A Thesis for the Degree of Ph.D. in Science

Initial-Boundary Value Problems for Motion of Inhomogeneous Incompressible Fluid-like Bodies

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Contents

Chapter 1

Introduction

This dissertation is concerned with mathematical analysis on motion of inhomogeneous incompressible fluid-like bodies, which arises from a study of continuum model for a flow of granular materials.

1.1 Background

1.1.1 Flows of granular materials

Granular materials can be found everywhere. For example, sand, powder, grain and crop are quite familiar to us in our life. We can see enormously huge amount of sand particles in deserts. We store the crops in silos, and mill them for the better use as flour. We use powdery medicines and chemicals, too. Thus granular materials indeed have the close relation to the lives of human beings in the geographical, industrial, agricultural, pharmaceutical fields, *etc.*

The nomenclature of "granular materials" are collection of a lot of granules. This concept implies the huge range of categories of materials. There are a lot variety of problems caused by phenomena of granular materials, for example, segregation, decomposition, fluidization, *etc.* Since each granular phenomenon is quite complex to understand, a solution to such a problem is often considered according to the situations. Thus, the universal theory for physics of granular materials has not been constructed so far.

In this study we are interested in the motion of granular materials. Of course, we can not and do not intend to consider all kinds of such materials by one model studied here. As the first step for observing the motion of them, we restrict the conditions which are needed to discuss mathematically.

1.1.2 Theoretical Models

There have been many researchers who attempt to characterize interesting granular phenomena. Since the particles of them have the visible-scale, at first we would consider the n-particle system to determine the motion. If we can precisely represent the interaction between the particles and also if we can solve the system, it could be the best way to study the motion. But, unfortunately, the n-particle system can not be solved mathematically in general by the work of Poincaré. Furthermore, such n can be taken a huge number. For example, the density of sugar is approximately 10^0g/cm^3 and the particle size is 10^{-2}cm. Thus there are 10^{10} sugar particles in the 1kg bag. This indicates that numerical analysis is still difficult to obtain good results for such n-body problems. Thus we need the alternative procedure to characterize the motion of them.

In order to overcome such difficulties the kinetic theory and continuum mechanics have been applied to granular materials. It seems to be natural to use the kinetic model because collision of particles could be the main factor for motion like yellow dust. On the other hand, in the macro-scopic scale flows of them can be modeled as non-linear continuum by the use of continuum approximation in the same way as in fluid-mechanics. In fact, when viewed from far away, a flow of sands can be seen as if water flows. In such a case, the appropriate constitutive equations for the stress are necessary to characterize the motion of them. Not only this kind of observation, but also some scientific evidence of fluidization of granular materials are also reported [32]. Thus the continuum models for granular flows are worth investigating in physically and even in mathematically. Certainly, the continuum approximation may be crude for the real granular motion, and it should be restricted for the flow concerned with. Even though in this situation, it is still important to consider the continuum model to make clear the mechanism of granular phenomena.

1.1.3 Transition of continuum models

The continuum models of granular materials were proposed by Goodman and Cowin study in thier prominent works [9, 10]. There they took into account the interstitial workings of the body. They used the conservation laws of mass, linear momentum and energy, and they adopted the constitutive equations for the stress $\mathbb{T}$ given by

$$\mathbb{T} = \bar{\mathbb{T}}(\zeta, \nabla\zeta, \mathbb{D}), \tag{1.1.1}$$

where ζ denotes the volume fraction at each point and $\mathbb{D}$ the symmetric part of the velocity gradient. Jenkins [16] and Savage [33] also considered this kind of constitutive equations to explain the peculiar behaviour of them because the normal stress difference occurs in their model. Of course, this model of the form (1.1.1) has limited applicability to, for example, the dense slow but moderately rapid flows of granular materials. However, it should be also emphasized that the models from kinetic theoretic approaches are applicable only to very rapid flows. Interestingly, the model (1.1.1) is similar to the one due to Korteweg that describes the mechanism of capillarity [18], and the density plays the role of volume fraction in this study. By requiring isotropic behaviour of the body, the most general isotropic representation for the stress is given by (1.1.1) in [23]. In 1990 Rajagopal and Massoudi [29] introduced a model as a subclass of (1.1.1) given by

