Transformations and linkages in triangulations on surfaces

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Preface

This thesis is written on the subject "Transformations and linkages in triangulations on surfaces" and is to be submitted for the degree of Doctor of Science at Keio University.

The basis of this thesis is formed by papers written during these seven years. After an introductory chapter, the reader will find five chapters. General terminology can be found in Chapter 1.

This thesis consists of two parts. In the first part, I will present my work about diagonal flips in triangulations on surfaces. In Chapter 2, we have decreased the number of diagonal flips needed to transform one spherical triangulation into the other with the same number of vertices. In Chapter 3, we have enhanced this result to the projective plane. We show that $O(n)$ diagonal flips are sufficent instead of $O(n^2)$ in the classical result.

In the second part, I will present my work about (m, n)-linked graphs on the surfaces. In Chapter 4, we give a necessary and sufficient condition for a planar graph to be $(3,3)$-linked. In Chapter 5, we present a sufficient condition that for a graph on a surface to be (k,k)-linked for $k = 3,4,5$.

Papers Underlying the thesis

[a] A. Nakamoto, K. Ota, R. Mori, Diagonal flips in Hamiltonian triangulations on the sphere, *Graphs Combinatorics.* **19** (2003) 413–418.

[b] A. Nakamoto, R. Mori, Diagonal flips in Hamiltonian triangulations on the projective plane, *Discrete Math.* **303** (2005) 142–153.

[c] R. Mori, (3,3)-Linked Planar graphs, *Discrete Math.* **308** (2008) 5280–5283.

[d] R. Mori, (k, k)-Linked triangulation on surface, submitted to *J. Graph Theory.*

Acknowledgment

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Contents

Introduction

A *triangulation* on a surface is a simple graph embedded on the surface such that each face is bounded by a 3-cycle. In this thesis, we study transformations and linkages in triangulations on surfaces.

A *diagonal flip* is an operation which replaces an edge e in the quadrilateral D formed by two faces sharing e with another diagonal of D. A diagonal flip can be applied only if the resulting graph is simple.

Wagner proved that any two spherical triangulations with the same number of vertices can be transformed into each other by a sequence of diagonal flips, up to isomorphism [16]. For the torus, the projective plane and the Klein bottle, Dewdney [4], Negami and Watanabe [10] proved the same facts. Moreover, Negami [12] proved that for any surface F^2, there exists an integer $N(F^2)$ such that any two triangulations G and G' on F^2 can be transformed into each other if $|V(G)| = |V(G')| \geq N(F^2)$. This result is the origin of a big stream of the researches concerning with diagonal flips in triangulations [13]. But there are only a few results on the number of diagonal flips. Let us consider the minimum number of diagonal flips needed to transformed one triangulation into the other.

From Wagner's proof, we can obtain the fact any two spherical triangulations with n vertices can be transformed into each other by at most $O(n^2)$ diagonal flips. However, Komuro [7] proved that $8n - 48$ diagonal flips are sufficient. We shall improve this result, focusing on a Hamilton cycle. Suppose that a spherical triangulation G has a Hamilton cycle C. Observe that G can decomposed into two spanning maximal outerplane graphs sharing C, and that each of the two maximal outerplane graphs can be transformed into our standard form by at most $\max\{n-5, 0\}$ diagonal flips. Since we can prove that these procedures in the two graphs can be done in G independently, we can prove the following theorem, preserving C.

THEOREM 1 *Any two Hamiltonian triangulations on the sphere with n vertices can be transformed into each other by at most max$\{4n-20, 0\}$ diagonal flips, preserving the existence of Hamilton cycles.*

How can we transform a given triangulation on the sphere into one with a Hamilton cycle? Tutte [15] has given a nice sufficient condition for plane graphs to have Hamilton cycles as in the following theorem.

Theorem 2 *Every 4-connected plane graph has a Hamilton cycle.*

In view of Theorem 2, let us estimate the number of diagonal flips needed to transform a given triangulation G into a 4-connected one. Since every 3-cut in G lies on a separating 3-cycle in G, we want to break all 3-cycles by applying diagonal flips. Since we can prove that every triangulation G on the sphere has at most $n-4$ separating 3-cycles and that each separating 3-cycle can be broken by a single diagonal flip without creating a new 3-cut, we need at most $n-4$ diagonal flips to transform G into a 4-connected one, and hence we have the following theorem by combining this and Theorem 1.

Theorem 4 *Any two triangulations on the sphere with n vertices can be transformed into each other by at most max$\{6n-30, 0\}$ diagonal flips, up to isomorphism.*

We would like to extend this result to the projective plane. Similarly to the spherical case, we observe that a *contractible* Hamilton cycle C in a triangulation G on the projective plane decomposes G into a maximal outer plane triangulation, and a triangulation on the Möbius band all of whose vertices lie on the boundary, which is called a *Catalan triangulation*. Edelman and Reiner [5] enumerated the Catalan triangulations on the Möbius band with n vertices, and it was proved that any two of them can be transformed into each other by diagonal flips, but the number of diagonal flips had never been estimated in their paper. In Chapter 3, we estimated how many diagonal flips suffice to transform any two Catalan triangulations on the Möbius band. Our theorem is the following.

Theorem 13 *Let G and G' be two triangulations on the projective plane with n vertices, each of which has a contractible Hamilton cycle. Then G and G' can be transformed into each other by at most $6n-12$ diagonal flips, preserving their Hamilton cycles.*

Since Thomas and Yu [14] have proved that every 4-connected graph on the projective plane has a contractible Hamilton cycle and we can prove that a projective planar triangulation with n vertices has at most $n-6$ separating 3-cycles, we can prove the following theorem, which is the first result to give a linear bound for the minimum number of diagonal flips to transform given two triangulations on a non-spherical surface with respect to n.

Theorem 12 *Any two triangulations on the projective plane with n vertices can be transformed into each other by at most $8n-26$ diagonal flips, up to isotopy.*

In the second part, we would like to consider a linkage in triangulations on surfaces. That is, we want to measure how rich linkage can be taken in a given triangulation. In order to do, we define an (m, n)-*linkage* of a graph, as follows. This notion is first defined in [3].

We say that a graph G is (m, n)-*linked* if for any two disjoint subsets R and B of $V(G)$ with $|R| \leq m$ and $|B| \leq n$, there are two disjoint subgraphs G_R and G_B in G containing R and B, respectively. Note that when $m = n = 2$, the $(2, 2)$-linkage is equivalent to the 2-linkage, where a graph G is said to be k-*linked* if for any $2k$ distinct vertices $s_1, \ldots, s_k$, $t_1, \ldots, t_k$ of G, there are k disjoint paths connecting s_i and t_i, for $i = 1, \ldots, k$. In Chapter 4, we prove the following theorem.

THEOREM 23 *Let G be a planar graph with at least six vertices. Then G is $(3, 3)$-linked if and only if G is maximal and 4-connected.*

In the above theorem, we can easily see that the maximality is necessary, since a graph with a non-triangular face is not 2-linked (hence it is not (2,2)-linked neither). So an essential argument to prove Theorem 23 is whether any spherical triangulation is $(3, 3)$-linked.

In Chapter 5, we shall generalize this result to triangulations on other surfaces in terms of the connectivity of the graph and the representativity of the embedding, where the *representativity* of an embedding G is the minimum number of intersecting points of G and any non-contractible simple closed curve on the surface. An essential argument in this generalization is that in a triangulation G, a minimal vertex cut lies on several cycles whose removal disconnects the surface. So, analyzing a relation between a minimal tree containing a specified vertex set S in G and a minimal cut set of G separating the tree, we obtain the following theorem, which also implies the sufficiency of Theorem 23 but whose proof is much shorter.

THEOREM 28 *Let k be a positive integer. Every $(k+1)$-connected $\lfloor \frac{k+4}{2} \rfloor$-representative triangulation on any surface is (k,k)-linked.*

Chapter 1

Foundation

In this chapter, we shall give the foundations of the thesis. That is, we shall present basic terminology and notation of graph theory and topology which will be needed in the following chapters.

1.1 Graphs

A *graph* G consists of a set $V(G)$ of *vertices*, a set of $E(G)$ of *edges*, and a mapping associating to each edge $e \in E(G)$ an unordered pair x and y of vertices called *endpoints* (or simply *ends*) of e. We say that an edge is *incident* with its ends, and that it *joins* its ends. In this case, x and y are called *adjacent* vertices of G. We allow $x = y$, in which the edge is called a *loop*. If at least two edges join x and y, then they are called *multiple*

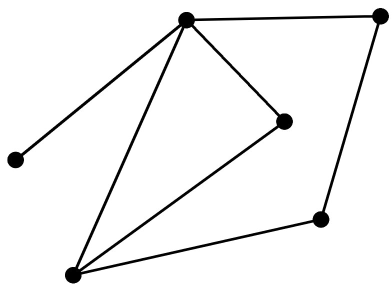

Figure 1.1: Graphs

edges. The *degree* of a vertex x is the number of edges incident with x and is denoted by $\deg_G(x)$. The set of vertices of G adjacent to a vertex $x \in V(G)$ is called the *neighborhood* of x in G and is denoted by $N_G(x)$.

A graph G is said to be *simple* if G has neither loops nor multiple edges, that is, there is no edge joining a vertex with itself and there is at most one edge between each pair of vertices of G. It is clear that for each $x \in V(G)$, $\deg_G(x) = |N_G(x)|$ if G is simple.

Two simple graphs G and G' are said to be *isomorphic* if there is a bijection $\rho : V(G) \to V(G')$ such that for any $x,\ y \in V(G)$, $xy \in E(G)$ if and only if $\rho(x)\rho(y) \in E(G')$. The bijection ρ is called an *isomorphism* between G and G'.

1.2 Subgraphs and operations on graphs

We say that a graph K is a *subgraph* of G if $V(K) \subset V(G)$ and $E(K) \subset E(G)$. In particular, if $V(G) = V(K)$, then K is a *spanning subgraph* of G.

Let G be a graph, let K be a subgraph of G and let S be a nonempty subset of $V(G)$. If $V(K) = S$ and $E(K)$ consists of the edges of G whose ends are both in S, then the subgraph K of G is said to be *induced* by S and is denoted $\langle S \rangle$.

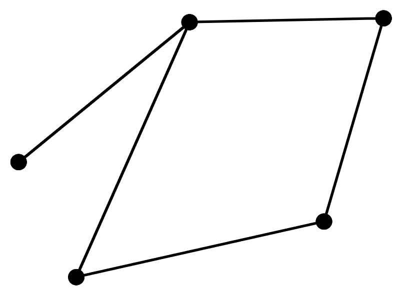

Figure 1.2: A induced subgraph of the graph in Fig: 5.1

We often construct new graphs from old ones by deleting or adding some vertices and edges. For a subset W of $V(G)$, we define $G - W = \langle V(G) - V(W) \rangle$. Similarly, for a subgraph H of G, we define $G - H = \langle V(G) - V(H) \rangle$.

Given an edge xy of a graph G, the graph G/xy is obtained from G by *contracting* the edge xy. To get G/xy, we identify the vertices x and y and remove all resulting loops and multiple edges. A graph obtained by a sequence of edge-contractions is called a *contraction* of G.

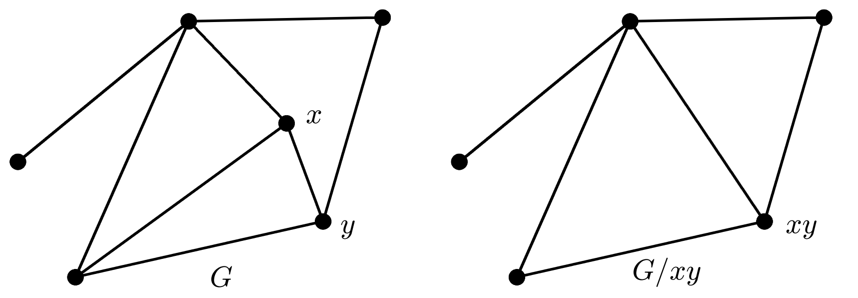

Figure 1.3: A graph G and its contraction G/xy

1.3 Paths and cycles

Let G be a graph and let

$$W := x_1e_1x_2e_2 \dots e_kx_{k+1}$$

where for $x_i \in V(G)$ and $e_i \in E(G)$, each e_i joins x_i and x_{i+1} for $i = 1, 2, \dots, k$. Then the sequence W is called a *walk* in G, and x_1 and x_{k+1} are called the *ends* of W. The number k is called the *length* of W and denoted by $|W|$. If $x_1, \dots, x_{k+1}$ are all distinct, then W is called a *path* in G.

