Ph.D. Dissertation

A Study of Statistical Issues in Selective Assembly:

Optimal Binning Strategies under Squared Error Loss

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Contents

Chapter 1

Preliminaries

1.1 Introduction

Selective assembly (also called match gauging) is an effective approach for improving a quality of an assembled product. Some statistical and mathematical issues arise in selective assembly. This thesis studies optimal binning strategies under squared error loss in selective assembly, based on the papers by Matsuura and Shinozaki (2007,2008,2009).

We consider a product composed of two mating components. Suppose that the quality characteristic of the product is the difference of the relevant dimensions of the mating components (i.e., the clearance) and that there is a target value for the clearance. Note that, although we use the clearance as the assembly dimension of interest, our discussion is equally valid for the case where we consider instead the sum of the dimensions of the mating components. In any production process, variability is inevitable and variation in the dimensions of component parts affects the quality of the assembled product. Assembling mating components in a random fashion may lead to an unacceptably large variance in the clearance. In these situations, selective assembly should be effective in reducing the variance.

In this approach, the components are measured and sorted (or binned) into several groups according to their dimensions, and the product is assembled by randomly selecting mating components from corresponding groups, as shown in Figure 1.1. This approach enables the assembly of high quality products from relatively low quality components (i.e., components with a wide variation in size), which may lead to a reduction in cost compared to the alternative of manufacturing the respective components with a lesser tolerance.

A piston and cylinder assembly (Figure 1.2) is an example from an automobile industry in Japan. There is a target value for the clearance, that is, the difference
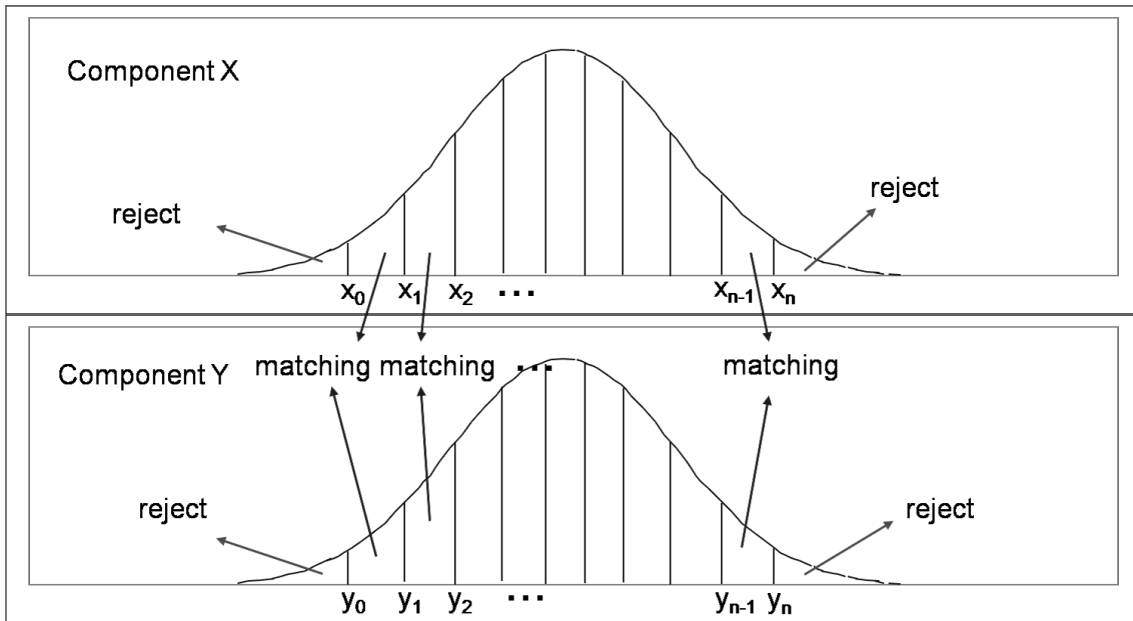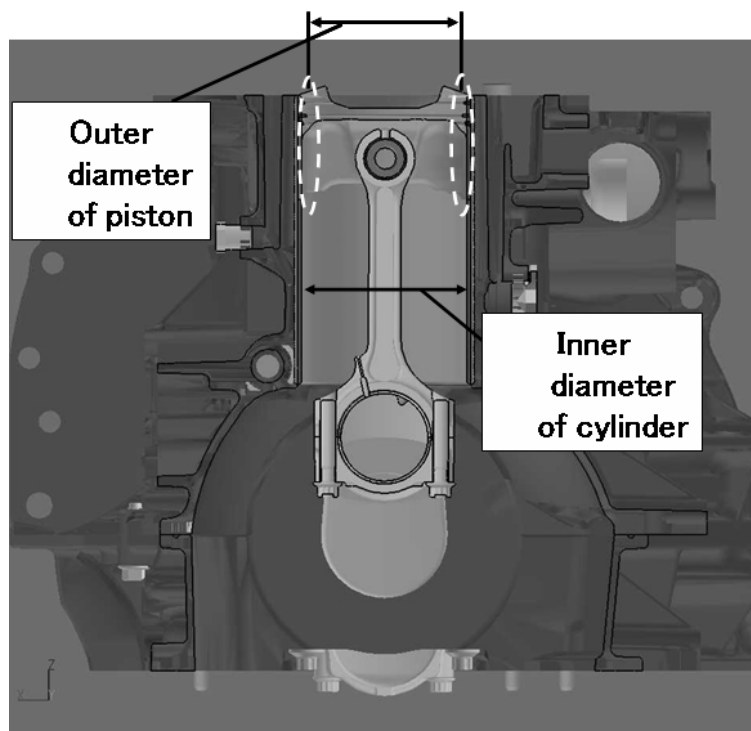
Figure 1.1. Selective assembly.

Figure 1.2. Piston and cylinder assembly.

between the inner diameter of the cylinder and the outer diameter of the piston. There is also a tolerance constraint on the clearance. If the clearance is too small, the oil film between the cylinder wall and piston becomes too thin and piston scuffing occurs. If the clearance is too large, the piston vibrates in the cylinder and abnormal noise occurs. Random assembling of pistons and cylinders leads to an unacceptably large variance in the clearance and also leads to an unacceptably large number of products which do not satisfy the tolerance constraint. Thus, the automobile industry has used selective assembly. Pistons and cylinders are sorted into several groups according to their outer and inner diameters, respectively. The smaller pistons are matched with the smaller cylinders and the bigger pistons with the bigger cylinders.

The second example (Figure 1.3) is a camshaft, valve, and valve lifter assembly, which Mease et al. (2004) also described. According to the distance from the bottom of a camshaft to the top of a valve, an appropriate valve lifter should be chosen to meet a given clearance specification. The Japanese automobile industry has used selective assembly and sorts the components into more than 20 groups.

Other applications of selective assembly include a valve body and spool assembly (Robinson and Mazharsolook (1993)), a hole and shaft matching (see Asha et al. (2008), for example), a gearbox shaft assembly (Kumar et al. (2007)), and a fan shaft assembly (Kumar and Kannan (2007)).

1.2   Main topics

There are some important statistical and mathematical issues which arise in selective assembly.

Determining optimal partition limits for the dimensional distributions of the components is one of the important issues. Equal width partitioning schemes, in which the dimensional distributions are partitioned so that all groups have equal widths, and equal area partitioning schemes, in which all groups have equal probabilities, are commonly used. However, these schemes are not necessarily optimal under some loss functions.
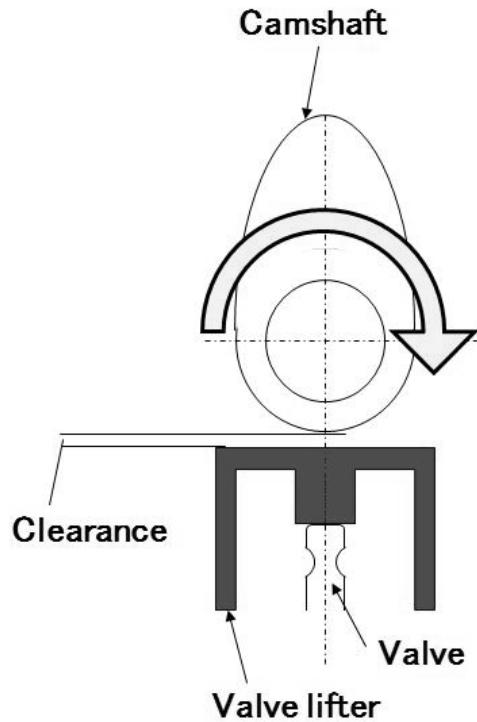
Figure 1.3. Camshaft, valve, and valve lifter assembly.

There has been little research and development effort on the optimal partitioning problem. Kwon et al. (1999) were the first to study optimal partitioning under squared error loss when the two component dimensions are identically and normally distributed after re-centering, and gave equations for the optimal partition limits. Mease et al. (2004) discussed optimal partitioning under several loss functions and distributional assumptions. They extensively developed optimal partitioning under squared error loss when the two component dimensions are identically distributed after re-centering. They gave equations for the optimal partition limits, and showed existence of unique solutions provided that the dimensional distribution is strongly unimodal (i.e., the density function of the dimensional distribution is log-concave). They also gave some numerical results which show that the optimal partition considerably reduces the expected squared error loss compared with equal width and equal area partitioning schemes.

Chapters 2 and 3 of this thesis present some extensions of the results of Mease et al. (2004), based on the papers by Matsuura and Shinozaki (2007,2009).

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Chapter 2 studies optimal partitioning under squared error loss when error is present in the measurements of component dimensions. The previous works implicitly assumed that measurement error is absent. However, most measurement processes have inherent variability which may not be negligibly small. Determining optimal partition limits is discussed and the effect of measurement error on the expected squared error loss is evaluated.

Chapter 3 studies optimal partitioning under squared error loss when the clearance is constrained by a tolerance parameter. Kwon et al. (1999) and Mease et al. (2004) did not deal with the problem of determining optimal partition limits under a tolerance constraint on the clearance, although such a constraint is usually present in practice. If some groups have larger widths than the tolerance limit in the case of unconstrained optimal partitioning, we have unacceptable products with positive probability. If this probability is not negligibly small and we remove the unacceptable products, additional measurement and inspection will be necessary, with correspondingly high cost. Thus, it is important to study optimal partitioning which minimizes the expected squared error loss under the condition that no group width should exceed the tolerance limit.

Another important issue in selective assembly is how to handle the case in which the two component dimensions have unequal variances. When the two component dimensions are identically distributed after re-centering, we can partition the dimensional distributions so that the expected clearance of the product from any group is equal to the target clearance. However, if there is a large difference between the variances of the two component dimensions, then this causes large differences between the target clearance and the expected clearances of the products that are assembled by selecting mating components from groups in the tails of the distributions, which leads to a large variance of the clearance. To cope with this difficulty, Mansoor (1961), Kannan et al. (1997), and Kannan and Jayabalan (2002) proposed a method of manufacturing the component with smaller variance in two (or more) versions, each version having a different mean value for its dimension by shifting the process mean. Mansoor (1961) proposed determining the number of versions according to the difference of the variances of the

two component dimensions. Kannan et al. (1997) and Kannan and Jayabalan (2002) proposed shifting the process mean so that the variance of the resulting dimensional distribution is equal to that of the other component dimension.

However, the determination of the optimal mean shift under a certain criterion has not been addressed in these papers. Based on the paper by Matsuura and Shinozaki (2008), Chapter 4 studies the problem of determining the optimal mean shift under squared error loss when the component with smaller variance is manufactured at two shifted means.

1.3 Literature review

Other important issues in selective assembly have been discussed by many authors. A literature review of the work on the issues is presented as follows.

Pugh (1986a) presented a computer program which generates an equal width partitioning and the desired number of groups. Using a simulation study, Pugh (1986b) showed that equal area partitioning reduces the number of unacceptable products compared with equal width partitioning when the number of groups is small. Pugh (1992) studied equal area partitioning for the case of dimensions with dissimilar variances, and proposed truncating the dimensional distribution with the larger variance to make the two variances equal. Desmond and Setty (1962) proposed an equal probability partitioning scheme in which corresponding groups have equal probabilities so that there are no surplus components, and Fang and Zhang (1995) presented an algorithm for such a scheme. Chan and Linn (1998) proposed rejecting components whose dimensions are within a certain range in order to reduce the total number of surplus components when the variances of the two component dimensions are very different. Kannan and Jayabalan (2001a) proposed using equal width partitioning if the tolerance limit on the clearance is smaller than the difference in the three times standard deviations of the two component dimensions, and using equal probability partitioning otherwise. Kannan and Jayabalan (2001b) proposed a partitioning method for a product assembled

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from three components.

Lee et al. (1990) studied the problem of determining optimal process means of the two components to maximize the number of acceptable products when the components are sorted into two groups. Iyama et al. (1995) described the behavior of the component flow using a Markov model. Thesen and Jantayavichit (1999) addressed the problem of limited buffer capacity. Mease and Nair (2006) studied optimal partitioning in one-sided selective assembly in which only one component is sorted into several groups and the other component is manufactured at the target means with negligibly small variance.

Recently, some authors have proposed methods for determining the best combination of selective groups using genetic algorithms, as follows. Kannan et al. (2003) proposed a method for minimizing the variation of a quality characteristic when three components are assembled linearly. Kannan et al. (2005) proposed a method for reducing clearance variation and the number of surplus components. Kumar and Kannan (2007) proposed a method for obtaining optimum manufacturing tolerance to reduce manufacturing cost and the number of surplus components. Kumar et al. (2007) proposed a new two-stage method. The surplus components that result from an equal width partitioning scheme in the first stage are sorted into three groups in the second stage and the best combination of groups is obtained using a genetic algorithm. Ponnambalam et al. (2006) proposed a Parallel Population Genetic Algorithm to minimize the variation of a quality characteristic, which is faster than normal genetic algorithms. Asha et al. (2008) proposed a method using a non-dominated sorting genetic algorithm for minimizing clearance variation in an assembled product with multiple quality characteristics. Kannan et al. (2008) proposed a method using a genetic algorithm for obtaining the best combination of selective groups under Taguchi's loss function.

Some authors have discussed methods of applying matching algorithms for selective assembly. In the methods, the components are not sorted into groups, but the best combination of components is determined according to their dimensions using a matching algorithm. The methods are suitable when the information on the component dimensions is available for a certain period of time. Fujino (1987) proposed a method

for reducing computing time with little loss of matching ratio when lot size is large. Yamada and Fujino (1992) showed that using a maximum matching algorithm considerably increases the matching ratio compared with the grouping method. See also Robinson and Mazharsolook (1993), Yamada and Kobayashi (1993), Yamada (1994), Coullard et al. (1998), Iwata et al. (1998), Iyama et al. (2003), and Iyama et al. (2007).

1.4 Summary of this thesis

In this thesis, we study optimal binning strategies under squared error loss in selective assembly. Chapters 2, 3, and 4 are based on the papers by Matsuura and Shinozaki (2007,2009,2008), respectively. The summary of this thesis is as follows.

Section 1.5 gives basic models, notation, and assumptions. Based on the paper by Mease et al. (2004), equations and uniqueness results for the optimal partition are given under the assumption that the two component dimensions are identically distributed after re-centering.

Chapters 2 and 3 also assume that the two component dimensions are identically distributed after re-centering.

Chapter 2 studies optimal partitioning when measurement error is present. Equations for the optimal partition limits are derived, and sufficient conditions under which the set of optimal partition limits is unique are given. We establish a relationship between the expected squared error losses based on the optimal partition limits for the cases when measurement error is present and not present. We give some numerical results to evaluate the effect of measurement error.