$$\mathbb{T} = \{\beta_0(\zeta) + \beta_1(\zeta)\mathrm{tr}\mathbb{M} + \beta_2(\zeta)\mathrm{tr}\mathbb{D}\}\mathbb{I} + \beta_3(\zeta)\mathbb{D} + \beta_4(\zeta)\mathbb{M}, \tag{1.1.2}$$

where

$$\mathbb{M} = \nabla\zeta \otimes \nabla\zeta, \tag{1.1.3}$$

$\mathrm{tr}\mathbb{M} = \sum_{i=1}^{3} M_{ii} = |\nabla\zeta|^2$, $\mathrm{tr}\mathbb{D} = \mathrm{div}\mathbf{v}$ and $\mathbb{I}$ the identity tensor. Furthermore, Boyle and Massoudi [5] derived a model with precise structural coefficients in (1.1.2) from the kinetic approach based on Enskog's dense gas theory. We should also remark that the stress depending on the density gradient was first considered by Korteweg [18]. Thus, the materials whose stress is given by the form (1.1.1) or (1.1.2) are called Korteweg type materials (see Hutter and Rajagopal [13] for detail). The relation (1.1.2) has been used to study a variety of problems (for example, [28, 43]), and its applicability to granular materials was also explained [43].

1.2 A model of inhomogeneous incompressible fluid-like bodies

1.2.1 Incompressible process

From a series of works of Rajagopal *et al.* (for example, [29, 43]) one can conclude that a continuum model of a flow of granular materials (1.1.2) is worth investigating. However, due to the complex structure of the stress, it is quite difficult to proceed the qualitative analysis mathematically for this model. Unfortunately, we have only a few mathematical works concerning (1.1.2) so far; even for the shear flow down an inclined plane there has been no complete result. The major difficulty might be the appearance of $\mathbb{M}$, namely the quadratic dependence on the density gradient in the stress. By substituting the stress of the form (1.1.2) for the conservation law of linear momentum, the quasi-linear partial differential equations whose principal terms include

$$\mathrm{div}\mathbb{M} = (\Delta\zeta\mathbb{I} + \mathbb{H}(\zeta))\nabla\zeta \tag{1.2.1}$$

are deduced, where $\mathbb{H}(\zeta) = (\frac{\partial^2\zeta}{\partial x_i \partial x_j})_{i,j=1,2,3}$ is the Hessian matrix of ζ. Of course, the term (1.2.1) may degenerate due to $\nabla\zeta$. Thus this degeneracy may cause severe difficulties unlike the Navier–Stokes equations. The difficulties come from the substantial character of the form (1.1.2), and it is not generally removable.

However, adding the limiting condition such as an incompressible constraint, we can make the problem somewhat easier. In this situation hyperbolicity caused by $\mathbb{M}$ can be weakened, and the effective linearized equations can be also deduced. Thus we may analyse the non-linear problem by the perturbation theory. Granular materials are substantially compressible due to the interstices between particles. On the other hand, if the particles are very fine so that the interlocking of the particles occurs, then they behave as if they are incompressible bodies. Usually, the interlocking never lasts for a long time when the body deforms, which is well-known as Reynold's dilatancy [31]. Hence, incompressible constraint seems to be relatively crude approximation of the real motion. But it can be applied to a slow and dense flow of granular materials with moderate deformation.

Following the study of Málek and Rajagopal [21] we introduce the notion of an incompressible body. We regard a material as incompressible when its compressibility is insignificant and more importantly this compressibility has insignificant consequences concerning the response of the body, namely the work of a fluid in the interstices can be neglected. Even though the body under consideration only undergoes the isochoric motion, the density distribution can be inhomogenous. Consequently, it is natural that granular materials are inhomogeneous incompressible fluid-like bodies (IIFB).