In a walk $W = x_1e_1x_2e_2 \dots e_kx_{k+1}$, if $x_1 = x_{k+1}$, then the walk W is called *closed*. A closed walk W is called a *cycle* if $x_1, \dots, x_k$ are all distinct and $e_1 \dots, e_k$ are all distinct. We call a cycle of length k an k-cycle. A edge

$x_i x_j$ is called a *chord* of C if $x_i x_j \notin E(C)$ and $x_i, x_j \in V(C)$. In particular, if C has no chord then we call it a *chordless* cycle.

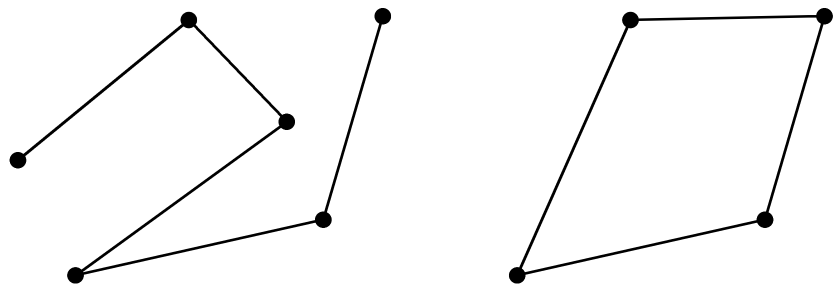

Figure 1.4: A path, A cycle

A cycle containing all vertices of a graph is called a *Hamilton cycle.* A graph G said to be a *Hamiltonian* if it has a Hamilton cycle.

1.4 Connectivity

A graph is *connected* if any two of its vertices can be joined by a path, and otherwise it is *disconnected.* A maximal connected subgraph of G is called a *component* of G. Let G be a connected graph and let S be a subset of $V(G)$. If $G - S$ is disconnected, then S is called *separating.* In particular, if $S - \{x\}$ is not separating for any $x \in S$, then S is called *minimal.*

Let G be a connected graph and let C be a subgraph of G. Let A be a one of the components of $G - C$ and let $x_1, \ldots, x_m \in V(C)$ be the vertices adjacent to vertices of A. Then the connected subgraph consisting of A together with the edges joining A and $\{x_1, \ldots, x_m\}$ is called a C-*bridge* with attachments $x_1, \ldots, x_m$. An edge $xy \in E(G) - E(C)$ with x and y on C is also called a C-bridge with attachments x and y.

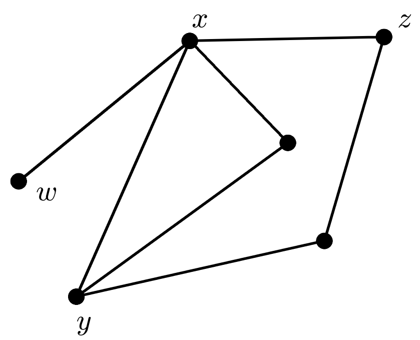

Figure 1.5: S_1-bridge

Let G be a graph shown in Fig 1.5. Let $S_1 = \{x, y, z\}$, $S_2 = \{x\}$ and $S_3 = \{y, z\}$ be subsets of $V(G)$. Then, S_1, S_2, S_3 are *separating*. In particular, S_2, S_3 are minimal.

Moreover, $G - \{w\}$ is one of S_2-bridges with attachment x. Similarly, $\langle\{w, x\}\rangle$ is one of S_1-bridges with attachment x.

1.5 Embedding of graphs into surfaces

Through this thesis, we shall call a connected compact 2-dimensional manifold without boundaries a *surface*.

A *closed curve* on a closed surface F^2 is a continuous function $\ell : S^1 \to F^2$ or its image, where S^1 is the 1-dimensional sphere, that is, $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. A closed curve ℓ is called *simple* if the function ℓ is an injection.

When we discuss embeddings of graphs into surfaces, we regard graphs as 1-dimensional topological spaces, not only as combinatorial objects. Roughly speaking, to *embed* a graph into a surface F^2 is to draw the graph on F^2 without crossing edges. Sometimes, it is effective to regard an embedding as an injective continuous map $f : G \to F^2$. We deal with G and $f(G)$ as the same object intuitively. However, to distinguish G from the embedded one $f(G)$, we often call G an *abstract graph* while we call $f(G)$ an *embedding*. In this thesis, we often denote an embedded graph by G. When G is embedded in a closed surface F^2, then G can be regarded as a subset of F^2. Each component of $F^2 - G$ is called a *face* of G embedded in F^2. A closed walk W of G which bounds a face F of G is called the *boundary walk* of F. An embedded graph G is said to be a *2-cell embedding*, or G is said to be *2-cell embedded* in F^2 if each face of G is homeomorphic to an *open 2-cell*, that is, $\{(x, y) \in R^2 | x^2 + y^2 < 1\}$. For a graph G embedded on a closed surface F^2, we denote the face set of G by $F(G)$, and denote the vertex set and edge sets of G by $V(G)$ and $E(G)$ respectively, as for abstract graphs. A graph

G is said to be *planar* if G is embeddable into the plane. If a graph G is a connected plane graph with all vertices lying on the boundary of its outer face, then we called an *outerplane* graph. Especially, if we cannot adding the edge preserving the condition of outerplane, then we called a *maximal outerplane*.

Let G_1 and G_2 be two graphs embedded in closed surfaces F_1^2 andF_2^2, respectively. Two graphs G_1 and G_2 are said to be *homeomorphic* to each other if there exists a homeomorphism $h : F_1^2 \to F_2^2$ with $h(G_1) = G_2$ which induce an isomorphism from G_1 to G_2. In this case, we also say that $G_1 \subset F_1^2$ and $G_2 \subset F_2^2$ are the same ones *up to homeomorphism*.

We say that a simple closed curve J on F^2 is *trivial* if J bounds a 2-cell on F^2, and *essential* otherwise. We apply these definitions to cycles of G by regarding them as simple closed curves on F^2. The *representativity* of a graph G on a surface is the minimum number of intersecting points of G and ℓ, where ℓ runs over all essential closed curves on the surface. (For convenience, we define the representativity of a plane graph to be the infinity.) A graph G is said to be *k-representative* if the representativity of G is at least k.

A *triangulation* G of a surface F^2 is a simple graph embedded in F^2 so that each face of G is triangular and so that any two faces of G share at most one edge. So it is easy to see every triangulation on any surface is 3-connected and 3-representative. It is to see that a triangulation G is k-representative if and only if every essential cycle of G has length at least k.

Chapter 2

Diagonal Transformations in Triangulations

In this chapter, we shall study the estimation problem for triangulations. It will be shown that any two Hamiltonian triangulations with n vertices on the sphere with $n \geq 5$ vertices can be transformed into each other by at most $4n - 20$. Moreover, using this result, we shall prove that at most $6n - 30$ diagonal flips are needed for any two triangulations on the sphere with n vertices to transform into each other.

2.1 Classical results

A *triangulation* G on a closed surface F^2 is a simple graph embedded on F^2 so that each face is triangular and any two faces meet along at most one edge. Let abd and bcd be two triangular faces of G which have an edge bd in common. The *diagonal flip* of bd is to replace the diagonal bd with ac in the quadrilateral $abcd$ (See Fiure 2.1). To avoid multiple edges, we do not carry out this diagonal flip, if there is an edge ac in G.

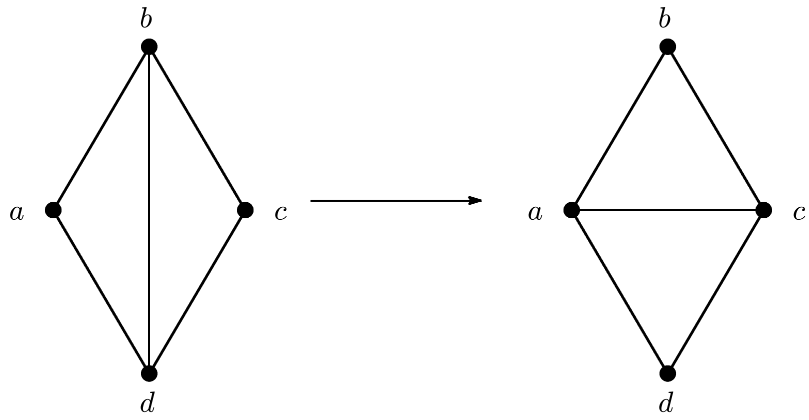

Figure 2.1: Diagonal flip

Classically, Wagner proved in [16] that any two triangulation on the sphere with the same number of vertices can be transformed into each other by a finite sequence of diagonal flips. Also, Dewdney [4], Negami and Watanabe [10] have shown the same result for the torus, the projective plane and the Klein bottle. The same fact does not hold for other surfaces in general, but Negami [12] has shown that there is a positive integer $N = N(F^2)$ for each surface F^2 such that two triangulations G_1 and G_2 can be transformed into each other by a finite sequence of diagonal flips if $|V(G_1)| = |V(G_2)| > N$. Moreover, there are several papers, for example [11], [2] and [1], describe interesting theorems on diagonal flips.

2.2 The minimum number of diagonal flips and the main theorem

From Wagner's proof, we can obtain the fact that any two spherical triangulations with n vertices can be transformed into each other by at most $O(n^2)$ diagonal flips. However, Komuro [7] proved that $8n - 48$ diagonal flips are sufficient, and he has constructed two spherical triangulations with n vertices which need at least $2n - 15$ diagonal flips to transform into each other. In the arguments on diagonal flips in triangulations, the *standard spherical triangulation* with n vertices, denoted by Δ_n, plays an essential role. (See Figure 2.2.) It is isomorphic to $P_{n-2} + K_2$ as a graph.

In this chapter, we focus on Hamiltonian spherical triangulations and consider diagonal flips in those preserving the existence of Hamilton cycles:

Theorem 1 *Any two Hamiltonian triangulations on the sphere with n*

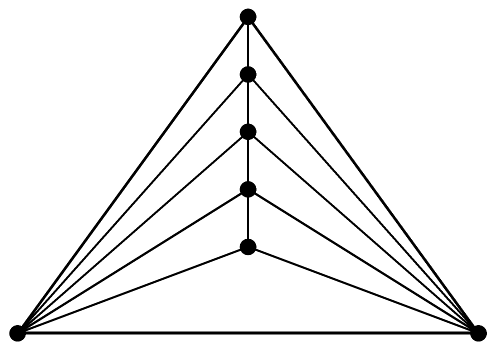

Figure 2.2: Standard spherical triangulation Δ_n

vertices can be transformed into each other by at most max$\{4n - 20, 0\}$ *diagonal flips, preserving the existence of Hamilton cycles.*

Tutte [15] has given a nice sufficient condition for plane graphs to have Hamilton cycles as in the following theorem.

THEOREM 2 (Tutte[15]) *Every 4-connected plane graph has a Hamilton cycle.*

Theorem 2 asserts that the number of diagonal flips needed to transform given two 4-connected spherical triangulations with n vertices is less than or equal to the number given in Theorem 1. Hence the following is obvious.

THEOREM 3 *Any two 4-connected triangulations on the sphere with* n *vertices can be transformed into each other by at most max*$\{4n-20, 0\}$ *diagonal flips, up to isomorphism.*

Note that Theorem 3 does not always guarantee the 4-connecedness of the triangulations appearing in the process of diagonal flips.

Finally, we shall prove the following theorem, estimating the number of diagonal flips to transform a given spherical triangulation into a 4-connected graph. In particular, the estimation in Theorem 4 is better than Komuro's result.