Chapter 3 studies optimal partitioning when the clearance is constrained by a tolerance parameter. Conditions for a set of constrained optimal partition limits are given, and uniqueness of the set is established provided that the dimensional distribution is strongly unimodal. We show some relations between constrained optimal partition limits and unconstrained optimal partition limits. Some numerical results are reported

that compare constrained optimal partitioning, unconstrained optimal partitioning, and equal width partitioning.

In Chapter 4, relaxing the assumption that the two component dimensions are identically distributed after re-centering, we discuss the case in which the variances of the two component dimensions are different. We deal with the problem of determining the optimal mean shift when the component with smaller variance is manufactured at two shifted means. Assuming that the two component dimensions are normally distributed, we show that the optimal mean shift is uniquely determined and also show some properties of the optimal mean shift. Some numerical results are given which show that using the optimal mean shift considerably reduces the expected squared error loss compared to the no shift case.

Chapter 5 gives conclusions and directions for future work.

1.5   Basic models, notation, assumptions

Let X and Y be continuous random variables which denote the two respective component dimensions. Suppose that the process means of X and Y can be adjusted so that their difference is equal to a given target clearance C. Since we can re-center the distribution of X (or Y) so that $E[X] = E[Y]$, we let $C = 0$ without loss of generality. Let x_L, y_L and x_U, y_U denote the lower and upper limits for X and Y, respectively. Components with X values in the intervals $(-\infty, x_L]$ and (x_U, ∞) and components with Y values in the intervals $(-\infty, y_L]$ and (y_U, ∞) are rejected before the assembly process. Let n denote the number of groups, and let $(x_0, x_1, x_2, \ldots, x_{n-1}, x_n)$ and $(y_0, y_1, y_2, \ldots, y_{n-1}, y_n)$ be the partition limits for X and Y, respectively, where $x_0 = x_L$, $y_0 = y_L$, $x_n = x_U$ and $y_n = y_U$. Throughout this thesis, we let the number of groups n and the lower and upper limits x_L, y_L, x_U, y_U be predetermined and not subject to optimization. Components with X values in the interval $(x_{i-1}, x_i]$ and components with Y values in the interval $(y_{i-1}, y_i]$ are sorted into the ith group of X and Y, respectively. Then, the product is assembled by randomly selecting mating components from

corresponding groups as shown in Figure 1.1.

Throughout this thesis, we assume that $Pr(x_{i-1} < X \le x_i) = Pr(y_{i-1} < Y \le y_i)$ holds for $i = 1, 2, \ldots, n$ so that there are no surplus components. The probability of the ith group,

$$\frac{Pr(x_{i-1} < X \le x_i)}{Pr(x_L < X \le x_U)} = \frac{Pr(y_{i-1} < Y \le y_i)}{Pr(y_L < Y \le y_U)},$$

is denoted by p_i. Let X_i and Y_i denote the truncated random variables of X and Y defined on the intervals $(x_{i-1}, x_i]$ and $(y_{i-1}, y_i]$, respectively, that is to say, $X_i \equiv [X|x_{i-1} < X \le x_i]$ and $Y_i \equiv [Y|y_{i-1} < Y \le y_i]$.

When mating components are randomly selected from corresponding groups, the expected squared error loss is expressed as

$$\sum_{i=1}^{n} E[(X_i - Y_i)^2]p_i. \tag{1.1}$$

We note that (1.1) also expresses the variance of the clearance.

Assuming that X and Y are identically distributed and that $(x_0, x_1, x_2, \ldots, x_{n-1}, x_n) = (y_0, y_1, y_2, \ldots, y_{n-1}, y_n)$, Mease et al. (2004) showed that the set of partition limits $(x_1, x_2, \ldots, x_{n-1})$ which minimizes (1.1) satisfies

$$x_i = \frac{E[X_i] + E[X_{i+1}]}{2} = \frac{E[X|x_{i-1} < X \le x_i] + E[X|x_i < X \le x_{i+1}]}{2}, \quad i = 1, 2, \ldots, n-1. \tag{1.2}$$

They gave the following algorithm for obtaining the partition limits which satisfy the equations (1.2).

1. Begin with an initial set of partition limits $(x_0^0, x_1^0, x_2^0, \ldots, x_{n-1}^0, x_n^0)$, where $x_0^0 = x_0$ and $x_n^0 = x_n$.

2. Put $x_i^1 = (E[X|x_{i-1}^0 < X \le x_i^0] + E[X|x_i^0 < X \le x_{i+1}^0])/2$, $i = 1, 2, \ldots, n-1$.

3. Finish the iteration if

$$|x_i^1 - x_i^0| < \epsilon, \quad i = 1, 2, \ldots, n-1,$$

where ϵ is a predetermined calculation error bound. Otherwise, repeat Step 2 with $(x_1^1, x_2^1, \ldots, x_{n-1}^1)$ in place of $(x_1^0, x_2^0, \ldots, x_{n-1}^0)$.

If the algorithm does converge, the resulting set of partition limits satisfies the equations (1.2). However, the equations (1.2) are necessary conditions for the optimal partition. In general the equations (1.2) have multiple solutions and the objective function (1.1) has multiple local minima and local maxima. Mease et al. (2004) showed that the solution to (1.2) is unique if the dimensional distribution satisfies

Condition (A) : $E[X|t < X \leq t + u] - t$ is nonincreasing in t for all $u > 0$,

which is guaranteed if the distribution is strongly unimodal (i.e., $\log f$ is concave where f denotes the density function of X). Note that normal, logistic, gamma, and uniform distributions are all strongly unimodal. Although strong unimodality may seem rather restrictive, they have given examples which show that the solution to (1.2) is not unique for some symmetric and unimodal distributions.

Chapter 2

Optimal Partitioning of the Distributions When Measurement Error Is Present

2.1 Introduction

Optimal partitioning of the dimensional distributions in selective assembly has been discussed by Kwon et al. (1999) and Mease et al. (2004). These previous works implicitly assumed that measurement error is absent. However, most measurement processes have inherent variability which may not be negligibly small. This chapter studies optimal partitioning when measurement error is present.

This chapter is organized as follows. After giving models, notation, and assumptions in Section 2.2, we give equations for the optimal partition limits in Section 2.3. In Section 2.4, it is shown that the solution to them is unique when the component dimension and the measurement error are normally distributed. In Section 2.5, assuming normal distribution, we give a relationship between the expected squared error losses based on the optimal partition limits for the cases when measurement error is present and not present. We also give some numerical results to evaluate the effect of measurement error on the expected squared error loss in Section 2.6.

2.2 Models, notation, and assumptions

Since we assume that the two component dimensions are identically distributed throughout this chapter, we let f denote the common density function of X and Y. Let the measurement errors of the two component dimensions, denoted by W^X and W^Y, be independent of X and Y, and let their distributions be continuous and common. g denotes the common density function of W^X and W^Y. Let Z^X and Z^Y be the random variables which denote the observations of the two component dimensions, that is, $Z^X \equiv X + W^X$ and $Z^Y \equiv Y + W^Y$. Therefore, Z^X and Z^Y are identically distributed,

and we denote their common density function by h. Note that

$$h(z) = \int_{-\infty}^{\infty} f(x)g(z-x)dx. \tag{2.1}$$

In selective assembly when measurement error is present, it is the distributions of Z^X and Z^Y, not those of X and Y that are partitioned into n groups because we only observe Z^X and Z^Y when each component dimension is measured. The two sets of partition limits for Z^X and Z^Y are the same so that there are no surplus components. Then we let

$$(z_0, z_1, z_2, \ldots, z_{n-1}, z_n)$$

be the common partition limits, where $z_0 = x_L$ and $z_n = x_U$. Components X and Y with their observations Z^X and Z^Y in the intervals $(-\infty, x_L]$ and (x_U, ∞) are rejected. Components X and Y with their observations Z^X and Z^Y in the interval $(z_{i-1}, z_i]$ are sorted into the ith group of X and Y, respectively. Here we put

$$p_i = \frac{P(z_{i-1} < Z^X \le z_i)}{P(x_L < Z^X \le x_U)}, \ i = 1, 2, \ldots, n-1.$$

Let Z_i^X and Z_i^Y be the truncated random variables of Z^X and Z^Y defined on the interval $(z_{i-1}, z_i]$, respectively. We further let X_i and Y_i denote the true dimensions of the two components X and Y conditioned so that the corresponding observations Z^X and Z^Y are on the interval $(z_{i-1}, z_i]$, respectively. Note that, although Z_i^X and Z_i^Y take values in the interval $(z_{i-1}, z_i]$, X_i and Y_i may take values outside that interval.

Then the problem is to find the set of partition limits $(z_1, z_2, \ldots, z_{n-1})$ which minimizes the expected squared error loss $\sum_{i=1}^{n} E[(X_i - Y_i)^2]p_i$.

2.3   Equations for the optimal partition limits

We give equations for the optimal partition limits in this section.

Since X_i and Y_i are independently and identically distributed, we see that

$$\sum_{i=1}^{n} E[(X_i - Y_i)^2]p_i = 2\sum_{i=1}^{n} \left\{ E[X_i^2] - (E[X_i])^2 \right\} p_i. \tag{2.2}$$

We note that the probability p_i is given as

$$p_i = \frac{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} f(x)g(z-x)dxdz}{\int_{z_0}^{z_n} \int_{-\infty}^{\infty} f(x)g(z-x)dxdz}. \tag{2.3}$$

We also note that

$$E[X_i] = \frac{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} xf(x)g(z-x)dxdz}{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} f(x)g(z-x)dxdz}. \tag{2.4}$$

From (2.2)-(2.4) and the equation $\sum_{i=1}^{n} E[X_i^2]p_i = E[X^2|z_0 < X + W^X \leq z_n]$, the expected squared error loss $\sum_{i=1}^{n} E[(X_i - Y_i)^2]p_i$ is rewritten as

$$2E[X^2|z_0 < X + W^X \leq z_n]$$

$$-2\sum_{i=1}^{n} \frac{\left\{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} xf(x)g(z-x)dxdz\right\}^2}{\left\{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} f(x)g(z-x)dxdz\right\}\left\{\int_{z_0}^{z_n} \int_{-\infty}^{\infty} f(x)g(z-x)dxdz\right\}}.$$

Thus, it is sufficient for us to maximize

$$\sum_{i=1}^{n} \frac{\left\{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} xf(x)g(z-x)dxdz\right\}^2}{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} f(x)g(z-x)dxdz}. \tag{2.5}$$

The partial derivative of (2.5) with respect to z_i ($1 \leq i \leq n-1$) is given by

$$\frac{2\left\{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} xf(x)g(z-x)dxdz\right\} \int_{-\infty}^{\infty} xf(x)g(z_i-x)dx}{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} f(x)g(z-x)dxdz}$$

$$-\frac{\left\{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} xf(x)g(z-x)dxdz\right\}^2 \int_{-\infty}^{\infty} f(x)g(z_i-x)dx}{\left\{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} f(x)f(z-x)dxdz\right\}^2}$$

$$-\frac{2\left\{\int_{z_i}^{z_{i+1}} \int_{-\infty}^{\infty} xf(x)g(z-x)dxdz\right\} \int_{-\infty}^{\infty} xf(x)g(z_i-x)dx}{\int_{z_i}^{z_{i+1}} \int_{-\infty}^{\infty} f(x)g(z-x)dxdz}$$

$$+\frac{\left\{\int_{z_i}^{z_{i+1}} \int_{-\infty}^{\infty} xf(x)g(z-x)dxdz\right\}^2 \int_{-\infty}^{\infty} f(x)g(z_i-x)dx}{\left\{\int_{z_i}^{z_{i+1}} \int_{-\infty}^{\infty} f(x)g(z-x)dxdz\right\}^2}.$$

Setting this partial derivative equal to 0, we obtain

$$2E[X_i] \int_{-\infty}^{\infty} xf(x)g(z_i-x)dx - (E[X_i])^2 \int_{-\infty}^{\infty} f(x)g(z_i-x)dx$$

$$-2E[X_{i+1}] \int_{-\infty}^{\infty} xf(x)g(z_i-x)dx + (E[X_{i+1}])^2 \int_{-\infty}^{\infty} f(x)g(z_i-x)dx = 0.$$

Thus, we see that the optimal partition limits satisfy

$$E[X|X + W^X = z_i] = \frac{E[X_i] + E[X_{i+1}]}{2}, \quad i = 1, 2, \ldots, n-1, \tag{2.6}$$

where $E[X|X + W^X = z_i] = \int_{-\infty}^{\infty} x f(x) g(z_i - x) dx / (\int_{-\infty}^{\infty} f(x) g(z_i - x) dx)$.

It seems quite difficult for us to discuss the uniqueness of the solution to (2.6) generally. However, we will show the uniqueness for the case when X (and Y) and W^X (and W^Y) are normally distributed in the next section.

2.4 Uniqueness of optimal partition

The optimal partition limits satisfy the equations (2.6), and thus in general they are dependent on both f and g, that is, the distributions of X and W^X. However, as is shown in the following proposition, the optimal partition limits depend only on the distribution of the observation Z^X if the conditional expectation of the component dimension given the value of the observation is a linear function of the value. Using this, we can show that the solution to the equations for the optimal partition limits is unique when the component dimension and the measurement error are normally distributed.

Proposition 2.1 *Suppose that for some $k \neq 0$ and b,*

$$E[X|X + W^X = z] = kz + b, \quad \text{for any } z \in (z_0, z_n]. \tag{2.7}$$

Then the equations (2.6) reduce to

$$z_i = \frac{E[Z_i^X] + E[Z_{i+1}^X]}{2}, \quad i = 1, 2, \ldots, n-1. \tag{2.8}$$

Proof From (2.1) and (2.3), we can rewrite the equations (2.6) as

$$\frac{\int_{-\infty}^{\infty} x f(x) g(z_i - x) dx}{h(z_i)}$$
$$= \frac{1}{2} \left\{ \frac{\int_{z_{i-1}}^{z_i} \int_{-\infty}^{\infty} x f(x) g(z - x) dx dz}{\int_{z_{i-1}}^{z_i} h(z) dz} + \frac{\int_{z_i}^{z_{i+1}} \int_{-\infty}^{\infty} x f(x) g(z - x) dx dz}{\int_{z_i}^{z_{i+1}} h(z) dz} \right\}. \tag{2.9}$$

Since the condition $E[X|X + W^X = z] = kz + b$ can be rewritten as

$$\int_{-\infty}^{\infty} xf(x)g(z-x)dx = kzh(z) + bh(z),$$

we can rewrite the equations (2.9) as the ones (2.8), which completes the proof. $\qquad\square$

From Proposition 2.1, we see that if the condition (2.7) is satisfied, then the optimal partition limits satisfy the equations (2.8) which are expressed by using only the marginal distribution of $Z^X = X + W^X$. We notice that the equations (2.8) are of the same form as those (1.2). Therefore, as is given in Mease et al. (2004), the solution to (2.8) is unique provided that the distribution of Z^X (and Z^Y) is strongly unimodal.

Example 2.1

The most important example is the one when the component dimension X and the measurement error W^X are normally distributed. Let X (and Y) be distributed as $N(\mu, \sigma^2)$ and let W^X (and W^Y) as $N(0, \tau^2)$. Then we see that

$$E[X|X + W^X = z] = \frac{\sigma^2}{\sigma^2 + \tau^2}z + \frac{\tau^2}{\sigma^2 + \tau^2}\mu. \qquad (2.10)$$

From Proposition 2.1, we see that the optimal partition limits satisfy the equations (2.8). Since the observation $Z^X = X + W^X$ (and Z^Y) follows $N(\mu, \sigma^2 + \tau^2)$, which is strongly unimodal, the solution to (2.8) is unique.

