$$

Claim 5.5 $s = s_1 = s_2 \geq 3$.

Proof. By Claim 5.3, we can choose x_1, x_2 so that $x_1, x_2 \in X_2$. By the degree condition and by Claim 5.4, we have

$$\begin{aligned}
n + 2m + 2 \ &\leq \ d_G(x_1) + d_G(x_2) + d_G(v_1) \\
&\leq \ n + 2m + s_1 + 1 - \varepsilon_S(x_1) - \varepsilon_S(x_2) \\
&\leq \ n + 2m + s_1 + 1,
\end{aligned}$$

and so $s_1 \geq 1$. By symmetry, $s_2 \geq 1$. Assume that $s_1 = 1$. Then we can choose x_1 and x_2 so that $x_1 \in S_2$ and $x_2 \in X_2$. Note that $\varepsilon_S(x_1) = 1$. Hence it follows from Claim 5.4 that

$$\begin{aligned} d_G(x_1) + d_G(x_2) + d_G(v_1) &\leq n + 2m + s_1 + 1 - \varepsilon_S(x_1) - \varepsilon_S(x_2) \\ &\leq n + 2m + s_1. \end{aligned}$$

Thus, we have $s_1 \geq 2$. By symmetry, we have $s_2 \geq 2$. Hence we can choose $x_1, x_2 \in S_2$. Note that $\varepsilon_S(x_1) = \varepsilon_S(x_2) = 1$. Again, by Claim 5.4, we obtain

$$\begin{aligned} n + 2m + 2 + \max\{s-3, 0\} &\leq d_G(x_1) + d_G(x_2) + d_G(v_1) \\ &\leq n + 2m + s_1 + 1 - \varepsilon_S(x_1) - \varepsilon_S(x_2) \\ &= n + 2m + s_1 - 1. \end{aligned}$$

This implies that $s_1 - 1 \geq 2 + \max\{s-3, 0\}$, or equivalently, $s_1 \geq s$ and $s_1 \geq 3$. Since $s_1 \leq s$, we have $s = s_1 \geq 3$. By the symmetry, we also obtain $s = s_2$. $\square$

By Claim 5.5, we can choose $x_1, x_2, x_3 \in S = S_1 = S_2$. Without loss of generality, we may assume x_1, x_2, x_3 appear in this order along $\overrightarrow{C}$. Let $D_i := x_i \overrightarrow{C} u_{i+1}$ (the indices are taken modulo 3) and $Z_i := W \cap V(D_i)$ for $i = 1, 2, 3$. Note that $\varepsilon_S(x_i) = 1$ for $i = 1, 2, 3$. In the rest of the proof, we use the similar argument as in the proof of Claim 5.4. Since $N_{D_1}(x_1)^- \cup N_{D_1}(x_2)^+ \cup N_{D_1}(x_3) \subset V(D_1) \cup \{x_2\}$, we have

$$|N_{D_1}(x_1)^- \cup N_{D_1}(x_2)^+ \cup N_{D_1}(x_3)| \leq |V(D_1)| + 1.$$

Let $U := N_{D_1}(x_1)^- \cap N_{D_1}(x_3)$ and $V := N_{D_1}(x_2)^+ \cap N_{D_1}(x_3)$. By Claim 5.2 (iii), we have $U \subset Z_1$ and $V \subset Z_1^+$. By Claims 5.1 and 5.2 (v), we obtain

$$\begin{aligned} |N_{D_1}(x_1)^- \cap N_{D_1}(x_2)^+| &\leq |Z_1 - U| + |Z_1^+ - V| + |S \cap V(D_1)| - \varepsilon_S(x_1) \\ &= |Z_1 - U| + |Z_1^+ - V| + |S \cap V(D_1)| - 1, \end{aligned}$$

and so

$$\begin{aligned} & d_{D_1}(x_1) + d_{D_1}(x_2) + d_{D_1}(x_3) \\ \leq\ & |V(D_1)| + 1 + |U| + |V| + |Z_1 - U| + |Z_1^+ - V| + |S \cap V(D_1)| - 1 \\ =\ & |V(D_1)| + 2|Z_1| + |S \cap V(D_1)|. \end{aligned}$$

Since we have the same inequalities on D_2 and D_3, it follows that

$$d_C(x_1) + d_C(x_2) + d_C(x_3) \leq |V(C)| + 2m + s.$$

On the other hand, by Claim 5.2 (i) and (ii),

$$\begin{aligned} d_{G-C}(x_1) + d_{G-C}(x_2) + d_{G-C}(x_3) &\le |V(G-C)| - |V(H)| \\ &\le |V(G-C)| - 2. \end{aligned}$$

Hence we obtain $d_G(x_1) + d_G(x_2) + d_G(x_3) \le n + 2m + s - 2$. This contradicts the degree condition and establishes Theorem 5.10. $\square$

5.3.3 The general case

Proof of Theorem 5.7.

We prove Theorem 5.7 by the induction on m. Let C be a longest cycle passing through F. If $E(F)$ forms a matching, then by Theorem 5.10, C is a dominating cycle, and the statement holds. Therefore we may assume that $E(F)$ is not a matching. Let $v_1v_2v_3$ be a subpath of F. Let G' be a graph obtained by $V(G') = V(G) - \{v_2\}$ and $E(G') = E(G - \{v_2\}) \cup \{v_1v_3\}$, let F' be a linear forest in G' by $F' = (F - \{v_2\}) \cup \{v_1v_3\}$, and let $C' := (C - \{v_2\}) \cup \{v_1v_3\}$. Note that C' is a longest cycle in G' passing through F'. Moreover, let $m' := |E(F')| = m - 1$ and $n' := |V(G')| = n - 1$. Then G' is $(m' + 2)$-connected and

$$\begin{aligned} \sigma_3(G') &\ge \sigma_3(G) - 3 \\ &\ge n + 2m + 2 + \max\{s-3, 0\} - 3 \\ &= (n-1) + 2(m-1) + 2 + \max\{s-3, 0\} \\ &= n' + 2m' + 2 + \max\{s-3, 0\}. \end{aligned}$$

By the induction hypothesis, C' is a dominating cycle in G'. Therefore for every $u \in V(G'-C') = V(G-C)$, $N_{G'}(u) \subset V(C')$ and so $N_G(u) \subset N_{G'}(u) \cup \{v_2\} \subset V(C)$. Hence C is a dominating cycle in G. $\square$

5.4 Hamilton cycles through given edges

5.4.1 Results

In 1960, Ore introduced a $\sigma_2(G)$ condition for a graph to be hamiltonian.

Theorem 5.11 (Ore [130]) *Let G be a graph of order $n \ge 3$. If $\sigma_2(G) \ge n$, then G is hamiltonian.*

Many researchers studied a hamilton cycle and a cycle passing through specified elements of a graph. For such a cycle, we refer the reader to Chapter 3 and the surveys [77, 78]. In this section, we focus on a hamiltonian cycle passing through a linear forest.

The following result is obtained by Pósa (for a $\delta(G)$ condition) and Kronk (for a $\sigma_2(G)$ condition).

Theorem 5.12 (Pósa [140], Kronk [101]) *Let G be a graph of order n, and F be a linear forest in G with m edges. If $\delta(G) \geq \frac{n+m}{2}$ (or $\sigma_2(G) \geq n+m$), then G contains a hamilton cycle passing through F.*

This result was improved by Gancarzewicz and Wojda [69] as follows; for a graph G of order n, and for a linear forest F in G with m edges, if $\max\{d_G(x), d_G(y)\} \geq \frac{n+m}{2}$ for any $x, y \in V(G)$ with $dist(x,y) = 2$, then G contains a hamilton cycle passing through F unless some edges in F form an edge cut of odd order. Amar, Flandrin and Gancarzewicz [6] showed that for a 3-connected graph G of order n, and for a matching M in G with m edges, if $\sigma_3(G) \geq 2n$, then G contains a hamilton cycle passing through M unless some edges in M form an edge cut of odd order.

For a bipartite graph, Amar, Flandrin, Gancarzewicz and Wojda [7] proved that for a balanced bipartite graph of order $2n$ with partite sets X and Y and for any matching M, if $d_G(x) + d_G(y) \geq \frac{4n+1}{3}$ for any $x \in X$ and $y \in Y$ with $xy \notin E(G)$, then G contains a hamilton cycle passing through M.

On the other hand, we consider a condition for a graph to be hamiltonian again. In 1989, Bauer, Broersma, Li and Veldman gave a $\sigma_3(G)$ condition with the connectivity.

Theorem 5.13 (Bauer et al. [13]) *Let G be a 2-connected graph of order n. If $\sigma_3(G) \geq n + \kappa(G)$, then G is hamiltonian.*

The purpose of this section is to give a $\sigma_3(G)$ condition for a hamilton cycle passing through a linear forest. Then we prove the following result.

Theorem 5.14 ([137]) *Let m be an integer with $m \geq 1$. Let G be an $(m+2)$-connected graph of order n, and F be a linear forest in G with m edges. If $\sigma_3(G) \geq n + \kappa(G) + 2m - 1$, then G contains a hamilton cycle passing through F.*

The connectivity condition in Theorem 5.14 is necessary by considering a graph G_1 and a linear forest F_1. The degree condition of Theorem 5.14 is sharp in a sense. Let k, m, s and t be positive integers with $k \geq m+2$, $s, t \geq m$ and $s+t = k+m-1$. Let $G_3 := K_s + kK_1 + K_t$, and let F_3 be a linear forest with $F_3 \subset K_s \cup K_t$ and $|E(F_3)| = m$. Then $\sigma_3(G_3) = 3(k+m-1) = |V(G_3)| + \kappa(G_3) + 2m - 2$ and G_3 contains no hamilton cycle passing through F_3. On the other hand, since

$\delta(G_3) = \kappa(G_3)+m-1$ and $|V(G_3)| = 2\kappa(G_3)+m-1$, one might expect the degree sum condition in Theorem 5.14 can be relaxed by adding a condition concerning minimum degree or order of a graph. Then, by considering the minimum degree condition, we show the following result, which is an extension of Theorem 5.13.

Theorem 5.15 ([137]) *Let m be an integer with $m \geq 0$. Let G be an $(m+2)$-connected graph of order n, and F be a linear forest with m edges. Suppose that $\delta(G) \geq \kappa(G) + m$. If $\sigma_3(G) \geq n + \kappa(G) + m$, then G contains a hamilton cycle passing through F.*

The conditions of Theorems 5.15 are sharp in a sense. The above graph G_3 show that the minimum degree condition in Theorem 5.15 cannot be relaxed. We consider the graph $G_4 := K_{s+m+1} + K_k + (K_{t+m} + (k+t)K_1)$, where $t \geq s \geq 0$, $m \geq 1$ and $k \geq m+2$. Let F_4 be a linear forest with $F_4 \subset K_k \cup K_{t+m}$ and $|E(F_4)| = m$. Then G_4 contains no hamilton cycle passing through F_4, and $|V(G_4)| = 2(k+t+m)+s+1 \geq 2\kappa(G_4)+2m+1$, $\delta(G_4) \geq k+m+s \geq \kappa(G_4)+m$ and $\sigma_3(G_4) = s+m+k+2(k+t+m) = |V(G_4)|+\kappa(G_4)+m-1$. Therefore both minimum degree condition and degree sum condition in Theorem 5.15 is sharp.

Next, we consider a graph G of sufficiently large order, or order at least $2\kappa(G)+|E(F)|$, where F is a given linear forest. The following graph G_5 shows that it is not able to decrease the value of degree sum for the graph G of order $2\kappa(G)+m+1$ or $2\kappa(G)+m+2$. Let k, m, r, s and t be positive integers with $s \geq k \geq m+2$, $r \leq 2$, and $s+1 = k+m$. Let $G_5 = K_r + K_s + kK_1 + K_1$, and let F_5 be a linear forest with $F_5 \subset K_s$ and $|E(F_5)| = m$. Then $\sigma_3(G_5) = s+r-1+2(k+m) = |V(G_5)|+\kappa(G_5)+2m-2$ and G_5 contains no hamilton cycle passing through F_5. Therefore we show the following theorem.

Theorem 5.16 ([137]) *Let m be a positive integer. Let G be an $(m+2)$-connected graph of order n, and F be a linear forest with m edges. Suppose that*

$$\sigma_3(G) \geq \begin{cases} n+\kappa(G)+2m-2 & \text{for } n = 2\kappa(G)+m, \text{ and } \kappa(G) \geq 4 \text{ or } m \geq 2, \\ n+\kappa(G)+2m-1 & \text{for } n = 2\kappa(G)+m,\ \kappa(G) = 3 \text{ and } m = 1, \\ n+\kappa(G)+2m-1 & \text{for } n = 2\kappa(G)+m+1, \\ n+\kappa(G)+2m-1-r & \text{for } n = 2\kappa(G)+m+2+r \text{ and } 0 \leq r \leq m-1, \\ n+\kappa(G)+m & \text{for } n \geq 2\kappa(G)+2m+1. \end{cases}$$

Then G contains a hamilton cycle passing through F.

The conditions of Theorem 5.16 are sharp. By the above graph G_5, the degree sum condition is best possible for the graph G of order at least $2\kappa(G)+2m+1$. To consider the sharpness for the graph G of order $2\kappa(G)+m+2+r$ with $0 \leq r \leq m-1$, we give the following graph G_6. Let $k \geq m+2$, $m \geq t+1$, $t \leq r$ and $s+t = k+m$.

Let G_6 be a graph obtained from $K_{r+1} + K_s + kK_1 + K_t$. Let F_6 be a linear forest with $F_6 \subset K_s \cup K_t$ and $|E(F_6)| = m$. Then $|V(G_6)| = 2k + m + r + 1$ and $\sigma_3(G_6) = 3(k + m) = |V(G_6)| + \kappa(G_6) + 2m - 1 - r$, but G_6 contains no hamilton cycle passing through F_6. Finally, we show the sharpness for the graph G of order $2\kappa(G)+m$. Let k, m be positive integers with $k \geq m+2$. Let G_7 be a graph obtained from $K_1+((k-1)K_1 \cup K_2)+K_{k+m-2}$ by deleting an edge joining a vertex of K_1 and a vertex of K_2. Let F_7 be a linear forest with $F_7 \subset K_{k+m-2}$ and $|E(F_7)| = m$. Then $\sigma_3(G_7) = 3(k + m - 1) = |V(G_7)| + \kappa(G_7) + 2m - 3$ and G_7 contains no hamilton cycle passing through F_7. Moreover, let $G_8 := K_1 + (2K_1 \cup K_2) + K_2$ and let F_8 be a linear forest consisting of one edge in the right side K_2. Then $\kappa(G_8) = 3$, $m = 1$, $n = 7 = 2\kappa + m$, $\sigma_3(G_8) = 10 = n + \kappa + 2m - 2$ and G_8 contains no hamilton cycle passing through F_8.

We do not know the sharp degree sum bound for a graph G of order at most $2\kappa(G) + |E(F)| - 2$, where F is a given linear forest. But, its behavior seems to be complicated depending on the connectivity of a graph and the size of a linear forest.

A graph G is called *hamilton-connected* if for every $u, v \in V(G)$, G has a hamilton path connecting u and v. The notion of hamilton-connectedness is related to a hamilton cycle passing through a prescribed edge. In fact, by using Theorem 5.14, we can show the following result.

Corollary 5.17 *Let G be a 3-connected graph of order n. If $\sigma_3(G) \geq n+\kappa(G)+2$, then G is hamilton-connected.*

Proof. Let G be a graph satisfying the assumption of Corollary 5.17, and let $u, v \in V(G)$. It suffices to find a hamilton path connecting u and v. If $uv \in E(G)$, there exists a hamilton cycle C passing through uv, because G satisfies the assumption of Theorem 5.14 for $m = 1$. On the other hand, suppose that $uv \notin E(G)$. Let $G' := G + uv$. Since $\kappa(G') \leq \kappa(G) + 1$, we have $\sigma_3(G') \geq \sigma_3(G) \geq n + \kappa(G) + 2 \geq n + \kappa(G') + 1$. Then again G' satisfies the assumption of Theorem 5.14, and hence there exists a hamilton cycle C passing through uv. In each case, $C - uv$ is a desired hamilton path. $\square$

5.4.2 Proof of Theorems 5.14, 5.15 and 5.16

The following lemma is easy to prove, and so we omit the proof.

Lemma 5.18 *Let G be a 2-connected graph of order n and F be a linear forest with m edges. Suppose that $u\overrightarrow{P}v$ is a path passing through F. If $d_G(u)+d_G(v) \geq n+m$, then there exists a cycle passing through $V(P) \cup F$.*

Proof of Theorems 5.14, 5.15 and 5.16.

Suppose that G is a graph satisfying the assumption of Theorem 5.14, 5.15 or 5.16, but G does not contain a hamilton cycle passing through F. Let $M := E(F)$, let V_0 be a vertex cut of G with $|V_0| = k = \kappa(G)$, let $H_1, \ldots, H_p$ be a component of $G - V_0$ and let $V_i := V(H_i)$ for $1 \le i \le p$. Since V_0 is a vertex cut, it follows that $p \ge 2$.

By Theorem 5.2, there exists a cycle passing through M. Let C be a longest cycle passing through M. Then C is a dominating cycle by Theorem 5.7. If $V(G-C) = \emptyset$, then we obtain the conclusion. Therefore suppose that $V(G - C) \neq \emptyset$, say $x_0 \in V(G - C)$. Choose C and x_0 so that (C1) $d_G(x_0)$ is as large as possible and (C2) $x_0 \in V_0$ if possible, subject to (C1). Let $T := N_G(x_0) = N_C(x_0) = \{u_1, \ldots, u_t\}$ and $u_{t+1} = u_1$. We may assume that $u_1, u_2, \ldots, u_t$ appear in this order along $\overrightarrow{C}$. Let $x_i := u_i^+$ and $z_{i-1} := u_i^-$ for $1 \le i \le t$. Let $W := \{w \in V(C) : ww^+ \in M\}$, $X := T^+ - W^+$ and $Z := T^- - W$. Let $X' := X - V_0$ and $Z' := Z - V_0$. Then it is easy to prove the following claim.

Claim 5.6 *Let $x_i, x_j \in X$, $1 \le i \neq j \le t$. Then the following statements hold.*

(i) $x_i \notin T$.

(ii) *There exists no C-path connecting x_i and x_j.*

(iii) $N_C(x_i)^- \cap N_C(x_j) \cap V(x_i^+ \overrightarrow{C} x_j^-) \subset W$.

Case 1. $d_G(x_0) \le k + m - 1$.

In this case, G satisfies the assumption of Theorem 5.14 or 5.16. Let $L = \{x_l \in T^+ : E(u_l \overrightarrow{C} u_{l+1}) \cap M = \emptyset\}$. Note that $|L| \ge d_G(x_0) - m \ge k - m \ge 2$, say $x_i, x_j \in L$. Choose $x_i \in L$ so that there exists $x_h \in X - \{x_i\}$ such that $x_i \in N_C(x_h)^-$ if possible.

First, suppose that G satisfies the assumption of Theorem 5.14. By Claim 5.6 (i) and (ii), $\{x_0, x_i, x_j\}$ is an independent set of order three. Hence $d_G(x_i) + d_G(x_j) \ge \sigma_3(G) - d_G(x_0) \ge n + k + 2m - 1 - (k + m - 1) \ge n + m$. On the other hand, $P = x_i \overrightarrow{C} u_j x_0 u_i \overleftarrow{C} x_j$ is a path such that $|V(P)| = |V(C)| + 1$ and $M \subset E(P)$. By Lemma 5.18, there exists a cycle passing through $V(P) \cup M$, a contradiction.

Next, suppose that G satisfies the assumption of Theorem 5.16. We may assume $n = 2k+m$ or $n = 2k+m+2+r$ $(r \ge 0)$. Let $C_1 = x_i \overrightarrow{C} u_j$ and $C_2 = x_j \overrightarrow{C} u_i$. Let $L' = L - \{x_i, x_j\}$, $L'_1 = L' \cap C_1$ and $L'_2 = L' \cap C_2$. Note that $|L'| = |L| - 2 \ge d_G(x_0) - m - 2$. Suppose that $N_{C_1}(x_i)^- \cap L'_1 \neq \emptyset$, say $x_a \in N_{C_1}(x_i)^- \cap L'_1$. Then, by the choice of x_i, there exists $x_h \in X - \{x_i\}$ such that $x_i \in N_C(x_h)^-$. We consider $C' = x_0 u_i \overleftarrow{C} x_h x_i^+ \overrightarrow{C} u_h x_0$ and $x_i \in V(G - C')$. Then $d_G(x_i) \le d_G(x_0)$ by the choice of x_0. Similarly, $d_G(x_a) \le d_G(x_0)$. By Claim 5.6 (i) and (ii), $\{x_0, x_i, x_a\}$ is an independent set of order three. Therefore $\sigma_3(G) \le d_G(x_0) + d_G(x_i) + d_G(x_a) \le 3(k + m - 1)$. If $n = 2k + m$ then $\sigma_3(G) \le n + k + 2m - 3$, a contradiction. If $n = 2k + m + 2 + r$, then $\sigma_3(G) \le n+k+2m-r-5$, a contradiction. Thus we have $N_{C_1}(x_i)^- \cap L'_1 = \emptyset$. Hence by Claim 5.6 (ii), $N_{C_1}(x_i)^- \cup N_{C_1}(x_j) \subset V(C_1) - L'_1$. By Claim 5.6 (iii), $N_{C_1}(x_i)^- \cap N_{C_1}(x_j) \subset W \cap V(C_1)$. Therefore $d_{C_1}(x_i) + d_{C_1}(x_j) \le |V(C_1)| + |W \cap V(C_1)| - |L'_1|$. By symmetry, $d_{C_2}(x_i) + d_{C_2}(x_j) \le |V(C_2)| + |W \cap V(C_2)| - |L'_2|$. Thus we have

$$\begin{aligned} d_C(x_i) + d_C(x_j) &\le |V(C)| + |W| - |L'| \\ &\le |V(C)| + 2m + 2 - d_G(x_0). \end{aligned}$$

By Claim 5.6 (i) and (ii), $\{x_0, x_i, x_j\}$ is an independent set of order three and we have $N_{G-C}(x_i) \cap N_{G-C}(x_j) = \emptyset$ and $N_{G-C}(x_i) \cup N_{G-C}(x_j) \subset V(G-C) - \{x_0\}$. This implies

$$d_{G-C}(x_i) + d_{G-C}(x_j) \le |V(G-C)| - 1.$$

Therefore $\sigma_3(G) \le d_G(x_0) + d_G(x_i) + d_G(x_j) \le n + 2m + 1$. If $n = 2k+m$, $k = 3$ and $m = 1$, then $\sigma_3(G) \le d_G(x_0) + d_G(x_i) + d_G(x_j) \le n + 3 < n + k + 2m - 1$; otherwise $\sigma_3(G) \le d_G(x_0) + d_G(x_i) + d_G(x_j) \le n + k + m - 1$, a contradiction. This completes the proof of Case 1. $\square$

Case 2. $d_G(x_0) \ge k + m$.

In this case, note that $|X| \ge k$ and $|T| \ge k+m$, and hence $n \ge 2k+m+1$. The following fact is obvious.

Fact 5.7 *If $|V_0 \cap (T \cup V(G-C))| \ge l$, then there exist l intervals $u_i \overrightarrow{C} u_{i+1}$ with $V(u_i \overrightarrow{C} u_{i+1}) \cap V_0 = \emptyset$ and $E(u_i \overrightarrow{C} u_{i+1}) \cap M = \emptyset$.*

Claim 5.8 *$X' \ne \emptyset$ or $Z' \ne \emptyset$.*

Proof. Suppose not. Since $|T| \ge d_C(x_0) \ge k+m$, $|W| = m$ and $|V_0| = k$, we have $|T| = k+m$, $x_0 \notin V_0$ and $V_0 = X$. By the symmetry, $V_0 = Z$. Without loss of generality, we may assume $x_1 \in X = Z$. We now consider the cycle $C' = x_0 u_1 \overleftarrow{C} u_2 x_0$. By Theorem 5.7, C' is a dominating cycle because C' is a longest cycle passing through M. By the choice of x_0 and by the assumption of Case 2, $d_G(x_1) = k+m$. Since $x_1 \in V_0$, this contradicts the choice of x_0. $\square$

Without loss of generality, we may assume that $X' \ne \emptyset$ and furthermore $x_1 \in X' \cap V_1$. Choose x_1 so that $u_1 \overrightarrow{C} u_2 \cap V_0 = \emptyset$ and $E(u_1 \overrightarrow{C} u_2) \cap M = \emptyset$ if possible.

Case 2.1. $\bigcup_{i=2}^{p} V_i \subset T \cup \{x_0\}$.

By the assumption of Case 2.1, $x_0 \in V_0$ or $u_1 \in V_0$. By Fact 5.7 and the choice of x_1, $V(u_1 \overrightarrow{C} u_2) \cap V_0 = \emptyset$ and $E(u_1 \overrightarrow{C} u_2) \cap M = \emptyset$, and so $z_1 \in Z' \cap V_1$.

Claim 5.9 *$X' - \{x_1\} \neq \emptyset$ or $Z' - \{z_1\} \neq \emptyset$.*

Proof. Assume not. If $x_0 \notin V_0$, then $u_1, u_2 \in V_0$ since $x_1, z_1 \in V_1$. By Fact 5.7, we see $X' - \{x_1\} \neq \emptyset$. Therefore $x_0 \in V_0$. Since $|T| = d_C(x_0) \geq k+m$, $|W| = m$ and $|V_0 \cap V(C)| \leq k-1$, we have $d_G(x_0) = |T| = k+m$ and $V_0 - \{x_0\} = X - \{x_1\}$. By the symmetry, $V_0 - \{x_0\} = Z - \{z_1\}$. Since $|X - \{x_1\}| = k-1 \geq m+2-1 \geq 2$, there exist $x_i, x_j \in X - \{x_1\}$ with $x_i \neq x_j$. We now consider the cycle $C' = x_0 u_i \overleftarrow{C} u_{i+1} x_0$. Then it follows from the choice of x_0 that $d_G(x_i) = k+m$, and $d_G(x_j) = k+m$ by the symmetry.

Suppose that G satisfies the assumption of Theorem 5.16 and $n = 2k+m+2+r$ ($r \geq 0$). By Claim 5.6 (i) and (ii), $\{x_i, x_j, x_0\}$ is an independent set of order three. Then we obtain

$$\begin{aligned} d_G(x_i) + d_G(x_j) + d_G(x_0) &\leq 3(k+m) \\ &\leq n+k+2m-2-r, \end{aligned}$$

a contradiction. Therefore we may assume that G satisfies the assumption of Theorem 5.14, 5.15 or Theorem 5.16 and $n = 2k+m+1$. By Claim 5.6 (i) and (ii), $N_G(x_1) \subset (V(G) - V_2) - (X \cup \{x_0\})$ and $\{x_1, x_i, x_0\}$ is an independent set of order three. By the assumption of Case 2.2, we obtain

$$\begin{aligned} d_G(x_1) + d_G(x_i) + d_G(x_0) &\leq n - |V_2| - |X| - 1 + 2(k+m) \\ &\leq n+k+2m-1-|V_2|. \end{aligned}$$

Because $V_2 \neq \emptyset$, this contradicts the assumption of Theorem 5.14, and Theorem 5.16 and $n = 2k+m+1$. Thus, G satisfies the assumption of Theorem 5.15. Let $v_2 \in V_2$. Then $N_G(v_2) \subset (V_2 - \{v_2\}) \cup V_0$. By the minimum degree condition, $|V_2| - 1 + |V_0| \geq d_G(v_2) \geq k+m$, or $|V_2| \geq m+1$. Hence we obtain

$$\begin{aligned} d_G(x_1) + d_G(x_i) + d_G(x_0) &\leq n+k+2m-1-|V_2| \\ &\leq n+k+m-2, \end{aligned}$$

a contradiction. $\square$

Without loss of generality, we may assume that $X' - \{x_1\} \neq \emptyset$, say $x_i \in X' - \{x_1\}$. Let $D_1 := x_1 \overrightarrow{C} u_i$ and $D_2 := x_i \overrightarrow{C} u_1$. By Claim 5.6 (ii), $N_{D_1}(x_1) \cap X = \emptyset$. Hence $N_{D_1}(x_1)^- \cap (V_2 - W) = \emptyset$, since $V_2 \subset T$. By the assumption of Case 2.1, we obtain $x_i \in V_1$. This yields $N_G(x_i) \cap V_2 = \emptyset$. Thus, we obtain $N_{D_1}(x_1)^- \cup N_{D_1}(x_i) \subset V(D_1) - (V_2 - W)$. By Claim 5.6 (ii) and (iii), $N_{D_1}(x_1)^- \cap N_{D_1}(x_i) \subset (W - V_2) \cap V(D_1)$. Hence we have

$$\begin{aligned} d_{D_1}(x_1) + d_{D_1}(x_i) &\leq |V(D_1) - (V_2 - W)| + |(W - V_2) \cap V(D_1)| \\ &\leq |V(D_1)| - |(V_2 - W) \cap V(D_1)| + |(W - V_2) \cap V(D_1)| \\ &\leq |V(D_1)| - |V_2 \cap V(D_1)| + |W \cap V(D_1)|. \end{aligned}$$

By the similar argument, $d_{D_2}(x_1)+d_{D_2}(x_i) \leq |V(D_2)|-|V_2 \cap V(D_2)|+|W \cap V(D_2)|$. On the other hand, $N_{G-C}(x_1) \cup N_{G-C}(x_i) \subset V(G-C)-\{x_0\}$. By Claim 5.6 (ii), $N_{G-C}(x_1) \cap N_{G-C}(x_i) = \emptyset$. Thus we deduce that

$$\begin{aligned} d_G(x_1)+d_G(x_i) &\leq |V(C)|-|V_2|+|W|+|V(G-C)|-1 \\ &= n-|V_2|+m-1. \end{aligned}$$

Let $y_1 \in V_2$. Then $d_G(y_1) \leq |V_2|+|V_0|-1 = |V_2|+k-1$. Since $x_1, x_i \in V_1$ and $y_1 \in V_2$, $\{x_1, x_i, y_1\}$ is an independent set of order three. Hence $\sigma_3(G) \leq d_G(x_1)+d_G(x_i)+d_G(y_1) \leq n+k+m-2$, a contradiction.

Case 2.2. $\bigcup_{i=2}^p V_i \not\subset T \cup \{x_0\}$.

Let $y_2 \in \bigcup_{i=2}^p V_i - (T \cup \{x_0\})$. Choose $y_2 \in \bigcup_{i=2}^p V_i \cap X'$ if possible. Without loss of generality, we may assume that $y_2 \in V_2$. Note that $x_1y_2 \notin E(G)$ because $x_1 \in V_1$ and $y_2 \in V_2$. Since $y_2 \notin T$ and C is dominating, it follows that $x_0y_2 \notin E(G)$. Therefore $\{x_0, x_1, y_2\}$ is an independent set of order three. By Claim 5.6 (ii), $N_C(x_1) \cap X = \emptyset$. Hence we obtain $N_C(x_1) \cap N_C(y_2) \subset V(C-X) \cap V_0$. On the other hand, $N_{G-C}(x_1) \cap N_{G-C}(y_2) \subset V(G-C) \cap V_0$.

First, suppose that $y_2 \notin X'$. Then the choice of y_2 yields $X' \subset V_1$, and hence $N_G(y_2) \cap X' = \emptyset$. Thus, we have $N_G(x_1) \cup N_G(y_2) \subset V(G)-(X' \cup \{x_0\})$. Next, suppose that $y_2 \in X'$. By Claim 5.6 (i) and (ii), $N_C(y_2) \cap X' = \emptyset$. Therefore we also have $N_G(x_1) \cup N_G(y_2) \subset V(G)-(X' \cup \{x_0\})$. Thus, we obtain

$$\begin{aligned} d_G(x_1)+d_G(y_2) &\leq |V(G)|-|X'|-|\{x_0\}|+|(V(C)-X) \cap V_0|+|V(G-C) \cap V_0| \\ &\leq |V(G)|-|X-V_0|-1+|V(C) \cap V_0|-|X \cap V_0|+|V(G-C) \cap V_0| \\ &= |V(G)|+|V_0|-|T^+-W^+|-1 \\ &= |V(G)|+|V_0|+|W^+|-|T^+|-1 \\ &= n+k+m-d_G(x_0)-1, \end{aligned}$$

and hence $\sigma_3(G) \leq d_G(x_0)+d_G(x_1)+d_G(y_2) \leq n+k+m-1 \leq n+k+2m-2$, a contradiction. $\square$

Chapter 6

Relative length

As in Chapter 4, we have focused on a dominating cycle, which is regarded as a "pre-hamilton" cycle. Extending this property "pre-hamiltonian," Enomoto, van den Heuvel, Kaneko and Saito proposed a new invariant, called *relative length*. They were interested in the property "relative length at most one," because that property implies that "any longest cycle of a graph is dominating." In this sense, they regard a graph with "relative length at most one" as a "pre-hamiltonian" graph. But recently, we show that not only "relative length at most one" but also the low relative, "relative length at most two or a little more" also implies some properties related to a hamilton cycle. So a graph with low relative length can be also regarded as a "pre-hamiltonian" graph, and hence we are interested in such a graph. In this chapter, we focus on it from two aspects; one of them is sufficient conditions to guarantee the low relative length, and another is what properties are implied by the low relative length.

The contents of this chapter are based on the paper [134] "On relative length of longest paths and cycles," jointwork with M. Tsugaki and T. Yamashita, and the paper [99] "Long cycles in graphs without hamiltonian paths," jointwork with K. Kawarabayashi and T. Yamashita.

6.1 Sufficient conditions for the low relative length

In this chapter, we focus on two invariants $p(G)$ and $c(G)$. Notice that $p(G)$ is the order of a longest path and $c(G)$ is that of a longest cycle. The main interest of this chapter is the difference $\mathrm{diff}(G) := p(G) - c(G)$, which is called *relative length*.

It is easy to see that a connected graph G is hamiltonian if and only if $\mathrm{diff}(G) = 0$. In [130], Ore gave a degree sum condition for the existence of a hamilton cycle.

Theorem 6.1 (Ore [130]) *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is hamiltonian, that is,* $\mathrm{diff}(G) = 0$.

Next, we consider a degree sum condition for a graph G to have $\mathrm{diff}(G) \leq 1$. It is also easy to see that any longest cycle of a graph G is dominating if $\mathrm{diff}(G) \leq 1$. In [26], Bondy showed that if G is a 2-connected graph of order n with $\sigma_3(G) \geq n+2$, then every longest cycle in G is dominating. Enomoto, van den Heuvel, Kaneko and Saito showed the following theorem, which is a generalization of Bondy's result.

Theorem 6.2 (Enomoto, van den Heuvel, Kaneko and Saito [46]) *Let G be a 2-connected graph of order n. If $\sigma_3(G) \geq n+2$, then $\mathrm{diff}(G) \leq 1$.*

In [111], Li, Saito and Schelp considered the concerning the property "$\mathrm{diff}(G) \leq 1$" and a $\sigma_4(G)$ condition. They proved that if G is a 3-connected graph of order n with $\sigma_4(G) \geq \frac{3}{2}n + 1$, then $\mathrm{diff}(G) \leq 1$ and also conjectured that the sharp coefficient of n is $\frac{4}{3}$. Lu, Liu and Tian gave a sharp bound on the $\sigma_4(G)$ condition, which is an extension of the result by Lu, Liu and Tian [116]; each longest cycle in a 3-connected graph G of order n with $\sigma_4(G) \geq \frac{1}{3}(4n+5)$ is dominating.

Theorem 6.3 (Lu, Liu and Tian [118]) *Let G be a 3-connected graph of order n. If $\sigma_4(G) \geq \frac{1}{3}(4n+5)$, then $\mathrm{diff}(G) \leq 1$.*

In [111], Li, Saito and Schelp showed that the property $\mathrm{diff}(G) \leq 1$ implies a number of cycle-related properties. But recently, as in Section 6.2, we know that when $\mathrm{diff}(G)$ is very small, it also can imply some cycle-related properties. In this sense, relative length is a very useful tool. Therefore, we will consider the following question.

Question 6.4 *For a positive integer k, what is a degree sum condition for $\mathrm{diff}(G) \leq k-1$?*

Now we focus on the case $k = 3$ of the above question, and we give a $\sigma_4(G)$ condition for $\mathrm{diff}(G) \leq 2$.

Theorem 6.5 ([134]) *Let G be a 3-connected graph of order n. If $\sigma_4(G) \geq n+6$, then $\mathrm{diff}(G) \leq 2$.*

We show the best possibility of Theorem 6.5. First, Theorem 6.5 does not hold for a 2-connected graph. Let $G_1 := K_2 + 3K_l$ where $l \geq 3$. Then $\mathrm{diff}(G_1) = l \geq 3$ while G_1 does not have an independent set of order 4. Therefore, the condition "3-connected" in Theorem 6.5 is necessary. Next, we present an example which shows that the degree sum condition is best possible. Let $G_2 := K_3 + 4K_m$, where $m \geq 3$. Then $|V(G_2)| = 4m+3$ and $\sigma_4(G_2) = 4(m+2) = |V(G_2)|+5$, but $\mathrm{diff}(G_2) = m \geq 3$. Thus, the lower bound of the degree sum condition in Theorem 6.5 is sharp.

In order to give an answer of Question 6.4, we propose the following conjecture, which has been verified for $k = 1$ (Theorem 6.1), $k = 2$ (Theorem 6.2) and $k = 3$ (Theorem 6.5).

Conjecture 6.6 *Let G be a k-connected graph of order n. If $\sigma_{k+1}(G) \geq n+k(k-1)$, then* $\mathrm{diff}(G) \leq k-1$.

The lower bound on $\sigma_{k+1}(G)$ is best possible in a sense. Let $G_3 := K_k + (k+1)K_m$. Suppose that $m \geq k$. Then $|V(G_3)| = (k+1)m + k$ and $\sigma_{k+1}(G_3) = n + k(k-1) - 1$, but $\mathrm{diff}(G_3) = m \geq k$ hold. Note that Conjecture 6.6 is a generalization of the famous conjecture due to Bondy.

Conjecture 6.7 (Bondy [26]) *Let G be a k-connected graph of order n, and let C be a longest cycle. If $\sigma_{k+1}(G) \geq n + k(k-1)$, then $p(G-C) \leq k-1$.*

Bondy [26] and Fraisse [65], respectively, established a weaker form and another variant of Conjecture 6.7. Bondy [26] showed that for any longest cycle C in a graph G satisfying the assumption of Conjecture 6.7, $G-C$ has no complete graph of order k, and Fraisse [65] proved that such a graph has a cycle, possibly not longest, such that removing all vertices of it results in a graph each of which component is of order at least $k-1$.

On the other hand, the following proposition shows that Conjecture 6.6 is a generalization of Conjecture 6.7.

Proposition 6.8 *Let G be a k-connected graph, and let C be a longest cycle of G. If* $\mathrm{diff}(G) \leq k-1$, *then $p(G-C) \leq k-1$.*

Proof of Proposition 6.8.

Let G be a k-connected graph with $\mathrm{diff}(G) \leq k-1$ and let C be a longest cycle in G. Assume that $G-C$ contains a path P of order k. Let x be an end-vertex of P. Since G is k-connected, there exists an xv-path R such that $V(R) \cap V(P) = \{x\}$ and $V(R) \cap V(C) = \{v\}$. Then there exists a path of order at least $c(G)+k$, contradicting $\mathrm{diff}(G) \leq k-1$. $\square$

We sometimes regard a graph with a hamilton path as having good property. So we often try to find sufficient conditions for a graph without a hamilton path to have the low relative length. In [46], Enomoto, van den Heuvel, Kaneko and Saito gave a $\sigma_3(G)$ condition of it.

Theorem 6.9 (Enomoto, van den Heuvel, Kaneko and Saito [46]) *Let G be a connected graph of order n. If $\sigma_3(G) \geq n$, then either* $\mathrm{diff}(G) \leq 1$ *or G has a hamilton path.*

Theorems 6.2 and 6.9 say that the connectivity and degree sum condition can be weakened for graphs without hamilton paths. Therefore, one might expect that the conditions of other results can be also weakened for graphs without hamilton paths. By the expectation of Theorem 6.3, we prove the following result.

Theorem 6.10 ([99]) *Let G be a 2-connected graph of order n. If $\sigma_4(G) \geq \frac{1}{3}(4n-2)$, then either $\mathrm{diff}(G) \leq 1$ or G has a hamilton path.*

On the other hand, in 2002, Schiermeyer and Tewes [148] investigated the relation between a $\sigma_4(G)$ condition and $\mathrm{diff}(G) \leq 2$ in a 2-connected graph G. A path P of a graph G is said to be *dominating* if removing all vertices of P from G results in a graph with no edges. They showed that for a 2-connected graph G of order n, if $\sigma_4(G) \geq n+3$, then either $\mathrm{diff}(G) \leq 2$ or every longest path in G is dominating. However, considering the relations between Theorems 6.2 and 6.9 and between Theorems 6.3 and 6.10, the conclusion of the above result seems to be weak. Therefore, we give an improvement of it.

Theorem 6.11 ([99]) *Let G be a 2-connected graph of order n. If $\sigma_4(G) \geq n+3$ then either $\mathrm{diff}(G) \leq 2$ or G has a hamilton path.*

The degree sum bounds of Theorems 6.10 and 6.11 are best possible. Let m be an integer with $m \geq 2$ and $G_4 := K_m + (K_1 \cup (m+1)K_2)$. Then $\sigma_4(G_4) = m + 3(m+1) = \frac{1}{3}(4n-3)$ and neither $\mathrm{diff}(G_4) \leq 1$ nor G_4 has a hamilton path. On the other hand, let $G_5 := K_m + (K_1 \cup (m+1)K_3)$. Then $\sigma_4(G_5) = m+3(m+2) = n+2$ and neither $\mathrm{diff}(G_5) \leq 2$ nor G_5 has a hamilton path.

6.2 Necessary conditions for the low relative length

In this section, we will mention some applications of the invariant "relative length." We shall show a new lower bound of the circumference of a graph G with low relative length, and establish a partial solution of Thomassen's conjecture. For the circumference of a graph, we also refer the reader to Chapter 7 We use the following lemma to prove these results.