THEOREM 4 *Any two triangulations on the sphere with* n *vertices can be transformed into each other by at most* $max\{6n - 30, 0\}$ *diagonal flips, up to isomorphism.*

2.3 Hamiltonian triangulations on the sphere

We begin with the following lemmas each of which obviously holds.

LEMMA 5 *For* n=4, 5*, there exists only one spherical triangulation with* n *vertices which are* Δ_4 *and* Δ_5*, respectively.* ■

LEMMA 6 *Every maximal outerplane graph has a vertex of degree* 2*.* ■

LEMMA 7 *Every maximal outerplane graph with at least* 5 *vertices has a vertex of degree at least* 4*.* ■

LEMMA 8 *Let G be a maximal outerplane graph with outer cycle C, and let e be any edge not contained in C. Then, e can be switched by a diagonal flip without breaking the simpleness of the graph.*

Proof. Suppose that $e = ac$ is a diagonal of a quadrilateral $abcd$, and it cannot be switched. Then b and d are adjacent in G. In this case, G has a subgraph isomorphic to K_4 with four vertices a, b, c and d. It is well-known that every outerplanar graph cannot include a subdivision of K_4. Therefore, we get a contradiction. ■

Consider the maximal outerplane graph with n vertices isomorphic to $P_{n-1} + K_1$. We call this the *standard maximal outerplane graph* and denote it by Γ_n. The unique vertex of degree $n-1$ of Γ_n is called the *apex*. (See Figure 2.3)

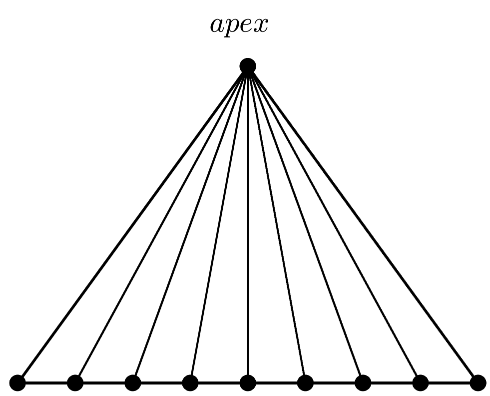

Figure 2.3: Standard maximal outerplane graph Γ_n

Proposition 9 *Let G be a Hamiltonian triangulation on the sphere with n vertices. Then G can be transformed into the standard spherical triangulation Δ_n by at most max$\{2n-10, 0\}$ diagonal flips, up to isomorphism. Moreover, if G is 4-connected, then at most max$\{2n-11, 0\}$ diagonal flips are enough.*

Proof. Let G be a Hamiltonian triangulation on the sphere with a Hamilton cycle C. Suppose that $|V(G)| = n$. By Lemma 5, Thus, we may assume that $n \geq 6$.

Clearly, G can be decomposed into two maximal outerplane graphs G_1 and G_2 such that $G_1 \cap G_2 = C$. By Lemma 6, G_1 has a vertex v of degree 2. Let v_1 and v_2 be the two neighbors of v in G_1.

Now we turn attention into the situation around v in G_2. Since $\deg_G(v) \geq 3$ by the 3-connectedness of G, we also have $\deg_{G_2}(v) \geq 3$. (Here, if G is 4-connected, then we have $\deg_{G_2}(v) \geq 4$.) If there is a triangular face vxy in G_2 with $xy \notin E(C)$, then xy can be switched into vz in the quadrilateral $vxyz$ formed by Lemma 8. Moreover, we have $vz \notin E(G_1)$ since $\deg_{G_1}(v) = 2$. Thus, the diagonal flip replacing xy with vz does not break the simpleness of the whole graph, either. Therefore, G_2 can be transformed into the standard maximal outerplane graph $S_2 \cong \Gamma_n$ with apex v by at most $n-4$ diagonal flips. (If G is 4-connected, then at most $n-5$ diagonal flips are enough.) Let G' be the Hamiltonian plane triangulation obtained from G by the sequence of diagonal flips transforming G_2 into S_2.

Now we consider the subgraph G'_1 of G obtained by removing v. Then G'_1 is $G'_1 = G - \{v\}$. We denote the outer cycle G'_1 by C'. Since no two vertices of G'_1 not adjacent in C' are adjacent in G'. We can freely apply a diagonal flip for any edge not on C', by Lemma 8.

In particular, since G'_1 has at least 5 vertices, G'_1 has a vertex u of degree at least 4, by Lemma 7. We can transform G'_1 into the standard maximal outerplane graph $S_1 \cong \Gamma_{n-1}$ with apex u by at most $n-6$ diagonal flips, since $\deg_{G_1}(u) \geq 4$ and $\deg_{S_1}(u) = n-2$. The resulting whole graph is nothing but the standard spherical triangulation Δ_n. The number of diagonal flips needed is at most $2n-10$. (If G is 4-connected, then at most $2n-11$ diagonal flips are enough.)

Note that no diagonal flips are applied to the edges on the fixed Hamilton cycle C. Hence the existence of Hamilton cycles is always preserved in the process of diagonal flips. Therefore, the proposition follows. ■

Now we shall prove Theorems 1 and 3.

Proof of Theorems 1 and 3. By Proposition 9, any two Hamiltonian triangulations on the sphere with n vertices can be transformed into each other by at most $\max\{4n - 20, 0\}$ diagonal flips, up to isomorphism, by the standard spherical triangulation Δ_n, preserving the existence of Hamilton cycles. Moreover, if they are 4-connected, then at most $\max\{4n - 22, 0\}$ diagonal flips are sufficient. ■

2.4 General spherical triangulations

In this section, we shall prove Theorem 4.

Lemma 10 *A spherical triangulation with n vertices has at most n-4 separating 3-cycles.*

Proof. Let G be a spherical triangulation with n vertices. We proceed by induction on n. In the case when $n = 4$, we have $G = \Delta_4$, by Lemma 5. Since Δ_4 has no separating 3-cycle, the lemma hold, and hence we suppose that $n \geq 5$.

We may assume that G has a separating 3-cycle $C = xyz$. Cutting along C, we can decompose G into two spherical triangulations G_1 and G_2 such that $G_1 \cap G_2 = C$. Let $n_1 = |V(G_1)|$ and $n_2 = |V(G_2)|$, and hence $n = n_1 + n_2 - 3$. By the induction hypothesis, G_1 has at most $n_1 - 4$ separating cycles, and G_2 has at most $n_2 - 4$ separating cycles. Therefore the number of separating cycles in G is at most

$$n_1 - 4 + n_2 - 4 + 1 = (n_1 + n_2 - 3) - 4 = n - 4.$$

Thus the lemma follows for any $n \geq 4$. (The standard spherical triangulation Δ_n attains the equality.)

Lemma 11 *Let G be a spherical triangulation with $n \geq 6$ vertices. Then G can be transformed into a 4-connected one by at most $n-4$ diagonal flips.*

Proof. It is easy to see that every spherical triangulation is 3-connected, and if $\{x, y, z\}$ is a set of vertices such that $G - \{x, y, z\}$, is disconnected, then x, y and z are contained in the same 3-cycle. Since G has at most $n - 4$ separating 3-cycles by Lemma 10, we shall show that G has an edge e such that the diagonal flip of e decreases the number of separating 3-cycles by at least one.

Let $C = xyz$ be a separating 3-cycle in G and $e = xy$. We may suppose that if G has an edge included in at least two separating 3-cycles, then we choose such an edge as e. Let $xayb$ be the quadrilateral formed by two triangular faces sharing e, where a and b lie in the interior and the exterior of C, respectively. Consider the cycle C in G has disappeared.

Now we show that no new separating 3-cycle has arisen in G'. Suppose that G' has a new separating 3-cycle C'. Then C' contains both a and b, since every three vertices separating the graph lies on a 3-cycle. Hence we can put $C' = abc$. Since a was contained in a component of $G - \{x, y, z\}$, we must have $z = c$. Since $|V(G)| \geq 6$, either xza, yza, xzb or yzb is separating. In these cases, xz or zy are included in at least two separating 3-cycles, but xy is contained in only one separating 3-cycle, which is contrary to the choice of e. Thus, no new separating 3-cycle has arisen in G'. ■

Now we shall prove Theorem 4.

Proof of Theorem 4. Let G_1 and G_2 be any two spherical triangulations with n vertices. By Lemma 5, we may assume $n \geq 6$. By Lemmas 10 and 11, for $i = 1, 2$, G_i can be transformed into a 4-connectde triangulation T_i by at most $n - 4$ diagonal flips. By Theorem 3, T_1 and T_2 can be transformed into each other by at most $4n - 22$ diagonal flips. Therefore, at most $6n - 30$ diagonal flips can transform G_1 and G_2 into each other. ■

Chapter 3

Extension to the Projective Plane

In this chapter, we enhanced to the result in Chapter 2 to the projective plane. That is, we shall prove that any two triangulation on projective plane with n vertices can be transformed by a linear number of diagonal flips with respect to n. This is the first result on non-spherical surfaces giving a linear bound for the number of diagonal flips.

3.1 Main theorem

In this chapter, we shall prove the following theorem:

Theorem 12 *Any two triangulations on the projective plane with n vertices can be transformed into each other by at most $8n - 26$ diagonal flips, up to isotopy.*

A cycle C of a graph G is embedded in a closed surface F^2 is said to be *contractible* if C bounds a 2-cell on F^2. In order to prove Theorem 3.1, we show the following theorem for triangulations on the projective plane with a contractible Hamilton cycle, as in the spherical case in Chapter 2.

Theorem 13 *Let G and G' be two triangulations on the projective plane with n vertices each of which has a contractible Hamilton cycle. Then G and G' can be transformed into each other by at most $6n - 12$ diagonal flips, preserving their Hamilton cycles.*

3.2 Triangulations with contractible Hamilton cycles

In the section, we deal only with triangulations which have contractible Hamilton cycles. Clearly, a contractible Hamilton cycle in a triangulation G on the projective plane separates G into two spanning subgraphs of G. One is a maximal outerplane graph, denoted by G_P, and the other is a triangulation of the Möbius band, denoted by G_M, in which all vertices appear on the boundary of the Möbius band. We call it a *Catalan triangulation* on the Möbius band.

Lemma 14 *Let P be a maximal outerplane graph with $n \geq 3$ vertices and let v be a vertex of degree $k \geq 2$ in P. Then P can be transformed into a maximal outerplane graph in which the degree of v is exactly $n-1$, by exactly $n-k-1$ $(\leq n-3)$ diagonal flips, through maximal outer-plane graphs.*

Proof. Let xy be an edge of P not in its outer cycle and let vxy and uxy be two faces sharing xy. Since $\deg_P(v) = k$, the number of vertices not adjacent to v is $n-k-1$. Since P has no subgraph isomorphic to K_4, u and v are not adjacent in P. Therefore, we can flip xy without making multiple edges. Hence we can increase the degree of v one by one, by diagonal flips. Therefore, the lemma follows. ■

In [5], the Catalan triangulations on the Möbius band with n vertices were enumerated and it was proved that any two of them can be transformed into each other by diagonal flips, but the number of diagonal flips had never been estimated yet.

Let M^2 denote the Möbius band and let ∂M^2 denote the boundary of M^2. Let K be a Catalan triangulation on M^2 with m vertices. Let $v_1, v_2, \ldots, v_m$ be the vertices of K lying on ∂M^2 in this cyclic order. An edge v_iv_j is said to be *trivial* if cutting along v_iv_j separates a disk D from M^2. Clearly, the subgraph of K induced by the vertices on D is a maximal outerplane graph, which is said to be *bounded by* v_iv_j. Edges of K which are not trivial are said to be *essential*.

Suppose that a Catalan triangulation K on the Möbius band M^2 has no trivial edge. An essential edge e of K incident to a vertex of degree 3 is called a *spoke*. The subgraph of K induced by the essential edges which are not spokes is said to be the *zigzag frame* of K, which is uniquely taken. It is easy to see that the zigzag frame of K is a cycle of an odd length homotopic to the center line of M^2. Moreover, if K has no trivial edge and no spoke, then K is 4-regular.