We should also notice that, we need not specify the values of the variances σ^2, τ^2 separately and we only need the values of the expectation and the variance of the observation Z^X, that is, the values of μ and $\sigma^2 + \tau^2$, to obtain the optimal partition limits. Moreover, we see from Proposition 2.1 that replacing X by Z^X, we can obtain the optimal partition limits by solving the equations (1.2) using the algorithm given in Section 1.5. As a matter of fact, this means that we can clearly obtain the optimal partition limits without worrying about whether measurement error is present or absent.

Example 2.2

We consider the case when X (and Y) and W^X (and W^Y) are gamma distributed with a common scale parameter. If X (and Y) follows $Ga(\nu_1, \alpha)$ and W^X (and W^Y) follows

$Ga(\nu_2, \alpha)$ where the density of $Ga(\nu, \alpha)$ is

$$f(x) = \frac{1}{\alpha^\nu \Gamma(\nu)} x^{\nu-1} e^{-\frac{x}{\alpha}}, \quad x > 0,$$

and $\Gamma(\nu)$ is the gamma function, then we obtain

$$E[X|X + W^X = z] = \frac{\nu_1}{\nu_1 + \nu_2} z, \quad z \geq 0.$$

From Proposition 2.1, we see that the optimal partition limits satisfy the equations (2.8). Since the observation $Z^X = X + W$ follows $Ga(\nu_1 + \nu_2, \alpha)$, which is strongly unimodal, the solution to (2.8) is unique.

2.5 Some properties of optimal partition for normal distribution

Here we assume normal distribution and give a relationship between the expected squared error losses when measurement error is present and not present.

For that purpose, we first consider the case when measurement error is not present and assume that the component dimension X (and Y) follows the normal distribution $N(0, \sigma^2)$. Suppose that $x_L = -\infty$ and $x_U = \infty$, and let the optimal partition limits which satisfy the equations (1.2) be

$$(-\infty, x_1^*, x_2^*, \ldots, x_{n-1}^*, \infty).$$

Also, we let X_i^* and Y_i^* denote the truncated random variables of X and Y defined on the interval $(x_{i-1}^*, x_i^*]$, respectively. We denote $P(x_{i-1}^* < X \leq x_i^*)$ by p_i^*. Then, the expected squared error loss $\sum_{i=1}^n E[(X_i^* - Y_i^*)^2] p_i^*$ is given as

$$2 \sum_{i=1}^n \{E[(X_i^*)^2] - (E[X_i^*])^2\} p_i^* = 2\sigma^2 - 2 \sum_{i=1}^n (E[X_i^*])^2 p_i^*. \tag{2.11}$$

Turning now to the case with the measurement error W^X (and W^Y), which is distributed as $N(0, \tau^2)$. Then, the optimal partition limits for $Z^X = X + W^X$ (and Z^Y)

are

$$(-\infty, z_1^*, z_2^*, \ldots, z_{n-1}^*, \infty)$$
$$= \left(-\infty, \frac{(\sigma^2 + \tau^2)^{\frac{1}{2}}}{\sigma} x_1^*, \frac{(\sigma^2 + \tau^2)^{\frac{1}{2}}}{\sigma} x_2^*, \ldots, \frac{(\sigma^2 + \tau^2)^{\frac{1}{2}}}{\sigma} x_{n-1}^*, \infty\right). \quad (2.12)$$

This is easily seen because we can show that if $(-\infty, x_1^*, x_2^*, \ldots, x_{n-1}^*, \infty)$ satisfies the equations (1.2), then $(-\infty, z_1^*, z_2^*, \ldots, z_{n-1}^*, \infty)$ satisfies the equations (2.8) by using a scale change.

In the following, we let Z_i^X and Z_i^Y be the truncated random variables of Z^X and Z^Y defined on the interval $(z_{i-1}^*, z_i^*]$, respectively. We also let X_i and Y_i be the random variables X and Y conditioned so that the corresponding observations Z^X and Z^Y are on the interval $(z_{i-1}^*, z_i^*]$, respectively. From (2.10), we first notice that

$$\int_{-\infty}^{\infty} x f(x) g(z - x) dx = \frac{\sigma^2}{\sigma^2 + \tau^2} z h(z). \quad (2.13)$$

From (2.3) and (2.13), we see that

$$E[X_i] = \frac{\sigma^2}{\sigma^2 + \tau^2} E[Z_i^X]. \quad (2.14)$$

Note that $P(z_{i-1}^* < Z^X \leq z_i^*) = P(x_{i-1}^* < X \leq x_i^*) = p_i^*$. Therefore, from (2.4) and (2.14), we see that the expected squared error loss when the measurement error is present is given as

$$\sum_{i=1}^{n} E[(X_i - Y_i)^2] p_i^* = 2E[X^2] - 2\sum_{i=1}^{n} (E[X_i])^2 p_i^*$$
$$= 2\sigma^2 - 2\sum_{i=1}^{n} \left(\frac{\sigma^2}{\sigma^2 + \tau^2} E[Z_i^X]\right)^2 p_i^* \quad (2.15)$$
$$= 2\sigma^2 - 2\frac{\sigma^2}{\sigma^2 + \tau^2} \sum_{i=1}^{n} (E[X_i^*])^2 p_i^*,$$

where in the last equality we have used the fact that

$$
\begin{aligned}
E[Z_i^X] &= E[Z^X | z_{i-1}^* < Z^X \leq z_i^*] \\
&= E\left[\frac{(\sigma^2 + \tau^2)^{1/2}}{\sigma} X \,\Big|\, x_{i-1}^* < X \leq x_i^*\right] \\
&= \frac{(\sigma^2 + \tau^2)^{1/2}}{\sigma} E[X_i^*].
\end{aligned}
$$

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Comparing (2.11) with (2.15), we see that the expected squared error loss for the case with the measurement error increases by

$$\frac{2\tau^2}{\sigma^2 + \tau^2} \sum_{i=1}^{n} (E[X_i^*])^2 p_i^* \qquad (2.16)$$

compared to that for the case without measurement error. We note that (2.16) increases when n increases since (2.11) (the expected loss when measurement error is not present) decreases when n increases. Thus, we see that the difference between the expected squared error losses when the measurement error is present and not present gets larger when n gets larger.

The expected squared error loss for the case without measurement error converges to 0 when n goes to infinity because we can show that $\sum_{i=1}^{n} (E[X_i^*])^2 p_i^*$ converges to σ^2 when n goes to infinity. However, the expected loss of $\frac{2\sigma^2 \tau^2}{\sigma^2 + \tau^2}$ remains even if n goes to infinity in the case when the measurement error is present.

From the above argument, we see that, even if n increases, we cannot obtain much loss reduction when considerable measurement error is present in contrast with the case when measurement error is not present.

2.6   Numerical results

Here we give some numerical results on the optimal partition limits and the expected squared error loss when the component dimension and the measurement error are normally distributed.

We let the component dimension be distributed as $N(0,1)$ without loss of generality. We set $x_L = -\infty$ and $x_U = \infty$.

Table 2.1 gives the results when measurement error is not present. Tables 2.2 and 2.3 give the results when the measurement error follows $N(0, 0.01)$ and $N(0, 0.1)$, respectively. Figure 2.1 compares the expected squared error losses when measurement error is present and not present.

Tables 2.1-2.3 show that the expected loss decreases when n increases, and endorse that (2.12) is satisfied for the optimal partition limits. Figure 2.1 endorses that the

difference between the expected squared error losses when measurement error is present and not present gets larger when n gets larger. We see that even if n increases, much loss reduction is not necessarily obtained when measurement error is present compared to the case when measurement error is not present.

Table 2.1. Optimal partition for $N(0,1)$ (without measurement error).

n	optimal partition limits	expected loss
1	–	2.0000
2	0	0.7268
3	±0.612	0.3803
4	0 ±0.982	0.2350
5	±0.382 ±1.244	0.1599
6	0 ±0.659 ±1.447	0.1160
7	±0.280 ±0.874 ±1.611	0.0880
8	0 ±0.501 ±1.050 ±1.748	0.0691
9	±0.222 ±0.681 ±1.198 ±1.866	0.0557
10	0 ±0.405 ±0.834 ±1.325 ±1.968	0.0459
11	±0.184 ±0.560 ±0.966 ±1.436 ±2.059	0.0384
12	0 ±0.340 ±0.694 ±1.081 ±1.534 ±2.141	0.0327
13	±0.157 ±0.476 ±0.813 ±1.184 ±1.623 ±2.215	0.0281
14	0 ±0.294 ±0.596 ±0.918 ±1.277 ±1.703 ±2.282	0.0245
15	±0.137 ±0.414 ±0.703 ±1.013 ±1.360 ±1.776 ±2.344	0.0215

Table 2.2. Optimal partition for $N(0,1)$ (with measurement error $N(0,0.01)$).

n	optimal partition limits	expected loss
1	–	2.0000
2	0	0.7394
3	±0.615	0.3964
4	0 ±0.986	0.2524
5	±0.384 ±1.251	0.1781
6	0 ±0.662 ±1.454	0.1346
7	±0.282 ±0.879 ±1.619	0.1069
8	0 ±0.503 ±1.055 ±1.757	0.0882
9	±0.223 ±0.685 ±1.204 ±1.875	0.0750
10	0 ±0.407 ±0.838 ±1.331 ±1.978	0.0652
11	±0.185 ±0.563 ±0.970 ±1.443 ±2.069	0.0579
12	0 ±0.342 ±0.698 ±1.087 ±1.542 ±2.151	0.0522
13	±0.158 ±0.478 ±0.817 ±1.190 ±1.631 ±2.226	0.0477
14	0 ±0.295 ±0.599 ±0.923 ±1.283 ±1.712 ±2.293	0.0440
15	±0.138 ±0.416 ±0.706 ±1.018 ±1.367 ±1.785 ±2.355	0.0411

Table 2.3. Optimal partition for $N(0,1)$ (with measurement error $N(0,0.1)$).

| n	optimal partition limits	expected loss
1	–	2.0000
2	0	0.8425
3	±0.642	0.5276
4	0 ±1.030	0.3954
5	±0.401 ±1.305	0.3272
6	0 ±0.691 ±1.517	0.2872
7	±0.294 ±0.917 ±1.689	0.2618
8	0 ±0.525 ±1.101 ±1.833	0.2446
9	±0.233 ±0.714 ±1.256 ±1.957	0.2325
10	0 ±0.425 ±0.875 ±1.389 ±2.064	0.2235
11	±0.193 ±0.587 ±1.013 ±1.506 ±2.160	0.2168
12	0 ±0.357 ±0.728 ±1.134 ±1.609 ±2.245	0.2115
13	±0.165 ±0.499 ±0.852 ±1.242 ±1.702 ±2.323	0.2074
14	0 ±0.308 ±0.625 ±0.963 ±1.339 ±1.786 ±2.393	0.2041
15	±0.144 ±0.435 ±0.737 ±1.062 ±1.427 ±1.863 ±2.458	0.2013
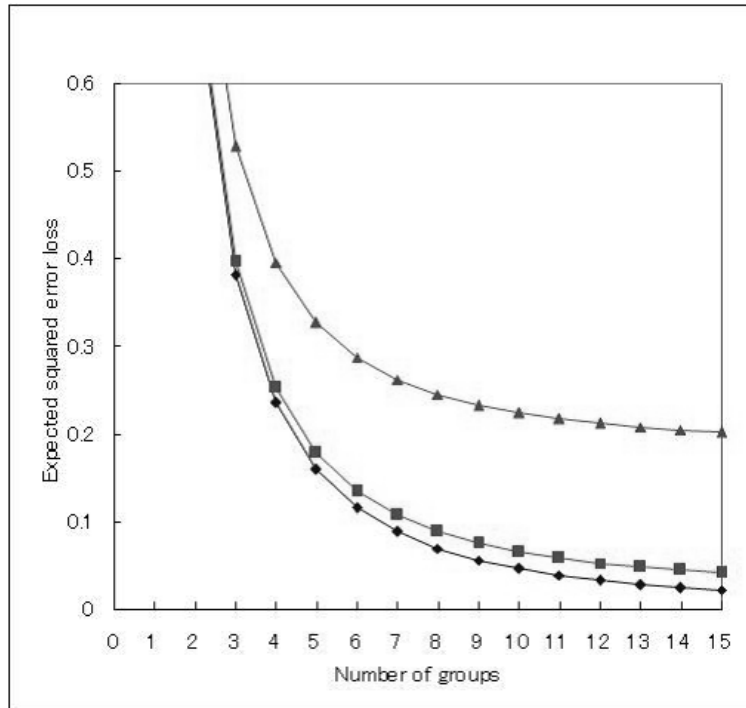
Figure 2.1. Expected squared error losses (without error;   error N(0,0.01);   error N(0,0.1)   .

Numerical example

Kannan and Jayabalan (2001a) have considered a hole and shaft assembly for analysis when measurement error is not present. Although the variances of the inner and

outer diameters of the hole and shaft are slightly different in their example, here we set them equal. The inner diameter of the hole (X) and the measurement error (W^X) are distributed as $N(35.006\text{mm}, (0.002\text{mm})^2)$ and $N(0\text{mm}, (0.0002\text{mm})^2)$, respectively. The outer diameter of the shaft (Y) and the measurement error (W^Y) are distributed as $N(34.994\text{mm}, (0.002\text{mm})^2)$ and $N(0\text{mm}, (0.0002\text{mm})^2)$, respectively. Then, the observation of the dimension of the hole (Z^X) is distributed as $N(35.006\text{mm}, (0.002^2 + 0.0002^2)\text{mm}^2)$, and that of the shaft (Z^Y) as $N(34.994\text{mm}, (0.002^2 + 0.0002^2)\text{mm}^2)$. The target of the clearance is 0.012mm. Tables 2.4 and 2.5 give the optimal partition limits, the expected squared error losses, and the probabilities of the groups for $n = 6$ and $n = 9$, respectively. We note that, since the ratio of the variance of the component dimensions to that of the measurement errors is 1 : 0.01, the results can be obtained from Table 2 ($n = 6$ and $n = 9$) by applying a location shift and a scale change. For example, the optimal partition limits for shaft in Table 2.4 are obtained by

$$\text{(the optimal partition limits for } n = 6 \text{ in Table 2.2)} \times 0.002 + 34.994,$$

such as

$$0.662 \times 0.002 + 34.994 = 34.99532 \text{ (the partition limit between 4th and 5th groups),}$$

and the expected squared error loss is obtained as

$$0.1346 \times (0.002)^2 = 0.538 \times 10^{-6}.$$

Table 2.4. Optimal partition for the shaft and hole example (n=6).

group	shaft (mm)		hole (mm)		expected loss	probability
	min	max	min	max	(10^{-6}mm^2)	
1	$-\infty$	34.99109	$-\infty$	35.00309	1.299	0.0740
2	34.99109	34.99268	35.00309	35.00468	0.468	0.1810
3	34.99268	34.99400	35.00468	35.00600	0.361	0.2450
4	34.99400	34.99532	35.00600	35.00732	0.361	0.2450
5	34.99532	34.99691	35.00732	35.00891	0.468	0.1810
6	34.99691	∞	35.00891	∞	1.299	0.0740
average					0.538	

Table 2.5. Optimal partition for the shaft and hole example (n=9).

group	shaft (mm)		hole (mm)		expected loss	probability
	min	max	min	max	(10^{-6}mm^2)	
1	-∞	34.99025	-∞	35.00225	1.051	0.0311
2	34.99025	34.99159	35.00225	35.00359	0.355	0.0845
3	34.99159	34.99263	35.00359	35.00463	0.252	0.1323
4	34.99263	34.99355	35.00463	35.00555	0.217	0.1644
5	34.99355	34.99445	35.00555	35.00645	0.208	0.1755
6	34.99445	34.99537	35.00645	35.00737	0.217	0.1644
7	34.99537	34.99641	35.00737	35.00841	0.252	0.1323
8	34.99641	35.99775	35.00841	35.00975	0.355	0.0845
9	34.99775	∞	35.00975	∞	1.051	0.0311
average					0.300	

Chapter 3

Optimal Partitioning of the Distributions When a Tolerance Constraint Is Given on the Clearance

3.1 Introduction

The problem of determining optimal partition limits has not been addressed under a tolerance constraint on the clearance, although such a constraint is usually present in practice. For the unconstrained optimal partitioning, the group width is smaller near the mode of the distribution and larger in the tails. If some groups in the tails have larger widths than the tolerance limit in the case of unconstrained optimal partitioning, we have unacceptable products with positive probability. If this probability is not negligibly small and we remove the unacceptable products, additional measurement and inspection will be necessary, with correspondingly high cost. This chapter studies optimal partitioning of the dimensional distributions when a tolerance constraint is given on the clearance. It is shown that the resulting constrained optimal partitioning is the partitioning which constrains the widths of some groups in the tails to the tolerance limit and which matches exactly the unconstrained optimal partitioning for the rest of the distribution.