Lemma 6.12 *Let G be a 2-connected graph, and C be a longest cycle of G. If $\mathrm{diff}(G) \leq 2$, the followings hold.*

(i) *Each component of $G - C$ has order at most 2.*

(ii) $|N_C(x)^+ \cap N_C(G-C)| \leq 1 - d_{G-C}(x)$ *for any $x \in V(G-C)$.*

Proof of Lemma 6.12.

Let H be a component of $G - C$ and suppose that $|V(H)| \geq 2$. Since G is 2-connected, there exist distinct vertices $x_1, x_2 \in V(H)$ such that $N_C(x_1) \neq \emptyset$ and $N_C(x_2) \neq \emptyset$. Since $\mathrm{diff}(G) \leq 2$, we have $N_H(x_1) = \{x_2\}$ and $N_H(x_2) = \{x_1\}$. This implies $V(H) = \{x_1, x_2\}$, that is, $|V(H)| = 2$. Thus, the statement (i) holds.

Let $x \in V(G-C)$. Then $d_{G-C}(x) \leq 1$ holds from the statement (i). First, suppose that $d_{G-C}(x) = 1$. Since $\mathrm{diff}(G) \leq 2$, we have $N_C(x)^+ \cap N_C(G-C) = \emptyset$, and hence the statement (ii) holds. Next, suppose that $d_{G-C}(x) = 0$ and $|N_C(x)^+ \cap N_C(G-C)| \geq 2$. Let $v, w \in N_C(x)^+ \cap N_C(G-C)$, $y \in N_{G-C}(v)$ and $z \in N_{G-C}(w)$. By the choice of C, we see that $y \neq x$ and $z \neq x$. Then $C' := yv\overrightarrow{C}w^-xv^-\overleftarrow{C}wz$ is a path of order $|V(C)|+3$ if $y \neq z$; otherwise C' is a longer cycle than C, contradicting the assumption. This completes the proof. $\square$

Bauer, Morgana, Schmeichel and Veldman gave a lower bound of the circumference of 2-connected graphs with large $\sigma_3(G)$.

Theorem 6.13 (Bauer, Morgana, Schmeichel and Veldman [17]) *Let G be a 2-connected graph of order n. If $\sigma_3(G) \geq n+2$, then $c(G) \geq \min\{n, n+\delta(G) - \alpha(G)\}$.*

Li, Saito and Schelp [111] showed that if $\mathrm{diff}(G) \leq 1$ then $c(G) \geq \min\{n, n+\delta(G)-\alpha(G)\}$. So, Theorem 6.13 can be easily proved by Theorem 6.2. On the other hand, Trommel investigated the relation between $\delta(G)$ and $c(G)$ in 3-connected graphs.

Theorem 6.14 (Trommel [156]) *Let G be a 3-connected graph of order n. If $\delta(G) \geq \frac{1}{4}(n+6)$, then $c(G) \geq \min\{n, n+2\delta(G) - 2\alpha(G) - 2\}$.*

For a graph G with $\mathrm{diff}(G) \leq 2$, we obtain the same lower bound for the circumference. Moreover, we can obtain the similar lower bound when $\mathrm{diff}(G)$ is small.

Theorem 6.15 *Let G be a connected graph of order n. If $\mathrm{diff}(G) \leq 2$, then $c(G) \geq \min\{n, n+2\delta(G) - 2\alpha(G) - 2\}$.*

Theorem 6.16 *Let G be a k-connected graph of order n. If $\mathrm{diff}(G) \leq k-1$, then $c(G) \geq \min\{n-(k-2)\alpha(G), n-(k-1)(\alpha(G)-\delta(G)+k-2)\}$.*

By Theorems 6.5 and 6.15, we give the following theorem, which is a generalization of Theorem 6.14. On the other hand, Theorem 6.15 implies a new sufficient condition for a graph to be hamiltonian.

Theorem 6.17 *Let G be a 3-connected graph of order n. If $\sigma_4(G) \geq n+6$, then $c(G) \geq \min\{n, n+2\delta(G) - 2\alpha(G) - 2\}$.*

Corollary 6.18 *Let G be a 2-connected graph with $\mathrm{diff}(G) \leq 2$. If $\delta(G) \geq \alpha(G)+1$, then G has a hamilton cycle.*

Theorem 6.15 is best possible in the following sense. Let $G_6 := K_k + mK_2$, where $m \geq k \geq 2$. Then $|V(G_6)| = k+2m$, $\mathrm{diff}(G_6) \leq 2$, $\alpha(G_6) = m$ and $\delta(G_6) = k+1$, but $c(G_6) = 3k = |V(G_6)| + 2\delta(G_6) - 2\alpha(G_6) - 2$. Thus, the lower bound of circumference cannot be improved. Next, recall the graph $G_3 = K_k + (k+1)K_m$. If

$m \geq 3$, then we see that $\mathrm{diff}(G_3) = m \geq 3$ and $c(G_3) = |V(G_3)| - m < |V(G_3)| = \min\{|V(G_3)|, |V(G_3)| + 2\delta(G_3) - 2\alpha(G_3) - 2\}$. Hence the upper bound of $\mathrm{diff}(G)$ also cannot be improved.

Proof of Theorem 6.15.

Let C be a longest cycle in G and $H := G - C$. If C is a hamilton cycle, the conclusion holds. Thus, we may assume that $V(H) \neq \emptyset$. Let $x \in V(H)$. By Lemma 6.12 (i), $d_{G-C}(x) \leq 1$ holds. Let S be an independent set of H with $|S| = \alpha(H)$.

If $N_C(x)^+ \cup S$ is an independent set, then $\alpha(G) \geq d_G(x) + |S|$. Suppose that $N_C(x)^+ \cup S$ is not an independent set. By Lemma 6.12 (ii), $|N_C(x)^+ \cap N_C(S)| = 1$, say $v \in N_C(x)^+ \cap N_C(S)$. Then $N_C(x)^+ - \{v\} \cup S$ is an independent set. In any case, we obtain $\alpha(G) \geq d_G(x) + |S| - 1$.

Again, by Lemma 6.12 (i), we have $|V(H)| \leq 2|S|$. Therefore $|V(H)| \leq 2(\alpha(G) - d_G(x) + 1)$ holds, and this implies that $|V(C)| \geq n - 2(\alpha(G) - d_G(x) + 1) \geq n - 2(\alpha(G) - \delta(G) + 1)$. $\square$

We use the following lemma in the proof of Theorem 6.16.

Lemma 6.19 *Let H be a graph with $p(H) \leq d$. Then $\alpha(H) \geq \frac{1}{d}|V(H)|$.*

Proof. Let $\mathcal{P}$ be a set of disjoint paths of H such that any vertex of H is contained in exactly one path in $\mathcal{P}$ and let $l = |\mathcal{P}|$. Choose $\mathcal{P}$ so that l is as small as possible. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_l\}$, and let x_i be an end-vertex of P_i for $1 \leq i \leq l$. By the choice of $\mathcal{P}$, $\{x_1, x_2, \ldots, x_l\}$ is independent. Since $p(H) \leq d$, we obtain $|V(H)| < d \cdot l \leq d \cdot \alpha(H)$. $\square$

Proof of Theorem 6.16.

Let C be a longest cycle in G and $H := G - C$. By Proposition 6.8, $p(H) \leq k-1$. We consider two cases depending on the value of $p(H)$.

Case 1. $p(H) \leq k - 2$.

By Lemma 6.19, we have $\alpha(H) \geq \frac{1}{k-2}|V(H)|$, so $|V(H)| \leq (k-2)\alpha(G)$. Thus, $c(G) = |V(C)| \geq n - (k-2)\alpha(G)$. $\square$

Case 2. $p(H) = k - 1$.

By Lemma 6.19, we have $\alpha(H) \geq \frac{1}{k-1}|V(H)|$. Let S be an independent set of H with $|S| = \alpha(H)$. We take a vertex $x \in V(H)$ so that $d_H(x) = \delta(H)$. Suppose that $d_H(x) \geq k - 1$. Then we can find a path P of order k, contradicting $p(H) \leq k - 1$. Thus, $d_H(x) \leq k-2$ and hence $|N_C(x)| = d_C(x) \geq \delta(G) - k + 2$. Since $\mathrm{diff}(G) \leq k-1$, $N_C(x)^+ \cup S$ is an independent set. Therefore

$$\begin{aligned} \alpha(G) &\geq d_C(x) + \alpha(H) \\ &\geq (\delta(G) - k + 2) + \frac{1}{k-1}|V(H)|, \end{aligned}$$

and so $|V(H)| \leq (k-1)(\alpha(G) - \delta(G) + k - 2)$. Thus, we have $c(G) \geq n - (k-1)(\alpha(G) - \delta(G) + k - 2)$. $\square$

Next, we consider the following conjecture due to Thomassen.

Conjecture 6.20 (Thomassen [63]) *Let G be a k-connected graph with $\alpha(G) \geq k \geq 2$. Then some cycle of G contains k independent vertices and their neighbors.*

The cases $k = 2, 3$ of this conjecture have already proved by Thomassen himself [63] and Manoussakis [119], respectively. J. Li [105] solved this conjecture for the case $\alpha(G) = k + 3$. We establish Conjecture 6.20 for a graph G with $\mathrm{diff}(G) \leq 2$.

Theorem 6.21 *Let G be a k-connected graph with $\alpha(G) \geq k \geq 2$. If $\mathrm{diff}(G) \leq 2$, then some cycle of G contains k independent vertices and their neighbors.*

Theorems 6.5 and 6.21 imply that Conjecture 6.20 is true for a graph G with large $\sigma_4(G)$.

Theorem 6.22 *Let G be a k-connected graph of order n with $\alpha(G) \geq k \geq 3$. If $\sigma_4(G) \geq n + 6$, then some cycle of G contains k independent vertices and their neighbors.*

Proof of Theorem 6.21.

We may assume that G is non-hamiltonian. Let C be a longest cycle in G, and $x \in V(G - C)$. By Lemma 6.12 (i), $d_{G-C}(x) \leq 1$ holds. Let H be a component of $G - C$ which contains x, and let $U := N_C(H)^+$. Since G is k-connected and C is longest, $U \cup \{x\}$ is an independent set of order at least $k+1$. If $U \cap N_C(G-C) = \emptyset$, then $U \cup N_G(U) \subseteq V(C)$, and hence C is a desired cycle. Therefore we may assume that $|N_C(x)^+ \cap N_C(G-C)| \geq 1$. By Lemma 6.12 (ii), $|N_C(x)^+ \cap N_C(G-C)| = 1$ and $d_{G-C}(x) = 0$, and hence $U = N_C(x)^+$, that is, $|U \cap N_C(G-C)| = 1$, say $U \cap N_C(G-C) = \{v\}$. Let $U' := U - \{v\} \cup \{x\}$. Then note that $N_G(U') = N_C(U')$ holds. Take $w \in N_C(U')$ so that $|V(v\overrightarrow{C}w)|$ is as small as possible, and let

$$C' := \begin{cases} xv^-\overleftarrow{C}wx & \text{if } w \in N_C(x), \\ xv^-\overleftarrow{C}uw\overrightarrow{C}u^-x & \text{otherwise,} \end{cases}$$

where $u \in U' - \{x\}$ such that $w \in N_C(u)$. By the choice of w, we have $U \cup N_G(U') \subset V(C')$, and it follows that C' is a desired cycle. $\square$

6.3 Chasing endable vertices of a longest path

In this section, we shall study a graph with $\mathrm{diff}(G) \geq 2$. Actually, in this section, we just assume that no cycles of length at least $p(G) - 1$, otherwise, we do not impose on any condition on our graph G.

Let Q be a longest path of G. Let C be a cycle and P_0 be a path with ends x and y_0 such that $V(C) \cup V(P_0) = V(Q)$, $V(C) \cap V(P_0) = \emptyset$ and $N_C(x) \neq \emptyset$. (Notice that there exist such a cycle C and a path P_0, because the end-vertex of Q has a neighbor in $V(Q)$.) Take such a cycle C and a path P_0 so that $|V(C)|$ is as large as possible. A vertex $y \in V(P_0)$ is called *endable for x* if there exists an xy-path P such that $V(P) = V(P_0)$. For example, a vertex in $N_{P_0}(y_0)^+$ in an endable vertex, where y_0 is the other end vertex of P_0. Let $L := \{y \in V(P_0) : y \text{ is endable for } x\}$ and let $L' := L \cup \{x\}$. We define $\mathcal{T} := \{(y, P) : y \in L \text{ and } P \text{ is an } xy\text{-path such that } V(P) = V(P_0)\}$. For $(y, P) \in \mathcal{T}$, $\overrightarrow{P}$ is an oriented path from x to y. By the maximality of $|V(Q)|$ and $|V(C)|$, the following two claims hold.

Claim 6.1 (i) $N_{G-Q}(L) = \emptyset$.

(ii) *For any $u \in N_C(L')$, $N_{G-C}(u^+) = N_{G-C}(u^-) = \emptyset$.*

Claim 6.2 *Suppose $u_1 \in N_C(L')$ and $u_2 \in N_C(G-C)$ $(u_1 \neq u_2)$. Let $C_1 = u_1^+ \overrightarrow{C} u_2$ and $C_2 = u_2^+ \overrightarrow{C} u_1$. Then the following statements hold.*

(i) $N_{C_1}(u_1^+)^- \cap N_{C_1}(u_2^+) = \emptyset$. *In particular, $u_1^+ u_2^+ \notin E(G)$.*

(ii) $N_{C_2}(u_1^+) \cap N_{C_2}(u_2^+)^- = \emptyset$.

For $u \in N_C(L')$ and $v \in N_C(G - C)$ with $u \neq v$ and $N_C(G - C) \cap V(u^+ \overrightarrow{C} v^-) = \emptyset$, we call $a \in V(u^+ \overrightarrow{C} v^-)$ is *insertible*, if there exists $b \in V(v \overrightarrow{C} u^-)$ such that $ab, ab^+ \in E(G)$; then bb^+ is called an *insertion edge* of a.

Claim 6.3 *For $u_1 \in N_C(L')$ and $u_2 \in N_C(G - C)$ $(u_1 \neq u_2)$, there exists a non-insertible vertex in $V(u_1^+ \overrightarrow{C} u_2^-)$.*

Proof. Assume that there exists no non-insertible vertex in $V(u_1^+ \overrightarrow{C} u_2^-)$. First, we consider the case $u_1 \in N_C(x)$. Take $u_2 \in N_C(G - C)$ as $|V(u_1^+ \overrightarrow{C} u_2)|$ is as small as possible. Note that $u_2 \neq u_1^+$ by Claim 6.1 (ii). Let $w \in N_{G-C}(u_2)$.

Suppose that $w \in V(P_0)$. Let $C' := wu_2 \overrightarrow{C} u_1 x \overrightarrow{P_0} w$, and if $w \neq y_0$, let $P' := w^+ \overrightarrow{P_0} y_0$. Then C' is a cycle such that $V(C') = V(x \overrightarrow{P_0} w \cup C) - V(u_1^+ \overrightarrow{C} u_2^-)$. By the definition of insertible, $V(u_1^+ \overrightarrow{C} u_2^-)$ can be inserted in $u_2 \overrightarrow{C} u_1$. Then, if $w \neq y_0$, we can obtain a cycle C'' and a path P' with ends w^+ and y_0 such that $V(C'') \cup V(P') = V(Q)$, $V(C'') \cap V(P') = \emptyset$, $N_{C''}(w) \neq \emptyset$ and $V(C'') = V(x \overrightarrow{P_0} w \cup C)$. This contradicts the maximality of $|V(C)|$. If $w = y_0$, we obtain a cycle C'' such that $V(C'') = V(Q)$, which contradicts $\mathrm{diff}(G) \geq 2$.

Suppose that $w \notin V(P_0)$. Let $Q' := wu_2 \overrightarrow{C} u_1 x \overrightarrow{P_0} y_0$. Then Q' is a path with $V(Q') = V(Q - u_1^+ \overrightarrow{C} u_2^-) \cup \{w\}$. By inserting $V(u_1^+ \overrightarrow{C} u_2^-)$ in $u_2 \overrightarrow{C} u_1$, we can obtain a path Q'' such that $V(Q'') = V(Q) \cup \{w\}$. This contradicts the maximality of $|V(Q)|$.

Therefore this claim is proved in the case $u_1 \in N_C(x)$. Since we only use the fact $N_C(x) \neq \emptyset$ in the above proof, we can show the case $u_1 \in N_C(L)$ in the same way. $\square$

For $u \in N_C(L')$, let u^* be the first non-insertible vertex along $\overrightarrow{C}$ and $I(u) := V(u^+ \overrightarrow{C} u^*)$. By Claim 6.3, we obtain the following claim.

Claim 6.4 *For any $u \in N_C(L')$, $N_{G-C}(I(u)) = \emptyset$.*

Claim 6.5 *Let $u_1, u_2 \in N_C(L')$ $(u_1 \neq u_2)$. Let $v_i \in I(u_i)$, and let $C_i := v_i \overrightarrow{C} u_{3-i}$ for $i = 1, 2$. Then the followings hold.*

(i) *For $i = 1, 2$, $N_{C_i}(v_i)^- \cap N_{C_i}(v_{3-i}) = \emptyset$, especially $v_1 v_2 \notin E(G)$.*

(ii) $N_C(v_2) \cap N_C(x)^+ \subseteq \{u_2^+\}$.

Proof. Assume that $N_{C_i}(v_i)^- \cap N_{C_i}(v_{3-i}) \neq \emptyset$, say $v \in N_{C_i}(v_i)^- \cap N_{C_i}(v_{3-i})$, for some $i = 1, 2$. Take such v_1, v_2 as $|V(u_1 \overrightarrow{C} v_1 \cup u_2 \overrightarrow{C} v_2)|$ is as small as possible. By the symmetry, we may assume that $i = 1$. Let $y_i \in L'$ such that $u_i \in N_C(y_i)$ for $i = 1, 2$.

Suppose that $y_1 = x$. Let $C' := y_2 u_2 \overleftarrow{C} v^+ v_1 \overrightarrow{C} v v_2 \overrightarrow{C} u_1 x \overrightarrow{P_0} y_2$. Note that C' is a cycle consisting of $V(x \overrightarrow{P_0} y_2 \cup C) - (V(u_1^+ \overrightarrow{C} v_1 \cup u_2^+ \overrightarrow{C} v_2) - \{v_1, v_2\})$. By the minimality of $|V(u_1 \overrightarrow{C} v_1 \cup u_2 \overrightarrow{C} v_2)|$, any edge in $E(u_{3-j} \overrightarrow{C} v_{3-j})$ is not an insertion edge of the vertex in $V(u_j^+ \overrightarrow{C} v_j) - \{v_j\}$ for $j = 1, 2$, and any edge in $E(v_1 \overrightarrow{C} u_2 \cup v_2 \overrightarrow{C} u_1)$ is not a common insertion edge of the vertex in $V(u_1^+ \overrightarrow{C} v_1) - \{v_1\}$ and the vertex in $V(u_2^+ \overrightarrow{C} v_2) - \{v_2\}$. Then $V(u_1^+ \overrightarrow{C} v_1 \cup u_2^+ \overrightarrow{C} v_2) - \{v_1, v_2\}$ can be inserted in C'. Hence we can obtain a cycle C'' and a path $P' := y_2^+ \overrightarrow{P_0} y_0$ (if $y_2 \neq y_0$) with ends y_2^+ and y_0 such that $V(C'') \cup V(P') = V(Q)$, $V(C'') \cap V(P') = \emptyset$, $N_{C''}(y_2^+) \neq \emptyset$ and $V(C'') = V(x \overrightarrow{P_0} y_2 \cup C)$, or obtain a cycle C'' such that $V(C'') = V(Q)$. This contradicts the maximality of $|V(C)|$ or $\mathrm{diff}(G) \geq 2$, respectively.

In the case $y_1 \neq x$, by changing the role of x and y_1, we can give a same proof. Moreover, we can show the statement (ii) in the similar way. $\square$

Claim 6.6 *Let $u_1 \in N_C(x)$, $u_2 \in N_C(L)$ $(u_1 \neq u_2)$, $v_i \in I(u_i)$ and $C_i := u_i^{*+} \overrightarrow{C} u_{3-i}$ for $i = 1, 2$. Then the followings hold.*

(i) $N_{C_i}(v_i)^- \cap N_{C_i}(v_{3-i})^+ = \emptyset$ *for $i = 1, 2$.*

(ii) $N_C(v_2)^- \cap N_C(x)^+ \subseteq \{u_2^+\}$.

Proof. Suppose that $N_{C_i}(v_i)^- \cap N_{C_i}(v_{3-i})^+ \neq \emptyset$ for some $i = 1, 2$. Take such v_1, v_2 as $|V(u_1 \overrightarrow{C} v_1 \cup u_2 \overrightarrow{C} v_2)|$ is as small as possible. Let $y \in L$ with $u_2 \in N_C(y)$, and let P be a path such that $(y, P) \in \mathcal{T}$. Since we will only use the fact that x and y are end-vertices of P, by symmetry, we may assume that $i = 1$. Let $v \in N_{C_1}(v_1)^- \cap N_{C_1}(v_2)^+$. Let $C' := y u_2 \overleftarrow{C} v^+ v_1 \overrightarrow{C} v^- v_2 \overrightarrow{C} u_1 x \overrightarrow{P} y$. Note that C' is a cycle consisting of $V(Q) - (V(u_1^+ \overrightarrow{C} v_1 \cup u_2^+ \overrightarrow{C} v_2) - \{v_1, v_2\}) - \{v\}$. By the choice of v_1, v_2 and Claim 6.5 (i), any edge in $E(u_1 \overrightarrow{C} v_1) \cup E(u_2 \overrightarrow{C} v_2) \cup \{v^- v, v v^+\}$ is not an insertion edge of the vertex in $V(u_1^+ \overrightarrow{C} v_1 \cup u_2^+ \overrightarrow{C} v_2) - \{v_1, v_2\}$ and any edge in $E(v_1 \overrightarrow{C} u_2 \cup v_2 \overrightarrow{C} u_1)$ is not a common insertion edge of the vertex in $V(u_1^+ \overrightarrow{C} v_1) - \{v_1\}$ and the vertex in $V(u_2^+ \overrightarrow{C} v_2) - \{v_2\}$. Hence $V(u_1^+ \overrightarrow{C} v_1 \cup u_2^+ \overrightarrow{C} v_2) - \{v_1, v_2\}$ can be inserted in C'. Therefore we can obtain a cycle containing all vertices of $V(Q) - \{v\}$. This contradicts $\mathrm{diff}(G) \geq 2$. Hence the statement (i) holds. We can similarly prove the statement (ii). $\square$

6.4 Proof of Theorem 6.5

Let G be a graph with $\mathrm{diff}(G) \geq 3$ and we use the same terminology in Section 6.3. In this section, we need the following claim in addition to Claims 6.1 – 6.6.

Claim 6.7 *Let $u_1 \in N_C(x)$, $u_2 \in N_C(L)$ $(u_1 \neq u_2)$, $v_i \in I(u_i)$ and $C_i := u_i^{*+} \overrightarrow{C} u_{3-i}$ for $i = 1, 2$. Then the followings hold.*

(i) $N_{C_i}(v_i)^{-2} \cap N_{C_i}(v_{3-i})^+ = \emptyset$ *for* $i = 1, 2$.

(ii) $N_C(v_2)^{-2} \cap N_C(x)^+ \subseteq \{u_2^+\}$.

Proof. Suppose that $N_{C_i}(v_i)^{-2} \cap N_{C_i}(v_{3-i})^+ \neq \emptyset$ for some $i = 1, 2$. Take such v_1, v_2 as $|V(u_1 \overrightarrow{C} v_1 \cup u_2 \overrightarrow{C} v_2)|$ is as small as possible. Let $y \in L$ with $u_2 \in N_C(y)$, and let P be a path such that $(y, P) \in \mathcal{T}$. Since we will only use the fact that x and y are end-vertices of P, by symmetry, we may assume that $i = 1$. Let $v \in N_{C_1}(v_1)^{-2} \cap N_{C_1}(v_2)^+$. Let $C' := y u_2 \overleftarrow{C} v^{+2} v_1 \overrightarrow{C} v^- v_2 \overrightarrow{C} u_1 x \overrightarrow{P} y$. Note that C' is a cycle consisting of $V(Q) - (V(u_1^+ \overrightarrow{C} v_1 \cup u_2^+ \overrightarrow{C} v_2) - \{v_1, v_2\}) - \{v, v^+\}$. By the choice of v_1, v_2 and Claim 6.5 (i), any edge in $E(u_1 \overrightarrow{C} v_1) \cup E(u_2 \overrightarrow{C} v_2) \cup E(v^- \overrightarrow{C} v^{+2})$ is not an insertion edge of the vertex in $V(u_1^+ \overrightarrow{C} v_1 \cup u_2^+ \overrightarrow{C} v_2) - \{v_1, v_2\}$ and any edge in $E(v_1 \overrightarrow{C} u_2 \cup v_2 \overrightarrow{C} u_1)$ is not a common insertion edge of the vertex in $V(u_1^+ \overrightarrow{C} v_1) - \{v_1\}$ and the vertex in $V(u_2^+ \overrightarrow{C} v_2) - \{v_2\}$. Hence $V(u_1^+ \overrightarrow{C} v_1 \cup u_2^+ \overrightarrow{C} v_2) - \{v_1, v_2\}$ can be inserted in C'. Therefore we can obtain a cycle containing all vertices of $V(Q) - \{v, v^+\}$. This contradicts $\mathrm{diff}(G) \geq 3$. Hence the statement (i) holds. We can similarly prove the statement (ii). $\square$

Now we divide the proof into two cases. Let $u_1 \in N_C(x)$.

Case 1. $|N_C(L) - \{u_1\}| \geq 2$.

Let $u_2, u_3 \in N_C(L) - \{u_1\}$. We may assume that u_1, u_2 and u_3 are arranged in this order along $\overrightarrow{C}$. For $1 \leq i \leq 3$, let $C_i := u_i^{*+}\overrightarrow{C}u_{i+1}$ and $D_i := I(u_i)$. By Claims 6.4 and 6.5 (i), $\{x, u_1^*, u_2^*, u_3^*\}$ is an independent set. Also, we obtain

$$d_{G-C}(x) + d_{G-C}(u_1^*) + d_{G-C}(u_2^*) + d_{G-C}(u_3^*) \leq |V(G-C)| - 1, \tag{6.1}$$

and for $1 \leq i \leq 3$,

$$d_{D_i}(x) + d_{D_i}(u_1^*) + d_{D_i}(u_2^*) + d_{D_i}(u_3^*) \leq |V(D_i)| - 1. \tag{6.2}$$

Claim 6.8 *For $1 \leq i \leq 3$, $d_{C_i}(x) + d_{C_i}(u_1^*) + d_{C_i}(u_2^*) + d_{C_i}(u_3^*) \leq |V(C_i)| + 3$*

Proof. First, we prove the case $i = 1$. By Claim 6.4, we have

(c-1) $N_C(x)^+ \cap N_C(x)^{+2} = \emptyset$.

Moreover, by Claims 6.5–6.7, we obtain;

(a-1) $N_{C_1}(u_1^*)^- \cap N_{C_1}(u_3^*) = \emptyset$ (by Claim 6.5 (i)),

(a-2) $N_{C_1}(u_2^*)^+ \cap N_{C_1}(u_3^*) = \emptyset$ (by Claim 6.5 (i)),

(a-3) $N_{C_1}(u_2^*)^+ \cap N_{C_1}(x)^{+2} = \emptyset$ (by Claim 6.5 (ii)),

(a-4) $N_{C_1}(u_1^*)^- \cap N_{C_1}(u_2^*)^+ = \emptyset$ (by Claim 6.6 (i)),

(a-5) $N_{C_1}(u_1^*)^{-2} \cap N_{C_1}(u_2^*)^+ = \emptyset$ (by Claim 6.7 (i)),

(a-6) $N_{C_1}(u_1^*)^{-2} \cap N_{C_1}(u_3^*) = \emptyset$ (by Claim 6.6 (i)), and

(a-7) $N_{C_1}(u_3^*) \cap N_{C_1}(x)^{+2} = \emptyset$ (by Claim 6.6 (ii)).

Let $A_1 := N_{C_1}(u_1^*)^- \cap N_{C_1}(x)^{+2}$, $A_2 := A_1^-$ and $A_3 := V(C_1) - (A_1 \cup A_2)$. By (c-1), $A_1 \cap A_2 = \emptyset$. Note that C_1 is partitioned into A_1, A_2 and A_3, and $|V(C_1)| = 2|A_1| + |A_3|$. For $1 \leq j \leq 3$, let

$$l_j := |N_{C_1}(u_1^*)^- \cap A_j| + |N_{C_1}(u_2^*)^+ \cap A_j| + |N_{C_1}(u_3^*) \cap A_j| + |N_{C_1}(x)^{+2} \cap A_j|.$$

Obviously, $|N_{C_1}(u_1^*)^- \cap A_1| + |N_{C_1}(x)^{+2} \cap A_1| = 2|A_1|$. Hence $l_1 = 2|A_1|$ by (a-1) and (a-4). Assume $N_{C_1}(u_1^*)^- \cap A_2 \neq \emptyset$, say $v \in N_{C_1}(u_1^*)^- \cap A_2$. Then $v^- \in N_{C_1}(x)$ and $v^+, v^{+2} \in N_{C_1}(u_1^*)$. This contradicts that u_1^* is a non-insertible vertex. Therefore $N_{C_1}(u_1^*)^- \cap A_2 = \emptyset$, and hence by (c-1), (a-5) and (a-6), we have $l_2 = 0$. Obviously, $N_{C_1}(u_1^*)^- \cap N_{C_1}(x)^{+2} \cap A_3 = \emptyset$. By (a-1)–(a-4) and (a-7), $N_{C_1}(u_1^*)^- \cap A_3$, $N_{C_1}(u_2^*)^+ \cap A_3$, $N_{C_1}(u_3^*) \cap A_3$ and $N_{C_1}(x)^{+2} \cap A_3$ are pairwise disjoint. Hence $l_3 \leq |A_3|$.

Thus, we obtain $|N_{C_1}(u_1^*)^- \cap V(C_1)| + |N_{C_1}(u_2^*)^+ \cap V(C_1)| + |N_{C_1}(u_3^*) \cap V(C_1)| + |N_{C_1}(x)^{+2} \cap V(C_1)| \leq \sum_{j=1}^{3} l_j \leq 2|A_1| + |A_3| = |V(C_1)|$. On the other hand, by (c-1), $|N_{C_1}(u_1^*)^- - V(C_1)| + |N_{C_1}(u_2^*)^+ - V(C_1)| + |N_{C_1}(u_3^*) - V(C_1)| + |N_{C_1}(x)^{+2} - V(C_1)| \leq 3$. Therefore we obtain the desired inequality for $i = 1$.

Next, we prove the case $i = 2$. By Claims 6.5–6.7, we obtain;

(b-1) $N_{C_2}(u_1^*) \cap N_{C_2}(u_2^*)^- = \emptyset$ (by Claim 6.5 (i)),

(b-2) $N_{C_2}(u_1^*) \cap N_{C_2}(u_3^*)^+ = \emptyset$ (by Claim 6.5 (i)),

(b-3) $N_{C_2}(u_2^*)^- \cap N_{C_2}(u_3^*) = \emptyset$ (by Claim 6.5 (i)),

(b-4) $N_{C_2}(u_3^*)^+ \cap N_{C_2}(x)^{+2} = \emptyset$ (by Claim 6.5 (ii)),

(b-5) $N_{C_2}(u_1^*)^- \cap N_{C_2}(u_3^*)^+ = \emptyset$ (by Claim 6.6 (i)),

(b-6) $N_{C_2}(u_1^*) \cap N_{C_2}(u_2^*)^{-2} = \emptyset$ (by Claim 6.6 (i)),

(b-7) $N_{C_2}(u_2^*)^- \cap N_{C_2}(x)^+ = \emptyset$ (by Claim 6.6 (ii)),

(b-8) $N_{C_2}(u_3^*) \cap N_{C_2}(x)^{+2} = \emptyset$ (by Claim 6.7 (ii)),

(b-9) $N_{C_2}(u_2^*)^- \cap N_{C_2}(x)^{+2} = \emptyset$ (by Claim 6.6 (ii)),

(b-10) $N_{C_2}(u_1^*)^- \cap N_{C_2}(u_1^*) = \emptyset$ (since u_1^* is a non-insertible vertex), and

(b-11) $N_{C_2}(u_3^*)^- \cap N_{C_2}(u_3^*) = \emptyset$ (since u_3^* is a non-insertible vertex).

Let $B_1 := N_{C_2}(u_1^*) \cap N_{C_2}(x)^{+2}$, $B_2 := B_1^-$, $B_3 := N_{C_2}(u_2^*)^- \cap N_{C_2}(u_3^*)^+$, $B_4 := B_3^-$ and $B_5 := V(C_2) - (B_1 \cup B_2 \cup B_3 \cup B_4)$. Then it follows from (b-10) and (b-11) that $B_1 \cap B_2 = \emptyset$ and $B_3 \cap B_4 = \emptyset$. By (b-1), (b-7) and (b-8), we have $B_1 \cap B_3 = B_2 \cap B_3 = B_1 \cap B_4 = \emptyset$, and so $B_2 \cap B_4 = \emptyset$. Note that C_2 is partitioned into B_1, B_2, B_3, B_4 and B_5, and $|C_2| = 2(|B_1| + |B_3|) + |B_5|$. For $1 \leq j \leq 5$, let

$$m_j := |N_{C_2}(u_1^*) \cap B_j| + |N_{C_2}(u_2^*)^- \cap B_j| + |N_{C_2}(u_3^*)^+ \cap B_j| + |N_{C_2}(x)^{+2} \cap B_j|.$$

Obviously, $|N_{C_2}(u_1^*) \cap B_1| = |N_{C_2}(x)^{+2} \cap B_1| = |B_1|$ and $|N_{C_2}(u_2^*)^- \cap B_3| = |N_{C_2}(u_3^*)^+ \cap B_3| = |B_3|$. By (b-1) and (b-2) (by (b-2) and (b-4)), we obtain $m_1 = 2|B_1|$ ($m_3 = 2|B_3|$, respectively). Furthermore, by (c-1), (b-5), (b-7) and (b-10) (by (b-3), (b-6), (b-8) and (b-11)), we have $m_2 = 0$ ($m_4 = 0$, respectively). By the definition of B_5, $N_{C_2}(u_1^*) \cap N_{C_2}(x)^{+2} \cap B_5 = N_{C_2}(u_2^*)^- \cap N_{C_2}(u_3^*)^+ \cap B_5 = \emptyset$. By (b-1), (b-2), (b-4) and (b-9), $N_{C_2}(u_1^*) \cap B_5$, $N_{C_2}(u_2^*)^- \cap B_5$, $N_{C_2}(u_3^*)^+ \cap B_5$ and $N_{C_2}(x)^{+2} \cap B_5$ are pairwise disjoint. Therefore we obtain $m_5 \leq |B_5|$. Thus, we obtain the desired inequality for $i = 2$ as in the proof of the previous case.

Finally, we prove the case $i = 3$. By Claims 6.5 and 6.6, $N_{C_3}(u_3^*)^-$, $N_{C_3}(u_2^*)$, $N_{C_3}(u_1^*)^+$ and $N_{C_3}(x)^{+2}$ are pairwise disjoint. Therefore we obtain the desired inequality for $i = 3$. $\square$

By (6.1), (6.2) and Claim 6.8, we deduce

$$d_G(x) + d_G(u_1^*) + d_G(u_2^*) + d_G(u_3^*) \leq n + 5,$$

a contradiction. $\square$

Case 2. $|N_C(L) - \{u_1\}| \leq 1$.

For convenience, we rename that $u = u_1$. Since G is 3-connected, we have $N_C(G - C) - \{u\} \neq \emptyset$. Take a vertex $w \in N_C(G - C) - \{u\}$ so that $|V(w\overrightarrow{C}u)|$ is as small as possible. By Claim 6.1 (ii), $w^+ \neq u$ and $N_{G-C}(u^+) = \emptyset$. By the choice of w, $N_{G-C}(w^+) = \emptyset$. Therefore, we obtain

$$d_{G-C}(u^+) + d_{G-C}(w^+) \leq |V(G - Q)|.$$

Let $C_1 := u^+\overrightarrow{C}w$ and $C_2 := w^+\overrightarrow{C}u$. By Claim 6.2 (i), $N_{C_1}(u^+)^- \cap N_{C_1}(w^+) = \emptyset$ and $u^+w^+ \notin E(G)$. Since $N_{C_1}(u^+)^- \cup N_{C_1}(w^+) \subseteq V(C_1)$, we obtain $d_{C_1}(u^+) + d_{C_1}(w^+) \leq |V(C_1)|$. Similarly, by Claim 6.2 (ii), $d_{C_2}(u^+) + d_{C_2}(w^+) \leq |V(C_2)|$. Therefore, we obtain

$$d_C(u^+) + d_C(w^+) \leq |V(C)|.$$

Summing the above inequalities, we deduce

$$d_G(u^+) + d_G(w^+) \leq |V(G - P_0)|. \tag{6.3}$$

Case 2.1. There exist $(y, P) \in \mathfrak{T}$ and $z \in L - \{y\}$ such that $z^+ \notin L$.

Take $(y, P) \in \mathfrak{T}$ and $z \in L - \{y\}$ so that $z^+ \notin L$. Then $yz \notin E(G)$. Hence $\{u^+, w^+, y, z\}$ is an independent set, because $N_P(u^+) = N_P(w^+) = \emptyset$. By Claim 6.1 (i), we obtain $d_{G-Q}(y) + d_{G-Q}(z) = 0$. By the assumption of Case 2, $|N_C(y) - \{u\}| \leq 1$ and $|N_C(z) - \{u\}| \leq 1$. This yields $d_C(y) + d_C(z) \leq 4$. Thus we have

$$d_{G-P}(y) + d_{G-P}(z) \leq 4. \tag{6.4}$$

Let $P_1 := x\overrightarrow{P}z$ and $P_2 := z^+\overrightarrow{P}y$. Suppose that there exists $a \in V(P_1)$ such that $a \in N_{P_1}(y) \cap N_{P_1}(z)^+$. Then we can find an xz^+-path $P' = x\overrightarrow{P}a^-z\overleftarrow{P}ay\overleftarrow{P}z^+$ with $V(P') = V(P) = V(P_0)$, and hence $z^+ \in L$, a contradiction. Therefore $N_{P_1}(y) \cap N_{P_1}(z)^+ = \emptyset$. Since $N_{P_1}(y) \cup N_{P_1}(z)^+ \subseteq V(P_1)$, it follows that

$$d_{P_1}(y) + d_{P_1}(z) \leq |V(P_1)|.$$

Suppose that there exists $b \in V(P_2)$ such that $b \in N_{P_2}(y)^+ \cap N_{P_2}(z)$. Then we can find an xz^+-path $P' = x\overrightarrow{P}zb\overrightarrow{P}yb^-\overleftarrow{P}z^+$, a contradiction again. Since $N_{P_2}(y)^+ \cup N_{P_2}(z) \subseteq V(P_2)$,

$$d_{P_2}(y) + d_{P_2}(z) \leq |V(P_2)|.$$

By these two inequalities,

$$d_P(y) + d_P(z) \leq |V(P)|. \tag{6.5}$$

By (6.3)–(6.5), we obtain

$$d_G(u^+) + d_G(w^+) + d_G(y) + d_G(z) \leq n + 4,$$

a contradiction.

Case 2.2. For any $(y, P) \in \mathfrak{T}$ and $z \in L - \{y\}$, $z^+ \in L$ holds.

If $|N_C(L) - \{u\}| = 1$, then let $u_2 \in N_C(L) - \{u\}$; otherwise let $u_2 = u$. By the assumption of Case 2 and Case 2.2 and by Claim 6.1 (i), we can obtain the following claim.