Lemma 15 *Let G be a triangulation on the projective plane with $n \geq 7$ vertices. If G has a contractible Hamilton cycle C, then G can be transformed into $K + K_1$ by at most $n-1$ diagonal flips, where K is some Catalan triangulation on the Möbius band.*

Proof. Let G_P and G_M be the maximal outerplane graph and the Catalan triangulation on the Möbius band, each of which is a spanning subgraph of G with boundary C.

We shall make a vertex of degree 2 in G_M by at most three diagonal flips, without breaking the simpleness of G. If G_M has a trivial edge xy, then xy bounds an outerplane graph L. It is easy to see that L has a vertex of degree 2 other than x and y. Thus, we have nothing to do, and hence we may suppose that G_M has no trivial edge.

First, if G_M has no trivial edge and no spoke, then G_M is 4-regular. Since G_P is outerplanar, G_P has a vertex of degree 2 in G_P, say v with two neighbors p and s. Suppose that G_M has faces pqv, qrv and rsv meeting at v, and faces vrs, rts and tus meeting at s in G_M. (See the left-hand of Figure 3.1.) Observe that since $\deg_{G_P}(v) = 2$, any diagonal flip in G_M increasing the degree of v yields no edge forming multiple edges with an edge in G_P. Moreover, since $n \geq 7$, we have $vt, vu \notin E(G_M)$; otherwise, we would have $u = q$ and $p = t$. Therefore rs can be replaced with vt, and next st can be replaced with vu. Now s has degree 2 in the resulting graph on M^2, which is obtained by two diagonal flips. (See the right-hand of Figure 3.1.)

Finally suppose that G_M has spokes but no trivial edges. We first suppose that G_M has two consecutive spokes pq and pr such that q and r are adjacent on C and $\deg_{G_M}(q) = \deg_{G_M}(r) = 3$. Let pqs, pqr and prt be three faces meeting at p. It is easy to see that a diagonal flip can replace an edge pq with sr without making multiple edges in G_M, but G_P might already have an edge sr. In this case, by the planarity of G_P, G does not

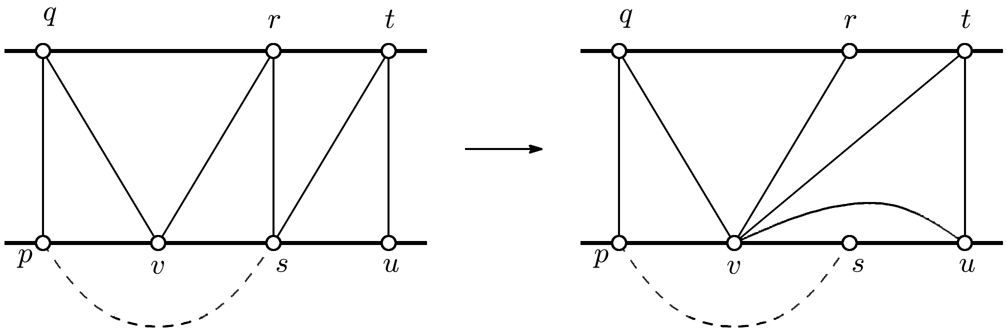

Figure 3.1: Two diagonal flips making a vertex of degree 2.

have an edge qt because of the obstruction of sr. Therefore, we can make r have degree 2 by one diagonal flip.

Now consider the case when the vertices of degree 3 in G_M are independent. Since $n \geq 7$, the zigzag frame of G_M has length at least 5. (For otherwise, i.e, if the zigzag frame has length 3 and all vertices of degree 3 are independent, then we have $n \leq 6$, a contradiction.) Let pq be a spoke with $\deg_{G_M}(q) = 3$ and shared by two faces pqs and pqt. Note that $4 \leq \deg_{G_M}(s), \deg_{G_M}(t) \leq 5$. Apply a diagonal flip of pq to make a vertex of degree 2 in G_M. If impossible, G_P already has an edge st. (Here, if G is assumed to be 4-connected, then this does not happen, because $G - \{p, s, t\}$ must be connected.) If G_P has an edge st, then we can make q have degree 5 or 6 and s have degree 2 by at most three diagonal flips, flipping the edges incident to s in G_M, not on ∂M^2, to make them be incident to q, similarly to the case when G_M is 4-regular. (Note that only the final case requires at most three diagonal flips to make a vertex of degree 2 and it does not happen in the 4-connected case. Hence this proves the following remark.)

We turn our attention to G_P. Let G'_M denote a Catalan triangulation with a vertex v of degree 2 obtained from G_M by at most three diagonal flips. Then we can apply any diagonal flip in G_P increasing the degree of v, without making multiple edges with an edge of G_M. Observe that $\deg_{G_P}(v) \geq 3$, since every vertex of a triangulation on a closed surface has degree at least 3. Therefore, at most $n-4$ diagonal flips can make v have degree $n-1$ in G_p, by Lemma 14. In the resulting graph, v is adjacent to all other vertices, and the graph with v removed is obviously a Catalan triangulation with $n-1$ vertices. ■

As shown in the above proof, we have the following remark.

Remark 16 *In Lemma* 15*, if we assume the* 4*-connectedness of* G*, then the number of diagonal flips can be improved to* $n-2$*.* ■

3.3 General projective planar triangulations

Consider a Catalan triangulation on the Möbius band shown in the left hand of Figure 3.2, which is a unique Catalan triangulation with five vertices isomorphic to K_5. Let $e = v_4v_5$ be an edge of the Catalan triangulation K_5 lying on the boundary of the Möbius band. Subdivide e by m vertices as shown in the right hand of Figure 3.2, where the Möbius band is obtained by identifying the arrows indicated in the left-hand and the right-hand sides of the rectangles. The resulting graph is called the *standard form* of the Catalan triangulations and denoted by Γ_m.

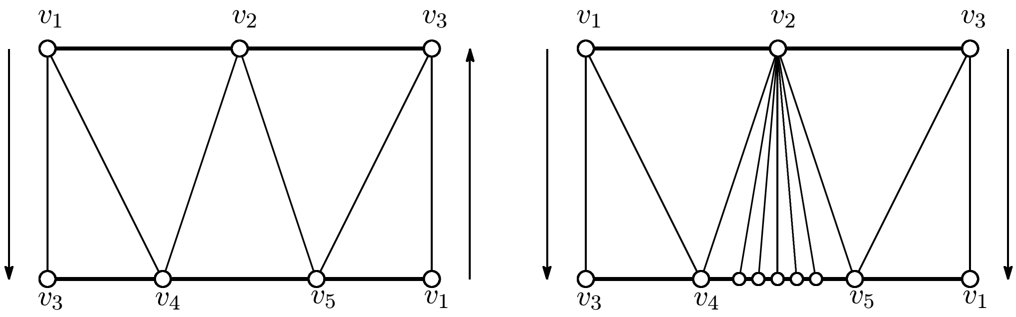

Figure 3.2: K_5 and the standard form Γ_m

The following is the most essential argument in this chapter.

Lemma 17 *Every Catalan triangulation K on the Möbius band with n vertices can be transformed into the standard form Γ_{n-5} by at most $2n-3$ diagonal flips.*

Proof. Suppose that K has p trivial edges. Then it is easy to see that the unique sub-Catalan triangulation, denoted by K', of K with no trivial edges is obtained from K by successively removing a vertex of degree 2. Clearly,

K' has exactly $n - p$ vertices.

Suppose that K' has q spokes and let $r = n - p - q$. Then the zigzag frame $v_1 \cdots v_r$, of K' has an odd length $r \geq 3$. Let q_i be the number of spokes of K' incident to v_i, for $i = 1, \dots, r$. We may suppose that $q_1 + q_3 + \cdots + q_r \geq q_2 + q_4 + \cdots + q_{r-1}$. (For otherwise, we can replace v_i by v_{i-1} for each i, because the subscripts are cyclic and taken modulo r). Let $q_2 + q_4 + \cdots + q_{r-1} = m$ and hence we have $2m \leq q$. (See Figure 3.3.)

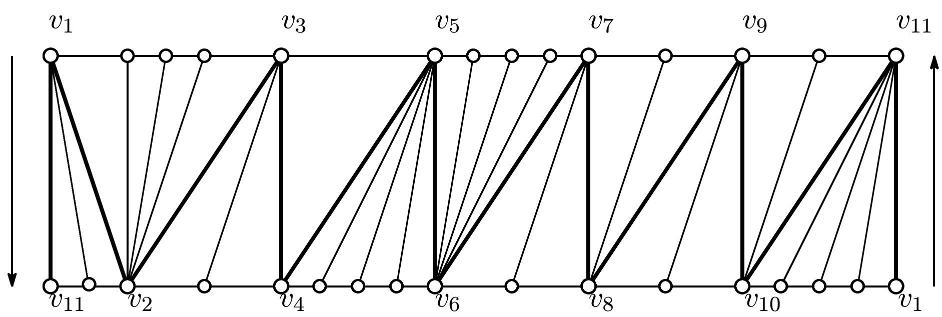

Figure 3.3: K' with zigzag frame $v_1 v_2 \dots v_r$

If $r = 3$, then by the simpleness of graphs, we have $q_2, q_3 \geq 1$. Hence we can flip an edge $v_2 v_3$ to make the zigzag frame have length 5. So, suppose that $r \geq 5$. Apply diagonal flips to all m spokes incident to $v_2, v_4, \dots, v_{r-1}$ to make them trivial one by one. The number of diagonal flips we did is exactly

$$q_2 + q_4 + \cdots + q_{r-1} = m. \tag{3.1}$$

Note that even if $r = 3$, the estimation (1) is true. Though we need one more diagonal flip of v_2v_3 to increase the length of the zigzag frame, this diagonal flip decreases q_2 and q_3 by one, respectively.

Next reduce the length of the zigzag frame from r to 5. In particular, we first apply a diagonal flip of v_4v_5, secondly flip q_5 spokes incident to v_5, and finally flip v_5v_6. (See Figure 3.4(1).) The number of diagonal flips we did is $q_5 + 2$. In the resulting graph, the zigzag frame has length $r - 2$, and exactly one new trivial edge v_3v_7 appears. As far as that the length of the zigzag frame is at least 7, we apply these operations. If its length is exactly 5, then we apply $q_1 + q_r$ diagonal flips, as shown in Figure 3.4(2). Then the total number of diagonal flips we did is

$$\begin{aligned} &(q_5 + 2) + (q_7 + 2) + \cdots + (q_{r-2} + 2) + q_1 + q_r \\ &\qquad \leq (q - m) + 2\left(\frac{r-5}{2}\right). \end{aligned} \tag{3.2}$$

Let H' be the current Catalan triangulation obtained from K'. The zigzag frame of H' has length exactly 5, and all spokes of H' are incident to v_3. Moreover, H' has $\frac{1}{2}(r - 5) + m$ trivial edges, since all m spokes incident to $v_2, v_4, \ldots, v_{r-1}$ in K' are replaced with trivial edges of H', and since decreasing the length of the zigzag frame of K' by two yields exactly one new trivial edge. Let H be the Catalan triangulation consisting of H' and all trivial edges of K. Then H has exactly $p + \frac{1}{2}(r - 5) + m$ trivial edges.