For example, we consider the case in which X (and Y) follow $N(0,1)$, $x_L = -3$, $x_U = 3$, the tolerance limit Δ is 0.7, and the number of groups n is 10. If we use the unconstrained optimal partition limits for X and Y (the common partition limits are shown in Figure 3.1), then we have unacceptable products with positive probability, since 1.134 (the width of the first right- and left-hand side groups) exceeds the tolerance limit $\Delta = 0.7$. We also see that 2.196 (the sum of the widths of the outer 3 groups on both sides) exceeds $\Delta \times 3 = 2.1$. However, if we use the constrained optimal partition limits (see Figure 3.2), then the widths of the outer 2 groups on both sides are equal

to the tolerance limit and the partitioning of the inner 6 groups is the unconstrained optimal partitioning for the distribution truncated at ± 1.6.
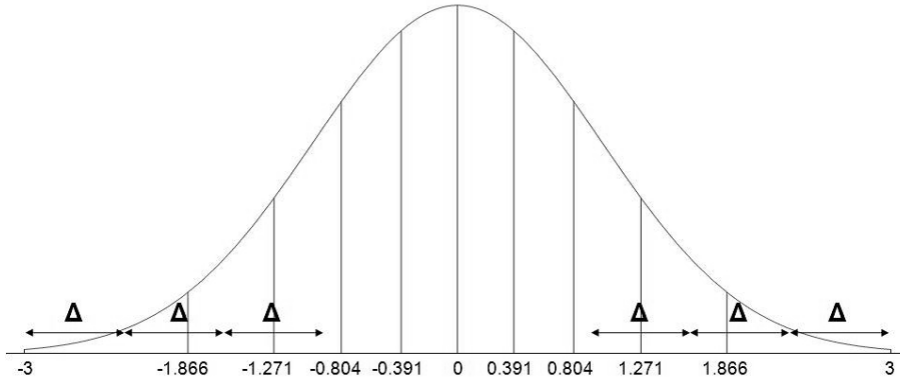

Figure 3.1. Unconstrained optimal partitioning for $n = 10$ and $\Delta = 0.7$.
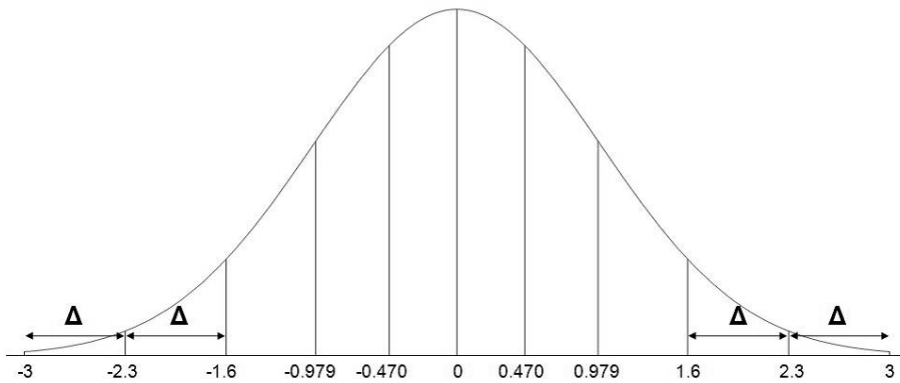
Figure 3.2. Constrained optimal partitioning for $n = 10$ and $\Delta = 0.7$.

This chapter is organized as follows. In Section 3.2, after presenting models, notation, and assumptions, we describe two existing strategies: unconstrained optimal partitioning and equal width partitioning schemes. In Section 3.3, conditions for a set of constrained optimal partition limits are derived, and a numerical algorithm for finding such a set is given. In Section 3.4, under the assumption that the dimensional distribution is strongly unimodal, we discuss some properties of unconstrained and constrained optimal partitioning schemes and show the uniqueness of constrained optimal partition limits. The final section gives some numerical results that enable us to compare the three strategies.

3.2 Models and two existing strategies

In this chapter, we use the notation given in Section 1.5. Let $\Delta > 0$ be the given tolerance limit on the clearance $X - Y$. Then, the width of any group is necessarily less than or equal to Δ, that is,

$$x_i - x_{i-1} \le \Delta, \quad i = 1, 2, \ldots, n.$$

We assume that $x_U - x_L < n\Delta$ so that the problem of choosing optimal partition limits makes sense. Let f denote the common density function of X and Y after truncation at x_L and x_U. We suppose that $f(x) > 0$ for $x_0 < x < x_n$, and $f(x) = 0$ otherwise. The expected squared error loss is given by $\sum_{i=1}^{n} E[(X_i - Y_i)^2]p_i$. We now describe two existing partitioning methods.

Equal width partitioning (EWP)

Equal width partition limits (EWP limits) are given by

$$x_i - x_{i-1} = \frac{x_n - x_0}{n}, \quad i = 1, 2, \ldots, n.$$

Apparently any product will satisfy the tolerance constraint in this case. However, EWP does not necessarily minimize the expected squared error loss.

Unconstrained optimal partitioning (UOP)

UOP is the partitioning which minimizes the expected squared error loss. Let $(x_0, x_1^\dagger, x_2^\dagger, \ldots, x_{n-1}^\dagger, x_n)$ denote the unconstrained optimal partition limits (UOP limits). Recall that Mease et al. (2004) showed that the UOP limits satisfy the equations (1.2) and also showed that the solution to (1.2) is unique if the distribution satisfies

Condition (A) : $E[X | t < X \le t + u] - t$ is nonincreasing in t for all $u > 0$,

which is guaranteed if the density f is strongly unimodal. With UOP, some groups have larger widths than the tolerance limit Δ in some cases. When mating components are randomly selected from the ith groups, the probability that the tolerance constraint

is not satisfied (the probability of non-acceptance) is given by

$$
Q_i = \begin{cases} \dfrac{2 \int_{x_{i-1}^\dagger + \Delta}^{x_i^\dagger} f(x) \int_{x_{i-1}^\dagger}^{x - \Delta} f(y) dy dx}{\left(\int_{x_{i-1}^\dagger}^{x_i^\dagger} f(x) dx \right)^2}, & \text{if } x_i^\dagger - x_{i-1}^\dagger > \Delta, \\[4mm] 0, & \text{if } x_i^\dagger - x_{i-1}^\dagger \le \Delta. \end{cases}
$$

The overall probability of non-acceptance for UOP is obtained by $\sum_{i=1}^n Q_i p_i$.

In Sections 3.3 and 3.4, we study optimal partitioning to minimize the expected squared error loss subject to the tolerance constraint. We show that constrained optimal partitioning is the partitioning which constrains the widths of some groups in the tails to the tolerance limit and which corresponds to UOP for the rest of the distribution. We also establish the uniqueness of the constrained optimal partition limits.

3.3 Constrained optimal partitioning (COP) and conditions for COP limits

COP is the partitioning which minimizes $\sum_{i=1}^n E[(X_i - Y_i)^2] p_i$ subject to the restrictions $x_i - x_{i-1} \le \Delta$, $i = 1, 2, \ldots, n$. In this section, we derive conditions for COP limits, and describe an algorithm for finding them.

Since X_i and Y_i are independently and identically distributed, $\sum_{i=1}^n E[(X_i - Y_i)^2] p_i$ can be rewritten as

$$
2 \int_{x_0}^{x_n} x^2 f(x) dx - 2 \sum_{i=1}^n (E[X_i])^2 p_i.
$$

Thus we may simply

$$
\text{maximize} \quad g(x_1, x_2, \ldots, x_{n-1}) = \sum_{i=1}^n (E[X_i])^2 p_i = \sum_{i=1}^n \frac{\left(\int_{x_{i-1}}^{x_i} x f(x) dx \right)^2}{\int_{x_{i-1}}^{x_i} f(x) dx},
$$
$$
\text{subject to} \quad 0 \le x_i - x_{i-1} \le \Delta, \quad i = 1, 2, \ldots, n.
$$

To derive necessary conditions for COP limits, we first note that the partial derivative of $g(x_1, x_2, \ldots, x_{n-1})$ with respect to each x_i ($1 \le i \le n - 1$) is given by

$$
\frac{\partial g(x_1, x_2, \ldots, x_{n-1})}{\partial x_i} = f(x_i)(E[X_{i+1}] - E[X_i])(E[X_i] + E[X_{i+1}] - 2x_i), \tag{3.1}
$$

as is shown in Mease et al. (2004). Let

$$h(x|x_{i-1}, x_{i+1}) = E[X|x_{i-1} < X \leq x] + E[X|x < X \leq x_{i+1}] - 2x. \qquad (3.2)$$

Then we see that the sign of (3.1) is the same as that of $h(x_i|x_{i-1}, x_{i+1})$. For any fixed $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{n-1}$ such that $x_{i+1} - x_{i-1} < 2\Delta$, if $x_i = x_i^\circ$ maximizes $g(x_1, x_2, \ldots, x_{n-1})$ subject to the constraints $x_i - x_{i-1} \leq \Delta$ and $x_{i+1} - x_i \leq \Delta$, then we have the following.

(i) If $x_{i+1} - \Delta < x_i^\circ < x_{i-1} + \Delta$, then $h(x_i^\circ|x_{i-1}, x_{i+1}) = 0$:

(ii) If $x_i^\circ = x_{i+1} - \Delta$, then $h(x_i^\circ|x_{i-1}, x_{i+1}) \leq 0$:

(iii) If $x_i^\circ = x_{i-1} + \Delta$, then $h(x_i^\circ|x_{i-1}, x_{i+1}) \geq 0$.

Therefore, we see that the COP limits, denoted by $(x_0, x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n)$, satisfy the following conditions.

$$E[X|x_{i-1}^* < X \leq x_i^*] + E[X|x_i^* < X \leq x_{i+1}^*] = 2x_i^*, \text{ if } x_{i+1}^* - \Delta < x_i^* < x_{i-1}^* + \Delta, \quad (3.3)$$

$$E[X|x_{i-1}^* < X \leq x_i^*] + E[X|x_i^* < X \leq x_{i+1}^*] \leq 2x_i^*, \text{ if } x_{i+1}^* - \Delta = x_i^* < x_{i-1}^* + \Delta, \quad (3.4)$$

$$E[X|x_{i-1}^* < X \leq x_i^*] + E[X|x_i^* < X \leq x_{i+1}^*] \geq 2x_i^*, \text{ if } x_{i+1}^* - \Delta < x_i^* = x_{i-1}^* + \Delta, \quad (3.5)$$

where we put $x_0^* = x_0$ and $x_n^* = x_n$. We can obtain partition limits satisfying the conditions (3.3)-(3.5) using the following iterative algorithm.

1. Begin with an initial set of partition limits $(x_0^0, x_1^0, x_2^0, \ldots, x_{n-1}^0, x_n^0)$ which satisfies $x_i^0 - x_{i-1}^0 \leq \Delta$, $i = 1, 2, \ldots, n$, where $x_0^0 = x_0$ and $x_n^0 = x_n$.

2. Repeat the following sub-steps for $k = 1, 2, \ldots, n-1$ sequentially with $x_0^1 = x_0$.

   (i) Compute $E[X|x_{k-1}^1 < X \leq x_{k+1}^0 - \Delta]$, $E[X|x_{k+1}^0 - \Delta < X \leq x_{k+1}^0]$, $E[X|x_{k-1}^1 < X \leq x_{k-1}^1 + \Delta]$, and $E[X|x_{k-1}^1 + \Delta < X \leq x_{k+1}^0]$.

   (ii) Put $x_k^1 = x_{k+1}^0 - \Delta$ if $E[X|x_{k-1}^1 < X \leq x_{k+1}^0 - \Delta] + E[X|x_{k+1}^0 - \Delta < X \leq x_{k+1}^0] \leq 2(x_{k+1}^0 - \Delta)$.

   Put $x_k^1 = x_{k-1}^1 + \Delta$ if $E[X|x_{k-1}^1 < X \leq x_{k-1}^1 + \Delta] + E[X|x_{k-1}^1 + \Delta < X \leq x_{k+1}^0] \geq 2(x_{k-1}^1 + \Delta)$.

   Put $x_k^1 = (E[X|x_{k-1}^1 < X \leq x_k^0] + E[X|x_k^0 < X \leq x_{k+1}^0])/2$ otherwise.

3. Finish the iteration if

$$|x_i^1 - x_i^0| < \epsilon, \ i = 1, 2, \ldots, n-1,$$

where ϵ is a predetermined calculation error bound. Otherwise, repeat Step 2 with $(x_1^1, x_2^1, \ldots, x_{n-1}^1)$ in place of $(x_1^0, x_2^0, \ldots, x_{n-1}^0)$.

We note that the last equation of step 2 (ii) is the one yielding the UOP limits (see the algorithm for finding UOP limits as given in Section 1.5). We can easily implement the algorithm using any software package which implements numerical integration. If the algorithm does converge, then the resulting set of partition limits satisfies the conditions (3.3)-(3.5). We show in the next section that, under the assumption that Condition (A) holds, the set of partition limits satisfying (3.3)-(3.5) is unique.

3.4 Some properties of COP and UOP and uniqueness of COP limits

In this section, we assume that Condition (A) is satisfied. We give a property of COP limits, which shows that the tolerance restriction is effective for the outer part of the distribution. We use it to prove the uniqueness of COP limits. Before that we give the corresponding property of UOP limits which shows that some groups in the tails of the dimensional distribution may have larger widths than Δ.