Claim 6.9 *For any $(y, P) \in \mathfrak{T}$ and $a \in L$, we have $V(a\overrightarrow{P}y) \subseteq L$ and $N_{G-P}(a\overrightarrow{P}y) \subseteq \{u, u_2\}$.*

Claim 6.10 *For any $y \in L$ and $v \in N_C(y)$, we have $|N_C(x) - \{v\}| \leq 1$.*

Proof. If $|N_C(x) - \{v\}| \geq 2$ holds for some $y \in L$ and $v \in N_C(y)$, then we can apply the proof of Case 1 by changing the role of x and y. $\square$

Claim 6.11 *For any $(y, P) \in \mathfrak{T}$, we have $u \notin N_C(y)$.*

Proof. Assume that there exists $(y, P) \in \mathfrak{T}$ such that $u \in N_C(y)$. Suppose that $N_P(x)^- \cap N_P(y) \neq \emptyset$. Then $x^+ \in L$ holds and so Claim 6.9 implies that $V(x^+\overrightarrow{P}y) \subseteq L$ and $N_{G-P}(x^+\overrightarrow{P}y) \subseteq \{u, u_2\}$. Hence both $\{x, u, u_2\}$ and $N_C(x) \cup \{u, u_2\}$ are cut sets of G. Therefore $u \neq u_2$ and $N_C(x) - \{u, u_2\} \neq \emptyset$, because G is 3-connected. This contradicts Claim 6.10. Therefore $N_P(x)^- \cap N_P(y) = \emptyset$, especially $xy \notin E(G)$. Since $N_P(x)^- \cup N_P(y) \subseteq V(P) - \{y\}$, it follows that $d_P(x) + d_P(y) \leq |V(P)| - 1$. By Claim 6.10 and the maximality of $|V(Q)|$, we have $|N_{G-P}(x)| \leq 2$, and hence $d_{G-P}(x) + d_{G-P}(y) \leq 4$. Thus, by the inequality (6.3), we deduce

$$\begin{aligned} & d_G(u^+) + d_G(w^+) + d_G(x) + d_G(y) \\ \leq \;& |V(P)| + |V(G-P)| + 3 \\ = \;& n + 3, \end{aligned}$$

a contradiction. $\square$

We choose $(y, P) \in \mathfrak{T}$ and $a \in N_P(y)$ so that $|V(x\overrightarrow{P}a)|$ is as small as possible. Note that $a^+ \in L$. Since G is 3-connected, $G - \{a, u_2\}$ is connected. Hence by Claims 6.9 and 6.11, $a \neq x$ and there exist $b \in V(a^+\overrightarrow{P}y)$ and $c \in V(x\overrightarrow{P}a^-)$ such that $bc \in E(G)$. Let $P_1 := x\overrightarrow{P}a^-$, $P_2 := a\overrightarrow{P}b^-$ and $P_3 := b\overrightarrow{P}y$. Then by the choice of a, we have $N_{P_1}(y) = \emptyset$, and hence

$$d_{P_1}(y) + d_{P_1}(b^-) \leq |V(P_1)|.$$

Suppose that there exists $d \in V(P_2)$ such that $d \in N_{P_2}(y) \cap N_{P_2}(b^-)^+$. Then we can find a path $P' := x\overrightarrow{P}d^-b^-\overleftarrow{P}dy\overleftarrow{P}b$. Then $V(P') = V(P)$, $c \in N_{P'}(b)$ and $|V(x\overrightarrow{P}a)| > |V(x\overrightarrow{P'}c)|$, contradicting the choice of (y, P) and a. Thus, we have $N_{P_2}(y) \cap N_{P_2}(b^-)^+ = \emptyset$, especially $yb^- \notin E(G)$. Hence $\{u^+, w^+, y, b^-\}$ is an independent set. Also, since $N_{P_2}(y) \cup N_{P_2}(b^-)^+ \subseteq V(P_2)$,

$$d_{P_2}(y) + d_{P_2}(b^-) \leq |V(P_2)|.$$

If there exists $d \in V(P_3)$ such that $d \in N_{P_3}(y)^+ \cap N_{P_3}(b^-)$, then we can find $P' := x\overrightarrow{P}b^-d\overrightarrow{P}yd^-\overleftarrow{P}b$. This contradicts the choice of (y, P) and a, again. Thus, we have $N_{P_3}(y)^+ \cap N_{P_3}(b^-) = \emptyset$. Since $N_{P_3}(y)^+ \cup N_{P_3}(b^-) \subseteq V(P_3)$,

$$d_{P_3}(y) + d_{P_3}(b^-) \leq |V(P_3)|.$$

Since $yb^- \notin E(G)$, it follows that $b^- \in V(a^+\overrightarrow{P}y)$. By Claims 6.9 and 6.11,

$$d_{G-P}(y) + d_{G-P}(b^-) \leq 2.$$

By these four inequalities,

$$d_G(y) + d_G(b^-) \leq |V(P)| + 2. \tag{6.6}$$

By the inequalities (6.3) and (6.6), we obtain

$$d_G(u^+) + d_G(w^+) + d_G(y) + d_G(b^-) \leq n + 2,$$

a contradiction. $\square$

6.5 Proofs of Theorems 6.10 and 6.11

In this section, in addition to the assumption in Section 6.3, we assume that G has no hamilton paths. This implies that there exists a vertex z in $V(H)$, where $H := G - Q$.

Claim 6.12 *Suppose $u_1 \in N_C(x)$ and $u_2 \in N_C(L)$ with $u_1 \neq u_2$. Let $C_1 = u_1^+\overrightarrow{C}u_2$ and $C_2 = u_2^+\overrightarrow{C}u_1$. Then the following statements hold.*

(i) $N_{C_1}(u_1^+)^- \cap N_{C_1}(z) = \emptyset$.

(ii) $N_{C_1}(u_2^+)^+ \cap N_{C_1}(z) = \emptyset$.

(iii) $N_{C_2}(u_1^+)^+ \cap N_{C_2}(z) = \emptyset$.

Proof. By the maximality of $|Q|$, it is easy to show the statement (i).

Assume that $N_{C_1}(u_2^+)^+ \cap N_{C_1}(z) \neq \emptyset$, say $v \in N_{C_1}(u_2^+)^+ \cap N_{C_1}(z)$. Let $(y, P) \in \mathfrak{T}$ such that $u_2 \in N_{C_1}(y)$. Then $zv\overrightarrow{C}u_2y\overleftarrow{P}xu_1\overrightarrow{C}v^-u_2^+\overrightarrow{C}u_1^-$ is a longer path than Q, a contradiction. Hence the statement (ii) holds. Similarly, we can prove (iii). $\square$

Hereafter, we divide the proof of Theorems 6.10 and 6.11 into two cases. Suppose there exists $(y, P) \in \mathfrak{T}$ such that there are two independent edges e_1, e_2 joining x and C, and y and C, respectively. Then we may think of x, y as symmetric vertices, with respect to $(y, P) \in \mathfrak{T}$. This is our first case. The second case is just dealing with the case where there are no such two independent edges.

Case 1. $|N_C(L) - \{u_1\}| \geq 1$.

In this case, we can use the symmetry between x and y. Now we will show that $d_{G-C}(y) + d_{G-C}(z) \leq |V(G - C)| - 2$. Note that $V(G - C) = V(P) \cup V(H)$. If there exists $a \in N_P(y)^+ \cap N_P(z)$, then a path $u^+\overrightarrow{C}ux\overrightarrow{P}a^-y\overleftarrow{P}az$ is a longer path than Q, where $u \in N_C(x)$. This contradiction yields $N_P(y)^+ \cap N_P(z) = \emptyset$.

Since $N_P(y)^+ \cup N_P(z) \subset V(P) - \{x\}$, it follows that $d_P(y) + d_P(z) \leq |V(P)| - 1$. By Claim 6.1 (i), we have $N_H(y) = \emptyset$. Therefore $d_H(y) + d_H(z) \leq |V(H)| - 1$. Hence we obtain

$$d_{G-C}(y) + d_{G-C}(z) \leq |V(G - C)| - 2. \tag{6.7}$$

Claim 6.13 (i) $|N_C(x)^+ \cap N_C(z)^-| \leq 1$ *or* $|N_C(y)^+ \cap N_C(z)^-| \leq 1$.

(ii) *If* $N_C(x)^+ \cap N_C(z)^- \neq \emptyset$ *and* $N_C(y)^+ \cap N_C(z)^- \neq \emptyset$, *then* $N_C(x)^+ \cap N_C(z)^- = N_C(y)^+ \cap N_C(z)^-$.

Proof. Assume that the statement (i) or (ii) does not hold. Then we can easily find two vertices $v_1 \in N_C(x)^+ \cap N_C(z)^-$ and $v_2 \in N_C(y)^+ \cap N_C(z)^-$ such that $v_1 \neq v_2$. Then $v_1^+\overrightarrow{C}v_2^-y\overleftarrow{P}xv_1^-\overleftarrow{C}v_2^+zv_1^+$ is a cycle of order $p(G) - 1$, a contradiction. $\square$

Let us now consider the proofs of Theorem 6.10 and 6.11, respectively.

6.5.1 Proof of Theorem 6.10 in Case 1

In this section, we use the following lemma shown by Enomoto, van den Heuvel, Kaneko and Saito.

Lemma 6.23 (Enomoto, van den Heuvel, Kaneko and Saito [46]) *Suppose that G is a graph of order n with $\mathrm{diff}(G) \geq 2$. Let P be a longest path in G and let $x, y \in V(G)$ be end-vertices of P. If there exists $z \in V(G - P)$, then $d_G(x) + d_G(y) + d_G(z) \leq n - 1$.*

Suppose now that G has a longest path Q, but Q is not a hamilton path. Furthermore, suppose G satisfies the assumption of Theorem 6, but $p(G) - c(G) \geq 2$. We shall prove that there are four independent vertices such that the degree sum of these vertices is at most $\frac{1}{3}(4n-3)$. This would be a contradiction to the assumption of Theorem 6.10.

If $N_C(x) \cap N_C(y) \neq \emptyset$, then by symmetry we may assume that $|N_C(x)| \geq |N_C(y)|$; otherwise, by Claim 6.13 (i) and (ii), $N_C(x)^+ \cap N_C(z)^- = \emptyset$ or $N_C(y)^+ \cap N_C(z)^- = \emptyset$ and by symmetry, we may assume that $N_C(y)^+ \cap N_C(z)^- = \emptyset$.

By the assumption of Case 1, there exist two distinct vertices $u \in N_C(x)$ and $w \in N_C(y)$. We choose u and w as follows: if $N_C(x) \cap N_C(y) \neq \emptyset$, then we can choose such vertices u and w so that $w \in N_C(x) \cap N_C(y)$; otherwise we choose such vertices u and w so that $|V(w^+ \overrightarrow{C} u)|$ is as small as possible. By Claim 6.1 (ii), we have $w^+ \neq u$. By Claims 6.1 and 6.5 (i), $\{u^+, w^+, y, z\}$ is an independent set. By applying Lemma 6.23 to paths $u^+ \overrightarrow{C} u x \overrightarrow{P} y$ and $u^+ \overrightarrow{C} w y \overleftarrow{P} x u \overleftarrow{C} w^+$, we obtain

$$d_G(u^+) + d_G(y) + d_G(z) \leq n - 1 \tag{6.8}$$

and

$$d_G(u^+) + d_G(w^+) + d_G(z) \leq n - 1, \tag{6.9}$$

respectively. Let $C_1 = u^+ \overrightarrow{C} w$ and $C_2 = w^+ \overrightarrow{C} u$.

Claim 6.14 $d_G(u^+) + d_G(w^+) + d_G(y) \leq n$.

Proof. By Claims 6.2 (i) and 6.6 (ii), we have $N_{C_1}(u^+)^- \cap N_{C_1}(w^+) = N_{C_1}(w^+) \cap N_{C_1}(y)^+ = N_{C_1}(u^+)^- \cap N_{C_1}(y)^+ = \emptyset$. Since $N_{C_1}(u^+)^- \cup N_{C_1}(w^+) \cup N_{C_1}(y)^+ \subset V(C_1) \cup \{w^+\}$, we obtain

$$d_{C_1}(u^+) + d_{C_1}(w^+) + d_{C_1}(y) \leq |V(C_1)| + 1.$$

Suppose that $N_C(x) \cap N_C(y) \neq \emptyset$, then $w \in N_C(x) \cap N_C(y)$. Using Claims 6.2 (ii) and 6.6 (ii), by the same argument as the case of C_1, we obtain $d_{C_2}(u^+) + d_{C_2}(w^+) + d_{C_2}(y) \leq |V(C_2)| + 1$.

On the other hand, suppose that $N_C(x) \cap N_C(y) = \emptyset$. Then by Claim 6.2 (ii), $N_{C_2}(u^+) \cap N_{C_2}(w^+)^- = \emptyset$ and hence $d_{C_2}(u^+) + d_{C_2}(w^+) \leq |V(C_2)|$. It follows from the choice of u and w that $N_{C_2}(y) = \emptyset$. Thus, in each case, we have

$$d_{C_2}(u^+) + d_{C_2}(w^+) + d_{C_2}(y) \leq |V(C_2)| + 1.$$

By the above inequalities, we obtain $d_C(u^+) + d_C(w^+) + d_C(y) \le |V(C)| + 2$.

By Claim 6.1 (ii), $N_{G-C}(u^+) = N_{G-C}(w^+) = \emptyset$, and by Claim 6.1 (i), $N_{G-C}(y) \subset V(P) - \{y\}$. Therefore, $d_{G-C}(u^+) + d_{G-C}(w^+) + d_{G-C}(y) \le |V(P)| - 1 \le |V(G - C)| - 2$. Thus, we obtain $d_G(u^+) + d_G(w^+) + d_G(y) \le n$. $\square$

Claim 6.15 $d_G(w^+) + d_G(y) + d_G(z) \le n - 1$.

Proof. If $u \in N_C(y)$, then $N_C(x) \cap N_C(y) \ne \emptyset$, and so $w \in N_C(x) \cap N_C(y)$. By applying Lemma 6.23 to a path $w^+ \overrightarrow{C} wx \overrightarrow{P} y$, we obtain $d_G(w^+) + d_G(y) + d_G(z) \le n - 1$. Hence we may assume that $u \notin N_C(y)$ and moreover $N_C(x) \cap N_C(y) = \emptyset$. Then, by the choice of y, $N_C(y)^+ \cap N_C(z)^- = \emptyset$.

By Claims 6.5 (ii) and 6.12 (ii), $N_{C_1}(w^+) \cap N_{C_1}(y)^+ = N_{C_1}(w^+) \cap N_{C_1}(z)^- = \emptyset$. By Claim 6.1 (ii), we have $u \notin N_C(z)^-$ and hence $N_{C_1}(w^+) \cup N_{C_1}(y)^+ \cup N_{C_1}(z)^- \subset V(C_1) \cup \{w^+\}$. Therefore

$$d_{C_1}(w^+) + d_{C_1}(y) + d_{C_1}(z) \le |V(C_1)| + 1.$$

By Claim 6.12 (i), $N_{C_2}(w^+)^- \cap N_{C_2}(z) = \emptyset$. By the choice of u and w, we have $N_{C_2}(y) = \emptyset$. Since $N_{C_2}(w^+)^- \cup N_{C_2}(z) \subset V(C_2)$, it follows that

$$d_{C_2}(w^+) + d_{C_2}(y) + d_{C_2}(z) \le |V(C_2)|.$$

By the above inequalities,

$$d_C(w^+) + d_C(y) + d_C(z) \le |V(C)| + 1. \tag{6.10}$$

By Claim 6.1 (ii) and by the inequalities (6.7) and (6.10), we obtain $d_G(w^+) + d_G(y) + d_G(z) \le n - 1$. $\square$

Thus, by the inequalities (6.8) and (6.9) and by Claims 6.14 and 6.15, we obtain

$$\begin{aligned} 3\sigma_4(G) &\le 3\big(d_G(u^+) + d_G(w^+) + d_G(y) + d_G(z)\big) \\ &\le 4n - 3, \end{aligned}$$

a contradiction. $\square$

6.5.2 Proof of Theorem 6.11 in Case 1

Suppose now that G has a longest path Q, but Q is not a hamilton path. Furthermore, suppose G satisfies the assumption of Theorem 6.11, but $p(G) - c(G) \ge 3$. We shall prove that there are four independent vertices such that the degree sum of these vertices is at most $n+2$, which would be a contradiction to the assumption of Theorem 6.11.

If $N_C(x) \cap N_C(y) \cap N_C(z)^{-2} = \emptyset$, then by Claim 6.13 (ii) and by the symmetry of x and y, we may assume that $N_C(y)^{+2} \cap N_C(z) = \emptyset$. In this case, we can take $u \in N_C(x)$ and $w \in N_C(y) - \{u\}$. If $N_C(x) \cap N_C(y) \cap N_C(z)^{-2} \neq \emptyset$, then by the assumption of Case 1 and by the symmetry of x and y, we may assume that there exist two distinct vertices $u \in N_C(x) \cap N_C(y) \cap N_C(z)^{-2}$ and $w \in N_C(y)$. In this case, $N_C(y)^{+2} \cap N_C(z) \subset \{u\}$ by Claim 6.13 (i) and (ii). In both cases, we choose such $w \in N_C(y)$ so that $|V(w^+ \overrightarrow{C} u)|$ is as small as possible. By Claims 6.1 and 6.2 (i), $\{u^+, w^+, y, z\}$ is independent. Let $C_1 = u^+ \overrightarrow{C} w$ and $C_2 = w^+ \overrightarrow{C} u$. We will show that $d_{C_i}(u^+) + d_{C_i}(w^+) + d_{C_i}(y) + d_{C_i}(z) \leq |V(C_i)| + 2$ for $i = 1, 2$.

First we show the case $i = 1$. Then we obtain $N_{C_1}(u^+)^-$, $N_{C_1}(z)$, $N_{C_1}(w^+)^+$ and $N_{C_1}(y)^{+2}$ are pairwise disjoint because

$N_{C_1}(u^+)^- \cap N_{C_1}(z) = \emptyset$ (by Claim 6.12 (ii)),

$N_{C_1}(u^+)^- \cap N_{C_1}(w^+)^+ = \emptyset$ (by Claim 6.6 (i)),

$N_{C_1}(u^+)^- \cap N_{C_1}(y)^{+2} = \emptyset$ (since $\mathrm{diff}(G) \geq 3$),

$N_{C_1}(z) \cap N_{C_1}(w^+)^+ = \emptyset$ (by Claim 6.2 (i)),

$N_{C_1}(z) \cap N_{C_1}(y)^{+2} = \emptyset$ (by the choice of y), and

$N_{C_1}(w^+)^+ \cap N_{C_1}(y)^{+2} = \emptyset$ (by Claim 6.2 (i)).

Since $N_{C_1}(u^+)^- \cup N_{C_1}(z) \cup N_{C_1}(w^+)^+ \cup N_{C_1}(y)^{+2} \subset V(C_1) \cup \{w^+, w^{+2}\}$, we have

$$d_{C_1}(u^+) + d_{C_1}(w^+) + d_{C_1}(y) + d_{C_1}(z) \leq |V(C_1)| + 2.$$

Next, we consider the case $i = 2$. By the choice of w, we have $N_{C_2}(y) \subset \{u\}$, and hence $d_{C_2}(y) \leq 1$. Moreover, by Claims 6.6 (i), 6.12 (i) and (ii), $N_{C_2}(w^+)^-$, $N_{C_2}(z)$ and $N_{C_2}(u^+)^+$ are pairwise disjoint. Since $N_{C_2}(w^+)^- \cup N_{C_2}(z) \cup N_{C_2}(u^+)^+ \subset V(C_2) \cup \{u^+\}$, we have

$$d_{C_2}(u^+) + d_{C_2}(w^+) + d_{C_2}(y) + d_{C_2}(z) \leq |V(C_2)| + 2.$$

By Claim 6.1 (ii) and by the inequality (6.7),

$$d_G(u^+) + d_G(w^+) + d_G(y) + d_G(z) \leq n + 2,$$

a contradiction. $\square$

6.5.3 Proofs of Theorems 6.10 and 6.11 in Case 2

Let us remind Case 2.

Case 2. $N_C(L) - \{u_1\} = \emptyset$.

Let $u \in N_C(x)$. Since G is 2-connected, there exists a path R connecting a vertex of P_0 and a vertex of $C - \{u\}$. Let $\{w\} = V(R) \cap (V(C) - \{u\})$. We take such a path R and a vertex w so that $|V(w^+ \overrightarrow{C} u)|$ is as small as possible. By Claims 6.1 (ii) and 6.2 (i), we have $w^+ \neq u$, $u^+ \neq w$ and $u^+w^+ \notin E(G)$.

We first show that we may assume $n \geq 9$. Then, since $n + 2 \leq \frac{1}{3}(4n - 3)$, we can prove Theorems 6.10 and 6.11 simultaneously. That is, we shall just prove that there are four independent vertices such that the degree sum of these vertices is at most $n + 2$. This would contradict the assumptions of both Theorems 6.10 and 6.11.

Suppose $n \leq 8$. Then it is easy to see that $|V(C)| \geq 4$. Since G is 2-connected, the assumption of Case 2 implies $|V(P_0)| \geq 3$. Since $|V(H)| \geq 1$, it follows that $|V(P_0)| = 3$. Let $P_0 = xx'x''$. Since G is 2-connected, we have $xx'', wx' \in E(G)$. Then $x' \in L$ and $w \in N_C(x')$, which contradicts the assumption of Case 2. Hence we may assume $n \geq 9$.

By the choice of w, we have $N_{P_0}(w^+) = \emptyset$, and so $N_{G-C}(w^+) \subset V(H)$. By Claim 6.1 (ii), $N_{G-C}(u^+) = \emptyset$. Hence we obtain

$$d_{G-C}(u^+) + d_{G-C}(w^+) \leq |V(H)|.$$

Let $C_1 := u^+ \overrightarrow{C} w$ and $C_2 := w^+ \overrightarrow{C} u$. By Claim 6.2 (i), $N_{C_1}(u^+)^- \cap N_{C_1}(w^+) = \emptyset$. Since $N_{C_1}(u^+)^- \cup N_{C_1}(w^+) \subset V(C_1)$, we obtain $d_{C_1}(u^+) + d_{C_1}(w^+) \leq |V(C_1)|$. Similarly, by Claim 6.2 (ii), $d_{C_2}(u^+) + d_{C_2}(w^+) \leq |V(C_2)|$. Therefore, we obtain

$$d_C(u^+) + d_C(w^+) \leq |V(C)|.$$

Summing the above inequalities, we have

$$d_G(u^+) + d_G(w^+) \leq |V(C)| + |V(H)|. \tag{6.11}$$

Case 2.1. There exist $(y, P) \in \mathfrak{T}$ and $v \in L - \{y\}$ such that $v^+ \notin L$.

Fix such a vertex $v \in L - \{y\}$. Suppose that $yv \in E(G)$. Then we can find a path $P' := x \overrightarrow{P} vy \overleftarrow{P} v^+$, and hence v^+ is endable for x, contradicting the assumption of Case 2.1. Thus, we have $yv \notin E(G)$. By the assumption of Case 2, $N_C(y) - \{u\} = N_C(v) - \{u\} = \emptyset$. Therefore, $\{u^+, w^+, y, v\}$ is an independent set and

$$d_C(y) + d_C(v) \leq 2. \tag{6.12}$$

Since $y, v \in L$, it follows from Claim 6.1 (i) that $N_H(y) = N_H(v) = \emptyset$, and so

$$d_H(y) + d_H(v) = 0. \tag{6.13}$$

Let $P_1 := x\overrightarrow{P}v$ and $P_2 := v^+\overrightarrow{P}y$. Suppose that there exists $a \in N_{P_1}(y) \cap N_{P_1}(v)^+$. Then we can find $P' := x\overrightarrow{P}a^-v\overleftarrow{P}ay\overleftarrow{P}v^+$, and hence v^+ is endable for x, which contradicts the definition of v. Therefore, $N_{P_1}(y) \cap N_{P_1}(v)^+ = \emptyset$. Since $N_{P_1}(y) \cup N_{P_1}(v)^+ \subset V(P_1)$, we obtain $d_{P_1}(y) + d_{P_1}(v) \le |V(P_1)|$. Suppose that there exists $b \in N_{P_2}(y)^+ \cap N_{P_2}(v)$. Then we can find $P' := x\overrightarrow{P}vb\overrightarrow{P}yb^-\overleftarrow{P}v^+$, and hence v^+ is endable for x, a contradiction again. Therefore $d_{P_2}(y) + d_{P_2}(v) \le |V(P_2)|$, since $N_{P_2}(y)^+ \cup N_{P_2}(v) \subset V(P_2)$. Thus, we obtain

$$d_P(y) + d_P(v) \le |V(P)|. \tag{6.14}$$

By the inequalities (6.11)–(6.14), we deduce

$$\begin{aligned} & d_G(u^+) + d_G(w^+) + d_G(y) + d_G(v) \\ \le \;& |V(P)| + |V(C)| + |V(H)| + 2 \\ = \;& n + 2, \end{aligned}$$

a contradiction. This contradiction completes the proof of Case 2.1.

Case 2.2. For any $(y, P) \in \mathcal{T}$ and any $v \in L - \{y\}$, $v^+ \in L$.

In this case, we have two claims.

Claim 6.16 *For any $(y, P) \in \mathcal{T}$ and any $a \in L$, $V(a\overrightarrow{P}y) \subset L$ and $N_{G-P}(a\overrightarrow{P}y) \subset \{u\}$.*

Proof. Suppose that there exists $v \in V(a\overrightarrow{P}y)$ such that $v \notin L$. Then, since $a \in L$, we can find a vertex $v_1 \in V(a\overrightarrow{P}v^-)$ such that $v_1 \in L$ and $v_1^+ \notin L$. This contradicts the assumption of Case 2.2. Therefore we have $V(a^+\overrightarrow{P}y) \subset L$. Moreover, by the condition of Case 2 and by Claim 6.1 (i), we obtain $N_{G-P}(a\overrightarrow{P}y) \subset \{u\}$. $\square$

Claim 6.17 *For any $(y, P) \in \mathcal{T}$, $N_C(y) = \emptyset$.*

Proof. Assume that there exists $(y, P) \in \mathcal{T}$ such that $N_C(y) \neq \emptyset$. Then $N_C(x) = N_C(y) = \{u\}$ by the assumption of Case 2. Suppose that $N_P(x)^- \cap N_P(y) \neq \emptyset$, say $a \in N_P(x)^- \cap N_P(y)$. Then $P' = xa^+\overrightarrow{P}ya\overleftarrow{P}x^+$ is a path such that $V(P') = V(P)$, which implies $x^+ \in L$. So all the vertices of P except for x are in L.

By Claim 6.16, $N_{G-P}(x^+\overrightarrow{P}y) \subset \{u\}$, and hence $N_{G-P}(P) \subset \{u\}$. This contradicts that G is 2-connected. Therefore $N_P(x)^- \cap N_P(y) = \emptyset$ and especially $xy \notin E(G)$. Since $N_P(x)^- \cup N_P(y) \subset V(P) - \{y\}$, we have $d_P(x) + d_P(y) \le |V(P)| - 1$.

Since x is endable for y, it follows from Claim 6.16 that $d_{G-P}(x) + d_{G-P}(y) \le 2$. Thus, by the inequalities (6.11), we obtain

$$\begin{aligned} & d_G(u^+) + d_G(w^+) + d_G(x) + d_G(y) \\ \le \ & |V(P)| + |V(C)| + |V(H)| + 1 \\ = \ & n + 1, \end{aligned}$$

a contradiction. $\square$

We choose $(y, P) \in \mathfrak{T}$ and $a \in N_P(y)$ so that $|V(x\overrightarrow{P}a)|$ is as small as possible. By Claims 6.16 and 6.17, $N_G(a^+\overrightarrow{P}y) \subset V(x\overrightarrow{P}a)$. If $a = x$, then $|V(x\overrightarrow{P}a)| = 1$, contradicting 2-connectedness. Thus we obtain $a \ne x$. Since $G - \{a\}$ is connected, there exists an edge bc such that $b \in V(a^+\overrightarrow{P}y)$ and $c \in V(x\overrightarrow{P}a^-)$. Suppose that $yb^- \in E(G)$. Let $P' := x\overrightarrow{P}b^-y\overleftarrow{P}b$. Then $V(P') = V(P)$, $c \in N_G(b)$ and $|V(x\overrightarrow{P}a)| > |V(x\overrightarrow{P'}c)|$, contradicting the choice of y and a. Thus, we have $yb^- \notin E(G)$ and $b^- \in V(a^+\overrightarrow{P}y)$. Since $N_P(u^+) \cup N_P(w^+) = \emptyset$, $\{u^+, w^+, y, b^-\}$ is an independent set. By Claims 6.16 and 6.17,

$$d_{G-P}(y) + d_{G-P}(b^-) = 0. \tag{6.15}$$

Let $P_1 := x\overrightarrow{P}a^-$, $P_2 := a\overrightarrow{P}b^-$ and $P_3 := b\overrightarrow{P}y$. Then the choice of y and a implies $N_{P_1}(y) = \emptyset$, and hence

$$d_{P_1}(y) + d_{P_1}(b^-) \le |V(P_1)|.$$

Suppose that there exists $d \in N_{P_2}(y) \cap N_{P_2}(b^-)^+$. Then we can find a path $P' := x\overrightarrow{P}d^-b^-\overleftarrow{P}dy\overleftarrow{P}b$. Then $V(P') = V(P)$, $c \in N_{P'}(b)$ and $|V(x\overrightarrow{P}a)| > |V(x\overrightarrow{P'}c)|$, contradicting the choice of y and a. Thus, we have $N_{P_2}(y) \cap N_{P_2}(b^-)^+ = \emptyset$. Since $N_{P_2}(y) \cup N_{P_2}(b^-)^+ \subset V(P_2)$,

$$d_{P_2}(y) + d_{P_2}(b^-) \le |V(P_2)|.$$

If there exists $d \in N_{P_3}(y)^+ \cap N_{P_3}(b^-)$, then we can find $P' := x\overrightarrow{P}b^-d\overrightarrow{P}yd^-\overleftarrow{P}b$. This contradicts the choice of y and a, again. Thus, we have $N_{P_3}(y)^+ \cap N_{P_3}(b^-) = \emptyset$. Therefore, since $N_{P_3}(y)^+ \cup N_{P_3}(b^-) \subset V(P_3)$, we obtain

$$d_{P_3}(y) + d_{P_3}(b^-) \le |V(P_3)|.$$

Summing the above inequalities, we have

$$d_P(y) + d_P(b^-) \le |V(P)|. \tag{6.16}$$

Therefore, it follows from the inequalities (6.11), (6.15) and (6.16) that

$$\begin{aligned} & d_G(u^+) + d_G(w^+) + d_G(y) + d_G(b^-) \\ \le \ & |V(P)| + |V(C)| + |V(H)| \\ = \ & n, \end{aligned}$$

a contradiction. $\square$

Chapter 7

Circumference of a graph

Similarly to a graph having a dominating cycle or the low relative length, we are interested in measuring how far given graphs are being from hamiltonian. One of the methods of it is the invariant "circumference," that is the length of a longest cycle. So we regard a graph with large circumference as "close" to hamiltonian. In this chapter, we survey some lower bounds of the circumference concerning with some particular classes of graphs or several graph invariants. In particular, we mention the relationship between the circumference and the relative length of a graph.

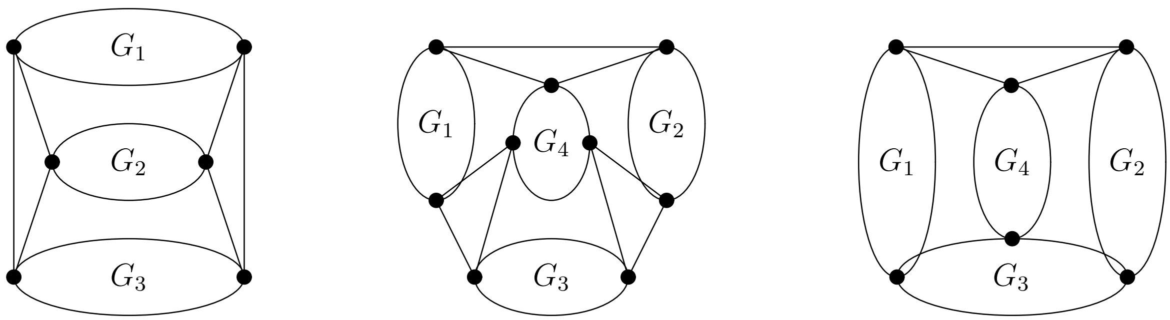
The contents of this chapter are based on the paper [138] "Length of longest cycles and paths and degree sum," jointwork with T. Yamashita.

7.1 Particular classes of graphs

In this section, we concentrate on the lower bound of the circumference of some particular classes.

7.1.1 Claw-free graphs

A study on the circumference of a claw-free graph originates the result by Matthews and Sumner [123]; for a 2-connected claw-free graph G of order n, $c(G) \geq \min\{2\delta(G)+4, n\}$. At the beginning of this result, the circumference of a 2-connected or 3-connected claw-free graph has been considered. Flandrin, Fournier and Germa [58] improved Matthews and Sumner's result to $c(G) \geq \min\{\sigma_2(G)+4, n\}$. On the other hand, it is known that the lower bound of $c(G)$ in Matthews and Sumner's result is best possible, but Li [106, 107] showed that avoiding some particular classes of graphs, we can increase the lower bound of $c(G)$ to $\min\{4\delta(G)-2, n\}$. He avoided three classes $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ in Figure 7.1, where G_1, G_2, G_3 and G_4 are connected graphs.



The set of graphs in $\mathfrak{J}_1$ The set of graphs in $\mathfrak{J}_2$ The set of graphs in $\mathfrak{J}_3$

Figure 7.1: The avoiding classes.

Analogously there are some results on the circumference of a 3-connected claw-free graph. Before mentioning them, we introduce some results on hamiltonicity of such graphs. Favaron and Fraisse [57] proved that if $\delta(G) \geq \frac{n+37}{10}$ for a 3-conneted claw-free graph G of order n, then G is hamiltonian. They also conjectured that we can decrease the lower bound of the minimum degree condition to $\frac{n+6}{10}$ when n is sufficiently large, and Lai, Shao and Zhan [104] gave a positive answer to this conjecture.

M.C. Li [108] showed that for a 3-connected d-regular claw-free graph G of order n, $c(G) \geq \min\{6d-17, n\}$ and M.C. Li and Xiong [110] proved the same conclusion holds even when G is not a regular graph but $\delta(G) \geq d$. Recently, M.C. Li, Cui, Xiong, Tian, Jiang abd Yuan [109] improved the lower bound of their result to $c(G) \geq \min\{6\delta - 15, n\}$.

For other results on a claw-free graph, we refer the reader to the claw-free survey [53].

7.1.2 Regular graphs

In 1980, Jackson [88] proved that any 2-connected d-regular graph of order $n \leq 3d$ is hamiltonian and Jackson and Li [89] showed that the upper bound $3d$ of order can be decreased to $6d - 38$ if the graph is bipartite. As an extension of Jackson's result on a general regular graph, Bondy made the following conjecture;

Conjecture 7.1 (Bondy [25]) *Let G be a 2-connected d-regular graph of order $n \leq rd$, where $r \geq 3$ and n is sufficient large. Then $c(G) \geq \frac{2n}{r-1}$.*

Recently, Wei [165] proved that when r is an integer, $c(G) \geq \frac{2n}{r-1} + \frac{2(r-3)}{r-1}$. So Conjecture 7.1 is true for an integer r.

On the other hand, the circumference of a regular graph with a higher connectivity than two was also considered; Fan [52] showed that $c(G) \geq \min\{3d, n\}$ for a

3-connected d-regular graph G, and Aung [9] showed that $c(G) \geq \min\{4d-4, \frac{1}{2}(n+3d-2), n\}$ for a 4-connected d-regular graph G of order $n \geq 3d+1$.

7.1.3 Bipartite graphs and triangle-free graphs

Voss and Zuluaga [162] proved that for a 2-connected bipartite graph G with bipartition (X, Y) and $|X| \geq |Y|$, $c(G) \geq \min\left\{4\delta(G)-4, 2|Y|\right\}$. This lower bound of $c(G)$ was improved to $c(G) \geq \min\left\{2\delta(X)+2\delta(Y)-2, 4\delta(X)-4, 2|Y|\right\}$ by Jackson [88], where $\delta(X)$ is the minimum degree of a vertex of X and $\delta(Y)$ is that of Y. Dang and Zhao [40] gave other improved lower bound $c(G) \geq \min\left\{4\sigma_2(G)-4\delta(G)-4, 2|Y|\right\}$ and Wang [163] proved $c(G) \geq \min\left\{2\sigma_2(G)-2, 2|Y|\right\}$ unless G belongs one of the particular classes.

There are some results on the circumference of a triangle-free graph. For any triangle-free graph G of order n, Bauer, Kahl, McGuire and Schemeichel [15] proved that $c(G) \geq \min\left\{4\delta(G)-4, n\right\}$ or every longest cycle of G is dominating, and Enomoto, Kaneko, Saito and Wei [48] proved that $c(G) \geq \min\left\{n-\alpha(G)+\kappa(G), n\right\}$.

7.2 Relationship to graph invariants

7.2.1 Girth and minimum degree

The circumference also concerns with the *girth* $g(G)$ of a graph G, that is the length of a shortest cycle of G. Many researchers have considered the relationship between them, and obtain the lower bound of the circumference, for example, [131, 161, 179]. The most general results is the following by Ellingham and Menser [44]; for a graph G with $g(G) \geq 3$ and $\delta(G) \geq 3$, $c(G) \geq \delta(G)\big(\delta(G)-1\big)^{\left\lfloor \frac{g(G)-3}{4} \right\rfloor}\left(p+\frac{4}{\delta(G)-2}\right) - g(G) - \frac{8}{\delta(G)-2}$, where $p \in \{1,2,3,4\}$ and $p \equiv g(G)+2 \pmod 4$.

7.2.2 Cocircumference

For a connected graph G, let $c^*(G)$ be the size of the largest *bond*, which is a minimal edge cut of G. $c^*(G)$ is sometimes called *cocircumference*. Wu [172] proved that for a 2-connected graph G, $c(G) \geq \frac{c^*(G)}{2}|E(G)|$, and Neumann-Lara, Rivera-Campo and Urrutia [129] improved the result as follows; any 2-connected graph G has $c(G)$ bonds, not necessarily disjoint, such that each edge of G is contained in at least two of them. Later, Wu [173] determined all 2-connected graphs with $c(G) = \frac{c^*(G)}{2}|E(G)|$.

7.2.3 Toughness and minimum degree

Bauer, Broersma, van den Heuvel and Veldman [12] showed that for any t-tough graph of order $n \geq 3$, if $\delta(G) \geq \frac{n-t}{t+1}$, then G is hamiltonian. Jung and Wittmann [94] improved it to a result on the circumference; for any 2-connected t-tough graph of order $n \geq 3$, $c(G) \geq \min\big\{(t+1)\delta(G) + t, n\big\}$.

7.2.4 Number of edges

Woodall [170] considered the relationship between the circumference and the number of edges. He proved that for a graph G of order n and for an integer d, if $|E(G)| \geq \frac{pd(d-1)}{2} + \frac{q(q+1)}{2}$, then $c(G) \geq d+1$, where p and q are non-negative integers satisfying $n = p(d-1) + q + 1$ and $q \leq d-2$. Later, Caccetta and Vijiayan [33] characterized the extremal graphs of the condition of the number of edges.

7.2.5 Neighborhood union

For a graph G, we define a *neighborhood union* of G as $NC(G) := \min\big\{|N_G(x) \cup N_G(y)| : xy \notin E(G)\big\}$ if G is not complete; otherwise $NC(G) := +\infty$. Faudree, Gould, Jacobson and Schelp [55] first showed the relationship between the neighborhood union and hamilton properties. Faudree, Gould, Jacobson and Schelp [54] also considered the lower bound of the circumference using the neighborhood union; for a 2-connected graph G of order $n \geq 3$, $c(G) \geq \min\{NC(G)+2, n\}$. Liu [113] showed the existence of a longer cycle when G has higher connectivity. He showed that $c(G) \geq \min\big\{\frac{3(NC(G)+2)}{2}, n\big\}$ if G is 3-connected, and $c(G) \geq \min\{2NC(G), n\}$ if G is 4-connected.

7.2.6 Length of a longest path

Dirac [41] showed that $c(G) \geq 2(p(G)-1)^{1/2}$ for a 2-connected graph G. Bondy and Locke [28] improved this result to $c(G) \geq 2(p(G)-1)/5$ for 3-connected graphs. Unfortunately, it is unknown whether the coefficient $2/5$ is sharp, but we know it is not greater than $3/4$. Later, Locke [114] improved Bondy and Locke's result for a graph with higher connectivity; for a k-connected graph G with $k \geq 3$, $c(G) \geq \frac{2k-4}{3k-4}(p(G)-1)$.

7.2.7 Degrees

For a graph G of order n (without assuming the connectivity condition), Alon [4] showed that if $\delta(G) \geq \frac{n}{k}$, then $c(G) \geq \lceil \frac{n}{k-1} \rceil$. Egawa and Miyamoto [43] improved this result to $\sigma_2(G)$ condition; if $\sigma_2(G) \geq \frac{2n}{k}$, then $c(G) \geq \frac{n}{k-1}$.

Now we consider graphs with connectivity conditions. Dirac [41] showed that for a 2-connected graph G of order n, $c(G) \geq \min\{2\delta(G), n\}$, and Bermond [21] and Linial [112], independently, improved Dirac's result to $c(G) \geq \min\{\sigma_2(G), n\}$. Fan [51] extended this result as follows; for a 2-connected graph G of order n, if $\max\{d_G(x), d_G(y)\} \geq \frac{c}{2}$ for any $x, y \in V(G)$ with $\mathrm{dist}(x, y) = 2$, then $c(G) \geq \min\{c, n\}$. Note that Tian [154, 155] gave short proofs of this result.

Fournier and Fraisse [64] showed another generalization of the result of Bermond and Linial, conjectured by Bondy [26]; if G is a k-connected graph of order n, then $c(G) \geq \min\{2\sigma_{k+1}(G)/(k+1), n\}$. This result was improved by Yamashita [176] as follows; $c(G) \geq \min\{\sigma_2^{k+1}(G), n\}$ for a k-connected graph of order n.

Let $\bar{\sigma_3}(G) := \min\left\{ \sum_{i=1}^3 d_G(x_i) - \left| \bigcap_{i=1}^3 N_G(x_i) \right| : \{x_1, x_2, x_3\} \text{ is an independent set} \right\}$ if $\alpha(G) \geq 3$; otherwise let $\bar{\sigma_3}(G) := +\infty$. Wei [164] showed that $c(G) \geq \min\{\bar{\sigma_3}(G), n\}$ for 3-connected graph. This result is an improvement of the result on a hamilton cycle by Flandrin, Jung and Li [59].

7.3 Dominating cycles and relative length

7.3.1 Results

Fraisse and Jung showed that any 3-conencted graph has a long cycle or a dominating cycle.

Theorem 7.2 (Fraisse and Jung [66]) *Let G be a 3-connected graph. Then $c(G) \geq \sigma_3(G) - 3$ or any longest cycle in G is dominating.*

Yamashita [174] showed that if any longest cycle in a 2-connected graph G of order n is dominating, then $c(G) \geq \min\{\sigma_3(G) - \kappa(G), n\}$. By combining this result and Theorem 7.2, we obtain $c(G) \geq \min\{\sigma_3(G) - \kappa(G), n\}$ for a 3-connected graph G.

On the othet hand, Bauer, McGuire, Trommel and Veldman [16] showed that if any longest cycle in G is dominating, then $c(G) \geq \min\{3\delta(G) - 1, n + \delta(G) - \alpha(G), n\}$. So by Theorem 7.2, we obtain $c(G) \geq \min\{3\delta(G) - 3, n + \delta(G) - \alpha(G), n\}$ for a 3-connected graph G. This result concerns with the lower bound of the circumference by Bauer, Morgana, Schmeichel and Veldman [17]; $c(G) \geq \min\{n + \frac{\sigma_3(G)}{3} - \alpha(G), n\}$ for any 2-connected graph G of order n with $\sigma_3(G) \geq n + 2$.

Recently, we improve Theorem 7.2 using the term relative length $\mathrm{diff}(G)$, where $\mathrm{diff}(G) := p(G) - c(G)$. Note that if $\mathrm{diff}(G) \leq 1$, then any longest cycle of a graph G is dominating. Therefore Theorem 7.3 is a generalization of Theorem 7.2.