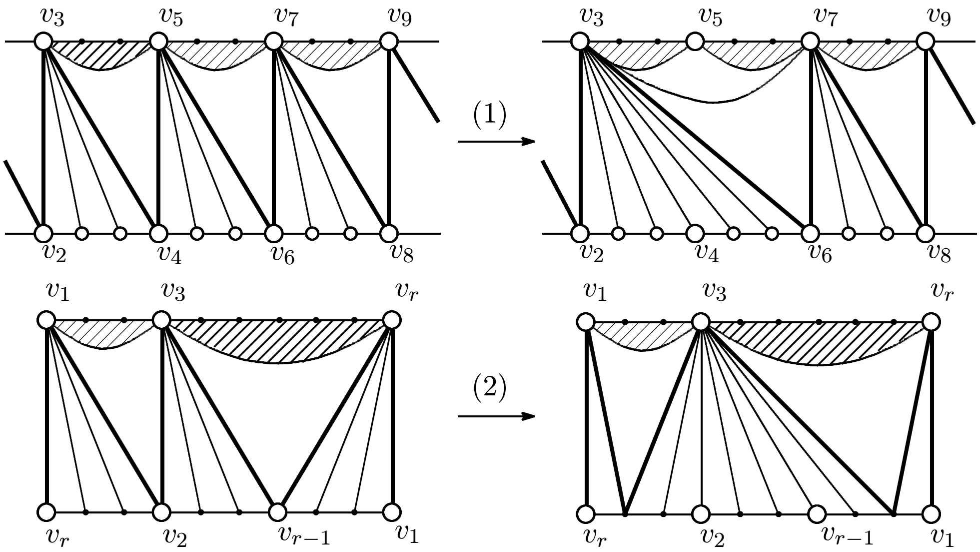

Figure 3.4: Reducing the length of zigzag frame.

Now, renaming vertices, we put H with the zigzag frame $u_1u_2u_3u_4u_5$ as shown in Figure 3.5, where $u_1 = v_1, u_3 = v_3$ and $u_5 = v_r$. The four triangular faces $u_1u_2u_5, u_1u_2u_3, u_3u_4u_5$ and $u_4u_5u_1$ of H come from K'. Let R_i denote the outer-plane graph bounded by an edge $u_{i-1}u_{i+1}$ and containing the edge $u_{i-1}u_{i+1}$, for $i \neq 3$. (Note that R_i might be just an edge.)

The *region* F_i of the zigzag frame of H is the union of the faces bounded by the two edges $u_{i-1}u_i, u_iu_{i+1}$ and the path on ∂M^2 connecting u_{i-1} and u_{i+1}, for each i, where the subscripts are taken modulo 5. Now we shall transform H into a Catalan triangulation in which all the regions of the zigzag frame, except one corresponding to F_3, consists of just one face.

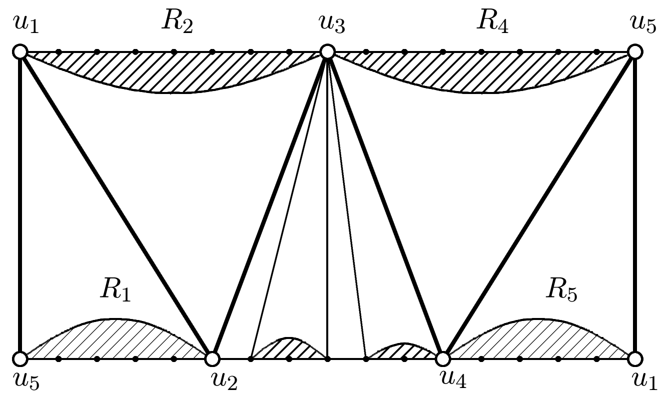

Figure 3.5: The Catalan triangulation H.

Here we focus on the outer-plane graph R_2 and first suppose that $|V(R_2)| \geq 3$. By Lemma 14, we can make u_1 have degree $|V(R_2)| - 1$ by at most $|V(R_2)| - 3$ diagonal flips. Let $u_1, x_1, \ldots, x_l, u_3$ be the vertices of R_2 lying on ∂M^2 in this order. Apply five diagonal flips of u_1u_3, u_1u_2, u_1x_l, u_1u_5 and u_4u_5 in this order, if $l \geq 2$. (See Figure 3.6.) If $l = 1$, then apply three diagonal flips of u_1u_3, u_1u_2, u_4u_5 in this order. In the resulting graph, each of two regions of the zigzag frame corresponding to F_2 and F_5 is just a face. The number of diagonal flips we did is at most $|V(R_2)| - 3 + 5$.

Secondly we suppose that $|V(R_2)| = 2$. If we also have $|V(R_5)| = 2$, then we have nothing to do for F_2 and F_5. So, suppose that $|V(R_5)| \geq 3$. Similarly to the above case, at most $|V(R_5)| - 3$ diagonal flips make u_4 have degree $|V(R_5)| - 1$ in R_5 and we apply two diagonal flips of u_1u_4 and u_4u_5.

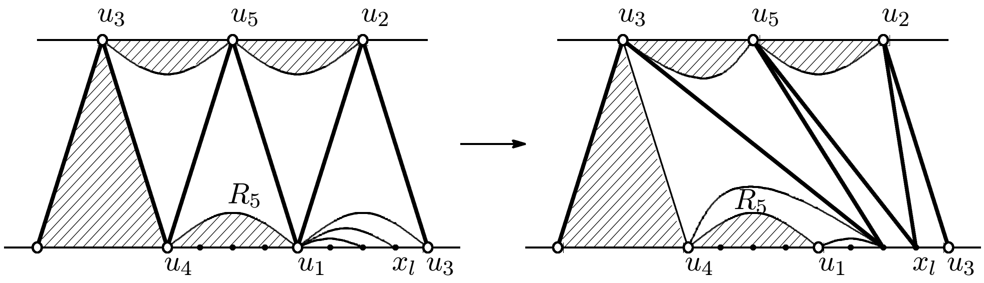

Figure 3.6: Moving vertices of R_4 and R_5.

In the resulting graph, the two regions corresponding to F_2 and F_5 are just faces. Then the number of diagonal flips we did is at most $|V(R_5)| - 3 + 2$. Note that by the above operations, the number of trivial edges decreases by one, if $|V(R_2)| \geq 3$ or $|V(R_5)| \geq 3$.

We can do the same procedures for the regions R_1 and R_4. Let L denote the resulting graph in which exactly four regions are just faces. Hence, the number of diagonal flips transforming H into L is at most

$$\max\{|V(R_2)| + 2, |V(R_5)| - 1\} + \max\{|V(R_4)| + 2, |V(R_1)| - 1\}$$

$$\leq p + \frac{1}{2}(r - 5) + m + 8, \tag{3.3}$$

since

$$(|V(R_1)|-2)+(|V(R_2)|-2)+(|V(R_4)|-2)+(|V(R_5)|-2) \leq p+\frac{1}{2}(r-5)+m.$$

Note that we can assume that the number of trivial edges of L is at most $p + \frac{1}{2}(r - 5) + m - 1$, since we may suppose that at least one of R_1, R_2,

R_4 and R_5 has at least three vertices. (For otherwise, we don't need to add (3.3) to the estimation of the maximum number of diagonal flips, and this case requires a few number of diagonal flips.)

Finally we flip all trivial edges of L, all of which are incident to u_3. Since the number of trivial edges of L is at most $p + \frac{1}{2}(r-5) + m - 1$, the number of diagonal flips transforming L into the standard form is at most

$$p + \frac{1}{2}(r-5) + m - 1 = p + \frac{r}{2} + m - \frac{7}{2}. \tag{3.4}$$

Therefore, by (3.1),(3.2),(3.3) and (3.4), the total number of diagonal flips is at most

$$m + (q - m + r - 5) + \left(p + \frac{r}{2} + m + \frac{11}{2}\right) + \left(p + \frac{r}{2} + m - \frac{7}{2}\right)$$

$$= 2p + q + 2m + 2r - 3 \leq 2(p + q + r) - 3 = 2n - 3,$$

since $q \geq 2m$. Therefore, the lemma follows. ■

3.4 Triangulations on the projective plane with contractible Hamilton cycles

In the previous section, we described only the result on triangulations with contractible Hamilton cycles. In this section, we shall mention how we can obtain triangulations with contractible Hamilton cycles from any triangulations.

The following gives an important sufficient condition for a graph on the projective plane to have a contractible Hamilton cycle.

Lemma 18 (Thomas and Yu [14]) *Every 4-connected graph on the projective plane has a contractible Hamilton cycle.*

The following lemma is essential.

Lemma 19 *Let G be a triangulation on the projective plane with n vertices. Then G can be transformed into a 4-connected triangulation by at most $n-6$ diagonal flips.*

Proof. Observe that a triangulation on the projective plane has no separating 3-cycle if and only if it is 4-connected. We first show that G has at most $n-6$ separating 3-cycles, by induction on n. It is well-known that the smallest triangulation on the projective plane is the unique triangular embedding of K_6, which has no separating 3-cycle. Therefore, the lemma holds when $n = 6$.

When $n \geq 7$, we may assume that G has a separating 3-cycle $C = xyz$, and it is *innermost*, that is, there is no separating 3-cycle in the 2-cell bounded by C. Cutting along C, we can decompose G into a plane triangulation G_1 with no separating 3-cycle and a triangulation G_2 on the projective plane. By induction hypothesis, G_2 has at most $|V(G_2)| - 6$ separating 3-cycles. Let M denote the number of separating 3-cycles in G. Then we have

$$M \leq |V(G_2)| - 6 + 1 = (n - |V(G_1)| + 3) - 5 \leq n - 6,$$

since $|V(G_1)| \geq 4$.

Now we shall show that there is a diagonal flip decreasing the number of separating 3-cycles by at least one. Let $C = xyz$ be a separating 3-cycle in G and $e = xy$. Let $xayb$ be the quadrilateral formed by two triangular faces sharing e, where a lies in the 2-cell region bounded by C. Consider the diagonal flip of e replacing xy with ab. In the resulting graph G', the separating cycle C in G has disappeared.

We shall show that no new separating 3-cycle arises in G', by possibly re-choosing e. Suppose that G' has a new separating 3-cycle C'. Then C' contains both a and b; otherwise, C' would be contained in G. We must have $C' = abz$, where we assume that x is contained in the 2-cell region bounded by C' in G'. This means that $V(G_1) = \{x, y, z, a\}$ since C is innermost in G. In this case, the edge yz can be flipped to destroy a 3-cycle byz and make no new separating 3-cycle, because byz separates a and other vertices outside byz. Therefore, at most $n - 6$ diagonal flips can make the graph be

4-connected. ■

3.5 Proof of theorems

It is well-known that the smallest triangulation on the projective plane is the unique triangular embedding of K_6. Let xy be one of its edges, and suppose that two faces xyz and xyw share xy. Subdivide xy by m vertices $v_1, \dots, v_m$ and add $2m$ edges v_iz, v_iw for $i = 1, \dots, m$. The resulting graph with $m+6$ vertices is called the *standard form* of triangulations on the projective plane and denoted by Ψ_m. (See Figure 3.7.) Clearly, we obtain the standard form Ψ_m from the standard form Σ_{m-1} of Catalan triangulations of the Möbius band M^2 by pasting a disk along ∂M^2, placing a vertex v at its center and joining v to all vertices of Σ_m.

We first prove the following theorem.

THEOREM 20 *Let G be a triangulation on the projective plane with n vertices which has a contractible Hamilton cycle. Then G can be transformed into Ψ_{n-6}, preserving the Hamilton cycle, by at most $3n-6$ diagonal flips. If G is 4-connected, then the number of diagonal flips is improved to $3n-7$.*

Proof. We may suppose that $n \geq 7$. By Lemma 15, G can be transformed into $K + K_1$ by at most $n-1$ diagonal flips, preserving the Hamilton cycle,

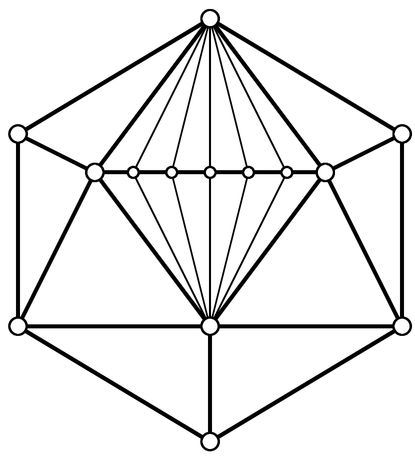

Figure 3.7: The standard form Ψ_m of triangulations on the projective plane.

where K is some Catalan triangulation on the Möbius band with $n-1$ vertices. (By Remark 16, if G is 4-connected, the number "$n-1$" of diagonal flips can be replaced with "$n-2$".)