Proposition 3.1 *For some $0 \le a < b \le n$, the UOP limits satisfy*

$$x_i^\dagger - x_{i-1}^\dagger \ge \Delta, \quad i = 1, 2, \ldots, a, \tag{3.6}$$

$$x_i^\dagger - x_{i-1}^\dagger < \Delta, \quad i = a+1, a+2, \ldots, b, \tag{3.7}$$

$$x_i^\dagger - x_{i-1}^\dagger \ge \Delta, \quad i = b+1, b+2, \ldots, n. \tag{3.8}$$

Note that some equations are vacuous when $a = 0$ or $b = n$. The proofs of Propositions 3.1 and 3.2 and the Remark to them are deferred to Appendix A, B, and E, respectively.

Proposition 3.2 *For some $0 \le a < b \le n$, the COP limits satisfy*

$$x_i^* - x_{i-1}^* = \Delta, \quad i = 1, 2, \ldots, a, \tag{3.9}$$

$$x_i^* - x_{i-1}^* < \Delta, \quad i = a+1, a+2, \ldots, b, \tag{3.10}$$

$$x_i^* - x_{i-1}^* = \Delta, \quad i = b+1, b+2, \ldots, n, \tag{3.11}$$

and

$$2x_i^* < E[X|x_{i-1}^* < X \le x_i^*] + E[X|x_i^* < X \le x_{i+1}^*], \quad i = 1, 2, \ldots, a-1, \tag{3.12}$$

$$2x_a^* \le E[X|x_{a-1}^* < X \le x_a^*] + E[X|x_a^* < X \le x_{a+1}^*], \tag{3.13}$$

$$2x_i^* = E[X|x_{i-1}^* < X \le x_i^*] + E[X|x_i^* < X \le x_{i+1}^*], \quad i = a+1, a+2, \ldots, b-1, \tag{3.14}$$

$$2x_b^* \ge E[X|x_{b-1}^* < X \le x_b^*] + E[X|x_b^* < X \le x_{b+1}^*], \tag{3.15}$$

$$2x_i^* > E[X|x_{i-1}^* < X \le x_i^*] + E[X|x_i^* < X \le x_{i+1}^*], \quad i = b+1, b+2, \ldots, n-1. \tag{3.16}$$

Note that some equations are vacuous when $a = 0, 1$ or $b = a+1, n-1, n$.

Since the equations (3.14) are of the same form as those (1.2) which UOP limits are required to satisfy, we see from Proposition 3.2 that COP is the partitioning which constrains the widths of some groups in the tails ($i = 1, 2, \ldots, a$ and $i = b+1, b+2, \ldots, n$) to the tolerance limit Δ and which yields UOP limits for the distribution truncated at x_a^* and x_b^*. We may anticipate that if (3.6)-(3.8) hold for some $0 \le a < b \le n$ in UOP and (3.9)-(3.11) with a, b replaced by c, d hold for some $0 \le c < d \le n$ in COP, then $a \le c$ and $b \ge d$. This is true and is shown by the following.

Remark *If $x_i^\dagger - x_{i-1}^\dagger \ge \Delta$ holds for some $1 \le i \le n$, then $x_i^* - x_{i-1}^* = \Delta$ holds.*

Now we can show the uniqueness of COP limits using Proposition 3.2. Since (3.14) is of the same form as (1.2) and the solution to (1.2) is unique, we see that for given x_a^* and x_b^*, $x_{a+1}^*, \ldots, x_{b-1}^*$ are uniquely determined. Thus, we need only show the uniqueness of the values of a and b for the COP limits. The proof is rather technical and deferred to Appendix C.

3.5 Numerical results

We present some numerical results for COP, UOP, and EWP applied to the standard normal distribution truncated at ± 3. We report on their partition limits and expected squared error losses, and the probabilities of non-acceptance of UOP for two values of the tolerance limit.

Table 3.1 gives the UOP limits and the expected squared error losses. Tables 3.2 and 3.3 give the COP limits and the expected squared error losses for $\Delta = 1.3$ and $\Delta = 1$, respectively.

From Tables 3.2 and 3.3, we see that for COP, the tolerance restriction is effective for the outer part of the distribution, as is shown analytically in Proposition 3.2.

Table 3.4 compares COP, UOP, and EWP in terms of expected squared error loss when $\Delta = 1.3$. The overall probability of non-acceptance for UOP, $\sum_{i=1}^{n} Q_i p_i$, is also given in Table 3.4. We give the results for $n = 5, 6, 7, 8$ because for $n \leq 4$ no set of partition limits satisfies the tolerance constraint ($x_n - x_0 > n\Delta$ holds), and for $n \geq 9$ COP and UOP are the same (see Tables 3.1 and 3.2). Table 3.5 compares the three methods when $\Delta = 1$. For the same reason, the results are given for $n = 6, 7, 8, 9, 10$ only. Based on these numerical results, we summarize the findings as follows. Although UOP minimizes the expected squared error loss, its probabilities of non-acceptance are not negligibly small in some cases. For EWP, the tolerance constraint is strictly satisfied, but its expected squared error losses are generally greater than the corresponding values for UOP. For COP, the tolerance constraint is also satisfied and the expected squared error loss is considerably reduced, in comparison with EWP (about 20-30% loss reduction for larger values of n).

Table 3.1. UOP for $N(0,1)$ truncated at ± 3.

n	UOP limits					expected loss
1	–					1.9467
2	0					0.6948
3	± 0.604					0.3579
4	0	± 0.964				0.2179
5	± 0.375	± 1.215				0.1464
6	0	± 0.643	± 1.405			0.1050
7	± 0.273	± 0.850	± 1.555			0.0789
8	0	± 0.486	± 1.017	± 1.677		0.0614
9	± 0.215	± 0.659	± 1.154	± 1.779		0.0491
10	0	± 0.391	± 0.804	± 1.271	± 1.866	0.0401

Table 3.2. COP for $N(0,1)$ truncated at ± 3 ($\Delta = 1.3$).

n	COP limits				expected loss
5	± 0.483	± 1.7			0.1895
6	0	± 0.745	± 1.7		0.1162
7	± 0.291	± 0.911	± 1.7		0.0809
8	0	± 0.491	± 1.027	± 1.7	0.0614
9	± 0.215	± 0.659	± 1.154	± 1.779	0.0491
10	0	± 0.391	± 0.804	± 1.271	± 1.866	0.0401

Table 3.3. COP for $N(0,1)$ truncated at ± 3 ($\Delta = 1$).

n	COP limits				expected loss
6	0	± 1	± 2		0.1540
7	± 0.323	± 1.022	± 2		0.0943
8	0	± 0.546	± 1.158	± 2	0.0678
9	± 0.232	± 0.713	± 1.261	± 2	0.0516
10	0	± 0.409	± 0.843	± 1.341	± 2	0.0409

Table 3.4. Comparison of the three methods for $N(0,1)$ truncated at ± 3 (Δ=1.3).

	UOP		COP	EWP
n	expected loss	$\sum_{i=1}^{n} Q_i p_i$ ($\times 10^{-3}$)	expected loss	expected loss
5	0.1464	5.057	0.1895	0.2145
6	0.1050	1.749	0.1162	0.1540
7	0.0789	0.410	0.0809	0.1155
8	0.0614	0.010	0.0614	0.0896

Table 3.5. Comparison of the three methods for $N(0,1)$ truncated at ± 3 ($\Delta=1$).

	UOP		COP	EWP
n	expected loss	$\sum_{i=1}^{n} Q_i p_i$ ($\times 10^{-3}$)	expected loss	expected loss
6	0.1050	8.919	0.1540	0.1540
7	0.0789	4.683	0.0943	0.1155
8	0.0614	2.372	0.0678	0.0896
9	0.0491	1.088	0.0516	0.0715
10	0.0401	0.399	0.0409	0.0583

Appendices

A   Proof of Proposition 3.1

Since we prove this proposition by contradiction, we suppose that for some $1 \leq c < d \leq n-1$,

$$x_c^{\dagger} - x_{c-1}^{\dagger} < \Delta,$$

$$x_i^{\dagger} - x_{i-1}^{\dagger} \geq \Delta, \quad i = c+1, c+2, \ldots, d,$$

$$x_{d+1}^{\dagger} - x_d^{\dagger} < \Delta.$$

From the definition of $h(x|x_{i-1}, x_{i+1})$ given in (3.2), using Condition (A), we have

$$
\begin{aligned}
h(x_c^{\dagger}|x_{c-1}^{\dagger}, x_{c+1}^{\dagger}) &= E[X|x_{c-1}^{\dagger} < X \leq x_c^{\dagger}] + E[X|x_c^{\dagger} < X \leq x_{c+1}^{\dagger}] - 2x_c^{\dagger} \\
&\geq E[X|x_{c-1}^{\dagger} + (x_d^{\dagger} - x_c^{\dagger}) < X \leq x_d^{\dagger}] \\
&\quad + E[X|x_d^{\dagger} < X \leq x_{c+1}^{\dagger} + (x_d^{\dagger} - x_c^{\dagger})] - 2x_d^{\dagger}.
\end{aligned}
$$

Noting that $x_{d-1}^{\dagger} < x_{c-1}^{\dagger} + (x_d^{\dagger} - x_c^{\dagger})$ and $x_{d+1}^{\dagger} < x_{c+1}^{\dagger} + (x_d^{\dagger} - x_c^{\dagger})$, we have

$$
\begin{aligned}
h(x_c^{\dagger}|x_{c-1}^{\dagger}, x_{c+1}^{\dagger}) &> E[X|x_{d-1}^{\dagger} < X \leq x_d^{\dagger}] + E[X|x_d^{\dagger} < X \leq x_{d+1}^{\dagger}] - 2x_d^{\dagger} \\
&= h(x_d^{\dagger}|x_{d-1}^{\dagger}, x_{d+1}^{\dagger}).
\end{aligned}
$$

However, from (1.2), we have $h(x_c^{\dagger}|x_{c-1}^{\dagger}, x_{c+1}^{\dagger}) = h(x_d^{\dagger}|x_{d-1}^{\dagger}, x_{d+1}^{\dagger}) = 0$, and this is a contradiction. This completes the proof.

B   Proof of Proposition 3.2

We first show that (3.9)-(3.11) hold for some $0 \leq a < b \leq n$ by contradiction. Suppose

that for some $1 \leq c < d \leq n-1$,

$$x_c^* - x_{c-1}^* < \Delta,$$
$$x_i^* - x_{i-1}^* = \Delta, \quad i = c+1, c+2, \ldots, d,$$
$$x_{d+1}^* - x_d^* < \Delta.$$

Using Condition (A), we have

$$
\begin{aligned}
h(x_c^*|x_{c-1}^*, x_{c+1}^*) &= E[X|x_{c-1}^* < X \leq x_c^*] + E[X|x_c^* < X \leq x_{c+1}^*] - 2x_c^* \\
&\geq E[X|x_{c-1}^* + (d-c)\Delta < X \leq x_c^* + (d-c)\Delta] \\
&\quad + E[X|x_c^* + (d-c)\Delta < X \leq x_{c+1}^* + (d-c)\Delta] - 2\{x_c^* + (d-c)\Delta\}.
\end{aligned}
$$

Since $x_{d-1}^* < x_{c-1}^* + (d-c)\Delta$, $x_d^* = x_c^* + (d-c)\Delta$, and $x_{d+1}^* < x_{c+1}^* + (d-c)\Delta$, we have

$$
\begin{aligned}
h(x_c^*|x_{c-1}^*, x_{c+1}^*) &> E[X|x_{d-1}^* < X \leq x_d^*] + E[X|x_d^* < X \leq x_{d+1}^*] - 2x_d^* \\
&= h(x_d^*|x_{d-1}^*, x_{d+1}^*).
\end{aligned}
$$

Since $h(x_c^*|x_{c-1}^*, x_{c+1}^*) \leq 0$ from (3.4), we have $h(x_d^*|x_{d-1}^*, x_{d+1}^*) < 0$. However, since $h(x_d^*|x_{d-1}^*, x_{d+1}^*) \geq 0$ from (3.5), we have a contradiction. Thus, we have shown that (3.9)-(3.11) are satisfied for some $0 \leq a < b \leq n$.

Next, we show that (3.12)-(3.16) are also satisfied for the same a and b for which (3.9)-(3.11) are satisfied. From (3.10) and (3.3), we have (3.14). When $a \geq 1$, from $x_{a+1}^* - \Delta < x_a^* = x_{a-1}^* + \Delta$ and (3.5), we have

$$h(x_a^*|x_{a-1}^*, x_{a+1}^*) = E[X|x_{a-1}^* < X \leq x_a^*] + E[X|x_a^* < X \leq x_{a+1}^*] - 2x_a^* \geq 0, \quad (3.17)$$

that is, we have (3.13). When $a \geq 2$, using Condition (A) and (3.9) and (3.10), we have

$$
\begin{aligned}
&E[X|x_{a-1}^* < X \leq x_a^*] + E[X|x_a^* < X \leq x_{a+1}^*] - 2x_a^* \\
\leq\ &E[X|x_{a-1}^* - \Delta < X \leq x_a^* - \Delta] + E[X|x_a^* - \Delta < X \leq x_{a+1}^* - \Delta] - 2(x_a^* - \Delta) \\
<\ &E[X|x_{a-2}^* < X \leq x_{a-1}^*] + E[X|x_{a-1}^* < X \leq x_a^*] - 2x_{a-1}^*.
\end{aligned}
$$

Thus, from (3.17), we have

$$E[X|x_{a-2}^* < X < x_{a-1}^*] + E[X|x_{a-1}^* < X < x_a^*] - 2x_{a-1}^* > 0.$$

34

Using this argument repeatedly, we obtain (3.12). Similarly we have (3.15) and (3.16), and this completes the proof.

C Continued proof of uniqueness of COP limits

Suppose that $(x_0, x_1^A, x_2^A, \ldots, x_{n-1}^A, x_n)$ is a set of COP limits. From Proposition 3.2, we see that for some $0 \le a < b \le n$, (3.9)-(3.11) and also (3.12)-(3.16) are satisfied with x_i^* replaced by x_i^A. Let $(x_0, x_1^B, x_2^B, \ldots, x_{n-1}^B, x_n)$ be another set of COP limits. We see that for some $0 \le c < d \le n$, (3.9)-(3.11) and also (3.12)-(3.16) are satisfied with a, b, and x_i^* replaced by c, d, and x_i^B, respectively. We notice that although some equations are vacuous when $a = 0, 1$ or $b = a + 1, n - 1, n$ or $c = 0, 1$ or $d = c + 1, n - 1, n$, we can apply the following argument. We show that if $d \ne b$, then we have a contradiction. Without any loss of generality, we suppose that $b < d$. Then we have $x_d^A = x_d^B$ and $x_{d-1}^A < x_{d-1}^B$ or $D_{d-1} > 0$, where $D_i = x_i^B - x_i^A$. From Condition (A), we have

$$
\begin{aligned}
E[X | x_{d-1}^B < X \le x_d^B] - x_{d-1}^B &\le E[X | x_{d-1}^A < X \le x_d^B - D_{d-1}] - x_{d-1}^A \\
&< E[X | x_{d-1}^A < X \le x_d^A] - x_{d-1}^A.
\end{aligned}
\tag{3.18}
$$

Now we need the following lemma whose proof is given in Appendix D.