Theorem 7.3 ([138]) *Let G be a 3-connected graph. Then $c(G) \geq \sigma_3(G) - 3$ or $\mathrm{diff}(G) \leq 1$.*

Moreover, it is also known that a graph G with small $\mathrm{diff}(G)$ has a long cycle. Li, Saito and Schelp [111] showed that if $\mathrm{diff}(G) \le 1$ for a 2-connected graph G of order n, then $c(G) \ge \min\{n-\alpha(G)+\delta(G), n\}$. By this result, we obtain a corollary of Theorem 7.3. This gives a new lower bound of the circumference of 3-connected graphs.

Corollary 7.4 *Let G be a 3-connected graph of order n. Then $c(G) \ge \min\{\sigma_3(G)-3, n-\alpha(G)+\delta(G), n\}$.*

On the other hand, Trommel [156], investigated the relation between $\delta(G)$ and $c(G)$ in a 3-connected graph G of order n with $\delta(G) \ge \frac{1}{4}(n+6)$. As mentioned in Chapter 6, we improved this result by considering that the property "$\mathrm{diff}(G) \le 2$."

Theorem 7.5 ([134]) *Let G be a 3-connected graph of order n. If $\sigma_4(G) \ge n+6$, then $c(G) \ge \min\{n, n+2\delta(G)-2\alpha(G)-2\}$*

Now we prove a slightly stronger result than Theorem 7.3. This improvement implies the following two corollaries.

Theorem 7.6 ([138]) *Let G be a 2-connected graph. Then* (i) $\mathrm{diff}(G) \le 1$, (ii) $c(G) \ge \sigma_3(G)-3$, *or* (iii) $\kappa(G)=2$ *and* $p(G) \ge \sigma_3(G)-1$.

Corollary 7.7 (Enomoto van den Heuvel, Kaneko and Saito [46]) *Let G be a 2-connected graph of order n. If $\sigma_3(G) \ge n+2$, then* $\mathrm{diff}(G) \le 1$.

Proof. Let G be a 2-connected graph of order n with $\sigma_3(G) \ge n+2$. By Theorem 7.6, we may assume that the conclusion (ii) or (iii) in Theorem 7.6 holds. If (ii) holds, then $c(G) \ge \sigma_3(G)-3 \ge n-1$. Since $p(G) \le n$, we obtain $\mathrm{diff}(G) = p(G)-c(G) \le n-(n-1) = 1$. If (iii) holds, then $p(G) \ge \sigma_3(G)-1 \ge n+1$, a contradiction. $\square$

Corollary 7.8 (Saito [146]) *Let G be a 2-connected graph. Then* $\mathrm{diff}(G) \le 1$ *or* $p(G) \ge \sigma_3(G)-1$.

Proof. Let G be a 2-connected graph. By Theorem 7.6, we may assume that $c(G) \ge \sigma_3(G)-3$ and $\mathrm{diff}(G) \ge 2$. Then $p(G) = c(G)+\mathrm{diff}(G) \ge \sigma_3(G)-1$. $\square$

In the rest of this section, we will show that Theorem 7.6 is best possible in a sense. First, we let l, m be integers with $l \ge m+1 \ge 4$, and let $G_1 := K_m + lK_1$. Then $\mathrm{diff}(G_1)=1$, $\kappa(G_1)=m \ge 3$, and $c(G_1)=2m < 3m-3 = \sigma_3(G_1)-3$. Thus, the conclusion (i) of Theorem 7.6 is best possible. Next, we let l, m be integers with $l \ge 4$ and $m \ge 2$, and let $G_2 := 3K_1 + lK_m$. Then $\mathrm{diff}(G_2) = m > 1$, $\kappa(G_2)=3$, and $c(G_2) = 3m+3 = 3(m+2)-3 = \sigma_3(G_2)-3$. Thus, the conclusion (ii) of

Theorem 7.6 is best possible. Finally, we let l, m be integers with $l \geq 3$ and $m \geq 2$, and let $G_3 := 2K_1 + lK_m$. Then $\mathrm{diff}(G_3) = m > 1$, $\sigma_3(G_3) = 3(m+1) = 3m+3$, $\kappa(G_3) = 2$, and $c(G_3) = 2m+2 < \sigma_3(G_3) - 3$. Since $p(G) = 3m+2 = \sigma_3(G_3) - 1$, the conclusion (iii) of Theorem 7.6 is also best possible.

7.3.2 Proof of Theorem 7.6

Notice again that an *endblock* of a graph is a block that has at most one cut vertex. For convenience, in this section, we consider K_1 and K_2 as 2-connected graphs, and we call a 2-connected graph itself an endblock. For a block B, we write by $I(B)$ the set of vertices of B which are not cut vertices.

For $x, y \in V(G)$, let $D'_G(x,y) = \{|V(P)| : P \text{ is a longest } xy\text{-path in } G\}$. For a 2-connected graph G, let $D'(G) = \min\{D'_G(x,y) : x, y \in V(G), x \neq y\}$. If G is connected and has cut vertices, we set $D'(G) = \max\{D'(B) : B \text{ is an endblock of } G\}$. For a trivial graph G, we define $D'(G) = 1$. (In fact, Fraisse and Jung [66] define an invariant $D(G)$ for a graph G. We can define $D'(G) = D(G) + 1$.)

Lemma 7.9 (Fraisse and Jung [66]) *Let G be a connected graph. Then there exist two vertices v_1, v_2 in G such that v_i is not a cut vertex of G and $D'(G) \geq d_G(v_i) + 1$ $(i = 1, 2)$. In particular, if $|V(G)| \geq 2$ then we can choose v_1 and v_2 such that $v_1 \neq v_2$.*

By the definition of $D'(G)$, we immediately obtain the following fact.

Fact 7.1 *Let G be a connected graph, and B be an endblock of G such that $D'(B) = D'(G)$. Let $u \in I(B)$ and $v \in V(G)$. Suppose that $u \neq v$ if $|V(G)| \geq 2$. Then there exists a uv-path in G of order at least $D'(G)$.*

Proof of Theorem 7.6.

Suppose that G satisfies the assumption of Theorem 7.6 and $\mathrm{diff}(G) \geq 2$. Let Q be a longest path of G. Let C be a cycle and P_0 be a path with ends x and y_0 such that $V(C) \cup V(P_0) = V(Q)$, $V(C) \cap V(P_0) = \emptyset$ and $N_C(x) \neq \emptyset$. (Notice that there exist such a cycle C and a path P_0, because the end-vertex of Q has a neighbor in $V(Q)$.) Take such a cycle C and a path P_0 so that $|V(C)|$ is as large as possible. Note that $|P_0| \geq 2$ because $\mathrm{diff}(G) \geq 2$. A vertex $y \in V(P_0)$ is called *endable for* (x, P_0) if there exists an xy-path P' such that $V(P') = V(P)$. Let $L := \{y \in V(P_0) : y \text{ is endable for } (x, P_0)\}$ and let $L' := L \cup \{x\}$. We define $\mathcal{T} := \{(y, P) : y \in L \text{ and } P \text{ is an } xy\text{-path such that } V(P) = V(P_0)\}$. For $(y, P) \in \mathcal{T}$, $\overrightarrow{P}$ is an oriented path from x to y. Let $u_0 \in N_C(x)$. By the maximality of $|V(Q)|$ and $|V(C)|$, the following three claims hold.

Claim 7.2 (i) *$N_{G-Q}(L) = \emptyset$. Moreover, if $N_C(L) \neq \emptyset$ then $N_{G-Q}(x) = \emptyset$.*

(ii) *If $u \in N_C(L')$, then $N_{G-C}(u^+) = N_{G-C}(u^-) = \emptyset$.*

Claim 7.3 *Suppose $u_1 \in N_C(L')$ and $u_2 \in N_C(G-C)$ with $u_1 \neq u_2$. Let $C_1 = u_1^+ \overrightarrow{C} u_2$ and $C_2 = u_2^+ \overrightarrow{C} u_1$. Then the following statements hold.*

(i) *$N_{C_1}(u_1^+)^- \cap N_{C_1}(u_2^+) = \emptyset$. In particular, $u_1^+ u_2^+ \notin E(G)$.*

(ii) *$N_{C_2}(u_1^+) \cap N_{C_2}(u_2^+)^- = \emptyset$.*

Claim 7.4 *Suppose $u_1 \in N_C(L)$ with $u_0 \neq u_1$. Let $C_0 = u_0^+ \overrightarrow{C} u_1$ and $C_1 = u_1^+ \overrightarrow{C} u_0$. Then $N_{C_0}(x)^+ \cap N_{C_0}(u_1^+)^- = \emptyset$ and $N_{C_1}(x)^+ \cap N_{C_1}(u_1^+)^- = \emptyset$.*

We divide the proof into two cases depending on $|N_C(L) - \{u_0\}|$.

Case 1. $|N_C(L) - \{u_0\}| \geq 2$.

Let $u_1, u_2 \in N_C(L) - \{u_0\}$ with $u_1 \neq u_2$. Choose u_0 and u_1 so that $|V(u_0 \overrightarrow{C} u_1)|$ is as small as possible under the assumption of Case 1. Take $v \in (N_C(u_1^+) \cup N_C(u_2^+)) \cap V(u_0^+ \overrightarrow{C} u_1)$ so that $|V(u_0^+ \overrightarrow{C} v)|$ is as small as possible. Since $u_1 \in N_C(u_1^+) \cap V(u_0^+ \overrightarrow{C} u_1)$, there exists such a vertex v. By Claim 7.2 (ii), $N_{G-C}(u_1^+) = N_{G-C}(u_2^+) = \emptyset$. Therefore, by Claim 7.3 (i), $\{x, u_1^+, u_2^+\}$ is independent.

Let $D_0 := u_0^+ \overrightarrow{C} v^-$, $D_1 := u_1^+ \overrightarrow{C} u_2$ and $D_2 := u_2^+ \overrightarrow{C} u_0 \cup v \overrightarrow{C} u_1$. By the choice of u_0, u_1 and v, we have $N_{D_0}(x) = N_{D_0}(u_1^+) = N_{D_0}(u_2^+) = \emptyset$. Let h be an integer such that $v \in N_C(u_h^+)$, let $y \in N_{P_0}(u_h) \cap L$ and let P be a path with $(y, P) \in \mathcal{T}$. By the choice of C, the cycle $x u_0 \overleftarrow{C} u_h^+ v \overrightarrow{C} u_h y \overleftarrow{P} x$ is not longer than C, which implies $|V(D_0)| \geq |V(P)|$. By Claim 7.2 (i), we have $N_{G-C}(x) = N_P(x)$. Since $N_P(x) \cup \{x\} \subseteq V(P)$, it follows that $|V(D_0)| \geq |N_{G-C}(x)| + 1$. By Claim 7.3 (i) and (ii), we have $N_{D_1}(x)^+ \cap N_{D_1}(u_2^+) = \emptyset$ and $N_{D_1}(u_1^+)^- \cap N_{D_1}(u_2^+) = \emptyset$. By Claim 7.4, $N_{D_1}(x)^+ \cap N_{D_1}(u_1^+)^- = \emptyset$. Hence $N_{D_1}(x)^+$, $N_{D_1}(u_1^+)^-$ and $N_{D_1}(u_2^+)$ are pairwise disjoint. Clearly, $N_{D_1}(x)^+ \cup N_{D_1}(u_1^+)^- \cup N_{D_1}(u_2^+) \subseteq V(D_1) \cup \{u_2^+\}$. Thus we obtain $|N_{D_1}(x)| + |N_{D_1}(u_1^+)| + |N_{D_1}(u_2^+)| \leq |V(D_1)| + 1$. Similarly, by Claim 7.3 (i) and (ii) and Claim 7.4, $N_{D_2}(x)^+$, $N_{D_2}(u_1^+)$ and $N_{D_2}(u_2^+)^-$ are pairwise disjoint. Since $N_{D_2}(x)^+ \cup N_{D_2}(u_1^+) \cup N_{D_2}(u_2^+)^- \subseteq V(D_2) \cup \{u_0^+, v^-, u_1^+\}$, we have $|N_{D_2}(x)| + |N_{D_2}(u_1^+)| + |N_{D_2}(u_2^+)| \leq |V(D_2)| + 3$. Thus we deduce

$$\begin{aligned} c(G) &\geq |V(C)| \\ &= |V(D_0)| + |V(D_1)| + |V(D_2)| \\ &\geq |N_{G-C}(x)| + 1 + |N_{D_1}(x)| + |N_{D_1}(u_1^+)| + |N_{D_1}(u_2^+)| - 1 \\ &\quad + |N_{D_2}(x)| + |N_{D_2}(u_1^+)| + |N_{D_2}(u_2^+)| - 3 \\ &\geq |N_G(x)| + |N_G(u_1^+)| + |N_G(u_2^+)| - 3 \\ &\geq \sigma_3(G) - 3. \qquad \square \end{aligned}$$

Case 2. $|N_C(L) - \{u_0\}| \leq 1$.

We first show that $p(G) \geq \sigma_3(G) - 1$. By the 2-connectedness of G, there exists a vertex $u_1 \in V(C) - \{u_0\}$ such that $N_{G-C}(u_1) \neq \emptyset$. Since $N_{G-C}(u_0^-) = \emptyset$, we can choose u_1 so that $N_{G-C}(u_1^+) = \emptyset$. By Claim 7.2 (ii) and by the choice of u_1, $N_{G-C}(u_0^+) = N_{G-C}(u_1^+) = \emptyset$. Hence, by Claim 7.3 (i), $\{y_0, u_0^+, u_1^+\}$ is independent.

Let $C_0 := u_0^+ \overrightarrow{C} u_1$ and $C_1 := u_1^+ \overrightarrow{C} u_0$. By Claim 7.3 (i) and (ii), we have $N_{C_0}(u_0^+)^- \cap N_{C_0}(u_1^+) = \emptyset$ and $N_{C_1}(u_0^+) \cap N_{C_1}(u_1^+)^- = \emptyset$. Clearly, $N_{C_0}(u_0^+)^- \cup N_{C_0}(u_1^+) \subseteq V(C_0)$ and $N_{C_1}(u_0^+) \cup N_{C_1}(u_1^+)^- \subseteq V(C_1)$. Thus we obtain $|N_{C_0}(u_0^+)| + |N_{C_0}(u_1^+)| \leq |V(C_0)|$ and $|N_{C_1}(u_0^+)| + |N_{C_1}(u_1^+)| \leq |V(C_1)|$. By Claim 7.2 (i), $N_{G-C}(y_0) = N_P(y_0)$. Since $N_P(y_0) \cup \{y_0\} \subseteq V(P)$, it follows that $|V(P)| \geq |N_{G-C}(y_0)| + 1$. By the assumption of Case 2, we have $|N_C(y_0)| \leq 2$. Hence we obtain

$$\begin{aligned}
p(G) &= |V(Q)| \\
&= |V(C_0)| + |V(C_1)| + |V(P)| \\
&\geq |N_{C_0}(u_0^+)| + |N_{C_0}(u_1^+)| + |N_{C_1}(u_0^+)| + |N_{C_1}(u_1^+)| + |N_{G-C}(y_0)| + 1 \\
&\geq |N_G(u_0^+)| + |N_G(u_1^+)| + |N_G(y_0)| - 2 + 1 \\
&\geq \sigma_3(G) - 1.
\end{aligned}$$

Since if $\kappa(G) = 2$ then the conclusion (iii) holds, henceforth we may assume that G is 3-connected.

Case 2.1. $|N_C(L) - \{u_0\}| = 1$.

Let $u_1 \in N_C(L) - \{u_0\}$ and $y \in N_{P_0}(u_1) \cap L$, and let P be a path with $(y, P) \in \mathcal{T}$. By the symmetry of x and y, we may assume that $N_C(x) \subseteq \{u_0, u_1\}$.

If $N_P(x)^- \cap N_P(y) \neq \emptyset$ then there exists $z \in L$ with $z^+ \notin L$. Assume not, and let $w \in N_P(x)^- \cap N_P(y)$. Then $P' = xw^+ \overleftarrow{P} yw \overleftarrow{P} x^+$ is a path with $V(P') = V(P)$, and so $x^+ \in L$. This implies $V(x^+ \overleftarrow{P} y) \subseteq L$. Since G is 3-connected, we obtain $N_C(L) - \{u_0, u_1\} \neq \emptyset$, which contradicts the assumption of Case 2. If $N_P(x)^- \cap N_P(y) \neq \emptyset$ then we take $z \in L$ so that $z^+ \notin L$; otherwise let $z := x$. In either case, note that $zy \notin E(G)$. By Claim 7.2 (ii), $\{z, y, u_1^+\}$ is independent.

By Claim 7.2 (i), $N_{G-C}(z) = N_P(z)$ and $N_{G-C}(y) = N_P(y)$. First assume $z = x$. Then $N_P(z)^- \cap N_P(y) = \emptyset$. Since $N_P(z)^- \cup N_P(y) \subseteq V(P) - \{y\}$, we have $|V(P)| \geq |N_{G-C}(z)| + |N_{G-C}(y)| + 1$. Next assume $z \neq x$, and let $P_1 := x \overrightarrow{P} z$ and $P_2 := z^+ \overrightarrow{P} y$. Since $z^+ \notin L$, it follows that $N_{P_1}(z)^+ \cap N_{P_1}(y) = \emptyset$ and $N_{P_2}(z) \cap N_{P_2}(y)^+ = \emptyset$. Clearly, $N_{P_1}(z)^+ \cup N_{P_1}(y) \subseteq V(P_1)$ and $N_{P_2}(z) \cup N_{P_2}(y)^+ \subseteq V(P_2)$. Thus, in either case, we obtain $|V(P)| \geq |N_{G-C}(z)| + |N_{G-C}(y)|$.

Choose $v \in N_C(u_1^+) \cap V(u_0^+ \overrightarrow{C} u_1)$ so that $|V(u_0^+ \overrightarrow{C} v)|$ is as small as possible. Let $D_0 := u_0^+ \overrightarrow{C} v^-$ and $D_1 := u_1^+ \overrightarrow{C} u_0 \cup v \overrightarrow{C} u_1$. By the choice of v and by Claim 7.2 (ii), $N_{D_0}(u_1^+) \cup N_{G-C}(u_1^+) = \emptyset$. The choice of C implies $|V(D_0)| \geq |V(P)| \geq |N_{G-C}(z)| + |N_{G-C}(y)|$. Since $N_{D_1}(u_1^+) \subseteq V(D_1) - \{u_1^+\}$, we have $|N_{D_1}(u_1^+)| \leq |V(D_1)| - 1$. By the assumption of Case 2, $N_C(z) \subseteq \{u_0, u_1\}$ and $N_C(y) \subseteq \{u_0, u_1\}$. Therefore we obtain

$$\begin{aligned} c(G) &\geq |V(C)| \\ &= |V(D_0)| + |V(D_1)| \\ &\geq |N_{G-C}(z)| + |N_{G-C}(y)| + |N_{D_1}(u_1^+)| + 1 \\ &\geq (|N_G(z)| - 2) + (|N_G(y)| - 2) + |N_G(u_1^+)| + 1 \\ &\geq \sigma_3(G) - 3. \qquad \square \end{aligned}$$

Case 2.2. $|N_C(L) - \{u_0\}| = 0$.

Let $y \in L$, and let P be a path with $(y, P) \in \mathcal{T}$. Since $G - \{u_0\}$ is 2-connected and $\min\{|V(P)|, |V(C) - \{u_0\}|\} \geq 2$, there exist two vertex disjoint paths R_i $(i = 1, 2)$ such that R_i connects z_i and u_i, where $\{z_i\} = V(R_i) \cap V(P)$ and $\{u_i\} = V(R_i) \cap (V(C) - \{u_0\})$. By the assumption of Case 2.2 and by Claim 7.2 (i), we have $z_i \neq y$. Choose such a path R_1 so that $|V(z_1 \overrightarrow{P} y)|$ is as small as possible.

By considering the reverse orientation of C if necessary, we may assume that $u_2 \in V(u_1^+ \overrightarrow{C} u_0^-)$.

First, we show the existence of a long path between z_1 and x or z_2. Let $P_1 := x \overrightarrow{P} z_1$ and $P_2 := z_1^+ \overrightarrow{P} y$. Note that $|V(P_1)| \geq 2$ by the choice of R_1. Also, note that $|V(P_2)| \geq 1$ by the assumption of Case 2.2. Let H_1 be a component of $G - (C \cup P_1)$ such that $V(P_2) \subseteq V(H_1)$ and let B_1 be an endblock of H_1 such that $D'(B_1) = D'(H_1)$. Note that $V(R_i) \cap V(H_1) = \emptyset$ for $i = 1, 2$ by the choice of R_1. By Lemma 7.9, there exist vertices $v_1, v_2 \in V(B_1)$ such that $D'(B_1) \geq d_{H_1}(v_i) + 1$ $(i = 1, 2)$.

Claim 7.5 *Let $z \in V(x \overrightarrow{P} z_1^-)$. For some i, there exists a zz_1-path in $P_1 \cup H_1$ of order at least $|N_G(v_i)|$.*

Proof. By the choice of R_1, we have $N_G(H_1) \subseteq V(P_1) \cup \{u_0\}$. Since $G - \{u_0\}$ is 2-connected, there exist two edges $e_1 := a_1 b_1$ and $e_2 := a_2 b_2$ such that one connects $I(B_1)$ and $V(P_1)$, another connects $V(H_1)$ and $V(P_1)$, $a_1, a_2 \in V(P_1)$ and $a_1 \neq a_2$. In particular, if $|V(H_1)| \geq 2$ then we can choose b_1 and b_2 such that $b_1 \neq b_2$. Since $z_1^+ z_1$ is an edge connecting $V(H_1)$ and $V(P_1)$, we can choose such two edges so that $a_1 \in V(z^+ \overrightarrow{P} z_1)$. Choose e_1 and e_2 so that (i) $|V(z^+ \overrightarrow{P} a_1)|$ is as small as possible and (ii) $|V(x \overrightarrow{P} a_2)|$ is as small as possible, subject to (i). If $|V(H_1)| \geq 2$, then we may assume that $v_1 \neq b_1$. Therefore it follows from the choice of e_1 and e_2 that $N_{P_1}(v_1) \cap V(z^+ \overrightarrow{P} a_1^-) = \emptyset$ and $N_{P_1}(v_1) \cap V(x \overrightarrow{P} a_2^-) \subseteq \{a_1\}$.

By Fact 7.1, there exists a $b_1 b_2$-path T_1 in H_1 with $|V(T_1)| \geq D'(B_1) \geq |N_{H_1}(v_1)| + 1$. Let

$$Q_1 := \begin{cases} z \overrightarrow{P} a_1 b_1 T_1 b_2 a_2 \overrightarrow{P} z_1 & \text{if } a_2 \in V(a_1^+ \overrightarrow{P} z_1), \\ z \overleftarrow{P} a_2 b_2 T_1 b_1 a_1 \overrightarrow{P} z_1 & \text{otherwise.} \quad \text{(See Figure 7.2.)} \end{cases}$$

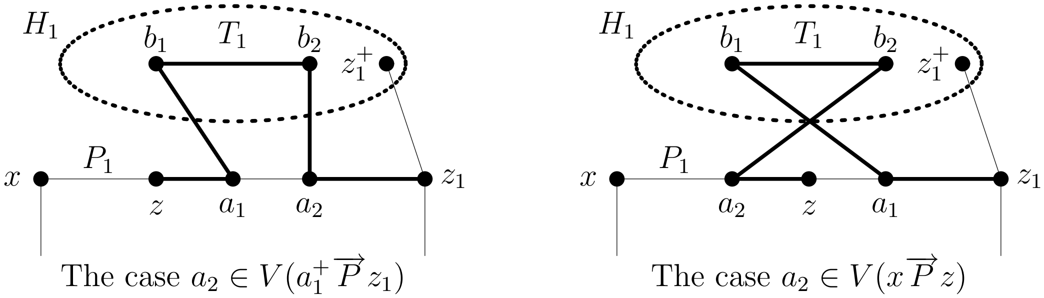


Figure 7.2: The path Q_1.

Then $N_{P_1}(v_1) \subseteq V(P_1) \cap V(Q_1)$ and hence $|N_{P_1}(v_1)| \leq \big|V(P_1) \cap V(Q_1)\big|$. Because $N_{G-P_1}(v_1) = N_{H_1}(v_1) \cup N_C(v_1)$ and $N_C(v_1) \subseteq \{u_0\}$, we have $|N_{G-P_1}(v_1)| \leq |N_{H_1}(v_1)| + 1$. Therefore we obtain

$$\begin{aligned} |V(Q_1)| &= |V(P_1) \cap V(Q_1)| + |V(T_1)| \\ &\geq |N_{P_1}(v_1)| + |N_{H_1}(v_1)| + 1 \\ &\geq |N_G(v_1)|, \end{aligned}$$

and so Q_1 is a zz_1-path of order at least $|N_G(v_1)|$. $\square$

By Claim 7.3, for some $i_x, i_z \in \{1, 2\}$ there exist an xz_1-path P_x and a z_2z_1-path P_z in H of order at least $|N_G(v_{i_x})|$ and $|N_G(v_{i_z})|$, respectively. Hereafter we never consider P_x and P_z at the same time, by the symmetry between v_1 and v_2, we may assume that $i_x = i_z = 1$.

Next, we will prove the existence of a cycle of length at least $\sigma_3(G) - 2$. To prove it, we focus on $u_2^+ \overrightarrow{C} u_0^-$.

Claim 7.6 *Suppose that there exists $w^* \in V(u_2^+ \overrightarrow{C} u_0^-)$ such that for every $w \in V(u_2^+ \overrightarrow{C} w^*)$, $N_{G-C}(w) \neq \emptyset$ or $N_C(w) \cap V(u_2^+ \overrightarrow{C} w) - \{w, w^-\} \neq \emptyset$. Then there exist a vertex $z \in V(G - C)$ and a zu_2-path R such that $V(R) = V(u_2 \overrightarrow{C} w^*) \cup \{z\}$.*

Proof. If $w^* = u_2^+$, then we can easily show the existence of a desired path because $N_{G-C}(w^*) \neq \emptyset$. Thus, we may assume that $w^* \neq u_2^+$. We define a function $f : V(u_2^+ \overrightarrow{C} w^*) \to V(u_2^+ \overrightarrow{C} w^*)$ as follows. For $w \in V(u_2^+ \overrightarrow{C} w^*)$, if $N_{G-C}(w) \neq \emptyset$ then let $f(w) = w$; otherwise let $u \in N_C(w) \cap V(u_2^+ \overrightarrow{C} w) - \{w, w^-\}$ and $f(w) = u^+$. Moreover, we define $w_0 = w^*$ and $w_{i+1} = f(w_i)$ for $i = 0, 1, 2, \cdots$. Since $w_{i+1} \in V(u_2^+ \overrightarrow{C} w_i)$ for any i, there exists j such that $w_{j+1} = w_j$. Take such j as small as possible. Note that $w_j \neq u_2^+$ because $w_j^- \in V(u_2^+ \overrightarrow{C} w^*)$ unless $w^* = w_0 = u_2^+$. Let $z \in N_{G-C}(w_j)$ and let

$$R := \begin{cases} zw_j \overleftarrow{C} w_{j-1}^- w_{j-2} \overleftarrow{C} w_{j-3}^- \cdots w_2^- w_1 \overleftarrow{C} w^* w_1^- \overrightarrow{C} w_2 \cdots w_j^- \overrightarrow{C} u_2 & \text{if } j\text{: odd,} \\ zw_j \overleftarrow{C} w_{j-1}^- w_{j-2} \overleftarrow{C} w_{j-3}^- \cdots w_2 \overleftarrow{C} w_1^- w^* \overrightarrow{C} w_1 w_2^- \cdots w_j^- \overrightarrow{C} u_2 & \text{if } j\text{: even,} \end{cases}$$

Then R is a desired path. $\square$

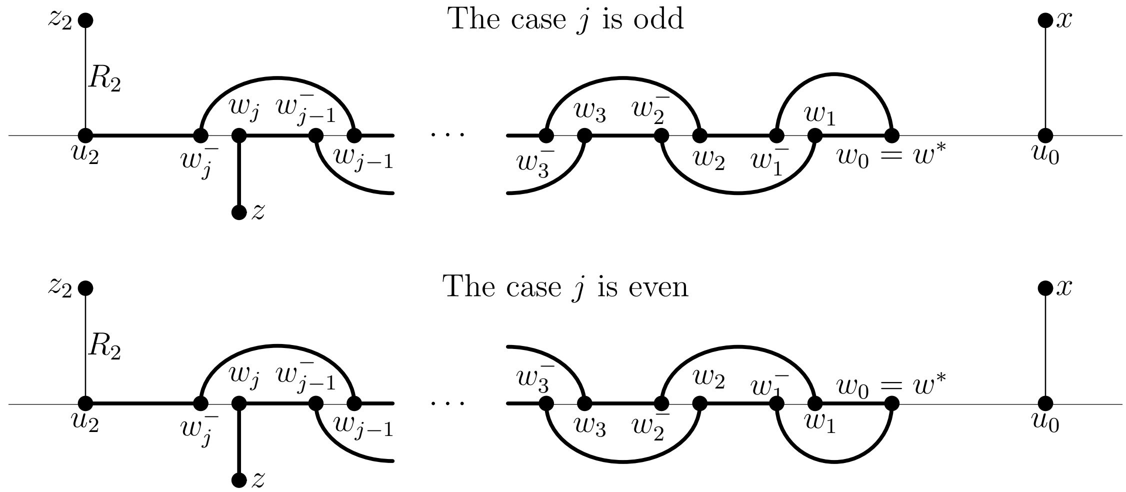


Figure 7.3: The path R.

We show that there exists a vertex $w_0 \in V(u_2^+ \overrightarrow{C} u_0^-)$ such that $N_{G-C}(w_0) = \emptyset$ and $N_C(w_0) \cap V(u_2^+ \overrightarrow{C} w_0) - \{w_0, w_0^-\} = \emptyset$. Assume not. By applying Claim 7.6 as $w^* = u_0^-$, there exists a zu_2-path R such that $z \in V(G-C)$ and $V(R) = V(u_2 \overrightarrow{C} u_0^-) \cup \{z\}$. Then $zRu_2 \overleftarrow{C} u_0 x \overrightarrow{P} y$ contradicts the choice of Q or C. Choose such a vertex w_0 so that $|V(u_2^+ \overrightarrow{C} w_0)|$ is as small as possible.

Choose $v \in (N_C(u_0^+) \cup N_C(w_0) \cup \{u_2\}) \cap V(u_1^+ \overrightarrow{C} u_2)$ so that $|V(u_1^+ \overrightarrow{C} v)|$ is as small as possible. Let

$$C' := \begin{cases} u_1 R_1 z_1 P_z z_2 R_2 u_2 \overrightarrow{C} u_1 & \text{if } u = u_2, \\ u_0^+ \overrightarrow{C} u_1 R_1 z_1 P_x x u_0 \overleftarrow{C} v u_0^+ & \text{if } v \in N_C(u_0^+), \\ w_0 \overrightarrow{C} u_1 R_1 z_1 P_z z_2 R_2 u_2 \overleftarrow{C} v w_0 & \text{if } v \in N_C(w_0). \end{cases}$$

Let $D_0 := u_0^+ \overrightarrow{C} u_1 \cup v \overrightarrow{C} u_2$, $D_1 := u_1^+ \overrightarrow{C} v^-$, $D_2 := u_2^+ \overrightarrow{C} w_0^-$ and $D_3 := w_0 \overrightarrow{C} u_0$. Note that $V(C) = V(D_0 \cup D_3 \cup D_1 \cup D_2)$ and $V(C' \cap C) \subseteq V(D_0 \cup D_3)$.

We prove $N_{D_0}(u_0^+)^- \cap N_{D_0}(w_0) = \emptyset$. Assume not, say $w \in N_{D_0}(u_0^+)^- \cap N_{D_0}(w_0)$. Then $w_0 \neq u_2^+$ by Claim 7.3. By applying Claim 7.6 as $w^* = w_0^-$, there exist a vertex $z \in V(G-C)$ and a zu_2-path R such that $V(R) = V(u_2 \overrightarrow{C} w_0^-) \cup \{z\}$. Then $zRu_2 \overleftarrow{C} w^+ u_0^+ \overrightarrow{C} w w_0 \overrightarrow{C} u_0 x \overrightarrow{P} y$ contradicts the choice of Q or C. Therefore $N_{D_0}(u_0^+)^- \cap N_{D_0}(w_0) = \emptyset$, and in particular, $u_0^+ w_0 \notin E(G)$. We can similarly prove $N_{D_3}(u_0^+) \cap N_{D_3}(w_0)^- = \emptyset$. Clearly, $N_{D_0}(u_0^+)^- \cup N_{D_0}(w_0) \subseteq V(D_0) \cup \{v^-\}$ and $N_{D_3}(u_0^+) \cup N_{D_3}(w_0)^- \subseteq V(D_3)$.

By the choice of w_0, $N_{G-C}(w_0) = \emptyset$. By Claim 7.2 (ii), we have $N_{G-C}(u_0^+) = \emptyset$. Therefore $\{v_1, u_0^+, w_0\}$ is independent.

By the choice of v, $N_{D_1}(u_0^+) = N_{D_1}(w_0) = \emptyset$. By the choice of w_0, $N_{D_2}(w_0) \subseteq \{w_0^-\}$. We show that $N_{D_2}(u_0^+) = \emptyset$. Assume not, say $w \in N_{D_2}(u_0^+)$. By Claim 7.3, we obtain $w \neq u_2^+$. By applying Claim 7.6 as $w^* = w^-$, there exists a vertex $z \in V(G - C)$ and a zu_2-path R such that $V(R) = V(u_2 \overrightarrow{C} w^-) \cup \{z\}$. Then $zRu_2 \overleftarrow{C} u_0^+ w \overrightarrow{C} u_0 x \overrightarrow{P} y$ contradicts the choice of Q or C. Thus we deduce

$$\begin{aligned} |V(C' \cap C)| &\geq |V(D_0)| + |V(D_3)| \\ &\geq |N_{D_0}(u_0^+)| + |N_{D_0}(w_0)| - 1 + |N_{D_3}(u_0^+)| + |N_{D_3}(w_0)| \\ &\geq |N_G(u_0^+)| + |N_G(w_0)| - 2. \end{aligned}$$

Since $V(P_x) \subseteq V(C' - C)$ or $V(P_z) \subseteq V(C' - C)$, it follows that $|V(C' - C)| \geq |N_G(v_1)|$. Therefore, we obtain $c(G) \geq |V(C')| \geq |N_G(u_0^+)| + |N_G(w_0)| + |N_G(v_1)| - 2 \geq \sigma_3(G) - 2$. $\square$

Chapter 8

k-trees containing specified vertices

Similarly to a hamilton cycle problem, a hamilton path problem are one of the most important topics in Graph Theory. Starting with a hamilton path problem, many researchers have been considered a spanning k-tree, which is a spanning tree with maximum degree at most k. Definitely a spanning 2-tree is equivalent to a hamilton path. So it is a relaxed concept of a hamilton path. On the other hand, as an extension of a hamilton cycle, cycles containing specified vertices have been considered. In this chapter, we focus on a k-tree containing specified vertices, which has above two properties.

The contents of this chapter are based on the paper [36] "A k-tree containing specified vertices," jointwork with S. Chiba, R. Matsubara and M. Tsugaki.

8.1 Results on a spanning k-tree

Let k be an integer with $k \geq 2$. A *k-tree* is a tree with maximum degree at most k. Since a spanning 2-tree is equivalent to a hamilton path, we can consider the existence of a spanning k-tree as an extension of hamilton properties.

There are many sufficient conditions for the existence of a hamilton cycle. Ore [130] showed a $\sigma_2(G)$ condition for the existence of a hamilton cycle. (See Theorem 3.2 in Chapter 3.) As a corollary of this result, we can obtain the result on a hamilton path; For a connected graph G of order n, if $\sigma_2(G) \geq n-1$ then G has a hamilton path, that is, a spanning 2-tree. Win generalized this result for a spanning k-tree.

Theorem 8.1 (Win [168]) *Let k be an integer with $k \geq 2$ and let G be a connected graph of order n. If $\sigma_k(G) \geq n-1$, then G has a spanning k-tree.*

Theorem 8.1 was extended to several directions. A tree T is called a *caterpillar* if there exists a path P in T such that every vertex of T which does not appear on P is adjacent to a vertex of P; the path P is called the *spine* of the caterpillar. Czygrinow, Fan, Hurlbert, Kierstead and Trotter [39] showed that the same condition implies a stronger conclusion; For a connected graph of order n, if $\sigma_k(G) \geq n-1$, then (i) for every longest path P, G has a caterpillar T with maximum degree k whose spine is P, and the set of vertices whose degree is at least three in T is independent, or (ii) G belongs to an exceptional class.

Caro, Krasikov and Roditty [35] considered a maximum k-tree, which is a k-tree of a graph G whose order is maximum among all k-tree of G. They showed that a connected graph of order n has a spanning tree T with $|T| \geq \min\{\sigma_k(G)+1, n\}$.

Flandrin, Jung and H. Li [59] showed that for a connected graph G of order n, if $\sum_{i=1}^{3} d(x_i) \geq n + |\bigcap_{i=1}^{3} N(x_i)| - 1$ for every independent set $\{x_1, x_2, x_3\}$, then G has a spanning k-tree. A. Kyaw [102] improved this result in terms of a *k-frame*. An independent set S of order k is a *k-frame* if $G - S'$ is connected for any $S' \subseteq S$, and let $N_i(S) = \{v \in V(G) : |N_G(v) \cap S| = i\}$. He showed that for a connected graph G of order n, if $\sum_{i=1}^{3} d(x_i) \geq n - \sum_{i=2}^{k+1}(k-i)|N_i(S)| - 1$ for every $(k+1)$-frame S, then G has a spanning k-tree.

On the other hand an independence number condition for a spanning k-tree is also considered. Chvátal and Erdős [37] showed the condition for the existence of a hamilton cycle, (See Theorem 3.3 in Chapter 3), and as a corollary of it, we can obtain the results on a hamilton path; For a connected graph G, if $\alpha(G) \leq \kappa(G)+1$, then G has a hamilton path, that is, a spanning 2-tree. Neumann-Lara and Rivera-Campo improved this result to a spanning k-tree with bounded number of vertices of degree k. Later Tsugaki showed the same result when $k = 3$. We obtain an independence number condition for the existence of a spanning k-tree as a corollary.

Theorem 8.2 (Neumann-Lara and Rivera-Campo [128] for $k \geq 4$, Tsugaki [157] for $k = 3$) *Let k, m and c be integers with $k \geq 3$, $m \geq 1$ and $m \leq c \leq 0$. Let G be an m-connected graph. If $\alpha(G) \leq (k-2)m + c + 1$, then G has a spanning k-tree such that the number of vertices whose degree are k is at most c.*

Corollary 8.3 *Let k and m be integers with $k \geq 3$ and $m \geq 1$. Let G be an m-connected graph. If $\alpha(G) \leq (k-1)m + 1$, then G has a spanning k-tree.*

As a common generalization of Theorem 8.1 and Corollary 8.3, Rivera-Campo [143] gave a $\sigma_{m(k-2)+c+2}(G)$ condition for an m-connected graph to have a spanning k-tree such that the number of vertices whose degree are k is at most c. Fujisawa, Matsumura and Yamashita improved this result to a $\sigma_k^{m(k-2)+c+2}(G)$ condition.

Theorem 8.4 (Fujisawa, Matsumura and Yamashita [67]) *Let k, m and c be integers with $k \geq 3$, $m \geq 1$ and $m \leq c \leq 0$. Let G be an m-connected graph of* *order n. If $\sigma_k^{m(k-2)+c+2}(G) \geq n - c - 1$ then G has a spanning k-tree such that the number of vertices whose degree is k is at most c.*

On the other hand, Rivera-Campo [142] considered a spanning k-tree containing a given matching. He showed that for integers $k \geq 3$ and $m \geq 1$ and for an m-connected graph G, if $\alpha(G) \leq m\left(\frac{3k}{2}\right) + 1$, then for any matching, G has a spanning k-tree containing it.

8.2 k-tree containing specified vertices

8.2.1 Results

A hamilton cycle is a cycle which has to pass through all vertices of a graph. In this sense, we can consider a cycle containing specified vertices as a relaxation of a hamilton cycle. Therefore, some researchers tried to obtain the results on the existence of such cycles by "localizing" the sufficient conditions for the existence of a hamilton cycle. In fact, as mentioned in Chapter 3, Shi showed a $\sigma_2(S)$ condition for the existence of cycles passing through specified vertices, which is a generalization of the result by Ore (Theorem 3.2 in Chapter 3). The most general extension of results on the independence number condition was proved in [135], see also [30, 63].

Theorem 8.5 (Shi [149]) *Let G be a 2-connected graph of order n and let $S \subseteq V(G)$. If $\sigma_2(S) \geq n$, then G has a cycle containing S.*

Theorem 8.6 (Ozeki and Yamashita [135]) *Let G be a 2-connected graph and let $S \subseteq V(G)$. If $\alpha(S) \leq \kappa(S)$, then G has a cycle containing S.*

There are many other results on a cycle passing through specified vertices, see Chapter 3. In this stream, Ota gave a degree sum condition for graphs with high connectivity to have such a cycle. Theorem 8.7 is a common generalization of Theorems 8.5 and 8.6.

Theorem 8.7 (Ota [132]) *Let G be a graph of order n and let $S \subseteq V(G)$ with $\kappa(S) \geq 2$. If $\sigma_{t+1}(S) \geq n + t^2 - t$ for every $t \geq \kappa(S) \geq 2$, then G has a cycle containing S.*

The condition of Theorem 8.7 seems complex and strange, however, it is, in a sense, a good one. In fact, Theorem 8.7 leads some applications, for example, the following two corollaries. Note that Harkat-Benhamdin et al. [82] independently proved the case $m = 3$ of Corollary 8.9. The proof of Corollary 8.9 is referred to Proposition 3.26 in Chapter 3.