Note that no two vertices of K are joined by an edge outside K. Therefore, it suffices to prove that K can be transformed into Σ_{n-6}. By Lemma 17, it can be done by at most $2(n-1)-3$ diagonal flips. Therefore, G can be transformed into Ψ_{n-6} by at most $3n-6$ ($3n-7$ when G is 4-connected) diagonal flips, preserving the Hamilton cycle. ■

Theorem 21 *Every triangulation on the projective plane with n vertices can be transformed into the standard form Ψ_{n-6} by at most $4n-13$ diagonal flips, up to isotopy.*

Proof. Let G be a triangulation on the projective plane with n vertices. By Lemma 19, at most $n-6$ diagonal flips transform G into a 4-connected triangulation, denoted by H. By Lemma 18, H has a contractible Hamilton cycle. Then apply Theorem 20. ■

Now we shall prove Theorems 12 and 13.

Proof of Theorems 1 and 2. Theorems 1 and 2 follow from Theorems 10 and 9, respectively, via the standard form Ψ_{n-6}. ■

Proof of Theorems 12 *and* 13*.* Theorems 12 and 13 follow from Theorems 21 and 20, respectively, via the standard form Ψ_{n-6}. ■

Finally we consider two triangulations G_1 and G_2 on the projective plane with n vertices which need many diagonal flips to transform them into each other. Let $G_1 = \Psi_{n-6}$, and let G_2 be a triangulation with maximum degree 6. For example, it is constructed from K_6 by putting a *triangular mesh* shown in Figure 3.8 to each face.

The maximum degree of G_1 is $n-1$ and it is attained by two vertices, say x and y. To transform G_1 into G_2, we have to decrease the degree of x and y to six or five. Since each diagonal flip decreases the degree of a fixed vertex at most by one, each of x and y requires at least $(n-1)-6$ diagonal flips. Observe that the degree of x and y decrease simultaneously by one diagonal

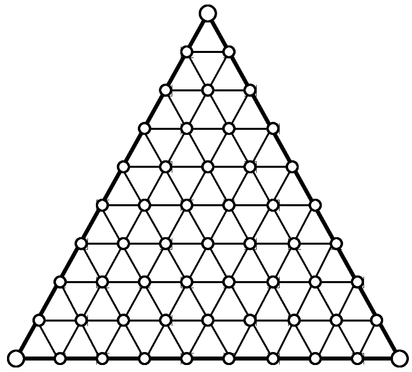

Figure 3.8: A triangular mesh.

flip, only if this diagonal flip is applied to the edge xy. If such diagonal flips are applied at least twice in the process from G_1 to G_2, then there must be a diagonal flip joining x and y, which increases $\deg(x) + \deg(y)$. Therefore, if the number $\deg(x) + \deg(y)$ is non-increasing in the process from G_1 to G_2, then the edge xy is flipped at most once, and the number of diagonal flips transforming G_1 to G_2 is at least

$$(n-1) - 6 + (n-1) - 6 - 1 = 2n - 13.$$

Therefore, the order of our estimation in Theorems 13 and 12 cannot be improved. Therefore, we have the following.

PROPOSITION 22 *For any integer N, there exists a pair of triangulations G_1 and G_2 on the projective plane with $n \geq N$ vertices such that at least $2n - 13$ diagonal flips are needed to transform them into each other.* ■

Chapter 4

(3,3)-Linked graphs on the sphere

In this chapter, we shall study $(3,3)$-linkage of graphs, in particular, we shall prove that a planar graph with at least six vertices is (3,3)-linked if and only if G is 4-connected and maximal.

4.1 Main theorem

A graph is said to be *k-linked* if for any distinct $2k$ vertices $a_1, \ldots, a_k$, $b_1, \ldots, b_k$, there are disjoint paths from a_i to b_i , for all i. A graph G is said to be *k-ordered* if for any distinct k vertices of G, there exists a cycle of G

through them in any specified order.

Recently, Chen et al. introduced the notion "(m,n)-linked" [3]. This derived from the Graph Minor argument related to a graph linkage problem. A graph G is said to be (m,n)-*linked* if for any two disjoint subsets $R, B \subset V(G)$ with $|R| \leq m$ and $|B| \leq n$, there are two disjoint connected subgraphs G_R and G_B containing R and B, respectively. Clearly, a graph is 2-linked if and only if it is $(2,2)$-linked. But there seems to be no relation between 3-linked graphs and $(3,3)$-linked graphs.

In this section, we shall prove the following theorem:

THEOREM 23 *Let G be a planar graph with at least six vertices. Then G is $(3,3)$-linked if and only if G is maximal and 4-connected.*

It is clear that if a graph is complete, then it is (3,3)-linked. Moreover, if a graph G is non-complete and has at most six vertices, then G is not (3,3)-linked (because G has a 3-cut).

It is easy to see that if a graph is 4-ordered, then it is 2-linked, and hence (2,2)-linked. Goddard proved that every 4-connected maximal planar graph is 4-ordered [6]. However, the converse does not necessarily hold, that is, the maximality is necessary but the 4-connectedness is not. We have the following corollary, combining Theorem 23 with the result on 4-ordered planar graphs. However, we don't know whether the corollary holds without the assumption on the planarity.

Corollary 24 *If a planar graph G with at least six vertices is* $(3,3)$*-linked, then G is* 4*-ordered.*

4.2 Proof of the theorem

In this section, we shall prove Theorem 23. In order to specify two disjoint subsets R and B of the vertices, we suppose that the vertices in R are colored *red*, those in B are colored *blue*, and other vertices are *white*. Therefore all vertices of the graph considered are distinguished with three colors, red, blue, and white. Each edge is classified according to the color of its end vertices. An edge joining two white vertices is called a *white edge*. An edge joining red and blue vertices are called a *vivid edge*.

To prove the theorem, we need the following lemmas.

Lemma 25 *Let G be a k-connected maximal planar graph and $S \subset V(G)$ with $|S| = k$ for $k = 3, 4, 5$. If S is separating, then there is a chordless k-cycle passing through S.*

Proof. We shall prove that if G is a k-connected plane graph and let S be a separating set with $|S| = k$, for any $k \geq 0$, then G admits a k-*curve* for S, that is, a simple closed curve l drawn on the plane which intersects G at exactly k vertices of S but has no other intersection, and both the interior and the exterior of l contain at least one vertex respectively.

We proceed by induction on k. In the case when $k = 0$ (that is, G is disconnected), there is a 0-curve, and hence the lemma holds. Then we suppose that $k \geq 1$.

For any fixed vertex $x \in S$, let $G' = G - \{x\}$ and $S' = S - \{x\}$. Then G' is a $(k-1)$-connected plane graph, and S' is a separating set with $|S'| = k-1$. By induction hypothesis, G' admits a $(k-1)$-curve γ for S'. Let F be the face of G' which is new face of G' according to remove x, and let Int F denote the interior of F. Observe that γ passes through Int F. (For otherwise, S' would be a separating set of G with $k - 1$ vertices, contrary to the k-connectivity of G.) Moreover, γ intersects Int F exactly once. (For otherwise, a subset of S' would form a smaller separating set, contrary to the $(k-1)$-connectivity of S'.) Now we put a vertex x on the segment of γ contained in Int F, and join x to all the neighbors of x in G. This can be done without crossing of edges added and γ, since γ intersects Int F exactly once. Therefore, G admits a k-curve for S.

Now let $k \in \{3, 4, 5\}$, and we prove that if G is a maximal plane graph and S is a separating set with $|S| = k$, then G admits a k-cycle passing through S. By the above observation, G admits a k-closed curve γ for S.

Since each face of G is triangular, we can find a k-cycle C along γ passing through only the vertices of S.

If C has a chord xy in G, then at least one of the two cycles C_1 and C_2 with $E(C_1) \cap E(C_2) = \{xy\}$ and $E(C_1) \cup E(C_2) = E(C) \cup \{xy\}$ has a separating cycle though a fewer vertices than those in S, contrary to the k-connectivity of G. Hence C is chordless. ■

Lemma 26 (Hama and Nakamoto [9]) *Every 4-connected maximal planar graph is transformed into the octahedron by a sequence of edge contractions, preserving the 4-connectedness.* ■

Lemma 27 *Let G be a 4-connected maximal planar graph and let e be an edge. If the graph G/e obtained from G by contracting e is not 4-connected, then e is contained in a separating 4-cycle in G.* ■

We shall prove Theorem 23.

Proof of Theorem 23. We first prove the necessity. Suppose that a plane graph G has a non-triangular face F with boundary walk $v_1 v_2 \cdots v_k$ $(k \geq 4)$. If $v_1, v_3 \in R$ and $v_2, v_4 \in B$, then G has no two connected subgraphs containing R and B respectively, by the planarity. Therefore, all faces of G must be triangular. If G has a cut set $S \subset V(G)$ with $|S| = 3$, then G must have a separating 3-cycle consisting of the three vertices of S. If $R = S$

and if two vertices of B are specified in two distinct components of $G - S$ respectively, then G has no connected subgraph containing B. Therefore, G must be 4-connected.

Now we shall prove the sufficiency. Let G be 4-connected maximal plane graph, which is a counterexample with a minimum number of vertices, throughout this proof.

Claim 1 *G has at least nine vertices.*

Proof. For convenience, a *double wheel*, denoted by DW_k, is a 4-connected plane triangulation consisting of a k-cycle $v_1 \ldots v_k$ $(k \geq 4)$ and other two vertices x and y lying on the interior and the exterior of the cycle with edges xv_i and yv_i, for $i = 1, \ldots, k$. An edge of DW_k not contained in the k-cycle is called a *spoke*. If $|V(G)| \leq 7$, then G is clearly a double, by lemma 26. If $|V(G)| = 8$, G is either DW_6 or DW_5 with one spoke subdivided again by lemma 26. It is easy to see that a double wheel is $(3,3)$-linked. Moreover, the other graph can be checked to be $(3,3)$-linked. $\diamond$

Let R and B be two fixed disjoint subsets of $V(G)$ with $|R| \leq 3$ and $|B| \leq 3$ which are arbitrarily specified in G. By the choice of G, there do not exist two disjoint connected subgraphs G_R and G_B containing R and B, respectively.

Claim 2 *Every non-vivid edge of G lies on a separating chordless 4-cycle.*

Proof. If a non-vivid edge $e = uv$ does not satisfy the claim, then the graph G/e obtained from G by contracting e is 4-connected, by Lemma 27. Let $[uv]$ be the vertex in G/e corresponding to the edge uv in G. Let R' and B' be two subsets of $V(G/e)$ corresponding to R and B in G. Since e is non-vivid, R' and B' are disjoint in G/e.

Then, by the minimality of G, G/e has two disjoint connected subgraphs $G_{R'}$ and $G_{B'}$ containing R' and B', respectively. Therefore, it is easy to see that the pre-images of $G_{R'}$ and $G_{B'}$ by the contraction are the required two disjoint connected subgraphs in G. This contradicts the minimality of G. $\diamond$

Claim 3 *Every white vertex of G has degree at least* 5.

Proof. If the claim does not hold, then G has a white vertex u of degree 4, by the 4-connectedness of G. Let v_1, v_2, v_3, v_4 be the neighbors of u lying in this cyclic order. Since each uv_i is non-vivid, it lies on an separating 4-cycle, by Claim 2. By Lemma 25, a separating 4-cycle through uv_1 contains v_3 and some vertex, say x. (Note that this 4-cycle contains neither v_2 nor v_4, since it is chordless.) Similarly, a separating 4-cycle for uv_2 contains v_4 and some vertex, say y. By the planarity, we must have $x = y$. Let K be the graph consisting of u, y, v_1, v_2, v_3 and v_4. Then K is isomorphic to the octahedron, which is a 4-connected maximal plane graph. If $G - V(K) \neq \emptyset$, then G would have a 3-cut contrary to the 4-connectedness of G, and hence $G = K$. This contradicts Claim 1. $\diamond$

Claim 4 *G has at least one white edge.*

Proof. Suppose that the white vertices are independent in G. By Claim 1, there are at least three white vertices, say u, v and w. Since any neighbor of each of u, v and w is not white, and since there are at most six non-white vertices in G, we have $|N_G(u) \cap N_G(v) \cap N_G(w)| \geq 3$, by Claim 3. Therefore, G has a subgraph isomorphic to $K_{3,3}$. By Kuratowski's theorem, this contradicts the planarity of G. $\diamond$

Let C be a separating 4-cycle of G, and let C_I and C_E be the connected components of $G - C$ lying in the interior and the exterior of C in G, respectively. Let $\overline{C_I} = G - C_E$ and let $\overline{C_E}$ be the plane graph which is a planar embedding of $G - C_I$ such that C is the outer cycle.