Lemma 3.1 *If for some $1 \le i \le n - 1$,*

$$
x_i^A < x_i^B,
$$

$$
x_i^B - E[X | x_i^B < X \le x_{i+1}^B] > x_i^A - E[X | x_i^A < X \le x_{i+1}^A],
\tag{3.19}
$$

$$
2x_i^B \le E[X | x_{i-1}^B < X \le x_i^B] + E[X | x_i^B < X \le x_{i+1}^B],
\tag{3.20}
$$

and

$$
2x_i^A \ge E[X | x_{i-1}^A < X \le x_i^A] + E[X | x_i^A < X \le x_{i+1}^A],
\tag{3.21}
$$

then

$$
x_{i-1}^A < x_{i-1}^B
$$

and

$$
x_{i-1}^B - E[X | x_{i-1}^B < X \le x_i^B] > x_{i-1}^A - E[X | x_{i-1}^A < X \le x_i^A].
\tag{3.22}
$$

We now return to the proof of uniqueness. Since (3.14)-(3.16) are satisfied when we replace x_i^* with x_i^A, (3.21) is satisfied for $i = d-1, d-2, \ldots, a+1$. Since (3.12)-(3.14) are satisfied when we replace a, b, and x_i^* with c, d, and x_i^B, respectively, (3.20) is satisfied for $i = d-1, d-2, \ldots, a+1$. Thus, from $D_{d-1} > 0$ and (3.18), we can apply Lemma 3.1 repeatedly for $i = d-1, d-2, \ldots, a+1$, and we obtain $x_a^A < x_a^B$. However, $x_a^A = x_0 + a\Delta \geq x_a^B$, and we have a contradiction. Therefore, we have $b = d$. In the same way, we have $a = c$. Thus we have shown that the set of COP limits is unique.

D Proof of Lemma 3.1

Using (3.20), (3.19), and (3.21), we see that

$$
\begin{aligned}
E[X|x_{i-1}^B < X \leq x_i^B] - x_{i-1}^B &= E[X|x_{i-1}^B < X \leq x_i^B] - x_i^B + (x_i^B - x_{i-1}^B) \\
&> E[X|x_{i-1}^A < X \leq x_i^A] - x_i^A + (x_i^B - x_{i-1}^B) \\
&= E[X|x_{i-1}^A < X \leq x_i^A] - (x_{i-1}^B - D_i).
\end{aligned}
$$

Using Condition (A) and this inequality, we obtain

$$
\begin{aligned}
E[X|x_{i-1}^B - D_i < X \leq x_i^A] - (x_{i-1}^B - D_i) &\geq E[X|x_{i-1}^B < X \leq x_i^B] - x_{i-1}^B \\
&> E[X|x_{i-1}^A < X \leq x_i^A] - (x_{i-1}^B - D_i).
\end{aligned}
$$

From this, we see that $x_{i-1}^B - D_i > x_{i-1}^A$, and we have

$$
D_{i-1} = x_{i-1}^B - x_{i-1}^A > D_i > 0. \tag{3.23}
$$

Using Condition (A) and (3.23), we have

$$
\begin{aligned}
E[X|x_{i-1}^B < X \leq x_i^B] - x_{i-1}^B &\leq E[X|x_{i-1}^A < X \leq x_i^B - D_{i-1}] - x_{i-1}^A \\
&< E[X|x_{i-1}^A < X \leq x_i^A] - x_{i-1}^A,
\end{aligned}
$$

which can be rewritten as (3.22). This completes the proof.

E Proof of the Remark to Propositions 3.1 and 3.2

Since we use Lemma 3.1 in the following by letting $x_i^A = x_i^\dagger$ and $x_i^B = x_i^*$, we let

$(x_0, x_1^A, x_2^A, \ldots, x_{n-1}^A, x_n)$ be the UOP limits, and also let $(x_0, x_1^B, x_2^B, \ldots, x_{n-1}^B, x_n)$ be the COP limits. Then we have (1.2) with $x_i^\dagger$ replaced by x_i^A, and we also have (3.6)-(3.8) with $x_i^\dagger$ replaced by x_i^A for some $0 \le a < b \le n$. Furthermore, we have (3.9)-(3.11) and (3.12)-(3.16) with a, b, and x_i^* replaced by c, d, and x_i^B, respectively, for some $0 \le c < d \le n$. We need only to show $b \ge d$ and $a \le c$. Here, we only show $b \ge d$ by contradiction. Suppose that $b < d$. Then we easily see that $x_d^A \le x_d^B$. Let $D_i = x_i^B - x_i^A$. Since $x_d^A - x_{d-1}^A \ge \Delta > x_d^B - x_{d-1}^B$, we have $x_{d-1}^A < x_{d-1}^B$ and $D_{d-1} > D_d \ge 0$. Thus, using Condition (A), we easily see that (3.18) holds. Applying Lemma 3.1 repeatedly $i = d-1, d-2, \ldots, a+1$, we obtain $x_a^A < x_a^B$. However, since $x_a^A \ge x_0 + a\Delta \ge x_a^B$, we have a contradiction. This completes the proof.

Chapter 4

Optimal Mean Shift of the Dimensional Distribution of the Component with Smaller Variance

4.1 Introduction

In this chapter, relaxing the assumption that the two component dimensions are identically distributed after re-centering, we discuss the case in which the variances of the two component dimensions are different. In selective assembly, if there is a large difference between the two variances, then this causes large differences between the target clearance and the expected clearances of the products that are assembled by selecting mating components from groups in the tails of the distributions. This leads to a large variance of the clearance.

To give an example, consider $X \sim N(0,1)$, $Y \sim N(0, 0.3^2)$, $x_L = -\infty$, $x_U = \infty$, and $n = 10$. Then $(0.3x_1, 0.3x_2, \ldots, 0.3x_9) = (y_1, y_2, \ldots, y_9)$ holds so that $p_i = q_i$, $i = 1, 2, \ldots, 10$. Since Mease et al. (2004) showed that the partition limits for X which minimize the expected squared error loss also satisfy the equations (1.2) when the distributions of X and Y differ only by a scale parameter, we see from Table 2.1 that the optimal partition limits for X are given as

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$$
$$= (-1.968, -1.325, -0.834, -0.405, 0, 0.405, 0.834, 1.325, 1.968).$$

The resulting expected squared error loss is 0.5038, which is much larger than the expected squared error loss 0.0459 for $Y \sim N(0,1)$. This deterioration is caused by the differences between the target clearance $C = 0$ and the expected clearances of the products, especially for groups in the tails of the distributions, as seen from Figure 4.1. The expected clearances of the products from the 1st and 10th groups are $E[X_1 - Y_1] = -2.345 - (-0.704) = -1.641$ and $E[X_{10} - Y_{10}] = 2.345 - 0.704 = 1.641$, respectively.
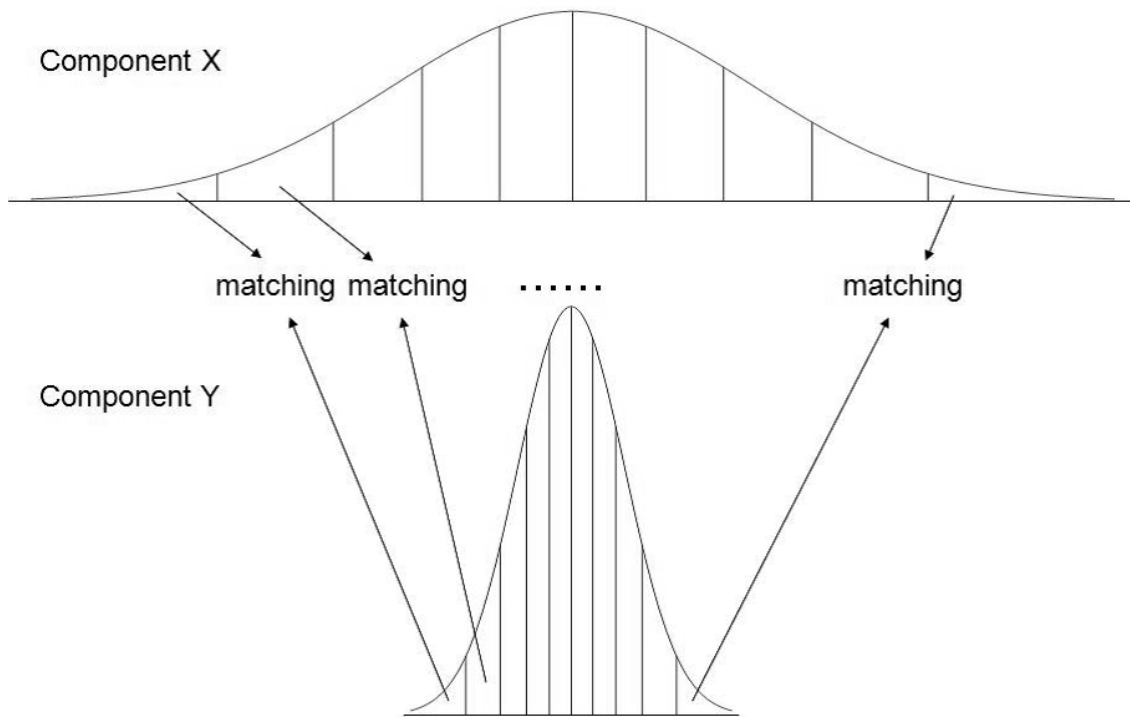
Figure 4.1. Selective assembly in the case of unequal variances.

Mansoor (1961), Kannan et al. (1997), and Kannan and Jayabalan (2002) proposed a method of manufacturing the component with smaller variance in two (or more) variants by shifting the process mean, as shown in Figure 4.2. However, the determination of the optimal mean shift under a certain criterion has not been addressed in these papers. This chapter deals with the problem of determining the optimal mean shift under squared error loss in selective assembly when the component with smaller variance is manufactured in two variants at two shifted means. It turns out that we can reduce the expected squared error loss from 0.5038 to 0.1301 by manufacturing the component Y at the two means ± 0.7963 (the resulting distribution of Y is the mixture distribution of the two normal distributions $N(-0.7963, 0.3^2)$ and $N(0.7963, 0.3^2)$) in the above example.

This chapter is organized as follows. Section 4.2 presents models, notation, and assumptions. Note that we assume normal distributions in this chapter. In Section 4.3, we show uniqueness and some properties of the optimal mean shift. Section 4.4 presents some numerical results which suggest that using the optimal mean shift can
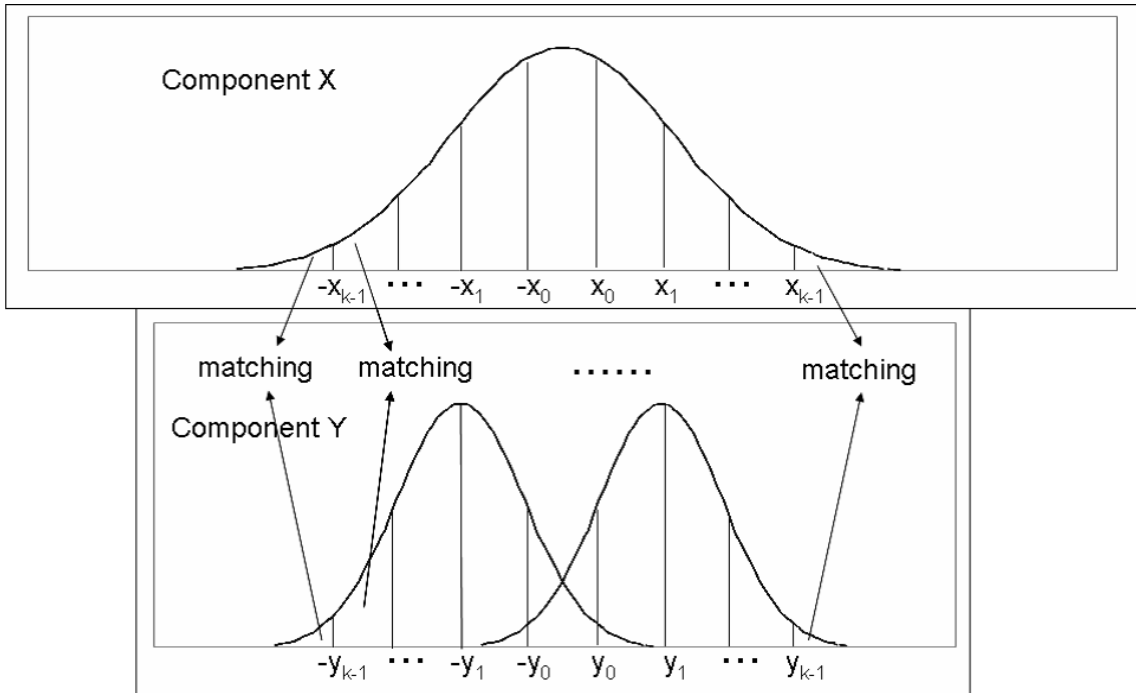
Figure 4.2. Selective assembly in the case with two shifted means.

considerably reduce the expected squared error loss compared to the situation of no shift, especially in the case where the variances of the two component dimensions are very different.

4.2 Models, notation, and assumptions

Let X and U denote the dimensions of the components with larger and smaller variance, respectively. The ratio of the smaller standard deviation to the larger one, $SD[U]/SD[X]$, is denoted by τ $(0 < \tau \leq 1)$. We assume that X and U are normally distributed throughout this chapter.

Without loss of generality, we may assume that $E[X] = 0$ and $SD[X] = 1$. Then, X is distributed as $N(0,1)$. Let $\phi(x)$ and $\Phi(x)$ denote the density and cumulative distribution functions of $N(0,1)$, respectively. The component with smaller variance is manufactured at two shifted means, and we let $\pm b$ be the two means. Then, the dimension of the component with smaller variance, denoted by Y, follows a mixture

distribution of $U's$ with the means $\pm b$. Its density and cumulative distribution functions are expressed as

$$f_Y(y) = \frac{1}{2\tau}\left\{\phi\left(\frac{y-b}{\tau}\right) + \phi\left(\frac{y+b}{\tau}\right)\right\},$$

$$F_Y(y) = \frac{1}{2}\left\{\Phi\left(\frac{y-b}{\tau}\right) + \Phi\left(\frac{y+b}{\tau}\right)\right\},$$

respectively. Suppose that $x_L = y_L = -\infty$ and $x_U = y_U = \infty$.

In this chapter, we assume that the partition limits for X are given and symmetric. The set of partition limits for X is denoted by $(-x_{k-1}, \ldots, -x_1, -x_0, x_0, x_1, \ldots, x_{k-1})$ where

$$k = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd,} \\ \frac{n}{2}, & \text{when } n \text{ is even.} \end{cases}$$

We note that $x_0 > 0$ holds if n is odd, and that we put $x_0 = 0$ if n is even. The set of partition limits for Y is denoted by $(-y_{k-1}, \ldots, -y_1, -y_0, y_0, y_1, \ldots, y_{k-1})$. In selective assembly, the components $X \in (x_{i-1}, x_i]$ are matched with the components $Y \in (y_{i-1}, y_i]$, $X \in (-x_i, -x_{i-1}]$ are matched with $Y \in (-y_i, -y_{i-1}]$, and $X \in (-x_0, x_0]$ are matched with $Y \in (-y_0, y_0]$ (if n is odd), as shown in Figure 4.2. We note that $x_k = \infty$ and $y_k = \infty$.

$\Phi(x_i) = F_Y(y_i)$, that is,

$$\Phi(x_i) = \frac{1}{2}\left\{\Phi\left(\frac{y_i-b}{\tau}\right) + \Phi\left(\frac{y_i+b}{\tau}\right)\right\}, \quad i = 0, 1, \ldots, k-1 \tag{4.1}$$

must hold so that there are no surplus components. Therefore, the partition limits for Y, $(-y_{k-1}, \ldots, -y_1, -y_0, y_0, y_1, \ldots, y_{k-1})$, are functions of b, denoted by $y_i(b)$, $i = 0, 1, \ldots, k-1$, since we fix $(-x_{k-1}, \ldots, -x_1, -x_0, x_0, x_1, \ldots, x_{k-1})$.

