SUMMARY OF Ph.D. DISSERTATION

School	Student Identification Number	SURNAME, First name
Fundamental Science and Technology		Kurosawa, Takeshi

Title

Transcendence criterion of Mahler functions

Abstract

For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, we put $|\boldsymbol{\lambda}| = \sum_{i=1}^m \lambda_i$ and $\boldsymbol{\alpha}^{\boldsymbol{\lambda}} = \prod_{i=1}^m {\alpha_i}^{\lambda_i}$. Let $r \geq 2$ be an integer. We define $\Omega_n \boldsymbol{z} := ({z_1}^{r^n}, \ldots, {z_m}^{r^n})$ for $\boldsymbol{z} = (z_1, \ldots, z_m)$ and consider the function

$$\Phi_0(\boldsymbol{z}) = \sum_{k \geq 0} \frac{E_k(\Omega_k \boldsymbol{z})}{F_k(\Omega_k \boldsymbol{z})} \in \boldsymbol{K}[[\boldsymbol{z}]] = \boldsymbol{K}[[z_1, \ldots, z_m]],$$

where $\boldsymbol{K}$ is an algebraic number field and

$$E_k(\boldsymbol{z}) = \sum_{1 \leq |\boldsymbol{\lambda}| \leq L} e_{k\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}}, \quad F_k(\boldsymbol{z}) = 1 + \sum_{1 \leq |\boldsymbol{\lambda}| \leq L} f_{k\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}} \in \boldsymbol{K}[\boldsymbol{z}]$$

are coprime. We assume that $\log \|e_{k\boldsymbol{\lambda}}\|, \log \|f_{k\boldsymbol{\lambda}}\| = o(r^k)$. For an algebraic number α, $\|\alpha\|$ is defined by $\max\{\overline{|\alpha|}, \mathrm{den}(\alpha)\}$, where $\overline{|\alpha|}$ and $\mathrm{den}(\alpha)$ are the maximum of the absolute values of the conjugates of α and the least positive integer such that $\mathrm{den}(\alpha)\,\alpha$ is an algebraic integer, respectively. The function $\Phi_0(\boldsymbol{z})$ satisfies a Mahler type functional equation.

The main theorems of the thesis are as follows:

Theorem 1. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\boldsymbol{K}^\times)^m$ with $0 < |\alpha_1|, \ldots, |\alpha_m| < 1$ such that $F_k(\Omega_k \boldsymbol{\alpha}) \neq 0$ for every $k \geq 0$. Assume that $|\alpha_1|, \ldots, |\alpha_m|$ are multiplicatively independent. Then $\Phi_0(\boldsymbol{\alpha})$ is algebraic if and only if $\Phi_0(\boldsymbol{z})$ is a rational function over $\boldsymbol{K}$.*

Theorem 1 insists the equivalence between the rationality of the Mahler function $\Phi_0(\boldsymbol{z})$ and the algebraicity of the value of the function at an algebraic point. Specializing Theorem 1, we get some criterions for the rationality over $\boldsymbol{K}$ of $\Phi_0(\boldsymbol{z})$. As an application of these criterions, we obtain transcendence results of reciprocal sums of binary linear recurrences. Let $\{R_n\}_{n\geq 0}$ be a binary linear recurrence satisfying

$$R_{n+2} = AR_{n+1} + BR_n, \tag{1}$$

where $A, B, R_0, R_1 \in \mathbb{Z}$ with $(A, B), (R_0, R_1) \neq (0, 0)$. Assume that $\Delta = A^2 + 4B$ is positive. Let $\sum_{k\geq 0}{}'$ be a sum taken over all $k \geq 0$ such that $R_{r^k} \neq 0$.

Theorem 2. *Let $\{R_n\}_{n\geq 0}$ be a binary linear recurrence defined by (1). Suppose that $\{R_n\}_{n\geq 0}$ be non–periodic and $R_{r^k} \neq 0$ for infinitely many k. Let $\{a_k\}_{k\geq 0}$ be a sequence in $\boldsymbol{K}$ such that $a_k \neq 0$ for infinitely many k and $\log \|a_k\| = o(r^k)$. Then*

$$\theta = \sum_{k\geq 0}{}' \frac{a_k}{R_{r^k}} \notin \overline{\mathbb{Q}}$$

except in the following two cases:

1) *Let $r = 2$, $a_n = a$ $(n \geq N)$ for some $a \in \boldsymbol{K}$ and $N \in \mathbb{N}$, $|B| = 1$, and $R_0 = 0$. Then $\theta \in \boldsymbol{K}(\sqrt{\Delta})$.*

2) *Let $r = 2$, $a_n = a2^n$ $(n \geq N)$ for some $a \in \boldsymbol{K}$ and $N \in \mathbb{N}$, $A = \pm(B - 1)$, and $AR_0 = 2R_1$. Then $\theta \in \boldsymbol{K}$.*