By Claim 4, there is at least one white edge xy. Moreover, there is a separating 4-cycle Γ containg xy, by Claim 2. Now we may assume that Γ is minimal, that is, there is no other separating 4-cycle through xy in $\overline{\Gamma_I}$.

Claim 5 *Γ_I has at least two vertices.*

Proof. If Γ_I has only one vertex, say v, then two edges vx and vy are non-vivid. By Claim 2, we can find two separating chordless 4-cycles through vx and vy, respectively. Similarly to Claim 3, we can conclude that G is the octahedron, contrary to Claim 1. $\diamond$

Claim 6 *Each of* Γ_I *and* Γ_E *has both red and blue vertices.*

Proof. Suppose that one of Γ_I and Γ_E, say Γ_I has no red vertices. Let $\tilde{G}$ be the graph obtained from G by contracting Γ_I into a single vertex, say v. Then $\tilde{G}$ is a 4-connected maximal plane graph with $|V(\tilde{G})| < |V(G)|$, by Claim 5. If Γ_I contains at least one blue vertex, then we specify that v is blue in $\tilde{G}$. Otherwise, we specify v to be white in $\tilde{G}$. By the assumption of G, $\tilde{G}$ is $(3,3)$-linked, and hence there are red and blue connected graphs, denoted by $\tilde{G}_r$ and $\tilde{G}_b$, in $\tilde{G}$, respectively. If v is contained in neither $\tilde{G}_r$ nor $\tilde{G}_b$, then we let $G_R = \tilde{G}_r$ and $G_B = \tilde{G}_b$ in G. If v is contained in $\tilde{G}_b$, then we let $G_R = \tilde{G}_r$ and let G_B be the graph obtained from $\tilde{G}_b$ by replacing v with Γ_I, therefore the claim holds. We can do similarly when v is contained in $\tilde{G}_r$. $\diamond$

Claim 7 Γ_I *has at most three vertices.*

Proof. We first prove that there is no white vertex in Γ_I. Suppose that there is a white vertex v in Γ_I. Since there is no white edge in $E(\overline{\Gamma_I}) - E(\Gamma)$, by

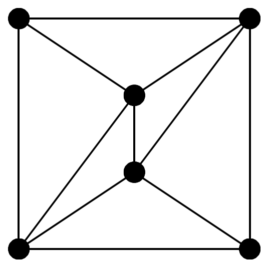

Figure 4.1: $\overline{\Gamma'_E}$ with $|\Gamma_E| = 2$

Claim 2 and the minimality of Γ, all neighbors of v are not white. Hence $\overline{\Gamma_I}$ must contain at least five non-white vertices, by Claim 3. However, this implies that Γ_E has at most one non-white vertex, contrary to Claim 6.

In order to prove the claim, we suppose that Γ_I has at least four vertices. For $\overline{\Gamma_E}$, let Γ' be a minimal separating 4-cycle of $\overline{\Gamma_E}$ through some white edge, say e, where possibly $\Gamma' = \Gamma$. Note that there is no white vertex in the interior Γ'_I of Γ', as proved similarly to the case for Γ_I. Therefore, by the assumption, there are at most two non-white vertices in Γ'_I, since G has at most six non-white vertices. By Claim 5, Γ'_I has exactly two vertices, and hence $\overline{\Gamma'_I}$ must be a graph shown in Figure 4.1, up to symmetry. However, we can clearly find separating 4-cycle thought e bounding a smaller number of vertices than Γ. This contradicts the minimality of Γ' in $\overline{\Gamma_E}$. $\diamond$

Claim 8 *Each vertex v of Γ has degree at least four in $\overline{\Gamma_I}$.*

Proof. Let $\Gamma = v_1v_2v_3v_4$, where v_1 and v_2 are white vertices. Suppose $N_{\overline{\Gamma_I}}(v_3) = \{v_2, x, v_4\}$. By Claim 5, $v_1v_2xv_4$ is a separating 4-cycle through white edge v_1v_2 in $\overline{\Gamma_I}$, other than Γ. This contradicts the minimality of Γ. Thus, we have $\deg_{\overline{\Gamma_I}}(v_3) \geq 4$. Similarly, we have $\deg_{\overline{\Gamma_I}}(v_4) \geq 4$. Now suppose $N_{\overline{\Gamma_I}}(v_1) = \{v_2, x, v_4\}$. Since v_1x is non-vivid, v_1x lies on a separating chordless 4-cycle, say C, by Claim 2. Since C must pass through v_3 and since C has length four, there exists an edge joining x and v_3 in G. Since G is 4-connected, Γ_I consists of only one vertex x. This contradicts Claim 5. Therefore, we have $\deg_{\overline{\Gamma_I}}(v_1) \geq 4$. Similarly, we have $\deg_{\overline{\Gamma_I}}(v_2) \geq 4$. $\diamond$

It is easy to see that Claim 8 implies that $|V(\Gamma_I)| \geq 4$, since G has no 3-cut. However, this contradicts Claim 7. Therefore, the counterexample G does not exist, and the theorem holds. ■

Chapter 5

(k, k)-Linked graphs on surfaces

In this chapter, we would like to generalize the result in Chapter 4 to triangulations on other surfaces with respect to the connectivity of graphs and the representativity of embeddings, where the *representativity* of an embedding G is the minimum number of intersecting points of G and any non-contractible simple closed curve on the surface. An essential argument in this generalization is that in a triangulation G, a minimal vertex cut of the graph lies on several cycles whose removal disconnects the surface. So, analyzing a relation between a minimal tree containing a specified vertex set in G and a minimal cut set of G separating the tree, we shall generalize

Theorem23. However, we note that the property “each face is triangular” is not necessary for a graph on a non-spherical surface to be (k, k)-linked, since such surfaces do not satisfy “Jordan Curve Theorem”. On the other hand, if we restrict a graph on a surface to be a triangulation, then we can use an important property called “a *k-separation property*”, which will play an immmportant role in proving our theorems.

5.1 Main theorem

In this chapter, we shall prove the following theorem:

THEOREM 28 *Let k be a positive integer. Every $(k + 1)$-connected $\lfloor \frac{k+4}{2} \rfloor$-representative triangulation on any surface is (k, k)-linked.*

Since every graph is obviously $(1, 1)$-linked, the case when $k \geq 2$ is essential in Theorem 28. Moreover, Theorem 28 for plane triangulations when $k = 3$ is equivalent to the sufficiency of Theorem 23, since the representativity of a plane triangulation is defined to be the infinity. Note that for any fixed surface F^2, there exist only finitely many 7-connected graphs embeddable in F^2, it might not be natural to consider triangulations with connectivity at least 7.

Now, let's consider the sharpness of Theorem 28 with respect to the connectivity and the representativity.

PROPOSITION 29 *The estimation for the connectivity and representativity cannot be relaxed.*

Proof. Since the $(k+1)$-connectedness is clearly necessary, we consider only the representativity. Figure 5.1 shows a triangulation on the annulus which is obtained from the rectangle by identifying its top and bottom, where the shaded part in the rectangle is arbitrarily triangulated and even the region might not be homeomorphic to a disk. Let G be a 5-connected triangulation on a surface containing an annular part shown in the figure. Since $C = xr_1r_2$ is an essential cycle of length 3, G is not 4-representative. We shall prove that G is not $(4,4)$-linked.

Let $R = \{r_1, r_2, r_3, r_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ be two disjoint vertex sets as in the figure, and we shall prove that G does not have two disjoint subgraphs containing R and B, respectively. For contradictions, we may suppose that G has a connected subgraph, denoted by H_R, containing all vertices of R but no vertices of B. Then H_R must contain the vertex x. In this case, the removal of $R \cup \{x\}$ from G separates G so that $\{b_1, b_2\}$ and $\{b_3, b_4\}$ are contained in distinct components of $G - (R \cup \{x\})$ respectively, and hence G cannot have a connected subgraph containing B but avoiding

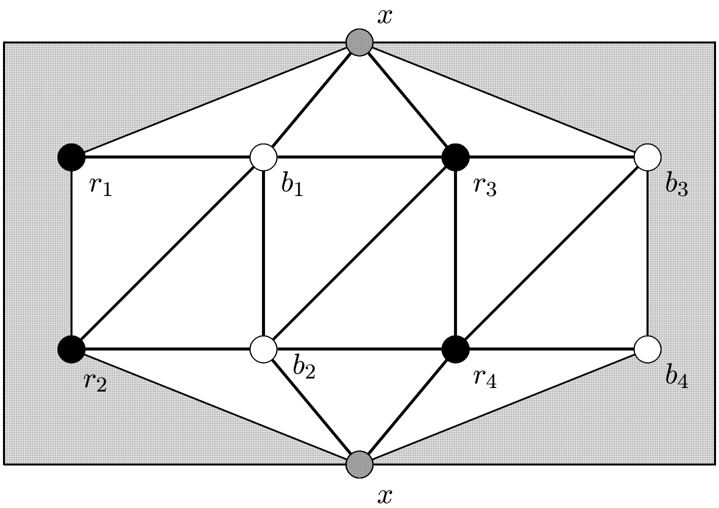

Figure 5.1: Not (4,4)-linked triangulations G on the torus

$V(H_R)$. Therefore, the 4-representativity cannot be omitted in the case when $k = 4$. ■

By similar arguments, one can easily construct examples showing the bound only for the representativity in the theorem is sharp. On the other hand, in case of $k \geq 6$, we have not yet proved that the bound is sharp, since the construction of triangulations with high connectivity seems to be difficult.

5.2 Proof of theorems

In this section, we shall prove Theorem 28. In order to prove the theorem, we need the following notion called "k-separation property" for abstract graphs.

Let G be a graph. For a nonempty subset $U \subset V(G)$, let $\langle U \rangle$ denote the subgraph of G induced by U. A subset S of $V(G)$ (resp., a subgraph K) is said to be *separating* if $G - S$ (resp., $G - V(K)$) is disconnected. We say that a separating set S of G is *minimal* if $S - \{x\}$ is not separating for any $x \in S$.

DEFINITION 30 *A graph G satisfies the k-*separation property *if for any minimal separating set $S \subset V(G)$, either of the following holds:*

(i) $\langle S \rangle$ has a cycle of length at least k, or

(ii) $\langle S \rangle$ contains a union of at least two cycles of length at least $\lfloor \frac{k+3}{2} \rfloor$ as a spanning subgraph.

Clearly, if a graph G satisfies the k-separation property, then G is k-connected. The readers might feel that the k-separation property of an abstract graph is unnatural and artificial in a sense, but the following lemma points out an important property which a k-connected $\lfloor \frac{k+3}{2} \rfloor$-representative triangulation on a surface has as an abstract graph.

LEMMA 31 *Let G be a k-connected $\lfloor \frac{k+3}{2} \rfloor$-representative triangulation on any surface F^2, where $k \geq 2$ is a positive integer. Then G satisfies the k-separation property.*

Proof. Let $S = \{v_1 \dots v_m\}$ be a minimal separating set of G. Then by the k-connectedness of G, we have $|S| = m \geq k$. We shall prove that the subgraph $\langle S \rangle$ of G induced by S has a spanning separating cycle or a union of at least two nonseparating essential cycles each of whose length is at least $\lfloor \frac{k+3}{2} \rfloor$ as a spanning subgraph.