We let X_i and Y_i be the truncated random variables of X and Y defined on the intervals $(x_{i-1}, x_i]$ and $(y_{i-1}(b), y_i(b)]$, $i = 1, 2, \ldots, k$, respectively, and also let X_0 and Y_0 (if n is odd) be the truncated random variables of X and Y defined on the intervals $(-x_0, x_0]$ and $(-y_0(b), y_0(b)]$, respectively. $Pr(x_{i-1} < X \leq x_i) = Pr(y_{i-1}(b) < Y \leq y_i(b))$ is denoted by p_i and $Pr(-x_0 < X \leq x_0) = Pr(-y_0(b) < Y \leq y_0(b))$ is denoted by p_0. If n is even, then $p_0 = 0$.

Then, the expected squared error loss can be expressed as

$$G(b) = 2\sum_{i=1}^{k} E[(X_i - Y_i)^2]p_i + E[(X_0 - Y_0)^2]p_0.$$

We discuss determining the optimal mean shift b^* which minimizes $G(b)$ in the next section.

4.3 Optimal mean shift

The problem is to minimize $G(b)$ subject to $b \geq 0$. We first present the following lemma.

Lemma 4.1 *Let $g(b)$ be the derivative function of $G(b)$ and $g'(b)$ be the derivative function of $g(b)$, and put*

$$h_i(b) = \frac{4\phi(\frac{y_i(b)-b}{\tau})\phi(\frac{y_i(b)+b}{\tau})}{\phi(\frac{y_i(b)-b}{\tau}) + \phi(\frac{y_i(b)+b}{\tau})}, \quad i = 0, 1, \ldots, k-1.$$

Then, it follows that

$$g(b) = 2b - \frac{2}{\tau}\sum_{i=1}^{k} E[X_i]\left\{\int_{y_{i-1}(b)-b}^{y_i(b)-b} \phi\left(\frac{y}{\tau}\right)dy - \int_{y_{i-1}(b)+b}^{y_i(b)+b} \phi\left(\frac{y}{\tau}\right)dy\right\}$$

and

$$g'(b) = 2 - \frac{2}{\tau}\left\{E[X_1]h_0(b) + \sum_{i=1}^{k-1}(E[X_{i+1}] - E[X_i])h_i(b)\right\}.$$

Further, $g'(b)$ is increasing in $b > 0$, $g(0) = 0$, and $g'(0) = 2 - 4\tau^{-1}\sum_{i=1}^{k}(E[X_i])^2 p_i$.

The proof of this lemma is deferred to Appendix A. Using this lemma, we prove the following proposition, which shows that the optimal mean shift b^* is uniquely determined.

Proposition 4.1 *Let $\tau_0 = 2\sum_{i=1}^{k}(E[X_i])^2 p_i$. If τ satisfies $\tau_0 \leq \tau \leq 1$, then $G(b)$ is increasing in $b > 0$ and $b^* = 0$ holds. If τ satisfies $0 < \tau < \tau_0$, then the equation $g(b) = 0$ has a unique solution in $b > 0$ and b^* is that solution.*

Proof From Lemma 4.1, we see that $g(0) = 0$ holds and that $g'(b)$ is increasing in $b > 0$.

In the case that $\tau_0 \leq \tau \leq 1$ holds, that is, $g'(0) \geq 0$ holds, we see that $G(b)$ is increasing in $b > 0$ since $g(b) > 0$ holds for any $b > 0$. Thus, we have $b^* = 0$.

In the case that $0 < \tau < \tau_0$ holds, that is, $g'(0) < 0$ holds, the equation $g(b) = 0$ has a unique solution in $b > 0$ since $\lim_{b \to \infty} g(b) = \infty$ holds. (For illustration, we show a graph of $g(b)$ (and $G(b)$) for $0 < \tau < \tau_0$ in Figure 4.3 (and Figure 4.4).) Letting the solution be denoted by b_0, we see that $g(b) < 0$ for $0 < b < b_0$ and $g(b) > 0$ for $b_0 < b$. Thus, we see that $b^* = b_0$. $\qquad \square$
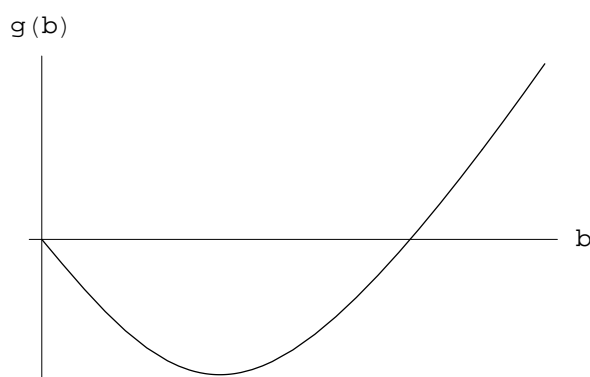

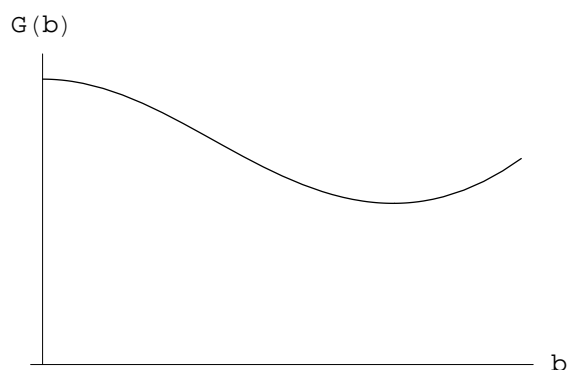Figure 4.3. A graph of $g(b)$ for $0 < \tau < \tau_0$.

Figure 4.4. A graph of $G(b)$ for $0 < \tau < \tau_0$.

Note that the optimal mean shift b^* can be easily obtained using a numerical software package, such as Mathematica. We see that if the variances of the two component dimensions are not very different ($\tau > \tau_0$ holds), then we do not need to shift the process mean ($b^* = 0$), and if they are very different ($\tau < \tau_0$ holds), then we do need

to shift the process mean ($b^* > 0$).

From the following proposition, we see that the optimal mean shift b^* increases as the difference between the variances of the two component dimensions increases. Its proof is deferred to Appendix C.

Proposition 4.2 b^* *is decreasing in* τ *when* $0 < \tau < \tau_0$.

4.4   Numerical results

We give the values of the optimal mean shift b^*, $G(b^*)$ (the expected squared error loss for $b = b^*$), $G(0)$ (the expected squared error loss for $b = 0$), and the improvement ratio $1 - G(b^*)/G(0)$ for $\tau = 0.8, 0.5, 0.3$ and $n = 1, 2, \ldots, 10$.

Let the set of partition limits for X be the solution to the equations (1.2). Recall that it is the optimal partition when $b = 0$, as is shown in Mease et al. (2004).

Tables 4.1-4.3 contain the numerical results for $\tau = 0.8, 0.5, 0.3$, respectively. Figure 4.5 illustrates the results presented in Table 4.2. From Tables 4.1-4.3, we see that using the optimal mean shift b^* considerably reduces the expected squared error loss $G(b)$ compared to the situation of no shift ($b = 0$).

Conversely, we can achieve a required accuracy of the clearance with a smaller number of groups by using the optimal mean shift. As an example, we consider the case in which $\tau = 0.5$ and the expected squared error loss is required to be less than 0.25. We see from Table 4.2 that, although the required accuracy is not achieved even when $n = 10$ if we do not shift the process mean, it is achieved for $n = 4$ if the component with smaller variance is manufactured at the two means ± 0.7324.

We see from Tables 4.1-4.3 that the improvement ratio is large especially for the case in which the two variances are very different ($\tau = 0.5, 0.3$). It is about $50 \sim 80\%$ for $\tau = 0.5, 0.3$ when the number of groups $n \geq 4$. We also see that the optimal mean shift b^* is about 0.5 for $\tau = 0.8$ and about $0.75 \sim 0.8$ for $\tau = 0.5, 0.3$ when $n \geq 4$.

Table 4.4 gives the values of $\tau_0 = 2 \sum_{i=1}^{k} (E[X_i])^2 p_i$ for $n = 1, 2, \ldots, 10$. We see that $\tau_0 = 0.6366$ for $n = 2$. We also see from Tables 4.1-4.3 that $b^* = 0$ holds for $\tau = 0.8$

and $b^* > 0$ holds for $\tau = 0.5, 0.3$ when $n = 2$. These results are compatible with the assertions in Proposition 4.1 that, $b^* = 0$ holds if $\tau_0 \leq \tau$ and $b^* > 0$ holds if $\tau < \tau_0$. The results of Tables 4.1-4.3 are also compatible with the assertion in Proposition 4.2 that the optimal mean shift b^* is decreasing in τ.

We notice that the value of τ_0 depends on the number of groups n and the partition limits $(-x_{k-1}, \ldots, -x_1, -x_0, x_0, x_1, \ldots, x_{k-1})$. From Table 4.4, we see that τ_0 is increasing in n. We can show that τ_0 converges to 1 when n goes to infinity. Therefore, when n is large, even if the variances of the two component dimensions are not very different, $b^* > 0$ holds, that is, we can reduce the expected squared error loss by manufacturing the component with smaller variance at two shifted means.

Table 4.1. Numerical results for $\tau = 0.8$.

n	b^*	$G(b^*)$	$G(0)$	Ratio (%)
1	0	1.64	1.64	0
2	0	0.6214	0.6214	0
3	0.1635	0.3441	0.3443	0.05
4	0.4225	0.2190	0.2280	3.95
5	0.4878	0.1508	0.1679	10.19
6	0.5192	0.1102	0.1328	17.03
7	0.5374	0.0840	0.1104	23.87
8	0.5491	0.0663	0.0953	30.42
9	0.5571	0.0537	0.0846	36.53
10	0.5629	0.0444	0.0767	42.13

Table 4.2. Numerical results for $\tau = 0.5$.

n	b^*	$G(b^*)$	$G(0))$	Ratio (%)
1	0	1.25	1.25	0
2	0.6351	0.5625	0.6134	8.29
3	0.6464	0.3242	0.4402	26.35
4	0.7324	0.2075	0.3675	43.52
5	0.7292	0.1539	0.3299	53.35
6	0.7542	0.1182	0.3080	61.62
7	0.7540	0.0976	0.2940	66.79
8	0.7637	0.0824	0.2845	71.03
9	0.7644	0.0724	0.2779	73.95
10	0.7689	0.0646	0.2729	76.34

Table 4.3. Numerical results for $\tau = 0.3$.

n	b^*	$G(b^*)$	$G(0)$	Ratio (%)
1	0	1.09	1.09	0
2	0.7912	0.4522	0.7080	36.14
3	0.6616	0.3608	0.6041	40.28
4	0.7942	0.2239	0.5605	60.05
5	0.7417	0.2135	0.5380	60.31
6	0.7953	0.1658	0.5249	68.41
7	0.7671	0.1645	0.5164	68.15
8	0.7959	0.1421	0.5107	72.17
9	0.7785	0.1425	0.5067	71.88
10	0.7963	0.1301	0.5038	74.17

Table 4.4. The value of $\tau_0 = 2 \sum_{i=1}^{k} (E[X_i])^2 p_i$.

n	k	τ_0
1	0	0
2	1	0.6366
3	1	0.8098
4	2	0.8825
5	2	0.9201
6	3	0.9420
7	3	0.9560
8	4	0.9655
9	4	0.9721
10	5	0.9771
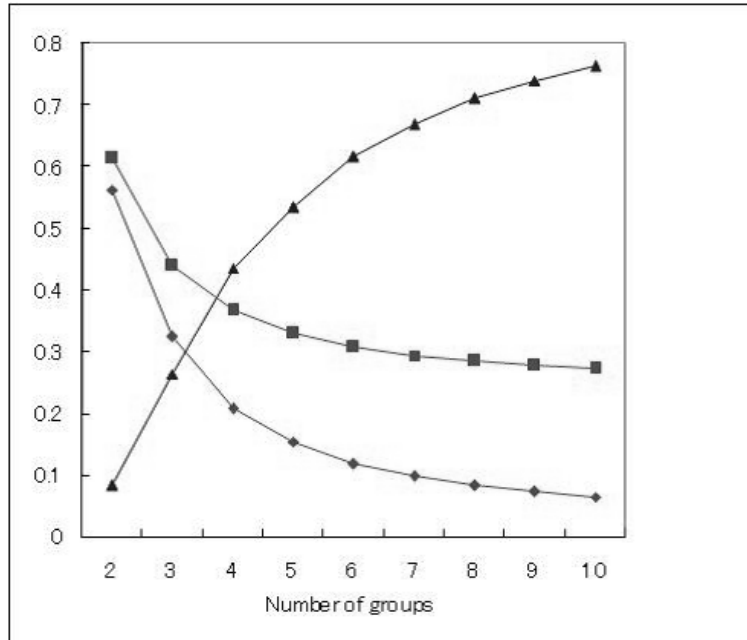
Figure 4.5. Numerical results for $\tau = 0.5$ ($\quad G(0)$; $\quad G(b^*)$; $\quad$ Improvement ratio).

Numerical example

We consider a pin and bushing assembly, which Pugh (1986b) also considered for analysis. The outer diameter of the pin is distributed as $N(2.500\text{cm}, 0.075^2\text{cm}^2)$. The inner diameter of the bushing is distributed as $N(2.750\text{cm}, 0.150^2\text{cm}^2)$. The target clearance is 0.250cm. The number of groups is 4. The partition limits for the dimensional distribution of the bushing are obtained by solving the equations (1.2) or by using

$$\text{(the partition limits for } n = 4 \text{ given in Table 2.1)} \times 0.150 + 2.750.$$

Since $\tau = 0.075/0.150 = 0.5$, we can obtain the optimal mean shift and the expected squared error loss from the results of Table 4.2 for $n = 4$ by applying a scale change.

If we do not shift the process mean, then the expected squared error loss is calculated as $0.3675 \times 0.150^2 = 8.268 \times 10^{-3}\text{cm}^2$. On the other hand, if the pin is manufactured at two shifted means, the optimal mean shift is obtained as $0.7324 \times 0.150 = 0.10986\text{cm}$. Then, the outer diameter of the pin follows the mixture distribution of $N((2.500 - 0.10986)\text{cm}, 0.075^2\text{cm}^2)$ and $N((2.500 + 0.10986)\text{cm}, 0.075^2\text{cm}^2)$. The partition limits for the mixture distribution are calculated so that the probability of any group is equal to that of the corresponding group of the bushing ($p_i = q_i$). Table 4.5 gives the results.

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The expected squared error loss is $4.670 \times 10^{-3} \text{cm}^2$.

Thus, using the optimal mean shift gives a 43.52% reduction in the expected squared error loss compared to the no shift case.