Corollary 8.8 (Ota [132]) *Let G be a graph of order n and let $S \subseteq V(G)$ with $\kappa(S) \geq m \geq 2$. Suppose that $n \geq (m+1)\alpha(S)-(m+2)$ and $\sigma_{m+1}(S) \geq n+m^2-m$. Then G has a cycle containing S.*

Corollary 8.9 *Let G be a graph of order n and let $S \subseteq V(G)$ with $\kappa(S) \geq m \geq 2$. If $\sigma_{m+1}(S) \geq n+(m-1)(\alpha(S)-1)$, then G has a cycle containing S.*

Similarly to considering a cycle passing through specified vertices, in this section, we focus on the existence of a k-tree containing specified vertices. By considering Theorems 8.5, 8.6 and 8.7, the followings are natural questions.

Question 8.10 *What is a degree sum condition for the existence of a k-tree containing specified vertices?*

Question 8.11 *What is an independence number condition for the existence of a k-tree containing specified vertices?*

In particular, we are interested in describing such conditions using invariants localized to specified vertices, for example, $\alpha(S)$, $\kappa(S)$ and $\sigma_k(S)$. As one of the answers to Question 8.10, Matsuda and Matsumura showed the sharp degree sum condition, which is a generalization of Theorem 8.1.

Theorem 8.12 (Matsuda and Matsumura [120]) *Let k be an integer with $k \geq 2$ and let G be a connected graph of order n. Let $S \subseteq V(G)$ with $\kappa(S) \geq 1$. If $\sigma_k(S) \geq n-1$, then G has a k-tree containing S.*

Later, Cutler [38] proved that for an integer t with $1 \leq t \leq n-2$, for a connected graph G of order n and for $S \subseteq V(G)$, if $\delta(S) \geq t$, then G has a tree containing S such that $d_T(x) \leq \lceil \frac{n-1}{t} \rceil$ for all $x \in S$. But by setting $k := \lceil \frac{n-1}{t} \rceil$, Cutler's result is implied by Theorem 8.12.

We defined $\sigma_k(S) = \infty$ if $\alpha(S) < k$, and hence we obtain that $\alpha(S) < k$ implies the existence of a k-tree containing S. So Theorem 8.12 is also one of the answers to Question 8.11. However, comparing Theorem 8.2, it seems that the condition "$\alpha(S) < k$" is too strong for graphs to have a k-tree containing S. In addition, although the degree sum bound of Theorem 8.12 is best possible, we may be able to decrease it if a graph has high connectivity. Motivated by these consideration, in this section, we show the following result, which is a k-tree analogy of Theorem 8.7.

Theorem 8.13 ([36]) *Let k be an integer with $k \geq 3$ and let G be a graph of order n. Let $S \subseteq V(G)$ with $\kappa(S) \geq 1$. If for every $l \geq \kappa(S)$, there exists an integer t such that $t \leq (k-1)l+2-\lfloor \frac{l-1}{k} \rfloor$ and $\sigma_t(S) \geq n+tl-kl-1$, then G has a k-tree containing S.*

For $k \geq 3$ and $l \geq 1$, let $G(k,l)$ be a complete bipartite graph $K_{l,(k-1)l+2}$. Suppose that $l < k^2 - k$ and let S_1 be a larger partite set of $G(k,l)$. Then $l = \kappa(S_1)$. Although $G(k,l)$ has no k-tree containing S_1, $\sigma_t(S_1) = tl = \left|V\big(G(k,l)\big)\right| - kl + tl - 2$, where $t = (k-1)l + 2 - \lfloor \frac{l-1}{k} \rfloor$. Moreover, for $t' = (k-1)(l+1) + 2 - \lfloor \frac{l}{k} \rfloor$, $\sigma_{t'}(S_1) = \infty$ since $\alpha(S_1) = (k-1)l + 2 < t'$. Hence the degree sum condition of Theorem 8.13 in case of $l = \kappa(S)$ is best possible for $\kappa(S) < k^2 - k$.

On the other hand, since $\sigma_t(S) = \infty$ if $\alpha(S) < t$, we also have the following corollary. This is an answer to Question 8.11.

Corollary 8.14 *Let k be an integer with $k \geq 3$ and let G be a graph. Let $S \subseteq V(G)$ with $\kappa(S) \geq m \geq 1$. If $\alpha(S) \leq (k-1)m + 1 - \lfloor \frac{m-1}{k} \rfloor$, then G has a k-tree containing S.*

However, we do not know whether the independence number condition of Corollary 8.14 is best possible or not.

By the same way as the case Theorem 8.7, we obtain some corollaries. In fact, Theorem 8.13 can imply Theorem 8.12 for $k \geq 3$ (The case $k = 2$ is implied by Theorem 8.5). We will prove such corollaries in Section 8.2.2. In Section 8.2.3, we show some lemmas, and by using them, we will prove a slightly stronger result than Theorem 8.13 in Section 8.2.4.

8.2.2 Corollaries of Theorem 8.13

In this section, we lead some corollaries of Theorem 8.13. First, we show that it implies Theorem 8.12 for $k \geq 3$.

Proof of Theorem 8.12.

Let G be a graph satisfying the assumption of Theorem 8.12. Since G is connected, $\kappa(S) \geq 1$. Let $t := k$. Note that $t = k \leq (k-1)l + 2 - \lfloor \frac{l-1}{k} \rfloor$ for any $l \geq 1$. Since $\sigma_t(S) \geq n - 1 = n + tl - kl - 1$, we see that G satisfies the assumption of Theorem 8.13. $\square$

Next, we will prove the following two corollaries by the same way as in [132] and [135], respectively. These two corollaries imply new results on a spanning k-tree.

Corollary 8.15 *Let k be an integer with $k \geq 3$ and let G be a graph of order n. Let $S \subseteq V(G)$ with $\kappa(S) \geq m$. Suppose that $n \geq \{(k-2)m+3\}\alpha(S) - 3m(k-3) - 10 - m$. If $\sigma_{(k-2)m+3}(S) \geq n + (k-2)m^2 - (k-3)m - 1$, then G has a k-tree containing S.*

Corollary 8.16 *Let k be an integer with $k \geq 3$ and let G be a graph of order n. Let $S \subseteq V(G)$ with $\kappa(S) \geq m$. If $\sigma_{(k-2)m+3}(S) \geq n + (\alpha(S) - 2)(m-1) + 3\left(\frac{\alpha(S)-3}{k-2} - m\right)$, then G has a k-tree containing S.*

Proof of Corollaries 8.15 and 8.16.

Let G be a graph satisfying the assumption of Corollaries 8.15 or 8.16. For any $l \geq 1$, $(k-2)l+3 \leq (k-1)l+2-\lfloor \frac{l-1}{k} \rfloor$. Therefore, the first assumption of Theorem 8.13 holds for $t=(k-2)l+3$, and hence we have only to show that $\sigma_{(k-2)l+3}(S) \geq n+tl-kl-1 = n+(k-2)l^2-(k-3)l-1$ for any $l \geq m$. Suppose that there exists an integer $l \geq m$ such that $\sigma_{(k-2)l+3}(S) < n+(k-2)l^2-(k-3)l-1$. Since $\alpha(S) < (k-2)l+3$ implies $\sigma_{(k-2)l+3}(S) = \infty$, we may assume that $\alpha(S) \geq (k-2)l+3$.

First, we prove Corollary 8.15. Let $X = \{x_1, x_2, \ldots, x_{(k-2)l+3}\} \subseteq S$ be an independent set such that $d_G(x_1) \leq d_G(x_2) \leq \cdots \leq d_G(x_{(k-2)l+3})$ and $\sum_{x \in X} d_G(x) = \sigma_{(k-2)l+3}(S)$. Let $Y := \{x_1, x_2, \ldots, x_{(k-2)m+3}\}$ and $Z := X - Y$. Then $|Y| = (k-2)m+3$ and $|Z| = (k-2)(l-m)$. By the assumption of Corollary 8.15,

$$\begin{aligned}
\sum_{x \in Y} d_G(x) &\geq n+(k-2)m^2-(k-3)m-1 \\
&\geq \big((k-2)m+3\big)(\alpha(S)+m-4)+1.
\end{aligned}$$

This implies that $d_G(x_j) \geq \alpha(S)+m-3 \geq (k-2)l+m$ for any $j \geq (k-2)m+3$. It follows from $k \geq 3$ and $l \geq m$ that

$$\begin{aligned}
\sigma_{(k-2)l+3}(S) &= \sum_{x \in Y} d_G(x) + \sum_{x \in Z} d_G(x) \\
&\geq n+(k-2)m^2-(k-3)m-1+(k-2)(l-m)\big((k-2)l+m\big) \\
&= n+(k-2)l^2+(k-2)(k-3)l(l-m)-(k-3)m-1 \\
&\geq n+(k-2)l^2-(k-3)l-1,
\end{aligned}$$

a contradiction.

Next, we prove Corollary 8.16. $l \geq m$ implies that $\big((k-2)m+3\big)\sigma_{(k-2)l+3}(S) \geq \big((k-2)l+3\big)\sigma_{(k-2)m+3}(S)$. Since $\alpha(S) \geq (k-2)l+3$, we obtain

$$\begin{aligned}
&\big((k-2)m+3\big)\big(n+(k-2)l^2-(k-3)l-1\big) \\
> &\big((k-2)l+3\big)\Big(n+(\alpha(S)-2)(m-1)+3\Big(\frac{\alpha(S)-3}{k-2}-m\Big)\Big) \\
\geq &\big((k-2)l+3\big)\Big(n+\big((k-2)l+1\big)(m-1)+3(l-m)\Big),
\end{aligned}$$

i.e.,

$$(k-2)l+1-\frac{6}{k-2} > n.$$

Consequently,

$$\alpha(S)-2-\frac{6}{k-2} > n,$$

a contradiction. $\square$

8.2.3 Notations and preliminary results

Let T be tree and let $U \subseteq V(T)$. For a component C of $T-U$, C is called a *leaf component of* $T-U$ if C has a leaf of T; otherwise C is called an *inside component of* $T-U$. Note that for any component C of $T-U$, $N_T(C) \subseteq U$, and for any inside component C, $|N_T(C)| \geq 2$. Let k be an integer. An inside component C is called a *path component of* $T-U$ if $|N_T(C)| \leq k$. We define $\mathcal{C}(T,U), \mathcal{L}(T,U), \mathcal{I}(T,U)$ and $\mathcal{P}(T,U)$ as follows:

$$\begin{aligned}
\mathcal{C}(T,U) &:= \{C : C \text{ is a component of } T-U\}, \\
\mathcal{L}(T,U) &:= \{C \in \mathcal{C}(T,U) : C \text{ is a leaf component of } T-U\}, \\
\mathcal{I}(T,U) &:= \{C \in \mathcal{C}(T,U) : C \text{ is an inside component of } T-U\}, \\
\mathcal{P}(T,U) &:= \{C \in \mathcal{I}(T,U) : C \text{ is a path component of } T-U\}.
\end{aligned}$$

Note that $\mathcal{C}(T,U) = \mathcal{L}(T,U) \cup \mathcal{I}(T,U)$ and $\mathcal{P}(T,U) \subseteq \mathcal{I}(T,U)$.

Claim 8.1 *Let T be a tree, $U \subseteq V(T)$ and $C \in \mathcal{P}(T,U)$. Then $\sum_{v \in V(C)} \big(d_T(v) - 2\big) \leq k-2$.*

Proof. Since $C \in \mathcal{P}(T,U) \subseteq \mathcal{I}(T,U)$, C has no leaf of T. Hence $k \geq |N_T(C)| = \sum_{v \in V(C)} \big(d_T(v) - 2\big) + 2$. This implies the statement of Claim 8.1. $\square$

In the rest of this section, we focus on the properties of a k-tree which contains as many specified vertices as possible. Let G be a graph and let $S \subseteq V(G)$. Suppose that G has no k-tree containing all vertices in S. We choose a k-tree T in G so that $|V(T) \cap S|$ is as large as possible. By the assumption, there exists $x_0 \in S$ which is not contained in T. Suppose that there exists an (x_0, T)-fan F with width l. Let $U := V(T) \cap V(F)$ be the set of all end-vertices of F. Note that $|U| = l$.

By the maximality of $|V(T) \cap S|$, we obtain the following fact. By using Fact 8.2 (i) and (ii), Fact 8.2 (iii) can be easily shown.

Fact 8.2 (i) *For any $u_1, u_2 \in U$, $u_1u_2 \notin E(T)$.*

(ii) *$d_T(u) = k$ for every $u \in U$.*

(iii) *$|\mathcal{C}(T,U)| = (k-1)l + 1$.*

In the following claim, we show the existence of specified vertices in each path component. It plays an important role in the proof of our theorem.

Claim 8.3 *For any $C \in \mathcal{P}(T,U)$, $V(C) \cap S \neq \emptyset$.*

Proof. Suppose that there exists $C \in \mathcal{P}(T,U)$ such that $V(C) \cap S = \emptyset$. By the definition of $\mathcal{P}(T,U)$, $|N_T(C)| \leq k$. Let F' be an (x_0, T)-fan which is obtained by restricting F to the union of paths whose end-vertices are in $N_T(C)$. Then $(T-C)\cup F'$ is a k-tree containing more vertices in S than T, a contradiction. $\square$

8.2.4 Proof of Theorem 8.13

We will prove Theorem 8.13 by induction on $|S|$. If $|S| \le 2$, then there is nothing to prove. Thus, we may assume that $|S| \ge 3$.

Let $x \in S$. By the definition of $\kappa(S)$ and $\sigma_t(S)$, $\kappa\big(S-\{x\}\big) \ge \kappa(S)$ and $\sigma_t\big(S-\{x\}\big) \ge \sigma_t(S)$ for every t, and hence $S-\{x\}$ satisfies the assumption of Theorem 8.13. Thus, by induction hypothesis, there exists a k-tree T containing $S-\{x\}$. If T contains x, then again there is nothing to prove. Hence we may assume that the following fact holds.

Fact 8.4 *For every $x \in S$, there exists a k-tree T containing $S-\{x\}$. In addition, every k-tree T containing $S-\{x\}$ does not contain x.*

We choose $x_0 \in S$ and a k-tree T containing $S-\{x_0\}$ so that

(T1) $\kappa(x_0, T)$ is as large as possible.

Fix $l := \kappa(x_0, T)$. We will show that $l \ge \kappa(S)$ and $\sigma_t(S) \le n + tl - kl - 2$ for any $t \le (k-1)l + 2 - \lfloor \frac{l-1}{k} \rfloor$, which contradicts the assumption of Theorem 8.13.

By the definition of $\kappa(x_0, T)$, there exists an (x_0, T)-fan F with width l. Let $U := V(T) \cap V(F)$ and for $u \in U$, let P_u be a path in F connecting x_0 and u. Let $S' := S - \{x_0\}$. By Fact 8.4, we can use results in Section 8.2.3.

For $C \in \mathcal{I}(T,U) - \mathcal{P}(T,U)$, we call C a *malignant component of* $T-U$ if there exist three vertices v_1, v_2, w in C such that $v_1w, v_2w \in E(T)$, $d_T(v_1) = d_T(v_2) = k$, $d_T(w) = 3$, $d_T(z) = 2$ for any $z \in V(C) - \{v_1, v_2, w\}$, and $w \notin S'$. We call w a *center* of C (see Figure 8.1).

Take above a k-tree T satisfying the choice (T1) and an (x_0, T)-fan F so that

(T2) $|V(T)|$ is as small as possible; subject to (T1),

(T3) $|L(T)|$ is as small as possible; subject to (T2),

(T4) $|\mathcal{P}(T,U)|$ is as small as possible; subject to (T3),

(T5) $\sum_{C\in\mathcal{P}(T,U)} |N_T(C)|$ is as large as possible; subject to (T4),

(T6) the number of malignant components is as small as possible; subject to (T5).

For convenience, we abbreviate $\mathcal{C}(T,U)$, $\mathcal{L}(T,U)$, $\mathcal{I}(T,U)$ and $\mathcal{P}(T,U)$, to $\mathcal{C}$, $\mathcal{L}$, $\mathcal{I}$ and $\mathcal{P}$ respectively, if there is no confusion.

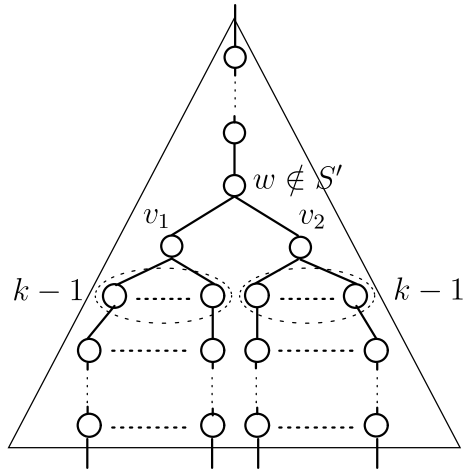


Figure 8.1: A malignant component.

Claim 8.5 $L(T) \subseteq S'$.

Proof. Suppose that there exists a vertex $v \in L(T)$ such that $v \notin S'$. Then $v \notin U$ by Fact 8.2 (ii). Hence $T - \{v\}$ is a k-tree containing S' such that $\kappa(x_0, T - \{v\}) = \kappa(x_0, T)$, which contradicts the choice (T2). $\square$

By Lemma 2.4, Fact 8.2 (ii) and Claim 8.5, $|S'| \geq |L(T)| \geq \sum_{u \in U} \big(d_T(u) - 2\big) + 2 = (k-2)l + 2 \geq l + 2 = \kappa(x_0, T) + 2 \geq \kappa(x_0, S') + 2 \geq \min\big\{|S'|, \kappa(S)\big\} + 2$. Hence $\kappa(S) < |S'|$ and $l \geq \kappa(S)$.

For each $C \in \mathcal{L} \cup \mathcal{P}$, we can take a vertex x_C in $V(C) \cap S'$ by Claims 8.3 and 8.5. Take x_C as $d_T(x_C)$ is as small as possible. (Note that x_C is a leaf of T in case $C \in \mathcal{L}$.) Let $X := \{x_C : C \in \mathcal{L} \cup \mathcal{P}\}$, and $X' := X \cup \{x_0\}$. Note that $|X| = |\mathcal{L}| + |\mathcal{P}|$. For $x \in X$, let C_x be a component of $\mathcal{L} \cup \mathcal{P}$ such that $x \in V(C_x)$.

Claim 8.6 $|X| \geq (k-1)l + 1 - \lfloor \frac{l-1}{k} \rfloor$.

Proof. By the definition of $\mathcal{P}$, $|N_T(C)| \geq k+1$ for any $C \in \mathcal{I} - \mathcal{P}$. Therefore, by Fact 8.2 (ii) and (iii),

$$\begin{aligned} kl = k|U| &= \sum_{C \in \mathcal{C}} |N_T(C)| \\ &\geq (k+1)\big(|\mathcal{I}| - |\mathcal{P}|\big) + |\mathcal{P}| + |\mathcal{L}| \\ &= (k+1)\big(|\mathcal{I}| - |\mathcal{P}|\big) + |\mathcal{P}| + |\mathcal{C}| - |\mathcal{I}| \\ &= k\big(|\mathcal{I}| - |\mathcal{P}|\big) + (k-1)l + 1. \end{aligned}$$

Hence

$$|\mathcal{I}| - |\mathcal{P}| \leq \left\lfloor \frac{l-1}{k} \right\rfloor.$$

This implies that

$$\begin{aligned} |X| &= |\mathcal{L}| + |\mathcal{P}| \\ &= |\mathcal{C}| - \big(|\mathcal{I}| - |\mathcal{P}|\big) \\ &\geq (k-1)l + 1 - \left\lfloor \frac{l-1}{k} \right\rfloor. \qquad \square \end{aligned}$$

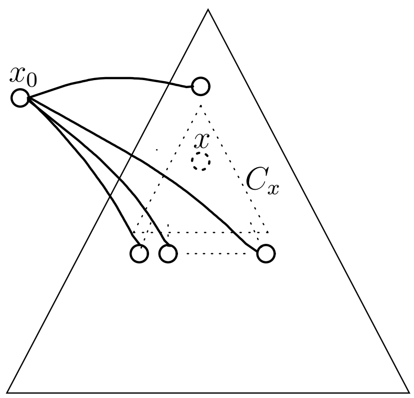


Figure 8.2: A tree $T(x \to x_0)$.

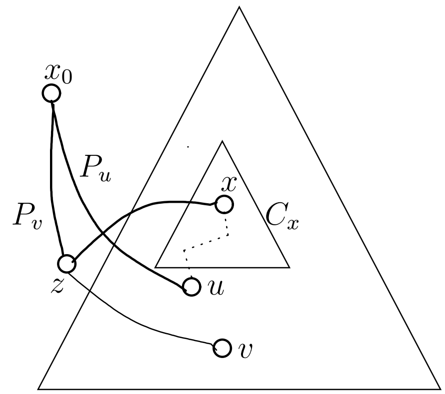


Figure 8.3: A k-tree used in the proof of Claim 8.7(iii).

Let $x \in X$ with $d_T(x) = k$. We define a tree $T(x \to x_0)$ as $T(x \to x_0) := \big(T - C_x\big) \cup \bigcup_{u \in N_T(C_x)} P_u$ (see Figure 8.2). Since $C_x \in \mathcal{P}$, $|N_T(C_x)| \leq k$, and hence $d_{T(x \to x_0)}(x_0) \leq k$. This implies that $T(x \to x_0)$ is a k-tree.

For $u \in N_T(C_x)$, we call a path xTu *removable* if $d_T(z) = 2$ and $z \notin S'$ for any $z \in V(xTu) - \{x, u\}$. Note that an edge xu is removable if $u \in N_T(x) \cap U$. The following two claims are useful when we deal with the vertices in X.

Claim 8.7 *For $x \in X$ with $d_T(x) = k$, the following statements hold.*

(i) *For any $u \in N_T(C_x)$, xTu is a removable path.*

(ii) *$T(x \to x_0)$ contains $S - \{x\}$.*

(iii) *$N_G(x) \cap \big(V(F) - U\big) = \emptyset$.*

Proof. (i) Suppose that there exists $u \in N_T(C_x)$ such that xTu is not a removable path. Then $d_T(z) \geq 3$ or $z \in S'$ for some $z \in V(xTu) - \{x, u\}$. If $d_T(z) \geq 3$, then $k - 2 \geq \big(d_T(x) - 2\big) + \big(d_T(z) - 2\big) \geq k - 1$ by Claim 8.1, a contradiction. Thus, $d_T(z) = 2$ and $z \in S'$. This implies that $d_T(z) < d_T(x)$, which contradicts the choice of x.

(ii) By the statement (i), a path xTu is removable for all $u \in N_T(C_x)$. This implies that $T(x \to x_0)$ contains $S - \{x\}$.

(iii) Suppose that $N_G(x)\cap\big(V(F)-U\big)\neq\emptyset$, say, $z\in N_G(x)\cap\big(V(F)-U\big)$. Let v be a vertex in U such that $z\in V(P_v)$. Since $|N_T(C_x)|=k\geq 3$, there exists a vertex $u\in N_T(C_x)$ such that $u\neq v$. By the statement (i), xTu is a removable path, and hence $\big(T-xTu\big)\cup P_u\cup x_0P_vz\cup\{zx\}$ is a k-tree containing S, where $T-xTu$ means the graph obtained from T by the deletion of all edges and all internal vertices of xTu, a contradiction (see Figure 8.3). $\square$

Claim 8.8 *Let $x\in X$ with $d_T(x)\leq k-1$, and let $v\in N_G(x)\cap V(T)$ such that $V(xTv)\cap U\neq\emptyset$. Consider x as a root of T. Then the followings hold.*

(i) $d_T(v)=k$.

(ii) $N_T(v)\cap U=\emptyset$.

(iii) $v^-\in S'$ *or* $d_T(v^-)\geq 3$. *Moreover, if* $x\in L(T)$, *then* $v^-\in S'$.

Proof. (i) If $d_T(v)\leq k-1$, then $\big(T-\{uu^-\}\big)\cup\{xv\}\cup P_u$ is a k-tree containing S, where $u\in V(xTv)\cap U$, a contradiction.

(ii) Suppose that there exists a vertex $u'\in N_T(v)\cap U$. Let $u\in V(xTv)\cap U$ such that $V(uTv)\cap U=\{u\}$. If $u'=u$, let $T':=\big(T-\{uv\}\big)\cup P_u\cup\{xv\}$; otherwise let $T':=\big(T-\{u^-u,vu'\}\big)\cup P_u\cup P_{u'}\cup\{xv\}$. Then T' is a k-tree containing S, a contradiction.

(iii) Suppose that $v^-\notin S'$ and $d_T(v^-)=2$. Let $T^v:=\big(T-\{vv^-\}\big)\cup\{xv\}$. Then T^v is a k-tree containing S' with $V(T^v)=V(T)$ and $v^-\in L(T^v)$. Note that $v^-\notin U$ by the statement (ii). Then $T^v-\{v^-\}$ contains S', contradicting the choice (T2). Therefore $v^-\in S'$ or $d_T(v^-)\geq 3$. Moreover, if $x\in L(T)$ and $d_T(v^-)\geq 3$, then $L(T^v)=L(T)-\{x\}$, contradicting the choice (T3). $\square$

By using these two claims, we study on the properties of X'.

Claim 8.9 *X' is an independent set of G.*

Proof. If there exists a vertex $x\in X$ such that $x_0x\in E(G)$, then $F'=F\cup\{x_0x\}$ is an (x_0,T)-fan with width $l+1$, which contradicts $\kappa(x_0,T)=l$. Thus we have $x_0x\notin E(G)$ for any $x\in X$.

Suppose that $xy\in E(G)$ for some $x,y\in X$. By Claim 8.8 (i), we may assume that $d_T(x)=k$. If $d_T(y)\leq k-1$, then by Claim 8.7 (ii), $T(x\to x_0)\cup\{xy\}$ is a k-tree containing S, a contradiction. Therefore $d_T(y)=k$. Let $u\in V(xTy)\cap N_T(C_x)$ and $v\in N_T(C_y)-V(xTy)$. By Claim 8.7 (i), two paths xTu and yTv are removable. This implies that $\big(T-(xTu\cup yTv)\big)\cup P_u\cup P_v\cup\{xy\}$ is a k-tree containing S, where $T-(xTu\cup yTv)$ means the graph obtained from T by the deletion of all edges and all internal vertices of xTu and yTv, a contradiction (see Figure 8.4). $\square$

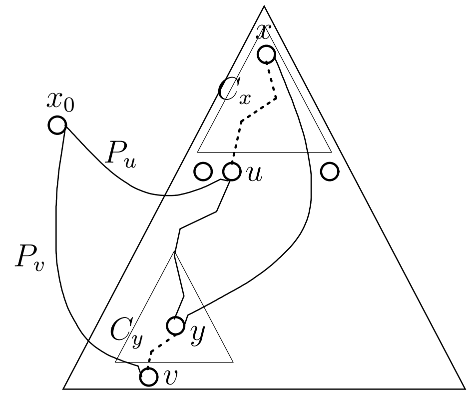


Figure 8.4: A k-tree used in the proof of Claim 8.9.

Claim 8.10 *For any $x, y \in X'$, $N_G(x) \cap N_G(y) \cap V(G - T) = \emptyset$.*

Proof. Suppose that there exists $z \in N_G(x) \cap N_G(y) \cap V(G-T)$ for some $x, y \in X'$. Without loss of generality, we may assume that $x \neq x_0$. Suppose that $z \in V(F)$ and let v be a vertex in U such that $z \in V(P_v)$. Since we can consider a path $x_0 P_v z x$ as an (x_0, T)-fan with width 1, $d_T(x) = k$ by Fact 8.2 (ii), which contradicts Claim 8.7 (iii). Therefore $z \notin V(F)$. Then by using a path xzy instead of an edge xy, we can use the similar discussion to the proof of Claim 8.9 and we lead a contradiction, again. $\square$

Now we show the degree bound of $x \in X$ in order to consider the degree sum of the vertices in X'. First, we consider the case $d_T(x) = k$.

Claim 8.11 *For $x \in X$ with $d_T(x) = k$, $\left|N_G(x) \cap V(T)\right| \leq |C_x| + l - 1$ holds.*

Proof. Let $x \in X$ with $d_T(x) = k$. Then by Claim 8.7 (ii), $T' := T(x \to x_0)$ is a k-tree containing $S - \{x\}$. By Claim 8.7 (iii) and by the choice (T1), $\left|N_G(x) \cap V(T - C_x)\right| = \left|N_G(x) \cap V(T')\right| \leq \kappa(x, T') \leq l$. Therefore $\left|N_G(x) \cap V(T)\right| = \left|N_G(x) \cap V(T - C_x)\right| + \left|N_G(x) \cap V(C_x)\right| \leq l + |C_x| - 1$. $\square$

Next, we focus on the properties of $x \in X$ with $d_T(x) \leq k-1$. In particular, we will show that $\left|N_G(x) \cap V(T)\right| \leq |C_x| + l - 1$. By constructing an $\left(x, S' - \{x\}\right)$-fan with width $\left|N_G(x) \cap V(T - C_x)\right|$, we can obtain the above degree bound of x by the similar way as in the proof of Claim 8.11. In order to show the existence of such a fan, we prove the following claim. In Claim 8.13, we can obtain the desired fan by using Claim 8.12 recursively.

Claim 8.12 *Let $x \in X$ with $d_T(x) \leq k-1$. Consider x as a root of T. Let $D \in \mathcal{C} - \{C_x\}$, $K := T\left[V(D) \cup N_T(D)\right]$ and $u \in N_T(D)$ such that $V(x \overrightarrow{T} u) \cap V(K) = \{u\}$.*

Then there exists a collection of disjoint paths $\mathcal{Q} = \{Q_v : v \in (N_G(x) \cap V(K)) \cup \{u\}\}$ *in* K *along* $\overrightarrow{T}$, *where* Q_v *starts from* v *and reaches a vertex in* $S' \cup (N_T(D) - \{u\})$. *(see Figure 8.5).*

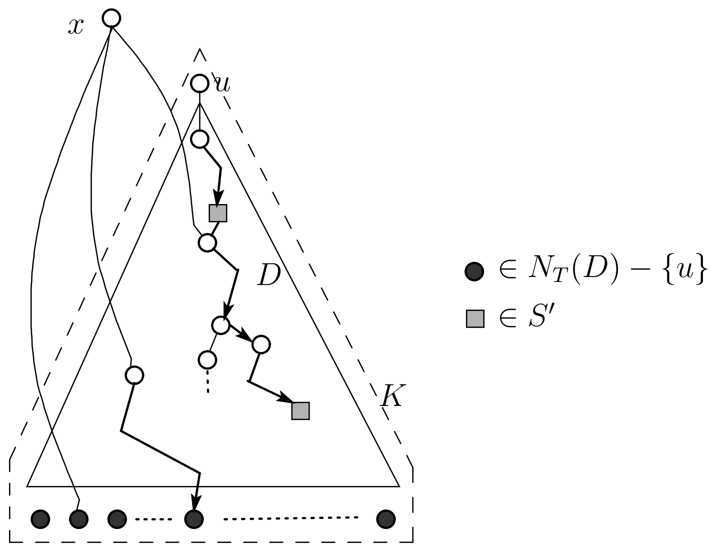


Figure 8.5: A collection of disjoint paths desired in Claim 8.12.

Proof. Let $x \in X$ with $d_T(x) \leq k-1$. For convenience, let $C := C_x$. For $v \in N_G(x) \cap V(K)$ with $v \neq u$, we call v a *bad vertex* if $v^- \notin S'$. By Claim 8.8 (iii), if v is a bad vertex, then $d_T(v^-) \geq 3$, and if there exists a bad vertex, then $C \in \mathcal{P}$ and $d_T(x) \geq 2$. For a bad vertex v, let D^v be the unique component of $D - \{vv^-\}$ containing v^-. By Claim 8.8 (ii), $V(D^v) \neq \emptyset$. Let $T^v := (T - \{vv^-\}) \cup \{xv\}$ and $C^v := C \cup \{xv\} \cup (D - D^v)$, (see Figure 8.6). Note that $C^v, D^v \in \mathcal{C}(T^v, U)$. If $v \in N_T(D) - \{u\}$, then $V(D^v) = V(D)$, $|N_{T^v}(C^v)| = |N_T(C)| + 1$ and $|N_{T^v}(D^v)| = |N_T(D)| - 1$. Since $d_T(x) \leq k-1$, T^v is also a k-tree containing S' with $V(T^v) = V(T)$ and F is also an (x_0, T^v)-fan. Moreover $L(T^v) = L(T)$ since $d_T(v^-) \geq 3$ and $d_T(x) \geq 2$. Thus, for any bad vertex v, T^v satisfies the choice (T1)–(T3).

Subclaim 8.12.1 *Suppose that* v *is a bad vertex. Then the following statements hold.*

(i) $D^v \in \mathcal{P}(T^v, U)$ *and* $V(D^v) \cap S' \neq \emptyset$.

(ii) $|N_G(x) \cap V(D^v) - \{v^-\}| \leq 1$. *Moreover, if there exists a vertex* $w \in N_G(x) \cap V(D^v) - \{v^-\}$, *then* $d_T(w) = d_{T^v}(w) = k$, $d_T(v^-) = 3$ *and* $d_T(z) = d_{T^v}(z) = 2$ *for any* $z \in V(D^v) - \{w, v^-\}$.

(iii) *For any bad vertex* w *such that* $w \in V(D^v) - \{v^-\}$, $w^- = v^-$

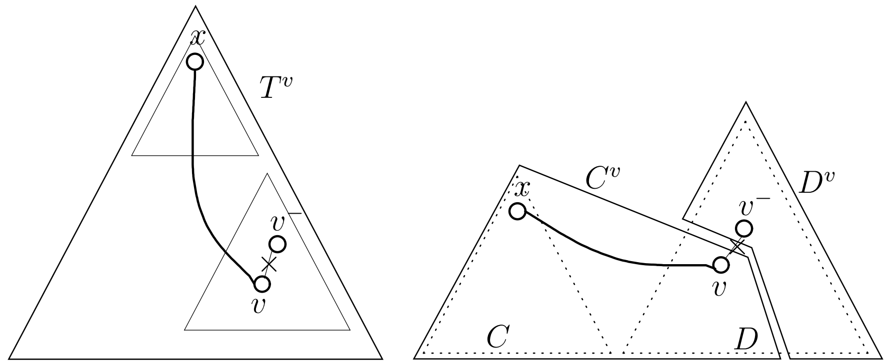


Figure 8.6: T^v, C^v and D^v.

Proof. (i) Since v is a bad vertex, $C \in \mathcal{P}(T, U)$. Suppose that $D^v \notin \mathcal{P}(T^v, U)$ (this implies that $D \notin \mathcal{P}(T, U)$). Then, by the choice (T4), $C^v \in \mathcal{P}(T^v, U)$, and hence $D - D^v$ has no leaf of T. This implies that $|N_{T^v}(C^v)| > |N_T(C)|$. Since $\mathcal{P}(T, U) - \{C\} = \mathcal{P}(T^v, U) - \{C^v\}$, and $N_{T^v}(C') = N_T(C')$ for any $C' \in \mathcal{P}(T, U) - \{C\}$, this contradicts the choice (T5). Hence $D^v \in \mathcal{P}(T^v, U)$ and by Claim 8.3, $V(D^v) \cap S' \neq \emptyset$.

(ii) Let $w \in N_G(x) \cap V(D^v) - \{v^-\}$. Note that $d_T(z) = d_{T^v}(z)$ for any $z \in V(D^v) - \{v^-\}$ and $d_T(v^-) = d_{T^v}(v^-) + 1$ by the definition of T^v. By Claim 8.8 (i), $d_T(w) = d_{T^v}(w) = k$. If there exists $z \in V(D^v) - \{w\}$ with $d_{T^v}(z) \geq 3$, then $|N_{T^v}(D^v)| \geq k + 1$ by Claim 8.1, and hence $D^v \notin \mathcal{P}(T^v, U)$, contradicting the statement (i). Since D^v has no leaf of T^v, we have $d_{T^v}(z) \geq 2$ for any $z \in V(D^v) - \{w\}$. These imply that $d_T(v^-) = d_{T^v}(v^-) + 1 = 3$ and $d_T(z) = d_{T^v}(z) = 2$ for any $z \in V(D^v) - \{w, v^-\}$. Moreover $\left|N_G(x) \cap V(D^v) - \{v^-\}\right| \leq 1$ by Claim 8.8 (i).

(iii) Suppose that w is a bad vertex such that $w \in V(D^v) - \{v^-\}$. By the statement (ii), $d_T(z) = 2$ for any $z \in V(D^v) - \{w, v^-\}$. On the other hand, $d_T(w^-) \geq 3$ by the definition of bad vertex, and hence $w^- = v^-$. $\square$

The following subclaim shows the method of taking a collection of disjoint paths desired in Claim 8.12.

Subclaim 8.12.2 *Let H be a subtree of K of order at least 2, and let h be a vertex in H such that $V(x\overrightarrow{T}h) \cap V(H) = \{h\}$. (We can consider H as a rooted subtree with root h.) Suppose that $L(H) \subseteq N_T(D) \cup L(T) \cup \{h\}$ and $w^- \in S'$ for any $w \in N_G(x) \cap V(H) - \{h\}$. Then we can take a collection of disjoint paths $\mathcal{R} := \{R_v : v \in (N_G(x) \cap V(H)) \cup \{h\}\}$ in H along $\overrightarrow{T}$, where R_v starts from v and reaches a vertex in $S' \cup (N_T(D) - \{h\})$.*

Proof. Let v be a vertex in $(N_G(x) \cap V(H)) \cup \{h\}$. If $v \in S' \cup (N_T(D) - \{h\})$, then let $R_v := v$ which is a path consisting of one vertex; otherwise we take a path R_v in H from v by recursively pursuing vertices along $\overrightarrow{T}$ till R_v first reaches a vertex in $S' \cup \big(L(H) - \{h\}\big) \cup \big(N_G(x) \cap V(H) - \{v\}\big)$. Since $|V(H)| \geq 2$, we can find a next vertex of v in R_v, and hence we can take such a path R_v. (If R_v arrives at a vertex of degree at least 3, we may choose an arbitrary adjacent vertex as a next vertex in R_v.)

Since $w^- \in S'$ for any $w \in N_G(x) \cap V(H) - \{h\}$, R_v must pass a vertex in S' before reaching a vertex in $N_G(x) \cap V(H) - \{v\}$. Thus, by the definition of R_v, R_v never reaches a vertex in $N_G(x) \cap V(H) - \{v\}$. This implies that every R_v reaches a vertex in $S' \cup \big(L(H) - \{h\}\big) \subseteq S' \cup (N_T(D) \cup L(T) - \{h\}) \subseteq S' \cup (N_T(D) - \{h\})$ by Claim 8.5. Moreover, since each R_v is a path along $\overrightarrow{T}$, $R_v \cap R_w = \emptyset$ for any $v, w \in \big(N_G(x) \cap V(H)\big) \cup \{h\}$. Hence $\mathcal{R} := \big\{R_v : v \in \big(N_G(x) \cap V(H)\big) \cup \{h\}\big\}$ is a desired collection of disjoint paths. $\square$

If there exists no bad vertex, then by applying Subclaim 8.12.2 to K, we can obtain a collection of disjoint paths which is desired in Claim 8.12. Thus, we may assume that there exists at least one bad vertex in K. Moreover, we obtain the following subclaim.

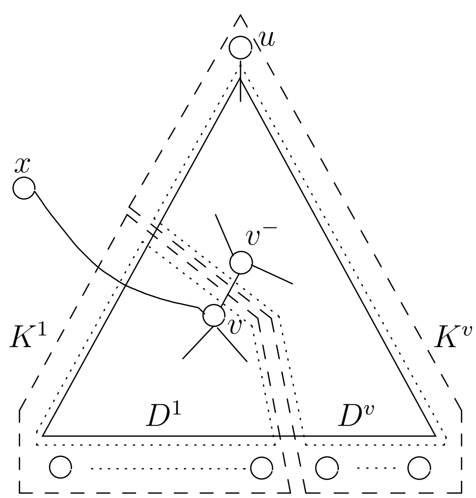


Figure 8.7: Division of K into K^v and K^1 (used in the proof of Subclaim 8.12.3).

Subclaim 8.12.3 *We may assume that there exist at least two bad vertices in K.*

Proof. Suppose that there exists only one bad vertex in K. Let v be the unique bad vertex in K and let $D^1 := D - D^v$. We define $K^v := T\big[V(D^v) \cup N_T(D^v) - \{v\}\big]$ and $K^1 := T\big[V(D^1) \cup N_T(D^1) - \{v^-\}\big]$ (see Figure 8.7). If $V(D^1) = \emptyset$ (in case $v \in U$), then let $K^1 := T[\{v\}]$. Since $d_T(v^-) \geq 3$, we have $v^- \notin L(K^v)$, and hence $L(K^v) \subseteq L(T) \cup (N_T(D^v) - \{v\}) \cup \{u\} \subseteq L(T) \cup N_T(D) \cup \{u\}$. On the other hand, $N_T(D^1) - \{v^-\} \subseteq N_T(D)$ implies that $L(K^1) \subseteq L(T) \cup (N_T(D^1) - \{v^-\}) \subseteq L(T) \cup N_T(D)$. In addition, since v is the unique bad vertex in K, both K^v and K^1 satisfy the assumption of Subclaim 8.12.2. Then we obtain a collection of disjoint paths $\mathcal{R}^v$ and $\mathcal{R}^1$ in K^v and K^1, which are described in Subclaim 8.12.2, respectively (if $V(D^1) = \emptyset$, then let $\mathcal{R}^1 := \{v\}$). Then $\mathcal{Q} := \mathcal{R}^v \cup \mathcal{R}^1$ is a collection of disjoint paths, which is desired in Claim 8.12. $\square$

In the following two subclaims, we consider the location of bad vertices.

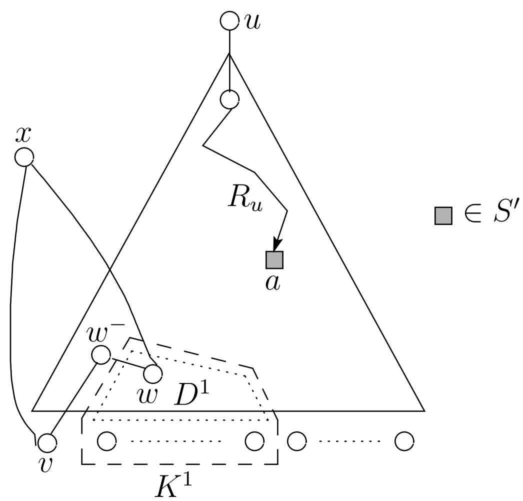


Figure 8.8: A subtree K^1 of K (used in the proof of Subclaim 8.12.4).