Since $G - S$ is a disconnected embedding on F^2, we can take a simple closed curve J or several simple closed curves $J_1, \dots, J_\ell$ on F^2 where $G - S$ is embedded, without intersecting $V(G-S)$ and $E(G-S)$, which separates F^2 so that each component of $F^2 - J$ (or $F^2 - \{J_1, \dots, J_\ell\}$) contains at least one vertex of $G - S$.

Now re-construct G from the embedding $G - S$ by adding $v_1, \dots, v_m$ to F^2 one by one and prove that G admits a simple closed curve on the surface intersecting only the vertices of S(or several simple closed curves). For any fixed vertex $v_i \in S$, let $G' = G - \{v_i\}$ and $S' = S - \{v_i\}$. Since S is a minimal separating set of G, S' is a minimal separating set of G', too. Let F be the face of G' which is a new face of G' arisen by removing v_i, and let Int F denote the interior of F. Observe that separating simple closed curve J (or some nonseparating essential simple closed curve) passes through Int F.

We first consider the case when we take J for $G - S$. Then v_i can be put on J so that any edge joining v_i and a vertex of $G - S$ does not cross J.

Therefore, J intersects G only at the vertices of S. Since each face of G is triangular, we can find an m-cycle C along J passing through only the vertices of S. Thus, $\langle S \rangle$ has an m-cycle with $m \geq k$.

The case when we take nonseparating essential cycles $J_1, \ldots, J_\ell$ for $G - S$ is essentially similar to the previous case for J. By the minimality of S, each vertex of S can be put on one of $J_1, \ldots, J_\ell$. We have only to find a cycle C_i of G along each J_i for $i = 1, \ldots, \ell$. (Note that some vertex of S might be contained in at least two of the cycles.) So, we have to note that G has no essential cycle whose length is less than $\lfloor \frac{k+3}{2} \rfloor$ by the assumption for the representativity of G. Hence we must have $|J_i| \geq \lfloor \frac{k+3}{2} \rfloor$ for $i = 1, \ldots, l$. ■

Let G be a graph and let C be a subgraph of G. Let A be one of the components of $G - V(C)$, or a *chord* of C, i.e., an edge xy of G such that $x, y \in V(C)$ but $xy \notin E(C)$, and let $x_1, \ldots, x_m \in V(C)$ be the vertices adjacent to vertices of A, or the end vertices of the chord. Then the connected subgraph of G induced by $V(A) \cup \{x_1, \ldots, x_m\}$ is called a *C-bridge* with attachments $x_1, \ldots, x_m$. (We say that the C-bridge obtained from a chord of C is *trivial*.)

The following theorem is the most essential argument for proving Theorem 28.

Theorem 32 *Let k be a positive integer. If a graph G satisfies the $(k+1)$-separation property, then G is (k,k)-linked.*

Proof. Let G be a graph satisfying the $(k+1)$-separation property. Then G is $(k+1)$-connected. Moreover, for every minimal separating set S of G, $\langle S \rangle$ has either a separating spanning cycle of length at least $k+1$ or a union of at least two cycles of length at least $\lfloor \frac{k+4}{2} \rfloor$ as a spanning subgraph. In order to prove the theorem, we shall prove that for any disjoint subsets $R, B \subset V(G)$ with $|R| \leq k$ and $|B| \leq k$, G has a connected subgraph H_R containing all vertices of R but no vertices of B such that $G - H_R$ is connected. Clearly, this implies that G is (k,k)-linked.

Since G is $(k+1)$-connected but $|B| \leq k$, $G - B$ must be connected. Hence, we can always take a connected subgraph H_R containing all vertices of R but avoiding B. A vertex $x \in V(H_R) - R$ is said to be *removable* if $H_R - \{x\}$ is still connected. We suppose that H_R is *minimal*, that is, H_R has no removable vertex. If $G - V(H_R)$ is connected, then the lemma immediately follows. Therefore, we suppose that $G - V(H_R)$ is disconnected.

We begin with the following claim.

Claim 9 *H_R has no cycle whose length is at least $k+1$.*

Proof. For contradictions, we suppose that H_R has a cycle $C = v_1v_2v_3 \cdots v_m$, where $m \geq k+1$. Then in the following argument, we shall find a removable vertex in H_R, which contradicts the minimality of H_R.

For each i, take a nontrivial C-bridge in H_R whose attachment is only v_i, if any. If there is, we let D_i be such a C-bridge, and let $D_i = \{v_i\}$ otherwise, for $i = 1, \ldots, m$. (In the latter case, we say that v_i is *bad*.) Since D_i and C intersect only at v_i in H_R, we have $D_i \cap D_j = \emptyset$ for any distinct $i, j \in \{1, \ldots, m\}$. Moreover, since $|R| \leq k$, we can find D_t containing no vertices in R for some $t \in \{1, \ldots, m\}$, by Pigeonhole Principle. Hence, by the minimality of H_R, we have $D_t = \{v_t\}$. Then v_t is removable in H_R since v_t is contained in C. A contradiction. $\diamond$

Claim 9 asserts that $\langle S \rangle$ has no cycle of length at least $k+1$. Hence, by the following two claims, we deny the other possibility for $\langle S \rangle$ described in Definition 30. Then we shall prove that if $\langle S \rangle$ has a union of cycles $C_1, \ldots, C_p$ as a spanning subgraph for some $p \geq 2$, then one of the cycles must have length less than $\lfloor \frac{k+4}{2} \rfloor$.

Claim 10 *$\langle S \rangle$ has two cycles C_i, C_j such that $|V(C_i) \cap V(C_j)| \leq 1$.*

Proof. For contradictions, we suppose that $|V(C_i) \cap V(C_j)| \geq 2$ for any distinct $i, j \in \{1, \ldots, p\}$. We first prove that $\langle S \rangle$ is 2-connected, by using Whitney's theorem [17] which states that a graph is 2-connected if and only if it has a cycle passing through any two distinct vertices specified in the graph. Let x_1, x_2 be any two distinct vertices in S. If x_1 and x_2 lie on the same cycle C_i, then the assertion clearly holds, and hence we may suppose that $x_i \in C_i$, for $i = 1, 2$, without loss of generality. Since $|C_1 \cap C_2| \geq 2$ by the assumption, we can take two distinct vertices $p, q \in V(C_1) \cap V(C_2) - \{x_1, x_2\}$. Observe that C_1 is decomposed into two paths with endvertices p, q, and let R be the path containing x_1 as an inner vertex. We may suppose that no vertices in $V(C_1) \cap V(C_2) - \{p, q\}$ are contained in R, by changing p, q suitably. Since p, q are two distinct vertices of C_1, we can take two paths P_1 and Q_1 from x_1 to p and q in C_1, respectively, and similarly, we can find two paths P_2 and Q_2 from x_2 to p and q in C_2, respectively, where P_i and Q_i intersect only at x_i, for $i = 1, 2$. Hence, $(P_1 \cup P_2) \cup (Q_1 \cup Q_2)$ contains a required cycle in $\langle S \rangle$, since R has no inner vertices of $V(C_1) \cap V(C_2)$. Therefore, we can conclude that $\langle S \rangle$ is 2-connected.

We proceed similarly to the proof of Claim 9. Since G is $(k+1)$-connected, we have $|S| \geq k + 1$, where we let $S = \{v_1, \ldots, v_m\}$ for some $m \geq k + 1$. For each i, consider a nontrivial $\langle S \rangle$-bridge whose attachment is only v_i. If there is, we let D_i be such a $\langle S \rangle$-bridge, and let $D_i = \{v_i\}$ otherwise. Since $D_i \cap D_j = \emptyset$ for any distinct i, j, and since $|R| \leq k$, we can find some D_t which has no vertex in R. Hence we have $D_t = \{v_t\}$ for some $t \in \{1, \ldots, m\}$ since H_R is minimal. On the other hand, since $\langle S \rangle$ is 2-connected as shown in the first paragraph, v_t is removable in H_R, a contradiction. $\diamond$

Without loss of generality, we may suppose that C_1 and C_2 satisfy Claim 10.

Claim 11 *At least one of C_1 and C_2 has length less than $\lfloor \frac{k+4}{2} \rfloor$.*

Proof. For contradictions, we suppose that the length k_i of C_i is at least $\lfloor \frac{k+4}{2} \rfloor$, for $i = 1, 2$. Since $|C_1 \cap C_2| = 0, 1$ by Claim 10, we consider the following two cases separately, depending on it.

We first suppose that C_1 and C_2 are disjoint. Let $V(C_1 \cup C_2) = \{v_1, \cdots v_m\}$. We proceed similarly to the proof of Claim 9. Define D_i as a $(C_1 \cup C_2)$-bridge whose attachment is only v_i, for $i = 1, \ldots, m$. (Note that a component W of $G - V(C_1 \cup C_2)$ which has only one foot in some vertex in $V(C_1)$ and some feets in the other C_2 must be neglected for the definition of D_i's, since W has at least two feets in $C_1 \cup C_2$ in this case.) Since

$$m = k_1 + k_2 \geq \left\lfloor \frac{k+4}{2} \right\rfloor + \left\lfloor \frac{k+4}{2} \right\rfloor \geq k + 3,$$

and since $|R| \leq k$, we can take three distinct D_i's containing no vertices of R, say D_p, D_q, D_r. Hence, by the minimality of H_R, the three vertices v_p, v_q, v_r are bad in H_R. Here we may suppose that $v_p, v_q \in V(C_1)$ without loss of generality. If v_p is removable (i.e., $H_R - \{v_p\}$ is still connected), then we are done, similarly to the earlier case. Hence we suppose that v_p is not removable in H_R. Since $v_p \notin R$, we may suppose that v_p is a cut vertex of H_R, and there is a C_1-bridge D_p containing C_2 whose attachment is only v_p. In this case, $v_q \in V(C_1)$ is removable, since v_q can no longer be a foot of the C_1-bridge containing C_2, and hence v_q is removable. This contradicts the minimality of S.

Finally, we suppose that C_1 and C_2 share exactly one vertex. Similarly to the above case, we define $D_1, \ldots, D_m$, where

$$m = k_1 + k_2 - 1 \geq \left\lfloor \frac{k+4}{2} \right\rfloor + \left\lfloor \frac{k+4}{2} \right\rfloor - 1 \geq k + 2.$$

Hence we can find at least two bad vertices, say v_p, v_q. Since at least one of v_p, v_q, say v_p, is not the unique vertex contained in both C_1 and C_2, v_p is removable, a contradiction. $\diamond$

By Claims 9, 10 and 11, we have proved that if $G - V(H_R)$ is disconnected, then G cannot satisfy the k-separation property, a contradiction. Therefore, $G - V(H_R)$ is connected, and hence we can take two disjoint connected subgraphs H_R and H_B containing two disjoint vertex sets R with $|R| \leq k$ and B with $|R| \leq k$, respectively, arbitrarily specified in G. $\diamond$

Proof of Theorem 28. Theorem 28 follows from Lemma 31 and Theorem 32. ■

Extending Thoerem 23, we have proved that every $(k+1)$-connected $\lfloor \frac{k+4}{2} \rfloor$-representative triangulation on any surface is (k,k)-linked. In the theorem, the assumption for the connectivity and the representativity is necessary. On the other hand, as I mentioned in Introduction, the condition "each face is triangular" is not necessary in our theorem. We do not know whether we can remove such an assumption from our theorem. Therefore, the following problem will be interesting.

Problem 33 *Can the condition with each face triangular be removed from our theorem?*

Finally, we would like to consider Theorem 23 with a sufficiently large integer k. In this paper, we essentially proved that any graph with the $(k+1)$-separation property is (k,k)-linked. On the other hand, a $(k+1)$-connected $\lfloor \frac{k+4}{2} \rfloor$-representative triangulation satisfies the $(k+1)$-separation property, and hence it is (k,k)-linked. Though we can take a sufficiently large k in this argument, we do not know whether there actually exists a $(k+1)$-connected $\lfloor \frac{k+4}{2} \rfloor$-representative triangulation for sufficiently large k. In order to construct such a triangulation, some algebraic method might be useful.

Index

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