Table 4.5. Partition when we use $b^* = 0.10986$.

group	bushing (cm)		pin (cm)		expected loss	probability
	min	max	min	max	(10^{-3}cm^2)	
1	$-\infty$	2.6028	$-\infty$	2.3563	7.228	0.1631
2	2.6028	2.7500	2.3563	2.5000	3.431	0.3369
3	2.7500	2.8972	2.5000	2.6437	3.431	0.3369
4	2.8972	∞	2.6437	∞	7.228	0.1631
average					4.670	

Appendices

A   Proof of Lemma 4.1

Noting that $E[Y^2] = \tau^2 + b^2$ and $E[X_0] = E[Y_0] = 0$, the expected squared error loss is rewritten as

$$
\begin{aligned}
G(b) &= E[X^2] + E[Y^2] - 4\sum_{i=1}^{k} E[X_i]E[Y_i]p_i \\
&= (1 + \tau^2) + b^2 - 4\sum_{i=1}^{k} E[X_i]E[Y_i]p_i.
\end{aligned}
$$

Since

$$
E[Y_i]p_i = \frac{1}{2\tau}\left\{\int_{y_{i-1}(b)-b}^{y_i(b)-b} (y+b)\phi\left(\frac{y}{\tau}\right)dy + \int_{y_{i-1}(b)+b}^{y_i(b)+b} (y-b)\phi\left(\frac{y}{\tau}\right)dy\right\}, \quad i = 1, 2, \ldots, k,
$$

we see that

$$
G(b) = (1 + \tau^2) + b^2 - \frac{2}{\tau}\sum_{i=1}^{k} E[X_i]\left\{\int_{y_{i-1}(b)-b}^{y_i(b)-b} (y+b)\phi\left(\frac{y}{\tau}\right)dy + \int_{y_{i-1}(b)+b}^{y_i(b)+b} (y-b)\phi\left(\frac{y}{\tau}\right)dy\right\}.
$$

Differentiating both sides of the equations (4.1) with respect to b, we have

$$
\left(\frac{dy_i(b)}{db} - 1\right)\phi\left(\frac{y_i(b)-b}{\tau}\right) + \left(\frac{dy_i(b)}{db} + 1\right)\phi\left(\frac{y_i(b)+b}{\tau}\right) = 0, \quad i = 0, 1, \ldots, k-1.
$$
(4.2)

Thus, we see that the derivative of $G(b)$ is given as

$$
g(b) = 2b - \frac{2}{\tau}\sum_{i=1}^{k} E[X_i]\left\{\int_{y_{i-1}(b)-b}^{y_i(b)-b} \phi\left(\frac{y}{\tau}\right)dy - \int_{y_{i-1}(b)+b}^{y_i(b)+b} \phi\left(\frac{y}{\tau}\right)dy\right\}.
$$

We easily see that $g(0) = 0$.

It follows from the equations (4.2) that

$$\frac{dy_i(b)}{db} = \frac{\phi(\frac{y_i(b)-b}{\tau}) - \phi(\frac{y_i(b)+b}{\tau})}{\phi(\frac{y_i(b)-b}{\tau}) + \phi(\frac{y_i(b)+b}{\tau})}, \quad i = 0, 1, \ldots, k-1. \tag{4.3}$$

Putting $h_k(b) = 0$ and using the equations (4.3), the derivative of $g(b)$ is given as

$$\begin{aligned}
g'(b) &= 2 - \frac{2}{\tau} \sum_{i=1}^{k} E[X_i] \left\{ \left(\frac{dy_i(b)}{db} - 1 \right) \phi\left(\frac{y_i(b) - b}{\tau} \right) - \left(\frac{dy_i(b)}{db} + 1 \right) \phi\left(\frac{y_i(b) + b}{\tau} \right) \right. \\
&\quad \left. - \left(\frac{dy_{i-1}(b)}{db} - 1 \right) \phi\left(\frac{y_{i-1}(b) - b}{\tau} \right) + \left(\frac{dy_{i-1}(b)}{db} + 1 \right) \phi\left(\frac{y_{i-1}(b) + b}{\tau} \right) \right\} \\
&= 2 - \frac{2}{\tau} \sum_{i=1}^{k} E[X_i]\{-h_i(b) + h_{i-1}(b)\}, \tag{4.4}
\end{aligned}$$

which can be rewritten as

$$g'(b) = 2 - \frac{2}{\tau} \left\{ E[X_1]h_0(b) + \sum_{i=1}^{k-1} (E[X_{i+1}] - E[X_i])h_i(b) \right\}.$$

Since we can show that $h_i(b)$ is decreasing in $b > 0$, whose proof is given in Appendix B, we see that $g'(b)$ is increasing in $b > 0$. Using (4.4), $y_i(0) = \tau x_i$, and $\phi(x_{i-1}) - \phi(x_i) = E[X_i]p_i$, we see that

$$\begin{aligned}
g'(0) &= 2 - \frac{4}{\tau} \sum_{i=1}^{k} E[X_i]\{\phi(x_{i-1}) - \phi(x_i)\} \\
&= 2 - \frac{4}{\tau} \sum_{i=1}^{k} (E[X_i])^2 p_i.
\end{aligned}$$

This completes the proof.

B Proof of the statement: $h_i(b)$ is decreasing in $b > 0$.

We first note that the derivative of $h_i(b)$ is given as

$$h_i'(b) = \frac{\frac{4}{\tau} \left(\frac{dy_i(b)}{db} - 1 \right) \phi'(\frac{y_i(b)-b}{\tau}) \left(\phi(\frac{y_i(b)+b}{\tau}) \right)^2 + \frac{4}{\tau} \left(\frac{dy_i(b)}{db} + 1 \right) \phi'(\frac{y_i(b)+b}{\tau}) \left(\phi(\frac{y_i(b)-b}{\tau}) \right)^2}{\left(\phi(\frac{y_i(b)-b}{\tau}) + \phi(\frac{y_i(b)+b}{\tau}) \right)^2}.$$

Using the equations (4.3), we see that

$$h_i'(b) = \frac{8}{\tau} \frac{\phi(\frac{y_i(b)-b}{\tau})\phi(\frac{y_i(b)+b}{\tau})}{\left(\phi(\frac{y_i(b)-b}{\tau}) + \phi(\frac{y_i(b)+b}{\tau})\right)^3} \left[\left\{\phi\left(\frac{y_i(b)-b}{\tau}\right)\right\}^2 \frac{\phi'(\frac{y_i(b)+b}{\tau})}{\phi(\frac{y_i(b)+b}{\tau})}\right.$$
$$\left. - \left\{\phi\left(\frac{y_i(b)+b}{\tau}\right)\right\}^2 \frac{\phi'(\frac{y_i(b)-b}{\tau})}{\phi(\frac{y_i(b)-b}{\tau})}\right].$$

Therefore, it is sufficient for us to show that

$$\left\{\phi\left(\frac{y_i(b)-b}{\tau}\right)\right\}^2 \frac{\phi'(\frac{y_i(b)+b}{\tau})}{\phi(\frac{y_i(b)+b}{\tau})} - \left\{\phi\left(\frac{y_i(b)+b}{\tau}\right)\right\}^2 \frac{\phi'(\frac{y_i(b)-b}{\tau})}{\phi(\frac{y_i(b)-b}{\tau})} < 0.$$

It follows from $y_i(b) \geq 0$ that $\phi(\frac{y_i(b)+b}{\tau}) \leq \phi(\frac{y_i(b)-b}{\tau})$. Since $\phi'(x)/\phi(x) = -x$ is decreasing in x, we see that

$$\left\{\phi\left(\frac{y_i(b)-b}{\tau}\right)\right\}^2 \frac{\phi'(\frac{y_i(b)+b}{\tau})}{\phi(\frac{y_i(b)+b}{\tau})} - \left\{\phi\left(\frac{y_i(b)+b}{\tau}\right)\right\}^2 \frac{\phi'(\frac{y_i(b)-b}{\tau})}{\phi(\frac{y_i(b)-b}{\tau})}$$
$$< \frac{\phi'(\frac{y_i(b)+b}{\tau})}{\phi(\frac{y_i(b)+b}{\tau})} \left[\left\{\phi\left(\frac{y_i(b)-b}{\tau}\right)\right\}^2 - \left\{\phi\left(\frac{y_i(b)+b}{\tau}\right)\right\}^2\right]$$
$$\leq 0.$$

Thus, we have shown that $h_i(b)$ is decreasing in $b > 0$.

C   Proof of Proposition 4.2

Let $\sigma = 1/\tau$. Since we do not need the derivative of $y_i(b)$ with respect to b in the proof, we simply denote $y_i(b)$ by y_i.

We note that y_i is a function of σ since the equations (4.1) are rewritten as

$$\Phi(x_i) = \frac{1}{2}\left\{\Phi(\sigma(y_i - b)) + \Phi(\sigma(y_i + b))\right\}, \quad i = 0, 1, \ldots, k-1. \tag{4.5}$$

The partial derivatives of both sides of the equations (4.5) with respect to σ are given as

$$0 = \frac{1}{2}\left\{(y_i - b)\phi(\sigma(y_i - b)) + \sigma\frac{dy_i}{d\sigma}\phi(\sigma(y_i - b)) + (y_i + b)\phi(\sigma(y_i + b))\right.$$
$$\left. + \sigma\frac{dy_i}{d\sigma}\phi(\sigma(y_i + b))\right\}, \quad i = 0, 1, \ldots, k-1.$$

Therefore, letting $\phi(\sigma(y_i - b)) = A_i$ and $\phi(\sigma(y_i + b)) = B_i$, we have

$$\sigma\frac{dy_i}{d\sigma} = -\frac{(y_i - b)A_i}{A_i + B_i} - \frac{(y_i + b)B_i}{A_i + B_i}, \quad i = 0, 1, \ldots, k-1. \tag{4.6}$$

It follows from the equations (4.6) that

$$(y_i - b)A_i + \sigma \frac{dy_i}{d\sigma} A_i - (y_i + b)B_i - \sigma \frac{dy_i}{d\sigma} B_i = -bh_i, \quad i = 0, 1, \ldots, k-1, \tag{4.7}$$

where $h_i = 4A_iB_i/(A_i + B_i), \; i = 0, 1, \ldots, k-1$.

Now, we can prove Proposition 4.2. Since $g(b^*) = 0$ and $g'(b^*) > 0$ hold when $\tau < \tau_0$, it is sufficient for us to show that $g(b)$ is increasing in τ (that is, $g(b)$ is decreasing in σ). We have

$$
\begin{aligned}
&g(b) \\
=\;& 2b - \frac{2}{\tau} \sum_{i=1}^{k} E[X_i] \left\{ \int_{y_{i-1}-b}^{y_i-b} \phi\left(\frac{y}{\tau}\right) dy - \int_{y_{i-1}+b}^{y_i+b} \phi\left(\frac{y}{\tau}\right) dy \right\} \\
=\;& 2b - 2 \sum_{i=1}^{k} E[X_i] \left\{ \Phi(\sigma(y_i - b)) - \Phi(\sigma(y_{i-1} - b)) - \Phi(\sigma(y_i + b)) + \Phi(\sigma(y_{i-1} + b)) \right\}.
\end{aligned}
$$

Putting $h_k = 0$ and using the equations (4.7), we see that

$$
\begin{aligned}
\frac{dg(b)}{d\sigma} &= -2 \sum_{i=1}^{k} E[X_i] \left\{ (y_i - b)A_i + \sigma \frac{dy_i}{d\sigma} A_i - (y_{i-1} - b)A_{i-1} - \sigma \frac{dy_{i-1}}{d\sigma} A_{i-1} \right. \\
&\qquad \left. - (y_i + b)B_i - \sigma \frac{dy_i}{d\sigma} B_i + (y_{i-1} + b)B_{i-1} + \sigma \frac{dy_{i-1}}{d\sigma} B_{i-1} \right\} \\
&= -2b \sum_{i=1}^{k} E[X_i](h_{i-1} - h_i) \\
&= -2b \left\{ E[X_1]h_0 + \sum_{i=1}^{k-1} (E[X_{i+1}] - E[X_i])h_i \right\} \\
&< 0.
\end{aligned}
$$

Thus, we have shown that b^* is decreasing in $\tau \in (0, \tau_0)$.

Chapter 5

Concluding Remarks

5.1 Conclusions

Selective assembly is a cost-effective approach to improve the quality of a product assembled from two components when the quality characteristic is the clearance between the mating components (or the sum of the relevant dimensions of the mating components). In this thesis, we have studied optimal binning strategies under squared error loss in selective assembly.

In Chapter 2, we have studied optimal partitioning of the distributions of the observations when measurement error is present and the two component dimensions are identically distributed after re-centering. It has been shown that if the component dimensions and the measurement errors are normally distributed, then the set of optimal partition limits is unique and we can obtain it without worrying about whether or not measurement error is present. It has also been shown that even if the number of groups increases, we cannot obtain much loss reduction when considerable measurement error is present compared to the case in which measurement error is not present.

In Chapter 3, we have studied optimal partitioning of the dimensional distributions when a tolerance constraint is given on the clearance and the two component dimensions are identically distributed after re-centering. We have shown that the set of constrained optimal partition limits is unique provided that the dimensional distribution is strongly unimodal. It has turned out that the resulting constrained optimal partitioning is the partitioning which constrains the widths of some groups in the tails of the distribution to the tolerance limit and which matches exactly the unconstrained optimal partitioning for the rest of the distribution. Some numerical results have shown that we have unacceptable products with positive probability in some cases of the unconstrained optimal partitioning, and we have also shown that for the constrained optimal parti-

tioning, the tolerance constraint is strictly satisfied and the expected squared error loss is considerably reduced in comparison with the equal width partitioning.

In Chapter 4, we have discussed the case in which the two component dimensions are normally distributed with unequal variances. We have dealt with the problem of determining the optimal mean shift when the component with smaller variance is manufactured at two shifted means. It has been shown that we can determine the optimal mean shift uniquely. It has also been shown that the optimal mean shift increases when the difference between the variances of the two component dimensions becomes larger. Some numerical results have shown that using the optimal mean shift considerably reduces the expected squared error loss compared to the situation of no shift, especially for the case where the two variances are very different.

5.2   Discussion and future work

There still remain some important issues to be addressed.

Although we have dealt with the squared error loss function, an alternative loss function is more appropriate in some cases. For example, in a piston and cylinder assembly, we should assume an asymmetric loss function since two different assemblies, with clearances above the target value and below the target value, result in different problems, as is described in Section 1.1.

We have fixed acceptable limits of the dimensional distributions (x_L, x_U, y_L, and y_U). Taking into account the quality loss of a sold product, the selling price of the assembled product, the manufacturing cost of the components, and the income from rejected components, the problem of choosing optimal acceptable limits should be considered. Recently, Matsuura and Shinozaki (to appear) have addressed this problem, assuming that the two component dimensions are identically distributed after re-centering.

We have assumed that the number of groups is predetermined. However, since the expected squared error loss decreases when the number of groups increases, a cost optimal choice of the number of groups will be possible by balancing out the cost of

partitioning with one more group against the reduction in expected loss that results from adding a group.

We have assumed that the probability distributions of the component dimensions (and the measurement errors) are known. However, we may estimate them using data of measurements of component dimensions in practice.

In Chapter 3, we have discussed the case in which a tolerance constraint is given, assuming that measurement error is not present. If it is present, then we will have some unacceptable products even when all group widths are less than the tolerance limit. We should choose the set of partition limits so that the probability of non-acceptance is not more than a certain specified value.

In Chapter 4, we have discussed the problem of determining the optimal mean shift under squared error loss when the component with smaller variance is manufactured at two shifted means. However, the determination of the optimal mean shift has not been addressed under a tolerance constraint on the clearance. We also note that Mansoor (1961), Kannan et al. (1997), and Kannan and Jayabalan (2002) proposed a method of manufacturing the component with smaller variance at three (or more) shifted means. Although normal distributions have been assumed in Chapter 4, extending the results to handle other distributions is an important issue.

Finally, we note that the optimal binning strategy for selective assembly of an assembled product with multiple quality characteristics may also be an important topic.

These topics are subjects for future research.

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