Subclaim 8.12.4 *For any two bad vertices v, w such that $vw \notin E(T)$, we may assume that $\{v, w\} \subseteq N_T(D)$.*

Proof. Suppose that $\{v, w\} \not\subseteq N_T(D)$. By symmetry of v and w, we may assume that $w \in V(D^v) - \{v^-\}$. Then by Subclaim 8.12.1 (ii) and (iii), $v^- = w^-$ and $N_G(x) \cap V(D^v) - \{w, v^-\} = \emptyset$. Moreover, $d_T(w) = d_{T^v}(w) = k$, $d_T(v^-) = 3$ and $d_T(z) = d_{T^v}(z) = 2$ for any $z \in V(D^v) - \{w, v^-\}$. These imply that there exists no bad vertex in $V(D^v) \cup N_T(D^v) - \{u, v, w\}$.

At first, suppose that $v \in N_T(D)$. Since $v^- \notin N_G(x)$ by Claim 8.8 (ii), $N_G(x) \cap V(D^v) = N_G(x) \cap V(D) = \{w\}$. Let $R_v := v$. By Subclaim 8.12.1 (i), we can take a path $R_u := u\overrightarrow{T}a$ in D^w, where $a \in S' \cap V(D^w)$ (there exists such a vertex a by Subclaim 8.12.1 (i)). Let $D^1 := D - D^w$ and $K^1 := T\big[V(D^1) \cup N_T(D^1) - \{w^-\}\big]$ (see Figure 8.8). Then $L(K^1) \subseteq L(T) \cup N_T(D)$ because $d_T(w) = k$. On the other hand, since there exists no bad vertex in $V(D^v) \cup N_T(D^v) - \{u, v, w\}$, K^1 satisfies the assumption of Subclaim 8.12.2, and hence there exists a collection of disjoint paths $\mathcal{R}^1$ in K^1 along $\overrightarrow{T}$ which is described in Subclaim 8.12.2. Then $\mathcal{Q} := \mathcal{R}^1 \cup \{R_u, R_v\}$ is a desired collection of disjoint paths.

Next, suppose that $v \in V(D)$. Then $v \in V(D^w) - \{w^-\}$ because $v^- = w^-$. By symmetry of v and w, $d_T(v) = d_{T^w}(v) = k$, and $d_T(z) = 2$ for any $z \in V(D) - \{w, v, v^-\}$. Then $|N_{T^v}(C^v)| = |N_T(C)| + d_T(v) - 1 \geq k+1$, and hence $C^v \notin \mathcal{P}(T^v, U)$. This implies that T^v satisfies the choice (T4). Moreover, $|N_{T^v}(D^v)| \geq d_T(w) = k$. Then T^v also satisfies the choice (T5). On the other hand, $v^- \notin S'$ because v is a bad vertex. Therefore D is a malignant component of $T - U$. Thus, by the choice (T6), C^v is a malignant component of $T^v - U$. Since $x \in S'$ and $d_{T^v}(x) \geq 3$, v must be a center of C^v and $k = 3$. However, for any $z \in V(D) - \{v, w, v^-\}$, $d_T(z) = 2$, which contradicts the definition of the malignant component. $\square$

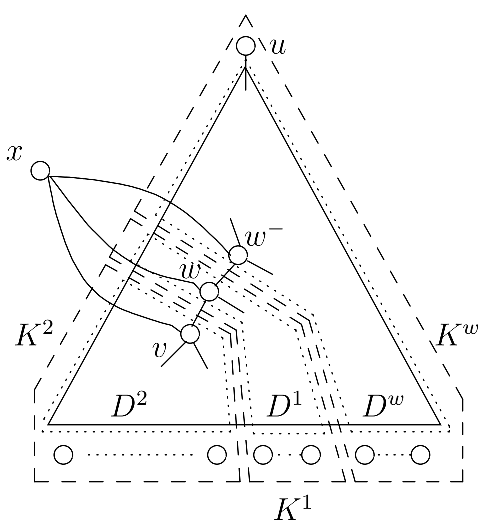


Figure 8.9: Division of K into K^w, K^1 and K^2 (used in the proof of Subclaim 8.12.5).

Subclaim 8.12.5 *For any two bad vertices v and w, we may assume that $vw \notin E(T)$.*

Proof. Suppose that $vw \in E(T)$ for some bad vertices v, w. By symmetry of v and w, we may assume that $w = v^-$. Note that $v \notin N_T(D)$ by Claim 8.8 (ii). Let D^1, D^2 be two components of $\big(D - D^w\big) - \{vv^-\}$ such that D^1 and D^2 contain w and v, respectively. Let $K^w := T\big[V(D^w) \cup N_T(D^w) - \{w\}\big]$, $K^1 := T\big[V(D^1) \cup N_T(D^1) - \{w^-, v\}\big]$, and $K^2 := T\big[V(D^2) \cup N_T(D^2) - \{w\}\big]$ (see Figure 8.9). Then $L(K^w) \subseteq L(T) \cup N_T(D) \cup \{u\}$, $L(K^1) \subseteq L(T) \cup N_T(D) \cup \{w\}$ and $L(K^2) \subseteq L(T) \cup N_T(D) \cup \{v\}$.

Note that $v, w \notin N_T(D)$. If there exists a bad vertex $z \in V(K) - \{u, v, w\}$, then we get a contradiction by applying Subclaim 8.12.4 to z and v or z and w. Thus we may assume that K^w, K^1 and K^2 do not contain a bad vertex except for v and w. Since v and w are roots of K^1 and K^2, respectively, all of K^w, K^1 and K^2 satisfy

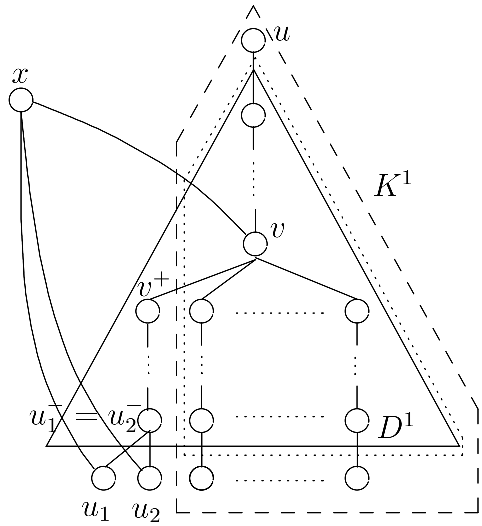


Figure 8.10: A subtree K^1 of K (used in the last part of the proof of Claim 8.12).

the assumption of Subclaim 8.12.2. Therefore, by considering the union of three collections of disjoint paths in K^w, K^1 and K^2, which are obtained by Subclaim 8.12.2 we have a desired collection of disjoint paths. $\square$

By Subclaims 8.12.3, 8.12.4 and 8.12.5, there exist at least two bad vertices, and any bad vertex is contained in $N_T(D)$. Let u_1, u_2 be two bad vertices. If $N_G(x) \cap V(D) = \emptyset$, then, by letting $Q_v := v$ for any $v \in N_G(x) \cap N_T(D) - \{u\}$, and letting $Q_u := u\overrightarrow{T}a$, where $a \in V(D^{u_1}) \cap S'$, (there exists such a vertex a by Subclaim 8.12.1 (i)), we obtain a collection of disjoint paths $\mathcal{Q} := \big\{Q_v : v \in \big(N_G(x) \cap V(K)\big) \cup \{u\}\big\}$, which is desired in Claim 8.12. Hence we may assume that $N_G(x) \cap V(D) \neq \emptyset$.

Let $v \in N_G(x) \cap V(D)$. Note that $v \in V(D^{u_i})$ and $v \neq u_i^-$ for $i = 1, 2$ by Claim 8.8 (ii). Then by Subclaim 8.12.1 (ii), $d_T(v) = k$, $d_T(u_i^-) = 3$ and $d_T(z) = 2$ for any $z \in V(D^{u_i}) - \{v, u_i^-\}$. Therefore $u_1^- = u_2^-$, v is the unique vertex in $N_G(x) \cap V(D)$, and there are no bad vertex except for u_1 and u_2.

Let v^+ be a vertex adjacent to v lying on $v\overrightarrow{T}u_1$. Then $D - \{vv^+\}$ has exactly two components because $v \neq u_1^-$. Let D^1 be the component of $D - \{vv^+\}$ containing v and let $K^1 := T\big[V(D^1) \cup N_T(D^1) - \{v^+\}\big]$ (See Figure 8.10). Note that $L(K^1) \subseteq L(T) \cup N_T(D) \cup \{u\}$. Then K^1 satisfies the assumption of Subclaim 8.12.2, and hence there exists a collection of disjoint paths $\mathcal{R}^1$ in K^1 along $\overrightarrow{T}$ which is described in Subclaim 8.12.2. On the other hand, let $R_{u_i} := u_i$ $(i = 1, 2)$. Then $\mathcal{Q} := \mathcal{R}^1 \cup \big\{R_{u_1}, R_{u_2}\big\}$ is a desired collection of disjoint paths. This completes the proof of Claim 8.12. $\square$

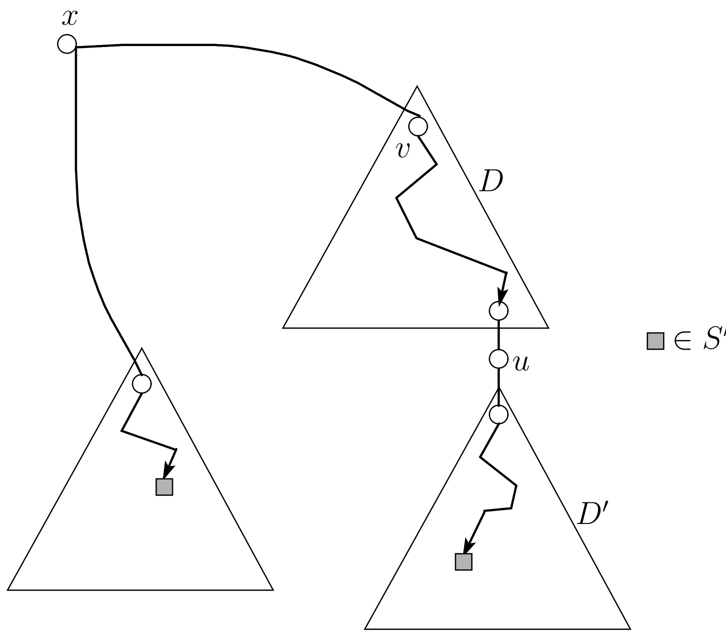


Figure 8.11: A method of taking an $(x, S' - \{x\})$-fan in the proof of Claim 8.13.

Claim 8.13 *For $x \in X$ with $d_T(x) \leq k-1$, $\big|N_G(x) \cap V(T)\big| \leq |C_x| + l - 1$.*

Proof. Let $x \in X$ with $d_T(x) \leq k-1$. Consider x as a root of T. By Claim 8.12, for each $D \in \mathcal{C} - \{C_x\}$, there exists a collection of disjoint paths $\mathcal{Q}^D$ in K^D along $\overrightarrow{T}$, which is described in Claim 8.12, where $K^D := T\big[V(D) \cup N_T(D)\big]$. We will construct an $\big(x, S' - \{x\}\big)$-fan with width $\big|N_G(x) \cap V(T - C_x)\big|$ using $\mathcal{Q}^D$'s.

For every $v \in N_G(x) \cap V(T - C_x)$, we take a path Q_v^D in $\mathcal{Q}^D$ starting from v, where D is chosen so that $v \in V(K^D)$. If $v \in U$, then there exist many such D's, in this case, we choose an arbitrary component D so that v is a root of K^D. If Q_v^D reaches a vertex in S', then we let $P_v' := Q_v^D$. Otherwise, Q_v^D reaches a vertex in $N_T(D)$. Let u be the end-vertex of Q_v^D other than v. Note that $u \notin N_G(x) \cap V(T - C_x)$ by the definition of $\mathcal{Q}^D$. Since u is contained in some $K^{D'}$ as a root, there exists a path $Q_u^{D'} \in \mathcal{Q}^{D'}$ starting from u. Connect $Q_u^{D'}$ to Q_v^D by u. Again $Q_u^{D'}$ reaches a vertex in S' or $N_T(D') - \{u\}$. If $Q_u^{D'}$ reaches a vertex in $N_T(D') - \{u\}$, then we take a path from a collection of disjoint paths in the next subtree and connect it to the previous path. Till the path reaches a vertex in S', we perform this operation and let P_v' be an obtained path as a consequence. (Since U is a finite set, P_v' must reach a vertex in S' and this operation is stopped.) Note that P_v' starts from v and reaches a vertex in S' along $\overrightarrow{T}$, and passing no vertex in $N_G(x)$ except for v. Then $F' := \bigcup_{v \in N_G(x) \cap V(T - C_x)} P_v'$ is an $\big(x, S' - \{x\}\big)$-fan with width $\big|N_G(x) \cap V(T - C_x)\big|$. (See Figure 8.11).

By Fact 8.4, there exists a k-tree T' containing $S - \{x\}$. Since all end-vertices of F' are contained in T', there exists an (x, T')-fan with width at least $\big|N_G(x) \cap V(T - C_x)\big|$. By the choice (T1), $\big|N_G(x) \cap V(T - C_x)\big| \leq \kappa(x_0, T) = l$. Therefore $\big|N_G(x) \cap V(T)\big| = \big|N_G(x) \cap V(C_x)\big| + \big|N_G(x) \cap V(T - C_x)\big| \leq |C_x| + l - 1$. $\square$

We will finish the proof of Theorem 8.13 by the degree calculation. For any $t \leq (k-1)l + 2 - \lfloor \frac{l-1}{k} \rfloor$, let Y be a subset of X with $|Y| = t - 1$. By Claim 8.6, we can take such a subset Y. By Claim 8.10,

$$\sum_{x \in Y \cup \{x_0\}} \big|N_G(x) \cap V(G - T)\big| \leq |V(G - T)| - 1 = n - |V(T)| - 1.$$

By Fact 8.2 (iii),

$$\begin{aligned}
\sum_{x \in Y} |C_x| &= |V(T)| - |U| - \sum_{C \in \mathcal{C} - \{C_x : x \in Y\}} |C| \\
&\leq |V(T)| - l - \big(|\mathcal{C}| - |Y|\big) \\
&= |V(T)| - l - \big((k-1)l + 1\big) + |Y| \\
&= |V(T)| + |Y| - kl - 1.
\end{aligned}$$

By the choice (T1), $|N_G(x_0) \cap V(T)| \leq |U| = \kappa(x_0, T) = l$. Therefore, by Claims 8.11 and 8.13, we obtain

$$\begin{aligned}
&\sum_{x \in Y \cup \{x_0\}} d_G(x) \\
&= \sum_{x \in Y} \big|N_G(x) \cap V(T)\big| + \big|N_G(x_0) \cap V(T)\big| + \sum_{x \in Y \cup \{x_0\}} \big|N_G(x) \cap V(G - T)\big| \\
&\leq \sum_{x \in Y} \big(|C_x| + l - 1\big) + l + \big(n - |V(T)| - 1\big) \\
&\leq |V(T)| + |Y| - kl - 1 + l|Y| - |Y| + l + n - |V(T)| - 1 \\
&= n + l|Y| - (k-1)l - 2 \\
&= n + tl - kl - 2.
\end{aligned}$$

By Claim 8.9, $Y \cup \{x_0\}$ is an independent set of G, which contradicts the degree sum assumption. This completes the proof of Theorem 8.13. $\square$

Chapter 9

Spanning f-trees

A k-tree is a tree such that the degree of each vertex is bounded by the constant k. Since a spanning 2-tree is equivalent to a hamilton path, many researchers have considered it. In this sense, we can consider more extended concept *f-tree*. For a graph G, let f be a mapping from $V(G)$ to positive integers. A tree T of a graph G is called an *f-tree* if $d_T(v) \leq f(v)$ for every $v \in V(T)$. In this chapter, we consider an independence number condition for the existence of a spanning f-tree.

The contents of this chapter are based on the paper [49] "The independence number condition for the existence of a spanning f-tree," jointwork with H. Enomoto.

9.1 Conjecture

For a graph G, let f be a mapping from $V(G)$ to positive integers. An *f-tree* T is defined as a subgraph of G which forms a tree such that $d_T(v) \leq f(v)$ for every $v \in V(T)$. When $V(T) = V(G)$, T is called a *spanning f-tree*. In this chapter, we concentrate on the existence of a spanning f-tree. When f is a constant mapping taking value k, then an f-tree T is called a *k-tree*. Neumann-Lara and Rivera-Campo showed the result on the existence of a spanning k-tree.

Theorem 9.1 (Neumann-Lara and Rivera-Campo [128]) *Let k and m be integers with $k \geq 3$, and let G be an m-connected graph. If $\alpha(G) \leq m(k-1)+1$, then there exists a spanning k-tree T.*

Matsuda and Matsumura gave the result on the existence of a spanning k-tree with specified leaves, which is an extension of Theorem 9.1.

Theorem 9.2 (Matsuda and Matsumura [121]) *Let m, k and s be integers with $k \geq 2$, $0 \leq s \leq k$ and $s+1 \leq m$ and let G be an m-connected graph. If $\alpha(G) \leq (m-s)(k-1)+1$, then for any s vertices of G, G has a spanning k-tree T such that the s specified vertices are contained in the set of leaves.*

Extending this result to an f-tree, we propose the following conjecture.

Conjecture 9.3 ([49]) *Let m be an integer, G be an m-connected graph and f be a mapping from $V(G)$ to positive integers. If $\sum_{x\in V(G)} f(x) \geq 2(|V(G)|-1)$ and $\alpha(G) \leq \min\{\sum_{x\in R}(f(x)-1) : R \subseteq V(G), |R| = m\} + 1$, then there exists a spanning f-tree.*

Suppose that there exists a spanning f-tree T. Then

$$\begin{aligned}\sum_{x\in V(G)} f(x) &\geq \sum_{x\in V(G)} d_T(x)\\ &= 2|E(T)|\\ &= 2(|V(G)|-1).\end{aligned}$$

Therefore for the existence of a spanning f-tree, the condition "$\sum_{x\in V(G)} f(x) \geq 2(|V(G)|-1)$" is a trivial necessary condition.

On the other hand, the following graph G_1 shows that the independence number condition in Conjecture 9.3 is sharp. Let S be a set of vertices with $|S| = m$ and f be a mapping from S to positive integers. Let $G_1 := S + tK_l$, where $t := \sum_{x\in S}(f(x)-1)+2$, and "+" means join of two graphs. Extend f to a mapping from $V(G_1)$ such that $f(x) \leq f(y)$ for any $x \in S$ and $y \in V(G) - S$. Then G_1 is m-connected, $\alpha(G_1) = t = \sum_{x\in S}(f(x)-1)+2 = \min\{\sum_{x\in R}(f(x)-1) : R \subseteq V(G_1), |R| = m\}+2$ and G_1 has no spanning f-tree.

9.2 Partial solution

9.2.1 Results

In this section, we show the following result, which gives a partial solution to Conjecture 9.3. For a mapping f, let $S_i(f) := \{x \in V(G) : f(x) = i\}$ and $s_i(f) := |S_i(f)|$.

Theorem 9.4 ([49]) *Let m be a positive integer, G be an m-connected graph and f be a mapping from $V(G)$ to positive integers. Suppose $s_1(f) + s_2(f) \leq m+1$, $\sum_{x\in V(G)} f(x) \geq 2(|V(G)|-1)$ and $\alpha(G) \leq \min\{\sum_{x\in R}(f(x)-1) : R \subseteq V(G), |R| = m\} + 1$. Then there exists a spanning f-tree in G.*

Let f_1 be a mapping on $V(G)$ which assigns 1 to s given vertices and k to other vertices. Then a spanning f_1-tree is a k-tree desired in Theorem 9.2. Moreover,

$$\begin{aligned}&\min\Big\{\sum_{x\in R}(f_1(x)-1) : R \subseteq V(G), |R| = m\Big\}+1\\ &\quad= s(1-1)+(m-s)(k-1)+1\\ &\quad= (m-s)(k-1)+1,\end{aligned}$$

and hence Theorem 9.2 is a special case of Conjecture 9.3. If $k \geq 3$, then $s_1(f_1) + s_2(f_1) = s \leq m+1$. This implies that Theorem 9.4 is a generalization of Theorem 9.2 for $k \geq 3$. Note that essential part of the proof of Theorem 9.2 is only the case $k \geq 3$, because the case $k = 2$ is contained in the following theorem by Chvátal and Erdős.

Theorem 9.5 (Chvátal and Erdős [37]) *Let m be an integer and M be an m-connected graph. If $\alpha(M) \leq m+1$, then there exists a hamilton path.*

Let g be a mapping from $V(G)$ to positive integers. If $d_T(x) = g(x)$, a vertex x in G is called *g-saturated* in a subgraph T, and let $A_T(g)$ be the set of g-saturated vertices in T. Let $\mathcal{G}$ be the set of graphs obtained from two disjoint complete graphs (of order $l_1, l_2 \geq 2$, respectively) with one connecting edge. (See Figure 9.1.)

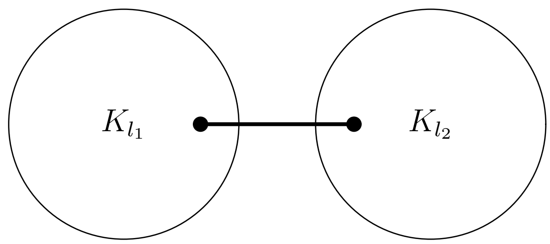


Figure 9.1: A graph contained in $\mathcal{G}$.

We call a pair (M, S) an *exception pair* if $M \in \mathcal{G}$ and S is the set of vertices which are end-vertices of the connecting edge in M.

Theorem 9.6 ([49]) *Let M be an m-connected graph and g be a mapping from $V(M)$ to positive integers. Suppose $s_1(g) = 0$, $s_2(g) \leq m+1$ and $\alpha(M) \leq \min\{\sum_{x \in R'}(g(x) - 1) : R' \subseteq V(M), |R'| = m\} + 1$. Then there exists a spanning g-tree T such that $|A_T(g)| \leq m$, unless $m = 1$, $s_2(g) = 2$ and $(M, S_2(g))$ is an exception pair.*

Remark that the condition "$s_2(g) \leq m+1$" in Theorem 9.6 is best possible. Let G_2 be the graph constructed by

$$\begin{aligned} V(G_2) &:= \{v_1, v_2, \cdots, v_m\} \cup \{u_1, u_2, \cdots, u_m\} \cup \{w_1, w_2\} \\ \text{and } E(G_2) &:= \{v_i u_j : 1 \leq i \leq m, 1 \leq j \leq m\} \\ &\quad \cup \{v_i w_j : 1 \leq i \leq m, j = 1, 2\} \cup \{w_1 w_2\}. \end{aligned}$$

We define the mapping g as follows;

$$g(x) := \begin{cases} 2 & \text{if } x = v_i \ (1 \leq i \leq m) \text{ or } x = w_j \ (j = 1, 2), \\ 3 & \text{otherwise.} \end{cases}$$

In this graph G_2, we have $\alpha(G_2) = m+1$, $s_2(g) = m+2$ and G_2 has no spanning g-tree T such that $|A_T(g)| \le m$.

We will prove Theorem 9.6 by considering two cases. First, we consider the case $s_2(g) \le m$. In this case, we will show a more general theorem.

Theorem 9.7 ([49]) *Let m and c be integers, M be an m-connected graph and g be a mapping from $V(M)$ to positive integers. Suppose $s_1(g) = 0$ and $s_2(g) \le c \le m$. If $\alpha(M) \le \min\{\sum_{x \in R'}(g(x)-2) : R' \subseteq V(M), |R'| = m\} + c + 1$, then there exists a spanning g-tree T such that $|A_T(g)| \le c$.*

This is a generalization of the following theorem on k-tree, which was showed by Neumann-Lara and Rivera-Campo for $k \ge 4$ and by Tsugaki for $k = 3$.

Theorem 9.8 (Neumann-Lara and Rivera-Campo [128], Tsugaki [157]) *Let k, m and c be integers with $k \ge 3$ and $c \le m$, and let M be an m-connected graph. If $\alpha(G) \le m(k-2) + c + 1$, then there exists a spanning k-tree T such that the number of vertices which have degree k in T is at most c.*

Secondly, we consider the case $s_2(g) = m+1$. To prove this case, we use the following theorem.

Theorem 9.9 ([49]) *Let m be an integer and M be an m-connected graph. If $\alpha(M) \le m+1$, then for every $S \subseteq V(M)$ with $|S| = m+1$, there exists a Hamilton path P such that at least one of the end-vertices of P is contained in S, unless $m = 1$ and (M, S) is an exception pair.*

Remark that Theorem 9.9 is an extension of Theorem 9.5.

9.2.2 Proof of Theorem 9.7

Before proving Theorem 9.7, we need some definitions and results. Let G be a graph. In [169], Win defined a *path and cycle system* as a spanning subgraph in which each component forms a path or a cycle. More generally, we define a *tree and cycle system* as a spanning subgraph in which each component is a tree or a cycle. For a mapping g from $V(G)$ to positive integers, a tree and cycle system F is called a *g-system* if $d_F(x) \le g(x)$ for any $x \in V(F)$.

For a tree or a cycle C, define

$$h(C) := \begin{cases} 1 & \text{if } C \text{ is a cycle, or a path which has at most 2 vertices,} \\ 2 & \text{otherwise,} \end{cases}$$

and for a tree and cycle system F, define

$$h(F) := \sum_{C \text{ is a component of } F} h(C).$$

Win showed the following theorem.

Theorem 9.10 (Win [169]) *Let m be an integer and M be an m-connected graph. If $\alpha(M) \leq m + l + 1$, there exists a path and cycle system W with $h(W) \leq l + 2$.*

Proof of Theorem 9.7.

Let $S_2 := S_2(g)$ and $s_2 := s_2(g)$. By Theorem 9.10, there exists a path and cycle system W with

$$h(W) \leq \min\Big\{ \sum_{x \in R'} (g(x) - 2) : R' \subseteq V(M), |R'| = m \Big\} - m + c + 2. \tag{9.1}$$

It is sufficient to prove the following claim. For a graph G, let $\omega(G)$ be the number of components of G and let $B_F(g) := \{x \in A_F(g) : g(x) \geq 3\}$.

Claim 9.1 *For every i with $1 \leq i \leq \omega(W)$, there exists a g-system F such that*

(i) $\omega(F) = \omega(W) - i + 1$,

(ii) $\sum_{x \in V(M)} \max\{d_F(x) - 2, 0\} \leq h(W) - h(F)$,

(iii) $|B_F(g)| \leq c - s_2$.

Proof. We prove Claim 9.1 by induction on i. Suppose that $i = 1$. Since each component of W is a path or a cycle, we have $d_W(x) \leq 2 \leq g(x)$, and hence W is a g-system. Moreover, (i) $\omega(W) = \omega(W) - i + 1$, (ii) $\sum_{x \in V(M)} \max\{d_W(x) - 2, 0\} = 0 = h(W) - h(W)$, and (iii) $|B_W(g)| = 0 \leq c - s_2$. Thus, W is a desired g-system.

Suppose that $i \geq 2$. By the induction hypothesis, there exists a g-system F' such that

(i) $\omega(F') = \omega(W) - (i - 1) + 1 = \omega(W) - i + 2$,

(ii) $\sum_{x \in V(M)} \max\{d_{F'}(x) - 2, 0\} \leq h(W) - h(F')$,

(iii) $|B_{F'}(g)| \leq c - s_2$.

Subclaim We may assume that $|B_{F'}(g)| < m - s_2$.

Proof. Assume that $|B_{F'}(g)| \geq m - s_2$. By the condition (iii), we have $m = c$ and $|B_{F'}(g)| = m - s_2$. Let $L := B_{F'}(g) \cup S_2$. Note that $|L| = m$. Then by the definition of $B_{F'}(g)$, we have

$$\begin{aligned}
&\sum_{x \in V(M)} \max\{d_{F'}(x) - 2, 0\} \\
&\quad \geq \sum_{x \in L} \max\{d_{F'}(x) - 2, 0\} \\
&\quad = \sum_{x \in L} (g(x) - 2). \qquad (9.2)
\end{aligned}$$

Thus, by the condition (ii) and the inequalities (9.1) and (9.2),

$$\begin{aligned}
&\sum_{x \in L} (g(x) - 2) \\
&\quad \leq \sum_{x \in V(M)} \max\{d_{F'}(x) - 2, 0\} \\
&\quad \leq h(W) - h(F') \\
&\quad \leq \min\left\{ \sum_{x \in R'} (g(x) - 2) : R' \subseteq V(M), |R'| = m \right\} - m + c + 2 - h(F') \\
&\quad \leq \sum_{x \in L} (g(x) - 2) + 2 - h(F'),
\end{aligned}$$

or

$$h(F') \leq 2.$$

On the other hand, by the condition (i), we have $\omega(F') = \omega(W) - i + 2 \geq 2$ and hence $h(F') \geq \sum_{C \text{ is a component of } F} h(C) \geq \omega(F') \geq 2$. Then

$$h(F') = \omega(F') = 2.$$

This implies that each component of F' is a cycle or a path of order at most 2, and hence $|B_{F'}(g)| = 0$. Since $|B_{F'}(g)| = m - s_2$, we have $m = s_2$. By the assumption of Theorem 9.7, we have $\alpha(M) \leq m + 1$. Then by Theorem 9.5, there exists a Hamilton path P in M and P is a desired g-system. □

Choose $L \subseteq V(M)$ so that $S_2 \subseteq L$, $|L| = m - 1$ and $\sum_{v \in L}(g(v) - d_{F'}(v))$ is as small as possible. Note that $A_{F'}(g) \subseteq L$ by Subclaim and by the definition of L.

We consider two cases. For $y \in V(M)$, let C_y be the component of F' such that $y \in V(C_y)$. When C_y is a cycle, let e_y be an edge of $E(C_y)$ such that e_y is incident to y.

Case 1. $M - L$ is formed by vertices of at least two components of F'.

In this case, since M is m-connected and $|L| = m-1$, there exists $yz \in E(M)$ such that $y, z \notin L$ and $C_y \neq C_z$. By the symmetry, we may assume that $g(z) - d_{F'}(z) \leq g(y) - d_{F'}(y)$.

Case 2. $M - L$ is formed by vertices of one component of F'.

Let C be the unique component of F' such that $(V(M) - L) \cap V(C) \neq \emptyset$. We take $y \in V(M - C)$ so that y is a leaf of C_y, if possible. By the assumption of Case 2, note that $y \in L$. Since F' is a tree and cycle system, if y is not a leaf of C_y, then C_y must be a cycle. Because M is m-connected, there exists $z \in V(C) - L$ such that $yz \in E(M)$.

In both cases, we define

$$F := \begin{cases} (F' - \{e_y\}) \cup \{yz\} & \text{if } C_y \text{ is a cycle and } C_z \text{ is not a cycle,} \\ (F' - \{e_z\}) \cup \{yz\} & \text{if } C_y \text{ is not a cycle and } C_z \text{ is a cycle,} \\ (F' - \{e_y, e_z\}) \cup \{yz\} & \text{if both } C_y \text{ and } C_z \text{ are cycles,} \\ F' \cup \{yz\} & \text{otherwise,} \end{cases}$$

and let C_{yz} be the component of F that contains y and z. Then $d_F(x) = d_{F'}(x) \leq g(x)$ for every $x \in V(M) - \{y, z\}$. Since $z \notin A_{F'}(g) \subseteq L$, we have $d_F(z) \leq d_{F'}(z) + 1 \leq g(z)$. By the definition of y, $d_F(y) \leq d_{F'}(y) + 1 \leq g(y)$ in Case 1 and $d_F(y) \leq 2 \leq g(y)$ in Case 2. Therefore F is a g-system. Now, we will check that F is a desired g-system.

(i) $\omega(F) = \omega(F') - 1 = \omega(W) - i + 1$, and hence F satisfies the condition (i).

(ii) For every $x \in V(M) - \{y, z\}$, we have $d_F(x) = d_{F'}(x)$, in particular, $\max\{d_F(x) - 2, 0\} = \max\{d_{F'}(x) - 2, 0\}$.

By the definition of F, if $h(C_y) = 1$, then $\max\{d_F(y) - 2, 0\} = \max\{d_{F'}(y) - 2, 0\} = 0$, and if $h(C_y) = 2$, then $d_F(y) \leq d_{F'}(y) + 1$. Thus, in both cases, $\max\{d_F(y) - 2, 0\} \leq \max\{d_{F'}(y) - 2, 0\} + h(C_y) - 1$. By the same way, $\max\{d_F(z) - 2, 0\} \leq \max\{d_{F'}(z) - 2, 0\} + h(C_z) - 1$.

Since $h(F) = h(F') - h(C_y) - h(C_z) + h(C_{yz}) \leq h(F') - h(C_y) - h(C_z) + 2$, we have

$$h(F') \geq h(F) + h(C_y) + h(C_z) - 2.$$

Therefore,

$$\begin{aligned} &\sum_{x \in V(M)} \max\{d_F(x) - 2, 0\} \\ &\leq \sum_{x \in V(M)} \max\{d_{F'}(x) - 2, 0\} + h(C_y) + h(C_z) - 2 \\ &\leq h(W) - h(F') + h(C_y) + h(C_z) - 2 \\ &\leq h(W) - h(F). \end{aligned}$$

Thus, F satisfies the condition (ii).

(iii) Assume that $|B_F(g)| > |B_{F'}(g)|$. Then by the definition of y and z, $z \in B_F(g) - B_{F'}(g)$ and hence $g(z) - d_{F'}(z) = 1$. By the definition of L, we obtain $d_F(x) \geq d_{F'}(x) \geq g(x) - 1$ for any $x \in L - S_2$. Let $L' := L \cup \{z\}$. Then we have $\max\{d_F(x) - 2, 0\} \geq g(x) - 3$ for every $x \in L' - B_F(g)$ and $\max\{d_F(x) - 2, 0\} = g(x) - 2$ for every $x \in B_F(g)$ by the definition of L and $B_F(g)$. Let $\varepsilon := 1$ if $y \in B_F(g) - B_{F'}(g)$; otherwise $\varepsilon := 0$. Since $B_F(g) \cup S_2 \subseteq L' \cup \{y\}$,

$$
\begin{aligned}
&\sum_{x \in V(M)} \max\{d_F(x) - 2, 0\} \\
&\quad \geq \sum_{x \in L'} \max\{d_F(x) - 2, 0\} + \varepsilon \\
&\quad \geq \sum_{x \in L' - (B_F(g) \cup S_2)} (g(x) - 3) + \sum_{x \in L' \cap (B_F(g) \cup S_2)} (g(x) - 2) + \varepsilon \\
&\quad = \sum_{x \in L'} (g(x) - 3) + |B_F(g)| + |S_2| \\
&\quad = \sum_{x \in L'} (g(x) - 2) - m + |B_F(g)| + s_2. \qquad (9.3)
\end{aligned}
$$

On the other hand, by the inequality (9.1),

$$
\begin{aligned}
&h(W) - h(F) \\
&\quad \leq \min\Big\{ \sum_{x \in R'} (g(x) - 2) : R' \subseteq V(M), |R'| = m \Big\} - m + c + 2 - h(F) \\
&\quad \leq \sum_{x \in L'} (g(x) - 2) - m + c + 2 - h(F) \qquad (9.4)
\end{aligned}
$$

Thus, by the condition (ii) and the inequalities (9.3) and (9.4),

$$
\begin{aligned}
&\sum_{x \in L'} (g(x) - 2) - m + |B_F(g)| + s_2 \\
&\quad \leq \sum_{x \in V(M)} \max\{d_F(x) - 2, 0\} \\
&\quad \leq h(W) - h(F) \\
&\quad \leq \sum_{x \in L'} (g(x) - 2) - m + c + 2 - h(F),
\end{aligned}
$$

or

$$|B_F(g)| \leq c + 2 - h(F) - s_2.$$

Because $h(F) \geq 2$, this implies that

$$|B_F(g)| \leq c - s_2.$$

Therefore F satisfies the condition (iii) and this completes the proof of Claim 9.1 and Theorem 9.7. $\square$

9.2.3 Proof of Theorem 9.9

Let M be a graph and S be a subset of $V(M)$ satisfying the assumption of Theorem 9.9 and (M, S) is not an exception pair. Let $\mathcal{P}$ be the set of paths such that at least one of the end-vertices is contained in S, and let P be a longest path in $\mathcal{P}$. If P contains all vertices of M, then there is nothing to prove. Therefore we may assume that there exists $x_0 \in V(M - P)$. Let v be the start vertex and u be the terminal vertex of P, respectively. Since at least one end-vertex of P is contained in S, we may assume that $v \in S$.

Since M is m-connected, there exist $l \geq m$ internally disjoint paths $\{Q_1, Q_2, \cdots, Q_l\}$ where Q_i connects x_0 and x_i with $\{x_i\} = V(Q_i \cap P)$. We may assume that $x_1, x_2, \cdots, x_l$ are along on $\overrightarrow{P}$. Let $X = \{x_1, x_2, \cdots, x_l\}$.

The following claim is obvious.

Claim 9.2 (i) $x_l \neq u$.

(ii) $|X^+| = l$.

(iii) *If* $x_1 = v$*, then* $|X^-| = l - 1$*; otherwise* $|X^-| = l$.

(iv) $X^+ \cup \{x_0\}$ *is an independent set.*

(v) $X^- \cup \{x_0, u\}$ *is an independent set.*

By Claim 9.2, the following claim is shown.

Claim 9.3 $l = m$ *and* $x_1 = v$.

Proof. By Claim 9.2 (iii) and (v), $X^- \cup \{x_0, u\}$ is an independent set of order $l+1$ or $l+2$ depending on $x_1 = v$ or $x_1 \neq v$. Since $l \geq m$ and $\alpha(M) \leq m+1$, we have $l = m$ and $x_1 = v$. $\square$

Claim 9.4 $u \notin S$ *and* $x_0 \notin S$.

Proof. Assume that $u \in S$ (or $x_0 \in S$). Then by Claim 9.3, the path $u\overleftarrow{P}x_1Q_1x_0$ (or $x_0Q_1x_1\overrightarrow{P}u$, respectively) is contained in $\mathcal{P}$, and longer than P, a contradiction. $\square$

Let K be the graph induced by $V(x_m^+\overrightarrow{P}u)$.

Claim 9.5 K *is a complete graph.*

Proof. Suppose that there exist $a, b \in V(K)$ such that $ab \notin E(M)$. Choose such vertices a and b so that $a\overrightarrow{P}b$ is as long as possible.

Since $(X - \{x_m\})^+ \cup \{x_0, a, b\}$ is of order $m+2$, this set is not independent. Since $ab \notin E(G)$, there exists $x_i \in X - \{x_m\}$ such that $x_i^+ a \in E(G)$ or $x_i^+ b \in E(G)$. By the definition of a and b, we have $a^- u \in E(G)$ or $a = x_m^+$, and $x_m^+ b^+ \in E(G)$ or $b = u$. Let

$$P' := \begin{cases} x_1 \overrightarrow{P} x_i Q_i x_0 Q_m x_m \overleftarrow{P} x_i^+ a \overrightarrow{P} u (a^- \overleftarrow{P} x_m^+) & \text{if } x_i^+ a \in E(G), \\ x_1 \overrightarrow{P} x_i Q_i x_0 Q_m x_m \overleftarrow{P} x_i^+ b \overleftarrow{P} x_m^+ (b^+ \overrightarrow{P} u) & \text{if } x_i^+ b \in E(G). \end{cases}$$

Then P' is contained in $\mathcal{P}$ and longer than P, a contradiction. $\square$

Let $K' := K - \{x_m^+\}$.

Claim 9.6 *$N_M(K') \cap V(M - K') \subseteq X \cup \{x_m^+\}$. Furthermore, if $|V(K)| = 1$ or $x_m \in N_M(K')$, then $N_M(K) \cap V(M - K) \subseteq X$.*

Proof. Clearly, $N_M(K') \cap V(M - K' - P) = \emptyset$.

Suppose that there exists $y \in V(P - K) - X$ such that $yw \in E(G)$ for some $w \in V(K')$. Note that $w^- \in V(K)$. Then $X^+ \cup \{x_0, y^+\}$ is not an independent set, and hence by Claim 9.2 (iv), $y^+ z \in E(G)$ for some $z \in X^+ \cup \{x_0\}$. Therefore let

$$P' := \begin{cases} x_1 \overrightarrow{P} y w \overrightarrow{P} u w^- \overleftarrow{P} y^+ x_0 & \text{if } z = x_0, \\ x_1 \overrightarrow{P} x_i Q_i x_0 Q_m x_m \overleftarrow{P} y^+ x_i^+ \overrightarrow{P} y w \overrightarrow{P} u w^- \overleftarrow{P} x_m^+ & \text{if } z = x_i^+ \in V(x_1 \overrightarrow{P} y), \\ x_1 \overrightarrow{P} y w \overrightarrow{P} u w^- \overleftarrow{P} x_i^+ y^+ \overrightarrow{P} x_i Q_i x_0 & \text{if } z = x_i^+ \in V(y^+ \overrightarrow{P} x_m). \end{cases}$$

Then $P' \in \mathcal{P}$ and P' is longer than P, a contradiction.

In the case $|V(K)| = 1$ or $x_m \in N_M(K')$, we can show $N_M(K) \cap V(M - K) \subseteq X$ by the similar way. $\square$

Claim 9.7 *We may assume that $x_1 u \in E(M)$.*

Proof. First, suppose that $|V(K)| = 1$ or $x_m \in N_M(K')$. Since M is m-connected, K has at least m neighbors in $V(M - K)$. By Claim 9.6, $N_M(K) \cap V(M - K) \subseteq X$ and $|X| = m$, and hence we have $x_1 \in N_M(w)$ for some $w \in V(K)$. In this case, since K is complete, we can change the path P so that x_1 and w are end-vertices.

Thus, we may assume that $|V(K)| \geq 2$ and $x_m \notin N_M(K')$. In this case, since M is m-connected, K' has at least m neighbors in $V(M - K) \cup \{x_m^+\}$. By Claim 9.6, $N_M(K') \cap V(M - K) \subseteq (X - \{x_m\}) \cup \{x_m^+\}$. If $m \geq 2$, then we have $x_1 \in N_M(w)$ for some $w \in V(K)$, and again, we can change the path P so that x_1 and w are end-vertices. Therefore, we may assume that $m = 1$. Then we have $\alpha(M) \leq 2$, and hence $M - P$ is complete. Since x_0 is an arbitrary vertex in $M - P$, we have $N_M(M - P) \cap V(P) \subseteq \{x_1\}$. If there exists $x' \in V(M - P)$ such that $x_1 \notin N_M(x')$, then $\{x', x_1, w\}$ is an independent set of order 3 for any $w \in V(K) - \{x_1^+\}$, a contradiction. Thus, we have $(M - P) \cup \{x_1\}$ is complete and this implies that $M \in \mathcal{G}$. Moreover, if $\{x_1^+\} \neq S - \{x_1\}$, then we can easily find a path which is contained in $\mathcal{P}$ and longer than P. Therefore $S = \{x_1, x_1^+\}$. This implies that (M, S) is an exception pair. $\square$

Since $|S| = m + 1$ and $|X| = m$, there exists $y \in S - X$. Then $X^+ \cup \{x_0, y^+\}$ is a set of order $m + 2$. By Claim 9.2 (iv), there exists $z \in X^+ \cup \{x_0\}$ such that $y^+ z \in E(G)$. Let

$$P' := \begin{cases} y\overleftarrow{P}x_1 u \overleftarrow{P} y^+ x_0 & \text{if } z = x_0, \\ y\overleftarrow{P}x_i^+ y^+ \overrightarrow{P} u x_1 \overrightarrow{P} x_i Q_i x_0 & \text{if } z = x_i^+ \in X \cap V(x_1 \overrightarrow{P} y), \\ y\overleftarrow{P}x_1 u \overleftarrow{P} x_i^+ y^+ \overrightarrow{P} x_i Q_i x_0 & \text{if } z = x_i^+ \in X \cap V(y^+ \overrightarrow{P} x_m). \end{cases}$$

Then P' is contained in $\mathcal{P}$ and longer than P, a contradiction. This completes the proof of Theorem 9.9. $\square$

9.2.4 Proof of Theorem 9.4

Let G be a graph and f be a mapping from $V(G)$ to positive integers satisfying the assumption of Theorem 9.4. Let $S_1 := S_1(f)$ and $s_1 := s_1(f)$. If $s_1 = m$ or $s_1 = m + 1$, then we have $\alpha(G) = 1$, and obviously the statement holds. Therefore we may assume that $m - s_1 \geq 1$. Let $m' := m - s_1$, $M := G - S_1$ and $g := f|_{V(M)}$. Then M is m'-connected, $s_1(g) = 0$ and $s_2(g) \leq m' + 1$. Moreover,

$$\begin{aligned} \alpha(M) &\leq \alpha(G) \\ &\leq \min\Big\{\sum_{x \in R}(f(x) - 1) : R \subseteq V(G), |R| = m\Big\} + 1 \\ &= \min\Big\{\sum_{x \in R'}(g(x) - 1) : R' \subseteq V(M), |R'| = m'\Big\} + 1. \end{aligned}$$

Thus, M satisfies the assumption of Theorem 9.6.

Case 1. There exists a spanning g-tree T such that $|A_T(g)| \leq m'$.

Let $Y := V(M) - A_T(g)$ and

$$Y' := \{y_i : y \in Y, 1 \leq i \leq g(y) - d_T(y)\}.$$

We construct a bipartite graph G' as follows;

$$\begin{aligned} V(G') &:= S_1 \cup Y' \\ \text{and } E(G') &:= \{x y_i : xy \in E(G)\}. \end{aligned}$$

Then by the assumption "$\sum_{x \in V(G)} f(x) \geq 2(|V(G)|-1)$," we obtain

$$\begin{aligned}
|Y'| &= \sum_{y \in Y} (g(y) - d_T(y)) \\
&= \sum_{y \in V(M)} g(y) - \sum_{y \in V(M)} d_T(y) \\
&= \sum_{y \in V(M)} g(y) - 2(|V(M)|-1) \\
&= \sum_{x \in V(G)} f(x) - s_1 - 2(|V(G)| - s_1 - 1) \\
&= \sum_{x \in V(G)} f(x) - 2(|V(G)|-1) + s_1 \\
&\geq s_1.
\end{aligned}$$

Since $G - A_T(g)$ is s_1-connected, G' is s_1-connected. Therefore there exists a matching E' between S_1 and Y' which covers S_1. Let $E := \{xy : xy_i \in E' \text{ for some } i \text{ with } 1 \leq i \leq f(y) - d_T(y)\}$. Then $T + E$ is a desired spanning f-tree. $\square$

Case 2. M has no spanning g-tree such that the number of g-saturated vertices is at most m'.

In this case, $m' = 1$, $|S_2(g)| = s_2(g) = 2$ and $(M, S_2(g))$ is an exception pair. Then we have $s_1 = m - 1$, and $\alpha(G) \leq 2$. If $\alpha(G) = 1$, then obviously the statement is true. Thus, we may assume that $\alpha(G) = 2$. Let $U := S_2(g) = \{u, u'\}$.

Let H_1 and H_2 be components of $M - U$ and $Y_i := V(H_i)$. Clearly, there exists a Hamilton path T in M. Note that $A_T(g) = U$. Let

$$Y_l' := \{y_i : y \in Y_l, 1 \leq i \leq g(y) - d_T(y)\}$$

for $l = 1, 2$. and let $Y' := Y_1' \cup Y_2'$. Note that $Y_1' \neq \emptyset$ and $Y_2' \neq \emptyset$. Again, we construct a bipartite graph G' as follows;

$$\begin{aligned}
V(G') &:= S_1 \cup Y' \\
\text{and } E(G') &:= \{xy_i : xy \in E(G)\}.
\end{aligned}$$

By the same argument as Case 1, we have $|Y'| \geq s_1 = |S_1|$. Again, we will show that there exists a matching E' between S_1 and Y' which covers S_1. Assume that there exists no such matching. Then by Hall's Theorem, there exists $\tilde{S} \subseteq S_1$ such that $|N_{G'}(\tilde{S})| < |\tilde{S}|$.

Claim 9.8 $Y_1' \subseteq N_{G'}(\tilde{S})$ *or* $Y_2' \subseteq N_{G'}(\tilde{S})$.

Proof. Suppose that there exist $y_i \in Y_1'$ and $z_j \in Y_2'$ such that $y_i, z_j \notin N_{G'}(\tilde{S})$. Then by the definition of G', for any $x \in \tilde{S}$, $\{x, y, z\}$ is an independent set of order

3, which contradicts $\alpha(G) = 2$. $\square$

By the symmetry, suppose that $Y'_1 \subseteq N_{G'}(\tilde{S})$ holds. Let $Z'_2 := N_{G'}(\tilde{S}) \cap Y'_2$ and $Z_2 := \{y : y_i \in Z'_2\}$. Since $|Y'_1| + |Z'_2| = |N_{G'}(\tilde{S})| < |\tilde{S}|$, we have

$$|Z'_2| \le |\tilde{S}| - |Y'_1| - 1 \le |\tilde{S}| - 2,$$

and hence

$$\begin{aligned} |Z'_2| + |S_1 - \tilde{S}| + |U| &\le |\tilde{S}| - 2 + |S_1 - \tilde{S}| + 2 \\ &= |S_1| \\ &= n - 1. \end{aligned} \tag{9.5}$$

On the other hand, since $|Y'| \ge |S_1|$ and $|Y'_1| + |Z'_2| < |\tilde{S}|$, we obtain

$$|Y'_2 - Z'_2| = (|Y'| - |Y'_1|) - |Z'_2| > |S_1| - |\tilde{S}| \ge 0.$$

Thus, $Y'_2 - Z'_2 \ne \emptyset$ and hence $Y_2 - Z_2 \ne \emptyset$. This implies that in the graph G, $Z_2 \cup (S_1 - \tilde{S}) \cup U$ separates Y_1 from $Y_2 - Z_2$, and hence

$$\begin{aligned} |Z'_2| + |S_1 - \tilde{S}| + |U| &\ge |Z_2| + |S_1 - \tilde{S}| + |U| \\ &\ge n, \end{aligned}$$

because G is n-connected. This contradicts the inequality (9.5).

Therefore there exists a matching E' between S_1 and Y' which covers S_1. Let $E := \{xy : xy_i \in E' \text{ for some } i \text{ with } 1 \le i \le g(y) - d_T(y)\}$. Again, $T + E$ is a desired spanning f-tree. $\square$

9.3 Conclusion

In this chapter, we pose Conjecture 9.3 and give a partial solution to it. The remaining part of this conjecture can be restated as follows:

Conjecture 9.11 *Let n be an integer, G be an m-connected graph and f be a mapping from $V(G)$ to positive integers. If $\sum_{x \in V(G)} f(x) \ge 2(|V(G)| - 1)$ and $\alpha(G) \le m - s_1(f) + 1$, then there exists a spanning f-tree.*

Chapter 10

Prism hamiltonian

In Chapter 8, we mentioned the concept of a spanning k-tree, which is a relaxed concept of a hamilton path. Similarly to extending a hamilton path to a spanning k-tree, we consider the concept of a spanning k-walk. It is known that the existence of a spanning 2-tree implies the existence of a spanning 2-walk.

Recently, we focus on the property of " being prism hamiltonian," which is between the properties "having a spanning 2-tree" and "having a spanning 2-walk." In fact, if a graph is prism hamiltonian, then it has a spanning 2-walk. So we are interested in the problem of determining whether a graph having a spanning 2-walk is also prism hamiltonian or not. In Section 10.1, we mention the relationship between the property of "being prism hamiltonian" and a spanning k-tree or a spanning k-walk. We show some sufficient conditions for satisfying the property of "being prism hamiltonian" in Sections 10.2 and 10.3. In particular, we focus on degree conditions in Section 10.3.

The contents of this chapter are based on the paper [133] "A degree sum condition for graphs to be prism hamiltonian."

10.1 Relationship to k-trees and k-walks

Let G be a connected graph, and let k be an integer with $k \geq 2$. A *k-tree of G* is a tree of G with maximum degree at most k, and a *k-walk of G* is a closed walk in G that passes through each vertex at most k times. Note that a spanning 1-walk and a spanning 2-tree are equivalent to a hamilton cycle and a hamilton path, respectively. It is known that the existence of a spanning k-tree implies the existence of a spanning k-walk, and that the existence of a spanning k-walk implies the existence of a spanning $(k+1)$-tree. Thus, the properties "having a spanning k-tree" and "having a spanning k-walk" provide a hierarchy for measuring how far a graph is from being hamiltonian.

A *prism over G* is defined as the Cartesian product of graphs G and K_2, denoted by $G\Box K_2$. Thus, it consists of two copies of G and a matching joining the corresponding vertices. A graph G is called *prism hamiltonian* if prism over G has a hamilton cycle. The property of "being prism hamiltonian" is between the properties "having a spanning 2-tree" and "having a spanning 2-walk," that is, if G has a spanning 2-tree then G is prism hamiltonian, and if G is prism hamiltonian then G has a spanning 2-walk. In other ward, as was shown in [95], the property of "being prism hamiltonian" is sharply "sandwiched" between "having a spanning 2-tree" and "having a spanning 2-walk." Thus, proving that a graph with some properties is prism hamiltonian yields a stronger result than proving that it has a spanning 2-walk. Therefore, the property of "being prism hamiltonian" can be added to the above "k-tree and k-walk" hierarchy.

10.2 Prism hamiltonicity of particular classes

10.2.1 Cubic graphs

Rosenfeld and Barnette [144] proved that any 3-connected cubic planar graph is prism hamiltonian. But their proof depended on Four Color Theorem, which was still unsolved at that time. For the proof not depending on Four Color Theorem, Goodey and Rosenfeld [75] showed that it is true for a 3-connected cyclically 4-connected cubic planar graph. Notice that a graph G is called *cyclically k-edge-connected* if the resultant graph removing any k edges does not have two components containing at least one cycle. Lastly, Fleischner [62] showed the same result as Rosenfeld and Barnette's one without using Four Color Theorem. The most general result on this direction is the following due to Paulraja [139]; any 3-connected cubic graph is prism hamiltonian. Recently, Goddard and Henning [74] pointed out that for any 3-connected cubic graph G, the prism $G\Box K_2$ is vertex even pancyclic; for any vertex $x \in V(G\Box K_2)$ and any even integer l from 4 up to the order of $G\Box K_2$, there exists a cycle of length l containing x. Furthermore, they showed pancyclicity of $G\Box K_2$ when G has a triangle; $G\Box K_2$ has a cycle of length from 3 up to $|G\Box K_2|$. Flandrin Li and Čada [61] considered pancyclicity of generalized prism, that is, the Cartesian product of graphs G and some other graphs, for example, paths or cycles.

On the other hand, Alspach and Rosenfeld [5] conjectured that the prism over any 3-connected cubic graph has two edge-disjoint hamilton cycles. This conjecture is still open and Čada, Kaiser, Rosenfeld and Ryjáček [34] showed that the conjecture is true for the prism over any 3-connected bipartite cubic planar graph.

10.2.2 Planar graphs

As mentioned above, if a graph is prism hamiltonian, then it has a spanning 2-walk. So we are interested in the problem of determining whether a graph having a spanning 2-walk is also prism hamiltonian or not. In this sense, one of the important classes is a class of "planar graphs." Barnette [11] and Gao and Richter [70] showed that any 3-connected planar graph has a spanning 3-tree, and a spanning 2-walk, respectively. Therefore it is natural to consider prism hamiltonicity of 3-connected planar graphs. In fact, Kaiser, Ryjáček, Král', Rosenfeld and Voss posed the following conjecture.

Conjecture 10.1 (Kaiser, Ryjáček, Král', Rosenfeld and Voss [95]) *Any* 3*-connected planar graph is prism hamiltonian.*

Conjecture 10.1 is still open and Biebighauser and Ellingham [22] proved that any plane triangulation is prism hamiltonian. Since all plane triangulation is 3-connected, this is a partial solution of Conjecture 10.1. They also showed that any triangulation of projective plane, torus, and Klein bottle is also prism hamiltonian.

10.2.3 Other classes

Horák, Kaiser, Rosenfeld and Ryjáček considered prism hamiltonicity of *middle-levels graph*. Let $\mathbf{B}_k$ be a bipartite graph whose vertices are all k or $k+1$ elements subsets of $\{1, 2, \ldots, 2k+1\}$, and whose edges corresponds to the inclusion between two such subsets. It is well-known conjecture that $\mathbf{B}_k$ is hamiltonian for all $k \geq 2$. This conjecture is still open and Horák, Kaiser, Rosenfeld and Ryjáček [86] showed that $\mathbf{B}_k$ is prism hamiltonian for all $k \geq 2$.

Let G be a connected graph. We define a *acyclic orientation graph* of G, denoted by $AO(G)$, whose vertices are the acyclic orientations of G and whose edges join two orientations that differ by reversing the direction of one edge. Pruesse and Ruskey [141] showed that for any connected graph G, $AO(G)$ is prism hamiltonian.

In [95], Kaiser, Ryjáček, Král', Rosenfeld and Voss studied prism hamiltonicity for other graphs, for example, 2-connected line graphs, and the square of graphs. They also proposed some conjectures. One of them is Conjecture 10.1 and we introduce some others.

Conjecture 10.2 ([95]) *Any* 4*-connected* 4*-regular graph is prism hamiltonian.*

Conjecture 10.3 ([95]) *There exists a constant value t such that any t-tough graph is prism hamiltonian.*

10.3 Degree conditions for prism hamiltonicity

10.3.1 Results

Ore [130] showed a $\sigma_2(G)$ condition for the existence of a hamilton cycle, which has a natural corollary the analogous result on the existence of a hamilton path.

Theorem 10.4 (Ore [130]) *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G has a hamilton cycle, that is, a spanning 1-walk.*

Corollary 10.5 (Ore [130]) *Let G be a graph of order $n \geq 2$. If $\sigma_2(G) \geq n-1$, then G has a hamilton path, that is, a spanning 2-tree.*

Jackson and Wormald in 1990, and Win in 1975 showed that the following degree sum conditions on graphs guarantee the properties "having a spanning k-walk" and "having a spanning k-tree," respectively. Theorems 10.6 and 10.7 are generalizations of Theorem 10.4 and Corollary 10.5, respectively, and both degree sum bounds are best possible.

Theorem 10.6 (Jackson and Wormald [92]) *Let G be a connected graph of order $n \geq 3$, and let k be an integer at least 1. If $\sigma_{k+1}(G) \geq n$, then G has a spanning k-walk.*

Theorem 10.7 (Win [168]) *Let G be a connected graph of order $n \geq 2$, and let k be an integer at least 2. If $\sigma_k(G) \geq n-1$, then G has a spanning k-tree.*

In particular, for a connected graph G of order n, $\sigma_2(G) \geq n-1$ implies having a spanning 2-tree (a hamilton path) and $\sigma_3(G) \geq n$ implies having a spanning 2-walk. Since the property of "being prism hamiltonian" is between "having a spanning 2-tree" and "having a spanning 2-walk," it is natural to pose the following problem.

Problem 10.8 *Determine a sharp degree sum condition for connected graphs to be prism hamiltonian.*

As an answer to this problem, in this section we show the following result: a connected graph G of order n with $\sigma_3(G) \geq n$ has not only the property "having a spanning 2-walk" but also "being prism hamiltonian." In this sense, the property of "being prism hamiltonian" is closer to the property "having a spanning 2-walk" than "having a spanning 2-tree."

Theorem 10.9 ([133]) *Let G be a connected graph of order $n \geq 2$. If $\sigma_3(G) \geq n$, then G is prism hamiltonian.*

The degree sum condition of Theorem 10.9 is best possible. Let $G_1 = K_1 + 3K_m$, and let $G_2 = K_{m,2m+1}$. Although $\sigma_3(G_i) = 3m = |V(G_i)| - 1$ for $i = 1, 2$, both G_1 and G_2 are not prism hamiltonian. Moreover, both have no spanning 2-walks.

The notion of closure can be used as an extension of the degree sum conditions. For example, Theorem 10.4 can be generalized to the following closure type result of Bondy and Chvátal.

Theorem 10.10 (Bondy and Chvátal [27]) *Let G be a graph of order n, and let x and y be two non-adjacent vertices such that the degree sum of x and y is at least n. Then G has a hamilton cycle if and only if $G + xy$ has a hamilton cycle.*

In 2006, Král' and Stacho showed an analogous closure result on prism hamiltonicity.

Theorem 10.11 (Král' and Stacho [100]) *Let G be a graph of order n, and let x and y be two non-adjacent vertices such that the degree sum of x and y is at least $\frac{4n}{3} - \frac{4}{3}$. Then G is prism hamiltonian if and only if $G + xy$ is prism hamiltonian.*

They also showed that the degree sum value $\frac{4n}{3} - \frac{4}{3}$ cannot be decreased to $\frac{4n}{3} - \frac{16}{3}$. Thus, the coefficient $\frac{4}{3}$ of n is sharp for the closure of prism hamiltonicity. On the other hand, Theorem 10.9 shows that the condition $\sigma_3(G) \geq n$ implies prism hamiltonicity. Hence, there is a large gap between the degree sums necessary for the property of "being prism hamiltonian" and for its closure. Thus, the situation of prism hamiltonicity is different from that of ordinary hamiltonicity with respect to closure.

In Section 10.3.2, we show some results that are used in our proof of Theorem 10.9. We will prove Theorem 10.9 in Section 10.3.3.

10.3.2 Lemmas used in the proof of Theorem 10.9

In the proof of Theorem 10.9, we use the invariant $\mathrm{diff}(G) := p(G) - c(G)$. It is known the following properties when $\mathrm{diff}(G)$ is small.

Proposition 10.12 *Let G be a connected graph. Then $\mathrm{diff}(G) = 0$ if and only if G has a hamilton cycle.*

Proposition 10.13 (Li Saito and Schelp [111]) *Let G be a graph with $\mathrm{diff}(G) = 1$ and let C be a longest cycle in G. Let $S := V(G) - V(C)$. Then $S \cup N_C(S)^+$ is an independent set.*

Enomoto, van den Heuvel, Kaneko and Saito showed that $\sigma_3(G) \geq n$ implies $\mathrm{diff}(G) \leq 1$ unless G belongs to one of exceptional classes. Observing those exceptional classes, we obtain the following result, which will play an important role in our proof of Theorem 10.9.

Theorem 10.14 (Enomoto, van den Heuvel, Kaneko and Saito [46]) *Let G be a connected graph of order $n \geq 3$. If $\sigma_3(G) \geq n$, then $\mathrm{diff}(G) \leq 1$ or G has a hamilton path.*

A connected graph F is called a *cactus* if each block of F is a cycle or an edge and every cut vertex of F is contained in exactly two blocks. A cactus with no odd cycle is called an *even cactus*. There is a close relationship between a spanning even cactus and prism hamiltonicity. The following proposition is shown in [139]: see also [34, 75].

Proposition 10.15 (Paulraja [139]) *If G has a spanning even cactus, then G is prism hamiltonian.*

In Section 10.3.3, we shall prove Theorem 10.9 by constructing a spanning even cactus. In fact, we consider a long even cycle C as a part of an even cactus and join the outside of C to the cycle C. In order to join the outside to distinct vertices of C, we need the following lemma.

Lemma 10.16 *Let G be a connected graph of order n, and let S be an independent set with $|S| \leq \frac{1}{3}n$. If $\sigma_3(G) \geq n$, then there exists a matching M that covers all the vertices in S.*

Proof of Lemma 10.16.

Suppose that there exists no matching that covers all vertices in S. Since S is an independent set, by removing edges between $V(G) - S$, we can consider the graph G as a bipartite graph with partite sets S and $V(G) - S$. By Hall's Theorem, there exists $X \subseteq S$ such that $|N_G(X)| < |X|$.

Let $S := \{x_1, x_2, \ldots, x_{|S|}\}$ with $d_G(x_1) \leq d_G(x_2) \leq \cdots \leq d_G(x_{|S|})$. Since $\sigma_3(G) \geq n$, we obtain $d_G(x_i) \geq \frac{1}{3}n \geq |S| \geq |X|$ for every $3 \leq i \leq |S|$. Hence $X \subseteq \{x_1, x_2\}$. On the other hand, the connectedness of G implies that $d_G(x_1) \geq 1$ and $d_G(x_2) \geq 1$, and hence $X = \{x_1, x_2\}$ and $N_G(x_1) = N_G(x_2) = \{y\}$ for some $y \in V(G) - S$. Since $n \geq 3|S| \geq 6$, there exists $z \neq x_1, x_2, y$. Then $N_G(z) \subseteq V(G) - \{x_1, x_2, z\}$ and hence $d_G(z) \leq n - 3$. This implies that $d_G(x_1) + d_G(x_2) + d_G(z) \leq n - 1$, contradicting the degree sum assumption. $\square$

10.3.3 Proof of Theorem 10.9

Let G be a graph satisfying the assumption of Theorem 10.9, that is, a connected graph of order n with $\sigma_3(G) \geq n$. By Theorem 10.14, $\mathrm{diff}(G) \leq 1$ or G has a hamilton path. If $\mathrm{diff}(G) = 0$ or G has a hamilton path, then by Proposition 10.15, G is obviously prism hamiltonian because a hamilton path is an even cactus. Thus, we may assume that $\mathrm{diff}(G) = 1$.

Let C be a longest cycle in G and let $S := V(G) - V(C)$. By Proposition 10.12, C is not a hamilton cycle, so $S \neq \emptyset$. By Proposition 10.13, S is an independent set.

If $n \leq 5$, then we can easily check that G is prism hamiltonian. So assume that $n \geq 6$. Suppose that $|S| > \frac{1}{3}n$. Then there are at least three vertices in S: hence we can choose a vertex $x \in S$ so that $d_G(x) \geq \frac{1}{3}n$. By Proposition 10.13, $N_C(x) \cap N_C(x)^+ = \emptyset$. This implies that $|V(C)| \geq 2d_C(x) \geq \frac{2}{3}n$. Then $n = |V(C)| + |S| > n$, a contradiction. Thus, $|S| \leq \frac{1}{3}n$. It follows from Lemma 10.16 that there exists a matching M that covers all vertices in S. We may assume that each edge of M contains a vertex in S.

We will find a spanning even cactus consisting of exactly one even cycle and some paths. If $|V(C)|$ is even, then $C \cup M$ is a spanning even cactus. So we may assume that $|V(C)|$ is odd.

We call each path of $C - N_C(S)$ a *segment*. If all segment has odd vertices, then $|C| = \sum_{I \in \mathcal{I}} |I| + |N_C(S)| \equiv 2|N_C(S)| \equiv 0 \pmod 2$, where $\mathcal{I}$ is the set of all segments, contradicting the fact that $|C|$ is odd. Thus, there exists a segment I with an even number of vertices. By Proposition 10.13, note that every segment with an even number of vertices has at least 2 vertices. Choose a longest cycle C and an even segment I such that $|V(I)|$ is as small as possible. Let $y_1, y_2 \in N_C(S)$ such that $y_1^+, y_2^- \in I$ and let $x_1, x_2 \in S$ such that $x_1y_1, x_2y_2 \in E(G)$. We take such vertices x_1, x_2 that $x_1 \neq x_2$ if possible. Then we have the following claim.

Claim 10.1 *If $x_1 = x_2$, then there is a spanning even cactus.*

Proof. Suppose that $x_1 = x_2$. Then let $M' := M - \{x_1w\}$ where $w \in N_M(x_1)$. By the definition of I and by the choice of x_1, x_2, we have $V(M') \cap \big(I \cup \{y_1, y_2\}\big) = \emptyset$. Let $C' := \big(C - I\big) \cup \{y_1x_1, x_1y_2\}$. Then $C' \cup \big(y_1y_1^+ \cup I\big) \cup M'$ is a spanning even cactus. $\square$

By Claim 10.1, we may assume that $x_1 \neq x_2$. Let $z_1 := y_1^{+2}$. By the definitions of y_1 and I, $z_1 \in I$ and $N_S(z_1) = \emptyset$. Let

$$\begin{aligned} A &:= N_C(z_1)^-, \\ B_1 &:= N_C(x_1), \\ \text{and} \qquad B_2 &:= N_C(x_1)^+. \end{aligned}$$

By Proposition 10.13, it is easily shown that $B_1 \cap B_2 = \emptyset$.

Claim 10.2 $A \cap B_1 = \{y_1\}$.

Proof. Suppose that there exists $w \in A \cap B_1$ such that $w \neq y_1$. Note that $w \notin I$. Let $C' := z_1 \overrightarrow{C} w x_1 y_1 \overleftarrow{C} w^+ z_1$. Then $|V(C')| = |V(C)|$ and $I - \{y_1^+, z_1\}$ is an even segment of C' if $N_G(y_1^+) \cap \big(I - \{y_1^+, z_1\}\big) = \emptyset$; otherwise $I - \{y_1^+, z_1\}$ contains smaller even segment than I, which contradicts the minimality of I. $\square$

Claim 10.3 *If $A \cap B_2 \neq \emptyset$, then there is a spanning even cactus.*

Proof. Suppose that there exists $w \in A \cap B_2$. By the definition of A, note that $w \neq y_1^+$. Let $C' := z_1 \overrightarrow{C} w^- x_1 y_1 \overleftarrow{C} w^+ z_1$. Then $|V(C')| = |V(C)| - 1$, and hence C' is an even cycle. Let $S' := \big(S - \{x_1\}\big) \cup \{w\}$. Note that $V(G) - V(C') = S' \cup \{y_1^+\}$. Since $|S'| = |S| \leq \frac{1}{3}n$ and S' is an independent set, there exists a matching M' that covers S' by Lemma 10.16. We may assume that every edge in M' contains a vertex in S'. By the definition of I, $y_1^+ \notin N_G(S)$ and $z_1 \notin N_G(S)$. By Proposition 10.13, $y_1^+ \notin N_G(w)$, because otherwise we can find an edge between two vertices in $N_C(x_1)^+$. By Claim 10.2, $z_1 \notin N_G(w)$. This implies that $V(M') \cap \{y_1^+, z_1\} = \emptyset$. Hence $C' \cup M' \cup \{y_1^+ z_1\}$ is a spanning even cactus. $\square$

By Claim 10.3, we may assume that $A \cap B_2 = \emptyset$. Since $A \cup B_1 \cup B_2 \subseteq V(C)$, it follows from Claim 10.2 that

$$\begin{aligned} d_G(z_1) + 2d_G(x_1) &= |A| + |B_1| + |B_2| \\ &= |A \cup B_1 \cup B_2| + 1 \\ &\leq |V(C)| + 1 \\ &\leq n - 1. \end{aligned} \tag{10.1}$$

Let $z_2 := y_2^{-2}$. By considering the reverse orientation of C, we have

$$d_G(z_2) + 2d_G(x_2) \leq n - 1. \tag{10.2}$$

By the inequalities (10.1) and (10.2),

$$d_G(z_1) + d_G(z_2) + 2d_G(x_1) + 2d_G(x_2) \leq 2(n-1),$$

and hence

$$\begin{aligned} & d_G(z_1) + d_G(x_1) + d_G(x_2) \leq n - 1 \\ \text{or} \quad & d_G(z_2) + d_G(x_1) + d_G(x_2) \leq n - 1. \end{aligned}$$

Since $\{z_i, x_1, x_2\}$ is independent for $i = 1, 2$, this yields a contradiction, and hence G has a spanning even cactus, so G is prism hamiltonian. $\square$

Chapter 11

Spanning trees with bounded number of leaves

In this chapter, we discuss about a spanning tree with few leaves. Since a hamiltonian path is as a spanning tree with two leaves, a spanning tree with bounded number of leaves can be considered as a generalization of a hamilton path. In particular, we consider an independence number condition and a degree sum condition for the existence of such a spanning tree. In Sections 11.1 and 11.2, we show some results on a general graph and on a claw-free graph, respectively.

The contents of this chapter are based on the paper [135] "Spanning trees with small number of leaves in a claw-free graph," jointwork with M. Kano, A. Kyaw, H. Matsuda, A. Saito and T. Yamashita.

11.1 On general graphs

Win [169] and Broersma and Tuinstra [31] gave an independence number condition and a $\sigma_2(G)$ condition for graphs to have a spanning tree with bounded number of leaves, respectively; For an m-connected graph G of order n, if $\alpha(G) \leq m + k - 1$, or if $\sigma_2(G) \geq n - k + 1$, then G has a spanning tree with at most k leaves. Note that Win's result on the independence number was conjectured by Las Vergnas. Notice also that these results are generalizations of results on a hamilton path by Chvátal and Erdős [37] and by Ore [130], respectively. Recently, Tsugaki and Yamashita obtained a common generalization.

Theorem 11.1 (Tsugaki and Yamashita [158]) *Let $m \geq 1$ and $k \geq 2$, and let G be an m-connected graph of order n. If $\sigma_2^{m+k}(G) \geq n - k + 1$, then G has a spanning tree with at most k leaves.*

We define the neighborhood union condition as follows:

$$N_k(G) = \min\{|N_G(X)| : X \text{ is an independent set of } G \text{ with } |X| = k\}$$

if $\alpha(G) \geq k$, $N_k(G) = +\infty$ if $\alpha(G) < k$. It is easy to obtain the following neighborhood union condition from the result for hamiltonianicity in [14]; For a connected graph G of order n, if $N_2(G) > \frac{2}{3}(n-2)$, then G has a hamiltonian path. Flandrin et al. [60] proved a generalization of this theorem.

Theorem 11.2 (Flandrin et al. [60]) *Let $k \geq 2$ and let G be a connected graph of order n. If $N_m(G) > \frac{k}{k+1}(n-k)$, then G has a spanning tree with at most k leaves.*

11.2 On claw-free graphs

11.2.1 Results

A graph G is said to be *claw-free* if it contains no $K_{1,3}$ as an induced subgraph. By Dirac's Theorem, every graph G on $n \geq 3$ vertices with $\delta(G) \geq \frac{1}{2}n$ has a hamilton cycle. As an immediate corollary, we can prove that every graph G of order n with $\delta(G) \geq \frac{1}{2}(n-1)$ has a hamilton path. For general graphs of order n, the bound $\frac{1}{2}(n-1)$ is sharp. For example, for a positive integer m, the complete bipartite graph with partite sets of order m and $m+2$ satisfies $\delta(G) = m = \frac{1}{2}(n-2)$, but G has no hamilton path. However, Matthews and Sumner proved that if we restrict ourselves to the class of claw-free graphs, a considerably smaller bound on minimum degree guarantees the existence of a hamilton path.

Theorem 11.3 (Matthews and Sumner [123]) *Let G be a connected claw-free graph of order n. If $\delta(G) \geq \big(n-2\big)/3$, then G has a hamilton path.*

Ore's Theorem states that every graph G on $n \geq 3$ vertices with $\sigma_2(G) \geq n$ has a hamilton cycle. It extends Dirac's Theorem, and implies as a corollary that every graph G of order n with $\sigma_2(G) \geq n-1$ has a hamilton path. The previous corollary of Ore's Theorem was extended by Broersma and Tuinstra [31] as mentioned before; For a connected graph G of order n, if $\sigma_2(G) \geq n-k+1$, then G has a spanning tree with at most k leaves. They also proved that the bound $n-k+1$ of $\sigma_2(G)$ is sharp. However, in view of Theorem 11.3, for claw-free graphs, a much weaker condition may yield the same conclusion as in Dirac's Theorem. Motivated by this observation, we study a degree sum condition for a claw-free graph to have a spanning tree with a bounded number of leaves, and give the following theorem.

Theorem 11.4 ([96]) *Let $k \geq 2$ be an integer and let G be a connected claw-free graph of order n. If $\sigma_{k+1}(G) \geq n-k$, then G has a spanning tree with at most k leaves.*

Note that Theorem 11.3 is a corollary of the case $k = 2$ of the above theorem. We first show that the bound $n - k$ of $\sigma_{k+1}(G)$ in Theorem 11.4 is sharp. Consider a graph G_1 constructed from one complete graph K_{k+1} and $k+1$ complete graphs K_m, $m \geq 2$, by identifying one vertex of each K_m with one distinct vertex of K_{k+1} (see Figure 11.1). Then G is claw-free and satisfies $\sigma_{k+1}(G_1) = n - k - 1$, but G has no spanning tree with at most k leaves.

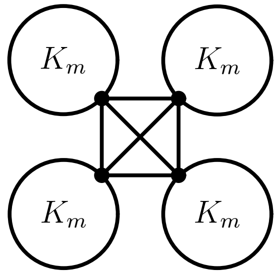


Figure 11.1: The graph G_1 for $k = 4$.

On the other hand, A. Kyaw [103] showed a result on $K_{1,4}$-free graphs.

Theorem 11.5 (Kyaw [103]) *Let G be a connected $K_{1,4}$-free graph of order n.* (i) *If $\sigma_3(G) \geq n$, then G has a hamilton path.* (ii) *If $\sigma_{k+2}(G) \geq n - k/2$ ($k \geq 3$), then G has a spanning tree with at most k leaves.*

11.2.2 Maximum Degree

Under the same assumption as that of Theorem 11.4, we can actually guarantee the existence of a 3-tree with at most k leaves. Moreover, we show that a $K_{1,t}$-free graph having a spanning tree with at most k leaves also has a spanning t-tree with at most k leaves.

Lemma 11.6 *Let $k \geq 2$ be an integer. If a connected $K_{1,t}$-free graph G has a spanning tree with at most k leaves, then G has a spanning t-tree with at most k leaves.*

Proof of Lemma 11.6.

Let u be an arbitrary vertex in G, and consider every spanning tree as a rooted tree with root u. Choose a spanning tree T with at most k leaves so that $\sum_{x \in V(T)} \mathrm{dist}_T(u, x)$ is as large as possible. Assume T has a vertex w of degree at least $t + 1$. Then w has at least $t + 1$ children, and since G is $K_{1,t}$-free, w has a pair of children v_1 and v_2 which are adjacent with each other in G. Let $T' = \big(T - wv_1\big) \cup \{v_1v_2\}$. Then T' is a spanning tree of G, and $d_{T'}(w) = d_T(w) - 1$, $d_{T'}(v_2) = d_T(v_2) + 1$ and $d_{T'}(x) = d_T(x)$ for each $x \in V(G) - \{w, v_2\}$. Since $d_T(w) \geq t + 1$, T' does not have the larger number of leaves than T.

Let $x \in V(G)$. Then T has a unique ux-path P. If P still exists in T', we have $\mathrm{dist}_T(u,x) = \mathrm{dist}_{T'}(u,x)$. If P does not exist in T', then $wv_1 \in E(P)$ and $P' = u\overrightarrow{P}wv_2v_1\overrightarrow{P}x$ is a unique ux-path in T'. This implies $\mathrm{dist}_{T'}(u,x) = \mathrm{dist}_T(u,x) + 1$. Therefore, $\mathrm{dist}_{T'}(u,x) \geq \mathrm{dist}_T(u,x)$ for each $x \in V(G)$ and $\mathrm{dist}_{T'}(u,v) > \mathrm{dist}_T(u,v)$. These imply $\sum_{x \in V(G)} \mathrm{dist}_{T'}(u,x) > \sum_{x \in V(G)} \mathrm{dist}_T(u,x)$. This contradicts the choice of T, and hence the maximum degree of T is at most t. $\square$

11.2.3 Remarks and conjecture

Matthews and Sumner [123] proved that a 2-connected claw-free graph of order n with minimum degree at least $\frac{1}{3}(n-2)$ has a hamilton cycle. This result was later extended by Zhang.

Theorem 11.7 (Zhang [180]) *A k-connected claw-free graph G of order n with $\sigma_{k+1}(G) \geq n-k$ has a hamilton cycle.*

Interpreting a hamilton cycle as a "spanning tree with one leaf" and comparing Theorems 11.4 and 11.7, we may make the following conjecture.

Conjecture 11.8 *For integers k and m with $k \geq 2$ and $m \leq \min\{6, k-1\}$, every m-connected claw-free graph G of order n with $\sigma_{k+1}(G) \geq n-k$ has a spanning tree with at most $k-m+1$ leaves.*

The assumption $m \leq 6$ in the above conjecture looks strange, but it comes from the following theorem by Ryjáček.

Theorem 11.9 (Ryjáček [145]) *Every 7-connected claw-free graph is hamiltonian.*

By the above theorem, a 7-connected claw-free graph has a spanning tree with two leaves without any degree sum condition.

11.2.4 Proof of Theorem 11.4

The graph constructed from two complete graphs K_m and K_n by identifying one vertex of K_m with one vertex of K_n is called a *double complete graph* and denoted by $DC(m,n)$, where $m, n \geq 2$. The common vertex of K_m and K_n is called the *center*, and the other vertices are called *non-central vertices* (See Figure 11.2). Note that the order of $DC(m,n)$ is $m+n-1$, and the path of order three is a double complete graph $DC(2,2)$. Let $\mathcal{D}$ denote the set of all double complete graphs.

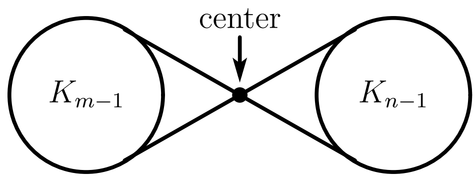


Figure 11.2: The double complete graph $DC(m, n)$.

Enomoto [45], Jung [93] and Nara [126] implicitly characterized the connected graphs G of order n such that G satisfies $d_G(x) + d_G(y) \geq n - 1$ for every pair of vertices x and y of G which are end-vertices of some hamilton path of G, but G has no hamilton cycle. The next lemma is a corollary of this characterization. We give its proof for the self-containedness.

Lemma 11.10 *Let G be a claw-free graph of order n having a hamilton path. Suppose that $\deg_G(x) + \deg_G(y) \geq n - 1$ for every pair of vertices x and y which are end-vertices of some hamilton path. Then G has a hamilton cycle, or G is a double complete graph.*

Proof of Lemma 11.10.

Assume G has no hamilton cycle. Let P be a hamilton path and let x and y be two end vertices of P. By the assumption, $xy \notin E(G)$. If $N_P(x)^- \cap N_P(y) \neq \emptyset$, then $x\overrightarrow{P}vy\overleftarrow{P}v^+x$, where $v \in N_G(x)^- \cap N_G(y)$, is a hamilton cycle, a contradiction. Thus, $N_P(x)^- \cap N_P(y) = \emptyset$. Since $N_P(x)^- \cup N_P(y) \subseteq V(G) - \{y\}$ and $\big|N_P(x)^- \cup N_P(y)\big| = d_G(x) + d_G(y) \geq n - 1$, we have $N_P(x)^- \cup N_P(y) = V(G) - \{y\}$ and $d_G(x) + d_G(y) = n - 1$. On the other hand, since $N_G(x) \cup N_G(y) \subseteq V(G) - \{x, y\}$ and $d_G(x) + d_G(y) \geq n - 1$, we have $N_G(x) \cap N_G(y) \neq \emptyset$. We consider two cases.

Case 1 $|N_G(x) \cap N_G(y)| = 1$.

In this case, $N_G(x) \cup N_G(y) = V(G) - \{x, y\}$. Let $N_G(x) \cap N_G(y) = \{z\}$. Since $N_P(x)^- \cap N_P(y) = \emptyset$ and $N_G(x) \cup N_G(y) = V(G) - \{x, y\}$, $v \in N_G(x) - \{x^+\}$ implies $v^- \in N_G(x)$. This implies $x^+\overrightarrow{P}z \subseteq N_G(x)$. Similarly, $z\overrightarrow{P}y^- \subseteq N_G(y)$. Since $N_G(x) \cap N_G(y) = \{z\}$, we have $N_G(x) = x^+\overrightarrow{P}z$ and $N_G(y) = z\overrightarrow{P}y^-$.

Let $x_1 \in x^+\overrightarrow{P}z^-$. Then $x_1^+ \in N_G(x)$ and $x_1\overleftarrow{P}xx_1^+\overrightarrow{P}y$ is a hamilton path of G. If $N_G(x_1) \cap z^+\overrightarrow{P}y \neq \emptyset$, then $x\overrightarrow{P}x_1y_1\overrightarrow{P}yy_1^-\overleftarrow{P}x_1^+x$, where $y_1 \in N_G(x_1) \cap z^+\overrightarrow{P}y)$, is a hamilton cycle of G, a contradiction. Therefore, $N_G(x_1) \subseteq x\overrightarrow{P}z - \{x_1\}$. Since $d_G(x_1) + d_G(y) \geq n - 1$ by the assumption, we have $N_G(x_1) \cap N_G(y) = \{z\}$. The we can apply the same argument as in the previous paragraph to x_1 and y, and obtain $N_G(x_1) = x\overrightarrow{P}z - \{x_1\}$. This implies that z is a cut vertex of G and $x\overrightarrow{P}z$ induces a complete graph. By symmetry, $z\overrightarrow{P}y$ also induces a complete graph. Therefore, G is a double complete graph. $\square$

Case 2 $|N_G(x) \cap N_G(y)| \geq 2$.

In this case, there exist $x_0 \in N_G(x)$ and $y_0 \in N_G(y)$ such that $x_0 \in y_0^+ \overrightarrow{P} y$. Choose such x_0 and y_0 so that $y_0 \overrightarrow{P} x_0$ is as short as possible. Since $N_P(x)^- \cap N_P(y) = \emptyset$, $y_0^+ \neq x_0$.

Since $xy \notin E(G)$, x_0^{-2} exists and $x_0^{-2} \in N_P(x)^- \cup N_P(y)$. Since $x_0^- \notin N_P(x)$ by the choice of x_0 and y_0, $x_0^{-2} \in N_P(y)$. Again by the choice of x_0 and y_0, we have $y_0 = x_0^{-2}$. Since $y_0^+ \overleftarrow{P} x x_0 \overrightarrow{P} y$ and $y_0^+ \overrightarrow{P} y y_0 \overleftarrow{P} x$ are both hamilton paths, we can apply the same argument as that for P to these paths, and obtain $d_G(y_0^+) + d_G(y) = d_G(y_0^+) + d_G(x) = d_G(y) + d_G(x) = n - 1$, which yields $d_G(x) = d_G(y) = d_G(y_0^+) = \frac{1}{2}(n-1)$.

Let $C = x \overrightarrow{P} y_0 y \overleftarrow{P} x_0 x$. Then $V(C) = V(G) - \{y_0^+\}$. Let $C = v_0 v_1 \dots v_{n-2} v_0$. If y_0^+ is adjacent to a consecutive vertices of C, then we can insert y_0^+ to this cycle to obtain a hamilton cycle of G, contradicting the assumption. Since $d_G(y_0^+) = \frac{1}{2}\big(n - 1\big)$, y_0^+ is adjacent to every other vertex of C. Let $v_i \in N_G(y_0^+)$. Then $v_{i-2} \in N_G(y_0^+)$. Since $\{v_{i-1}, v_{i+1}, y_0^+\} \subseteq N_G(v_i)$ and G is claw-free, we have $v_{i-1}v_{i+1} \in E(G)$. Then by replacing $v_{i-2}v_{i-1}v_i v_{i+1}$ in C with $v_{i-2} y_0^+ v_i v_{i-1} v_{i+1}$, we have a hamilton cycle of G. This is a contradiction, and the lemma follows. $\Box$

Win [169] introduced a *k-ended system* to prove the existence of a spanning tree with at most k leaves. In this chapter, we modify the definition of a k-ended system and define a *k-extended system*. It plays an important role in the proof of Theorem 11.4.

Let G be a connected claw-free graph of order n, and F be a subgraph of G. The set of components of F is denoted by $\mathcal{C}(F)$. We call F an *extended system* if each component of F is a path, a cycle or a double complete graph. For an extended system F, we define a mapping f from $\mathcal{C}(F)$ to $\{1, 2\}$ as follows. For every $C \in \mathcal{C}(F)$,

$$f(C) = \begin{cases} 1 & \text{if } C \text{ is } K_1,\ K_2, \text{ a cycle or a double complete graph,} \\ 2 & \text{otherwise (i.e., a path of order at least four),} \end{cases}$$

and define

$$f(F) = \sum_{C \in \mathcal{C}(F)} f(C).$$

Let $\mathcal{C}_i(F) = \{C \in \mathcal{C}(F) : f(C) = i\}$ for $i = 1, 2$. An extended system F is called a *k-extended system* if $f(F) \leq k$.

The following lemma is an easy but important observation.

Lemma 11.11 *Let G be a claw-free graph and D be an induced double complete subgraph of G. If a vertex $v \in V(G) - V(D)$ is adjacent to the center of D, then v is also adjacent to a non-central vertex of D.*

Proof of Lemma 11.11.

Let D_1 and D_2 be the two blocks of D. Then both D_1 and D_2 are complete graphs. Let x be the center of D and let $x_i \in D_i - \{x\}$ $(i = 1, 2)$. Since D is an induced subgraph of G, $x_1x_2 \notin E(G)$. Since $\{x_1, x_2, v\} \subseteq N_G(x)$ and G is claw-free, $\{x_1v, x_2v\} \cap E(G) \neq \emptyset$. $\square$

The next lemma shows a relationship between a k-extended system and a spanning tree with at most k leaves in a claw-free graph.

Lemma 11.12 *Let $k \geq 2$ be an integer and G be a connected claw-free graph. If G has an extended system F_0, then G has a spanning tree with at most $f(F_0)$ leaves. In particular, if G has a k-extended system, then G has a spanning tree with at most k leaves.*

Proof of Lemma 11.12.

Take a spanning extended system F with $f(F) \leq f(F_0)$ so that the number of double complete graphs is as small as possible. Then every double complete graph of F is an induced subgraph of G since if two non-central vertices of a double complete graph D of F are joined by an edge e of G, then $D + e$ has a hamilton cycle, and so D should be replaced by this hamilton cycle.

Since G is connected, there exists a minimal set X of edges such that F together with X forms a connected spanning subgraph of G. We shall construct a spanning tree with at most k leaves consisting of F and X. By Lemma 11.11, we may assume that no edge in X is incident with the center of a double complete graph. For any double complete graph D of F, there exists an edge $e_D \in X$ incident with a vertex v_D of D, where v_D is not the center of D. Then D has a hamilton path starting at v_D, and we replace D with this hamilton path.

For any cycle C of F, there exists an edge $e_C \in X$ incident with a vertex v_C of C. Delete an edge of C incident with v_C. By repeating the above procedure for every double complete graph and every cycle of F, we obtain a spanning tree T. By the construction, for each $C \in \mathcal{C}(F)$, the number of leaves of T contained in C is at most $f(C)$.

Hence T has at most $f(F) \leq f(F_0)$ leaves. $\square$

We call a k-extended system F of G a *maximal k-extended system* if G has no k-extended system F' such that $V(F)$ is a proper subset of $V(F')$. In order to prove our theorem, we need the following lemma.

Lemma 11.13 *Suppose that a graph G does not have a spanning k-extended system. Let F be a maximal $(k+1)$-extended system of G. Then G does not have* *a k-extended system F' with $V(F') = V(F)$. In particular, F is not a k-extended system, and so $f(F) = k+1$.*

Proof of Lemma 11.13.

Let F be a maximal $(k+1)$-extended system of G. Assume that G has a k-extended system F' with $V(F') = V(F)$. Since G does not have a spanning k-extended system, there exists a vertex $v \in V(G) - V(F')$, and thus G has a $(k+1)$-extended system $F' \cup \{v\}$, which contradicts the maximality of F. $\square$

By Lemma 11.12, in order to prove our Theorem 11.4, it suffices to prove the following theorem.

Theorem 11.14 *Let $k \geq 2$ be an integer and G be a claw-free graph of order n. If $\sigma_{k+1}(G) \geq n-k$, then G has a spanning k-extended system.*

Proof of Theorem 11.14.

Suppose that G has no spanning k-extended system. Take a maximal $(k+1)$-extended system F so that

(F1) $\sum_{P \in \mathcal{C}_2(F)} |V(P)|$ is as large as possible,

(F2) The number of cycles in $\mathcal{C}_1(F)$ is as large as possible subject to (F1), and

(F3) $\sum_{P \in \mathcal{C}_2(F)} \big(d_P(x_P) + d_P(y_P)\big)$ is as small as possible, subject to (F1) and (F2), where x_P and y_P are the end-vertices of P.

By Lemma 11.13, $f(F) = k+1$. We begin with a simple but important observation.

Fact 11.1 *For each $D \in \mathcal{C}_1(F)$ and for each $v \in V(D)$ that is not the center of D if D is a double complete graph, D has a hamilton path containing v as one of its end-vertices.*

The next fact follows from the condition (F2) and the same argument as in the first paragraph of the proof of Lemma 11.12.

Fact 11.2 *Every double complete graph D of F is an induced subgraph of G.*

Next, we investigate the adjacency between the components of F.

Claim 11.3 *The following three statements hold.*
(i) *No two components of $\mathcal{C}_1(F)$ are connected by an edge of G.*
(ii) *No end-vertex of a path in $\mathcal{C}_2(F)$ is connected to a component of $\mathcal{C}_1(F)$ by an edge of G.*
(iii) *No two end-vertices of two distinct paths or of the same path in $\mathcal{C}_2(F)$ are joined by an edge of G*

Proof. (i) Assume that two components Q_1 and Q_2 of $\mathcal{C}_1(F)$ are joined by an edge e of G. By Lemma 11.11, we may assume that no end-vertex of e is the center of a double complete graph. So $Q_1 \cup Q_2$ contains a hamilton path P_0. By replacing Q_1 and Q_2 of F by P_0, we obtain another maximal $(k+1)$-extended system F' on $V(F)$. If $|P_0| \geq 4$ this contradicts the condition (F1). If $|P_0| \leq 3$, then $f(P_0) = 1$ and hence F' is a k-extended system, which contradicts Lemma 11.13.
(ii) If an end-vertex of a path $P \in \mathcal{C}_2(F)$ is joined to a component $Q \in \mathcal{C}_1(F)$ by an edge e of G, then by an argument similar to the one in (i), we see that $P \cup Q$ has a hamilton path. Thus, we can derive a contradiction by Lemma 11.13.
(iii) If two end-vertices of two paths or of the same path in $\mathcal{C}_2(F)$ are joined by an edge of G, then we can obtain a k-extended system with vertex set $V(F)$, which contradicts Lemma 11.13. $\square$

For every component $Q \in \mathcal{C}_1(F)$, we take one vertex x_Q from Q so that x_Q is a non-central vertex of Q if Q is a double complete graph. For every path $P \in \mathcal{C}_2(F)$, let x_P and y_P be the two end-vertices of P. Define X by

$$X := \bigcup_{Q \in \mathcal{C}_1(F)} \{x_Q\} \cup \bigcup_{P \in \mathcal{C}_2(F)} \{x_P, y_P\}.$$

Then $|X| = f(F) = k+1$ by Lemma 11.13. Claim 11.3 and Lemma 11.11 yield the next two claims.

Claim 11.4 *X is an independent set of G.*

Claim 11.5 *For every component $Q \in \mathcal{C}_1(F)$ of F and the vertex $\{x_Q\} = X \cap V(Q)$, it follows that*

$$\sum_{x \in X} d_Q(x) = d_Q(x_Q) \leq |V(Q)| - 1 = |V(Q)| - f(Q).$$

Now we measure the neighborhood of X in a path of $\mathcal{C}_2(F)$.

Claim 11.6 *Let P be a path in $\mathcal{C}_2(F)$. Then for each distinct pair of vertices z, w in $X - \{x_P, y_P\}$, the following statements hold.*
(i) $N_P(z) \cap N_P(w) = \emptyset$.
(ii) $N_P(x_P)^- \cap N_P(y_P) = \emptyset$.
(iii) $N_P(z)^- \cap N_P(y_P) = \emptyset$ *and* $N_P(z)^+ \cap N_P(x_P) = \emptyset$.
(iv) $N_P(z) \cap N_P(x_P) = \emptyset$.

Proof. Let Q and R be the components of F containing z and w, respectively.
(i) Suppose $N_P(z) \cap N_P(w) \neq \emptyset$ and take a vertex $v \in N_P(z) \cap N_P(w)$. Then $v \neq x_P, y_P$ by Claim 11.4. Since $\{z, w, v^-\} \subseteq N_G(v)$ and G is claw-free, $zv^- \in E(G)$ or $wv^- \in E(G)$. By symmetry, we may assume that $zv^- \in E(G)$. If $Q \neq R$,

then replace P, Q, R of F by two hamilton paths Q' and R' in $x_P \overrightarrow{P} v^- z \overrightarrow{Q}$ and $y_P \overleftarrow{P} vw \overrightarrow{R}$, respectively. Then we obtain a new $(k+1)$-extended system F' on $V(F)$. If $f(Q') + f(R') < f(P) + f(Q) + f(R)$, then F' is a k-extended system, which contradicts Lemma 11.13. Thus, $f(Q') + f(R') \geq f(P) + f(Q) + f(R)$. This is possible only if $\{Q', R'\} \subseteq \mathcal{C}_2(F')$ and $\{Q, R\} \subseteq \mathcal{C}_1(F)$. However, this contradicts the condition (F1). If $Q = R$, then Q is a path whose end-vertices are z and w and $x_P \overrightarrow{P} v^- z \overrightarrow{Q} wv \overrightarrow{P} y_P$ is a hamilton path of a graph induced by $V(P) \cup V(Q)$, and by replacing P and Q with this path, we have a k-extended system on $V(F)$, contradicting Lemma 11.13.

(ii) If $N_P(x_P)^- \cap N_P(y_P) \neq \emptyset$, then $G[P]$ has a hamilton cycle, and so G has a k-extended system with vertex set $V(F)$, which contradicts Lemma 11.13.

(iii) By symmetry, it suffices to show that $N_P(z)^- \cap N_P(y_P) = \emptyset$. Assume that there exists a vertex $v \in N_P(z)^- \cap N_P(y_P)$. Then $x_P \overrightarrow{P} v y_P \overleftarrow{P} v^+ z \overrightarrow{Q}$ has a hamilton path of $G[V(P) \cup V(Q)]$, and so by replacing P and Q of F with this path, we have a k-extended system on $V(F)$. This contradicts Lemma 11.13.

(iv) Suppose that there exists a vertex v in $N_P(z) \cap N_P(x_P)$. Then $v \neq y_P$ by Claim 11.4. Since $\{v^+, x_p, z\} \subseteq N_G(v)$ and G is claw-free, we have $v^+ z \in E(G)$ by (iii) and Claim 11.4. Suppose that Q is a path of order at least four. If $v \neq x_P^+$, then replace P and Q by the cycle $x_P \overrightarrow{P} v x_P$ and a hamilton path of $y_P \overleftarrow{P} v^+ z \overrightarrow{Q}$. If $v = x_P^+$, replace P and Q with $x_P v$ and a hamilton path of $y_P \overleftarrow{P} v^+ z \overrightarrow{Q}$. In either case, G has a k-extended system on $V(F)$, which contradicts Lemma 11.13.

Next suppose that Q is a cycle. Let us denote the two vertices of Q adjacent to z by z^- and z^+. Then since $\{v, z^-, z^+\} \subseteq N_G(z)$ and G is claw-free, we may assume that $z^- v \in E(G)$ or $z^- z^+ \in E(G)$ by symmetry. If $z^- v \in E(G)$, then $x_P \overrightarrow{P} v z^- \overleftarrow{Q} z v^+ \overrightarrow{P} y_P$ has a hamilton path, and by replacing P and Q with this path, we again have a k-extended system on $V(F)$, a contradiction. Therefore we may assume that $z^- z^+ \in E(G)$. If the order of Q is at least four, replace P and Q with the path $P' = x_P \overrightarrow{P} v z v^+ \overrightarrow{P} y_P$ and the cycle $z^- \overleftarrow{Q} z^+ z^-$. If the order of Q is three, replace P and Q with the path P' and $z^- z^+$. Then in either case, we obtain a maximal $(k+1)$-extended system with $\sum_{P \in \mathcal{C}_2(F')} |V(P)| > \sum_{P \in \mathcal{C}_2(F)} |V(P)|$. This contradicts the condition (F1).

We finally consider the case that Q is K_1, K_2 or a double complete graph. In this case, consider $Q - z$ and the path $P' = x_P \overrightarrow{P} v z v^+ \overrightarrow{P} y_P$. Note that $Q - z$ is empty, K_1, K_2, a double complete graph or a complete graph of order at least three. In the last case, $Q - z$ has a hamilton cycle. Therefore, by replacing P and Q with P' and a certain subgraph of $Q - z$, we obtain a maximal $(k+1)$-extended system F' with $\sum_{P \in \mathcal{C}_2(F')} |V(P)| > \sum_{P \in \mathcal{C}_2(F)} |V(P)|$. This contradicts the choice (F1) of

F. $\square$

Claim 11.7 *For each $P \in \mathcal{C}_2(F)$,*

$$\sum_{x \in X} d_P(x) \leq |V(P)| - f(P).$$

Proof. First assume that $N_P(z) = \emptyset$ for every $z \in X - \{x_P, y_P\}$. Let $H = G[V(P)]$. By the condition (F3), for each hamilton path P^* of H,

$$\sum_{Q \in \mathcal{C}_2(F) - \{P\}} \big(d_Q(x_Q) + d_Q(y_Q)\big) + d_H(x_{P^*}) + d_H(y_{P^*}) \quad \geq \quad \sum_{Q \in \mathcal{C}_2(F)} \big(d_Q(x_Q) + d_Q(y_Q)\big),$$

which implies $d_H(x_{P^*}) + d_H(y_{P^*}) \geq d_H(x_P) + d_H(y_P)$. Thus, if $d_H(x_P) + d_H(y_P) \geq |V(H)| - 1$, then by Lemma 11.10, either H has a hamilton cycle or H is a double complete graph. Then whichever occurs, we can replace P with an appropriate subgraph of H to obtain a k-extended system on $V(F)$, which contradicts Lemma 11.13. Therefore,

$$\begin{aligned} \sum_{x \in X} d_P(x) &= d_P(x_P) + d_P(y_P) \\ &= d_H(x_P) + d_H(y_P) \leq |V(H)| - 2 = |V(P)| - f(P). \end{aligned}$$

Next we assume that $N_P(z_1) \neq \emptyset$ for some vertex $z_1 \in X - \{x_P, y_P\}$. Let $v \in N_P(z_1)$, $P_1 = x_P \overrightarrow{P} v^-$ and $P_2 = v^+ \overrightarrow{P} y_P$. Then $|V(P)| = |V(P_1)| + |V(P_2)| + 1$. By Claim 11.6 (i)–(iv), $N_{P_1}(x_P)^-$, $N_{P_1}(y_P)$ and

$$\Big(N_{P_1}(z)^-\Big)_{z \in X - \{x_P, y_P\}}$$

are well-defined and these $k+1$ sets are pairwise disjoint. Moreover, they do not contain v^- by Claim 11.6 (iii). Thus

$$\sum_{z \in X} d_{P_1}(z) \leq |V(P_1)| - 1.$$

By symmetry of P_1 and P_2, we obtain $\sum_{z \in X} d_{P_2}(z) \leq |V(P_2)| - 1$. By Claim 11.6 (i) and (iv), v is not adjacent to any vertex in $X - \{z_1\}$, and so $\sum_{z \in X} |N_G(z) \cap \{v\}| = 1$. By summing these three inequalities, we have

$$\begin{aligned} \sum_{z \in X} d_P(z) &= \sum_{z \in X} d_{P_1}(z) + \sum_{z \in X} d_{P_2}(z) + \sum_{z \in X} \big|N_G(z) \cap \{v\}\big| \\ &\leq |V(P_1)| - 1 + |V(P_2)| - 1 + 1 \\ &= |V(P)| - 2 = |V(P)| - f(P). \qquad \square \end{aligned}$$

We now prove Theorem 11.14. If $N_{G-F}(z) \cap N_{G-F}(w) \neq \emptyset$ for some z, $w \in X$ with $z \neq w$. Let P and Q be the components of F that contain z and w, respectively (possibly $P{=}Q$). Let $a \in N_{G-F}(z) \cap N_{G-F}(w)$. If $P \neq Q$, then since P and Q have hamilton paths which contain z and w as an end-vertex, respectively, $\overrightarrow{P}zaw\overrightarrow{Q}$ contains a hamilton path. By replacing P and Q with this path, we obtain a new $(k+1)$-extended system F' with $V(F') = V(F) \cup \{a\}$. This contradicts the maximality of F. If $P = Q$, then we may assume $z = x_{P_0}$ and $w = y_{P_0}$ for some P_0. Then by replacing P with a cycle $\overrightarrow{P}azw$, we again obtain a $(k+1)$-extended system F' with $V(F') = V(F) \cup \{a\}$, a contradiction. Therefore, we have $N_{G-F}(z) \cap N_{G-F}(w) = \emptyset$ for each distinct pair of vertices z and w in X. Hence

$$\sum_{z \in X} d_{G-F}(z) \leq |V(G)| - |V(F)|.$$

Then by Claims 11.5 and 11.7, we obtain

$$\begin{aligned}
\sum_{z \in X} \deg_G(z) &= \sum_{C \in \mathcal{C}(F)} \sum_{z \in X} d_C(z) + \sum_{z \in X} d_{G-F}(z) \\
&\leq \sum_{C \in \mathcal{C}(F)} \big(|V(C)| - f(C)\big) + |V(G)| - |V(F)| \\
&= |V(F)| - f(F) + |V(G)| - |V(F)| \\
&= n - k - 1.
\end{aligned}$$

This contradicts the condition $\sigma_{k+1}(G) \geq n - k$, and Theorem 11.14 follows. $\square$

Chapter 12

Spanning trees with bounded number of branch vertices

In this chapter, we discuss a spanning tree with few branch vertices. A vertex of a tree is called a *branch vertex* if the degree of it is at least three. Since a hamilton path is a spanning tree with no branch vertices, a spanning tree with bounded number of branch vertices can be considered as a generalization of a hamilton path. In particular, we consider an independence number condition and a degree sum condition for the existence of such a spanning tree. In Sections 12.1 we show some results on a general graph and on a bipartite graph, and in Section 12.2, we consider some results on a claw-free graph.

The contents of this chapter are based on the paper [122] "A spanning trees with bounded number of branches in a claw-free graph," jointwork with H. Matsuda and T. Yamashita.

12.1 On general graphs and bipartite graphs

Let T be a tree. For $v \in V(T)$, v is called a *branch vertex in* T if $d_T(v) \geq 3$. In this chapter, we consider a spanning tree with small number of branch vertices. Note that a hamilton path is a spanning tree with no branch vertices. In this sense, it is an extended property of a hamilton path. One of the interest in the existence of a spanning tree with bounded branch vertices arises in the realm of multicasting in optical networks; see [71, 72, 73].

A *spider* is a tree with at most one branch vertex. A unique branch vertex of a spider is called a *center* if such a vertex exists; otherwise the spider must be a path, so we can regard any vertex other than leaf as a center. For the existence of a spanning spider with specified center, Flandrin, Kaiser, Kužel, Li and Ryjáček proved the following result, and Gargano and Hammar considered the bipartite graph case.

Theorem 12.1 (Flandrin et al. [60]) *Let G be a connected graph of order n and let $x \in V(G)$. If $d_G(y) + d_G(z) \geq n - 1$ for any independent vertices x, y, z, then G has a spanning spider such that x is a center of it.*

Theorem 12.2 (Gargano and Hammar [71]) *Let G be a bipartite graph with bipartition (X, Y) and $|X| \geq |Y|$. If $d_G(x) + d_G(y) \geq |X|$ and $d_G(y) \geq \frac{|X||Y|}{|X|+|Y|}$ for any $x \in X$ and $y \in Y$, then for any vertex $u \in X$, G has a spanning spider with the center u.*

When we do not specify a center of a spanning spider, Gargano, Hammar, Hell, Stacho and Vaccaro gave a $\sigma_3(G)$ condition for the existence of a spanning spider. Gargano and Hammer considered a degree condition for bipartite graphs.

Theorem 12.3 (Gargano, Hammar, Hell, Stacho and Vaccaro [72]) *Let G be a connected graph of order n. If $\sigma_3(G) \geq n - 1$, then G has a spanning spider.*

Theorem 12.4 (Gargano and Hammar [71]) *Let G be a bipartite graph with partition (X, Y) and $|X| \geq |Y|$. If $d_G(x) + d_G(y) \geq |Y|$ for any $x \in X$ and $y \in Y$, then G has a spanning spider.*

Considering the spider case as in Theorem 12.3, Gargano, Hammar, Hell, Stacho and Vaccaro [72] posed the following conjecture.

Conjecture 12.5 (Gargano, Hammar, Hell, Stacho and Vaccaro [72]) *Let k be a non-negative integer and let G be a connected graph of order n. If $\sigma_{k+2}(G) \geq n - 1$, then G has a spanning tree with at most k branch vertices.*

However, it is not known whether the degree sum condition of Conjecture 12.5 is sharp or not even when it is true. In fact, the following stronger statement may hold.

Conjecture 12.6 *Let k be a non-negative integer and let G be a connected graph of order n. If $\sigma_{k+3}(G) \geq n - k$, then G has a spanning tree with at most k branch vertices.*

Note that the degree sum condition of Conjecture 12.6 is best possible if it is true. Let $P = x_0 x_1 \dots x_k$ be a path and let G_1 be the graph obtained from P by joining $(k + 3)$ complete graphs to vertices of P as follows; joining two complete graphs K_m to x_0 and x_k, and one complete graph to x_i for any $1 \leq i \leq k - 1$, respectively. (See Figure 12.1.) Note that $\sigma_{k+1}(G_1) = |V(G_1)| - k - 1$. Since $(k + 1)$ vertices $x_0, x_1, \dots, x_k$ have to be branch vertices of any spanning trees, G has no spanning tree with at most k branch vertices. Thus, the condition "$\sigma_{k+3}(G) \geq n - k$" is best possible.

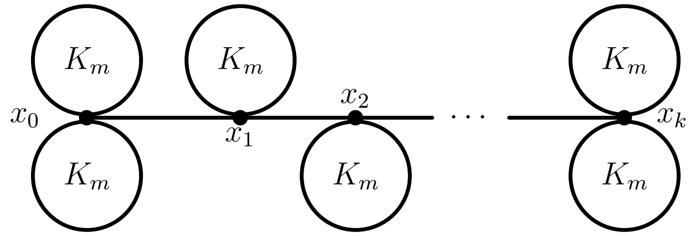


Figure 12.1: The graph G_1.

12.2 On claw-free graphs

12.2.1 Results and Conjecture

Matthews and Sumner [123] showed that for a claw free graph G of order n, if $\sigma_3(G) \geq n-2$, then G has a hamilton path. Gargano, Hammar, Hell, Stacho and Vaccaro extended this result for the existence of a spanning tree with bounded branch vertices.

Theorem 12.7 (Gargano, Hammar, Hell, Stacho and Vaccaro [72]) *Let k be a non-negative integer and let G be a connected claw-free graph of order n. If $\sigma_{k+3}(G) \geq n-k-2$, then G has a spanning tree with at most k branch vertices.*

Corollary 12.8 *Let G be a connected claw-free graph of order n. If $\sigma_4(G) \geq n-3$, then G has a spanning spider.*

However, it is not known whether the degree sum condition of Theorem 12.7 is sharp or not. In fact, we propose the following stronger conjecture.

Conjecture 12.9 *Let k be a non-negative integer and let G be a connected claw-free graph of order n. If $\sigma_{2k+3}(G) \geq n-2$, then G has a spanning tree with at most k branch vertices.*

The degree sum condition of Conjecture 12.9 is best possible if it is true. Let $T_1, T_2, \ldots, T_k$ be k triangles with $T_i = x_i y_i z_i$ for $0 \leq i \leq k$. The graph G_2 is obtained from $T_1 \cup T_2 \cup \cdots \cup T_k$ by adding k edges $z_i x_{i+1}$ for $0 \leq i \leq k-1$ and by joining $(k+3)$ complete graphs K_m to each vertices x_0, y_0, y_1, $\ldots$,y_k and z_k, respectively. (See Figure 12.2.) Note that $(k+3)$ vertices in each added complete graph and k vertices $x_1, x_2 \ldots, x_k$ constitute an independent set and the degree sum of these $(2k+3)$ vertices is $|V(G_2)| - 3$. Note also that $\sigma_{2k+3}(G_2) = |V(G_2)| - 3$. Since at least one vertex in each triangle T_i with $1 \leq i \leq k$ has to be a branch vertex of any spanning trees, G has no spanning tree with at most k branch vertices. Thus, the condition "$\sigma_{2k+3}(G) \geq n-2$" is best possible if Conjecture 12.9 is true.

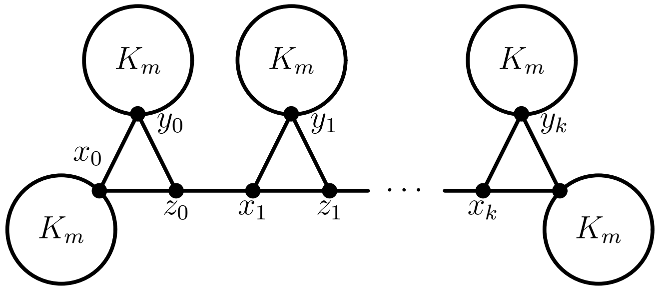


Figure 12.2: The graph G_2.

Conjecture 12.9 still open. But we know that when we cannot take $(2k+3)$ pairwise nonadjacent vertices from a graph G, that is, when $\sigma_{2k+3}(G) = +\infty$, this conjecture holds.

Theorem 12.10 ([122]) *Let k be a positive integer and let G be a connected claw-free graph. If $\alpha(G) \leq 2k+2$, then there exists a spanning tree with at most k branch vertices.*

Note that the graph G_2 also shows the sharpness of the independence number condition of 12.10. In Section 12.2.2, we prove Theorem 12.10.

12.2.2 Proof of Theorem 12.10

We denote by $L(T)$ the set of leaves of T. For $u, v \in V(T)$, the unique path in T connecting u and v is denoted by uTv. We define

$$\begin{aligned} B(T) &:= \{v \in V(T) : v \text{ is a branch vertex in } T\}, \\ B_i(T) &:= \{v \in B(T) : d_T(v) = i\}, \\ \text{and} \quad B_{\geq i}(T) &:= \{v \in B(T) : d_T(v) \geq i\}. \end{aligned}$$

Let T be a rooted tree with root r, and let $u \in V(T)$. The *parent* of u is a unique neighbor of u on rTu, and is denoted by u^-; its *children* are its other neighbors. The *ancestors* of u are vertices of $rTu - \{u\}$, and are denoted by $An(u)$.

Proof of Theorem 12.10.

Let G be a graph satisfying the assumption of Theorem 12.10 and having no spanning tree T with $|B(T)| \leq k$. We choose such a spanning tree T of G so that

(T1) $|B(T)|$ is as small as possible,

(T2) $|L(T)|$ is as small as possible; subject to (T1),

(T3) $|B_3(T)|$ is as small as possible; subject to (T2).

Note that $|B(T)| \geq k+1$. By the choice (T2), we have following fact.

Fact 12.1 *$L(T)$ is an independent set.*

If $B_3(T) = \emptyset$, then $d_T(v) \geq 4$ for every $v \in B(T)$ and hence

$$\begin{aligned} |L(T)| &= 2 + \sum_{v \in B(T)} \big(d_T(v) - 2\big) \\ &\geq 2 + 2|B(T)| \\ &\geq 2k+4, \end{aligned}$$

a contradiction. Therefore there exists a vertex r in $B_3(T)$. We consider a spanning tree T as a rooted tree with root r. Let $B_3^* := B_3(T) - \{r\} := \{v_1, v_2, \ldots, v_m\}$, and let $b_i := \mathrm{dist}_T(r, v_i)$ for $i = 1, 2, \ldots, m$. We may assume that $b_1 \leq b_2 \leq \cdots \leq b_m$. We call a sequence $(b_1, b_2, \ldots, b_m)$ *a distance sequence of T*. Note that spanning trees satisfying (T1)–(T3) have the same order of vertices of degree three. We also choose T so that

(T4) the distance sequence $(b_1, b_2, \ldots, b_m)$ of T is as small as possible in lexicographic order; subject to (T1)–(T3).

Claim 12.2 *For every $v \in B_3^*$, there exists an edge between two children of v.*

Let $X := L(T) \cup B_3^*$.

Claim 12.3 *Let $u_1, u_2 \in X$ such that $u_1u_2 \in E(G)$. Then the following statements hold.*

(i) *$u_1 \notin V(rTu_2)$ and $u_2 \notin V(rTu_1)$.*

(ii) *For any $w \in V(u_1Tu_2) - \{u_1, u_2\}$, $d_T(w) = 2$ or $d_T(w) \geq 4$.*

(iii) *Let $w \in V(u_1Tu_2) \cap B_{\geq 4}(T)$, and let $\{w_i\} = N_T(w) \cap V(u_iTw)$ for $i = 1, 2$. Then $N_G(w_i) \cap N_T(w) = \{w_{3-i}\}$ for $i = 1, 2$ and $N_T(w) - \{w_1, w_2\}$ is a clique.*

For $u, v \in X$ such that $uv \in E(G)$, let $f(u, v)$ be a unique vertex in $V(uTv) \cap V(rTu) \cap V(rTv)$. By Claim 12.3 (i) and (ii), $f(u, v) \in B_{\geq 4}(T)$ and $f(u, v) \in An(u) \cap An(v)$.

Claim 12.4 *$G[X]$ has no P_3 as a subgraph.*

Proof. Suppose that there exists a path $u_1u_2u_3$ in $G[X]$. By Claim 12.3 (i), $u_h \notin V(u_iTu_j)$ for $\{h, i, j\} = \{1, 2, 3\}$. Hence let w be a unique vertex in $V(u_1Tu_2) \cap V(u_2Tu_3) \cap V(u_3Tu_1)$, and let $\{w_i\} = N_T(w) \cap V(wTu_i)$ for $i = 1, 2, 3$. Note that $w_i \neq w_j$ for $i, j \in \{1, 2, 3\}$. By applying Claim 12.3 (iii) to u_1 and u_2, we have $w_1w_2 \in E(G)$ and $w_2w_3 \notin E(G)$. Moreover, by applying Claim 12.3 (iii) to u_2 and u_3, we obtain $w_2w_3 \in E(G)$, a contradiction. $\square$

Claim 12.5 *Let $u_1, u_2, u_3, u_4 \in X$ with $\{u_1, u_2\} \neq \{u_3, u_4\}$. If $u_1u_2 \in E(G)$ and $u_3u_4 \in E(G)$, then $f(u_1, u_2) \neq f(u_3, u_4)$.*

Proof. Suppose that $f(u_1, u_2) = f(u_3, u_4) = z$. By Claim 12.4, $\{u_1, u_2\} \cap \{u_3, u_4\} = \emptyset$. By Claim 12.3 (i), let w be a unique vertex in $V(u_1Tu_2) \cap V(u_2Tu_3) \cap V(u_3Tu_1)$. Without loss of generality, we may assume that $w \in V(zTu_1)$. Let $\{w_i\} = N_T(w) \cap V(wTu_i)$ for $i = 1, 2, 3$. By applying Claim 12.3 (iii) to u_1 and u_2, we obtain $N_G(w_i) \cap N_T(w) = \{w_{3-i}\}$ for $i = 1, 2$.

By Claim 12.3 (ii), we have $w \neq u_4$. Let $\{w_4\} = N_T(w) \cap V(wTu_4)$. Since $z \in V(u_3Tu_4)$, it follows that $w \in V(u_3Tu_4)$, and hence $w_3 \neq w_4$. By applying Claim 12.3 (iii) to u_3 and u_4, we obtain $N_G(w_i) \cap N_T(w) = \{w_{7-i}\}$ for $i = 3, 4$. Therefore $w_1 \neq w_4$ and $w_2 \neq w_4$, and moreover we have $w = z$. Since $z \neq r$ and $z \in An(u_i)$ for $i = 1, 2, 3, 4$, there exists w^- and $w^- \neq w_i$. Hence $w^- \notin N_G(w_1) \cup N_G(w_3)$. Thus, $\{w, w^-, w_1, w_3\}$ is an induced claw, a contradiction. $\square$

Let Y be a maximum independent set of X. By the maximality of Y and by Claim 12.4, for every $x \in X - Y$, we can find a unique vertex $y \in Y$ such that $xy \in E(G)$, and moreover a unique vertex $f(x, y)$. Therefore, by Claim 12.5, we can define an injection mapping g from $X - Y$ to $B_{\geq 4}(T)$ by $g(x) = f(x, y)$. Hence $|X - Y| = |g(X - Y)| \leq |B_{\geq 4}(T)|$. Let $Z := Y \cup \big(g(X - Y) \cap B_4(T)\big)$.

Claim 12.6 $|Z| \geq 2k + 3$.

Proof. Since T is a tree,

$$\begin{aligned} |L(T)| &= 2 + \sum_{v \in B(T)} \big(d_T(v) - 2\big) \\ &\geq 2 + |B_3(T)| + 2|B_4(T)| + 3|B_{\geq 5}(T)|, \end{aligned}$$

and hence

$$\begin{aligned} |Z| &= |Y| + |g(X - Y) \cap B_4(T)| \\ &\geq |Y| + |X - Y| - |B_{\geq 5}(T)| \\ &= |L(T)| + |B_3(T)| - 1 - |B_{\geq 5}(T)| \\ &\geq 1 + 2\big(|B_3(T)| + |B_4(T)| + |B_{\geq 5}(T)|\big) \\ &= 1 + 2|B(T)| \\ &\geq 2k + 3. \qquad \square \end{aligned}$$

Claim 12.7 *For any $u_1 \in Y$ and for any $w \in g(X - Y) \cap B_4(T)$, $u_1w \notin E(G)$.*

Proof. Suppose that $u_1w \in E(G)$ for some $u_1 \in Y$ and for some $w \in g(X - Y) \cap B_4(T)$. Let $w = f(u_2, u_3)$ and let $\{w_i\} = N_T(w) \cap V(u_iTw)$ for $i = 1, 2, 3$. By Claim 12.3 (iii), $w_2w_3 \in E(G)$.

We prove that $u_2, u_3 \notin An(u_1) \cup \{u_1\}$. Assume that $u_2 \in An(u_1) \cup \{u_1\}$. Then $T' := T - \{ww_2, ww_3\} \cup \{wu_1, w_2w_3\}$ is a spanning tree with $|B(T)| = |B(T')|$, and $d_{T'}(w) = 3$. Note that $u_1 \in Y \subseteq L(T) \cup B_3^*$. If $u_1 \in L(T)$, then $L(T') = L(T) - \{u_1\}$, which contradicts (T2). Therefore we have $u_1 \in B_3^*$, and hence $|L(T)| = |L(T')|$, $|B_3(T)| = |B_3(T')|$ and $d_{T'}(u_1) = 4$. Since $u_1 \in B_3^*$ and $\mathrm{dist}_T(r, w) < \mathrm{dist}_T(r, u_1)$, there exists an integer l such that $b_{l-1} \leq \mathrm{dist}_T(r, w) < b_l$. Let $(b'_1, b'_2, \ldots, b'_m)$ be the distance sequence of T'. By the definition of T', $b'_i = b_i$ for every $1 \leq i \leq l-1$ and $b'_l = \mathrm{dist}_{T'}(r, w) = \mathrm{dist}_T(r, w) < b_l$, contradicting (T4). Therefore $u_2 \notin An(u_1) \cup \{u_1\}$, and by symmetry, $u_3 \notin An(u_1) \cup \{u_1\}$.

Let z be a unique vertex in $V(u_1Tu_2) \cap V(u_1Tu_3) \cap V(u_2Tu_3)$ and let $\{z_i\} = N_T(z) \cap u_iTz$ for $i = 1, 2, 3$. By Claim 12.3 (ii), $u_1 \notin V(u_2Tu_3)$. Therefore we have $z \neq u_1$. By Claim 12.3 (ii), we know $z \in B_{\geq 4}(T)$. Hence Claim 12.3 (iii) implies $z_2z_3 \in E(G)$.

Suppose that $d_T(z) = 4$ or 5. Let $z_4 \in N_T(z) - \{z_1, z_2, z_3\}$. By Claim 12.3 (iii), $z_1z_4 \in E(G)$. Then $T' = T - \{zz_1, zz_3, zz_4\} \cup \{u_1w, z_1z_4, u_2u_3\}$ is a spanning tree with $B(T') = B(T) - \{z\}$, a contradiction.

Suppose that $d_T(z) \geq 6$. Then $T' = T - \{zz_2, zz_3\} \cup \{z_2z_3, u_1w\}$ is a spanning tree with $B(T') = B(T)$, since $d_{T'}(z) = d_T(z) - 2 \geq 4$. If $u_1 \in L(T)$, then $L(T') = L(T) - \{u_1\}$, contradicting (T2). If $u_1 \in B_3^*$, then $L(T') = L(T)$ and $B_3(T') = B_3(T') - \{u_1\}$, contradicting (T3). This completes the proof of Claim 12.7. $\square$

Claim 12.8 *For any $w, z \in g(X - Y) \cap B_4(T)$, $wz \notin E(G)$.*

Proof S. uppose that $wz \in E(G)$ for some $w, z \in g(X - Y) \cap B_4(T)$. Let $w = f(u_1, u_2)$ and $z = f(u_3, u_4)$. By Claim 12.5, we have $w \neq z$.

Since $w \notin An(z)$ or $z \notin An(w)$, without loss of generality, we may assume that $w \notin An(z)$, that is, $w^- \in V(zTw)$. Let $\{w_i\} = N_T(w) \cap V(wTu_i)$ for $i = 1, 2$ and let $w_3 = N_T(w) - \{w^-, w_1, w_2\}$. By Claim 12.3 (iii), $w_1w_2 \in E(G)$ and $w^-w_3 \in E(G)$. Then $T' := T - \{ww_1, ww^-, ww_3\} \cup \{zw, w^-w_3, u_1u_2\}$ is a spanning tree with $B(T') = B(T) - \{w\}$, a contradiction. $\square$

Since Y is independent, it follows from Claims 12.6–12.8 that Z is an independent set of order at least $2k + 3$. This contradicts the assumption, and completes the proof of Theorem 12.10. $\square$

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