1.2.2 Constitutive equation for the Cauchy stress

Hereafter we concentrate our interest on the IIFB model. Following the procedure initiated in in Rajagopal [27], Rajagopal and Srinivasa [30], Málek and Rajagopal [21] derived the following constitutive equation for the Cauchy stress

$$\mathbb{T} = -p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{v}) - \varrho\left(\Psi_{,\mathbf{z}} \otimes \nabla\varrho - \frac{1}{3}(\Psi_{,\mathbf{z}} \cdot \nabla\varrho)\mathbb{I}\right). \tag{1.2.2}$$

Here, $p = -\frac{1}{3}\mathrm{tr}\mathbb{T}$ is the mean normal stress (the pressure), ϱ the density of the body which plays the role of ζ in (1.1.2), $\mathbb{D}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + [\nabla\mathbf{v}]^{\mathrm{T}})$ the symmetric part of the velocity gradient, $\nu = \nu(p, \varrho, \mathbb{D}(\mathbf{v}))$ the viscosity, $\Psi = \Psi(\varrho, \nabla\varrho)$ the Helmholtz potential satisfying the symmetric condition

$$\Psi_{,\mathbf{z}} \otimes \nabla\varrho = \nabla\varrho \otimes \Psi_{,\mathbf{z}}, \tag{1.2.3}$$

and $\Psi_{,\mathbf{z}}(\varrho, \mathbf{z}) = \nabla_{\mathbf{z}}\Psi(\varrho, \mathbf{z})$.

To derive the relation (1.2.2) they considered the body with the specific Helmholtz potential depending on the density and the gradient of the density, and took into accont the maximization of entropy production.

We should emphasize that if Ψ depends on $|\mathbf{z}|$, the symmetric condition (1.2.3) holds. Moreover, if

$$\Psi(\varrho, \mathbf{z}) = \bar{\Psi}(\varrho, |\mathbf{z}|^2), \tag{1.2.4}$$

then Ψ satisfies (1.2.3), and

$$\Psi_{,\mathbf{z}}(\varrho, \nabla\varrho) = 2\bar{\Psi}_{,z}(\varrho, |\nabla\varrho|^2)\nabla\varrho,$$

where $\bar{\Psi}_{,z}(\varrho, z) = \frac{\partial\bar{\Psi}}{\partial z}(\varrho, z)$. In this case (1.2.2) becomes

$$\mathbb{T} = -p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{v}) - 2\varrho\bar{\Psi}_{,z}(\varrho, |\nabla\varrho|^2)\left(\mathbb{M} - \frac{\mathrm{tr}\mathbb{M}}{3}\,\mathbb{I}\right) \tag{1.2.5}$$

with $\mathbb{M} = \nabla\varrho\otimes\nabla\varrho$. Thus, the model (1.2.2) with (1.2.4) interestingly coincides with the special form of the model (1.1.2) with the incompressible constraint. From this point we infer that (1.2.2) (or (1.2.5)) is worth studying for incompressible flows of granular materials.

This particular form of Helmholtz potential (1.2.4) is natural when one takes into account the isotropic potential and the material objectivity [42].

The simplest form of the specific Helmholtz potential belonging to the subclass of (1.2.4) is

$$\Psi(\varrho, \nabla\varrho) = \beta^*(\varrho) + \frac{\beta}{2\varrho}|\nabla\varrho|^2 \tag{1.2.6}$$

with β^* being a function of ϱ and β a positive constant. For simplicity we consider that the viscosity depends only on ϱ, *i.e.*, $\nu = \nu(\varrho)$. Consequently, (1.2.5) becomes

$$\mathbb{T} = -p\mathbb{I} + 2\nu(\varrho)\mathbb{D}(\mathbf{v}) - \beta\left(\mathbb{M} - \frac{\mathrm{tr}\mathbb{M}}{3}\,\mathbb{I}\right). \tag{1.2.7}$$

1.3 Initial-boundary value problem for IIFB model

1.3.1 System of governing equations

Taking the form (1.2.7) in the continuum model of IIFB, we have the following system of partial differential equations for the velocity vector field $\mathbf{v} = (v_1, v_2, v_3)(X,t)$, the pressure $p = p(X,t)$ and the density $\varrho = \varrho(X,t)$ in a bounded domain $\Omega\,(\subset \mathbb{R}^3)$ for $t > 0$:

$$\begin{cases} \dfrac{\mathrm{D}\varrho}{\mathrm{D}t} = 0, \qquad \operatorname{div}\mathbf{v} = 0 \qquad \text{for } X \in \Omega,\ t > 0, \\ \varrho\dfrac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \operatorname{div}\mathbb{T} + \varrho\mathbf{b} \qquad\qquad \text{for } X \in \Omega,\ t > 0. \end{cases} \tag{1.3.1}$$

Here $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)$ is the material derivative, $\mathbb{T}$ the Cauchy stress tensor given by (1.2.7), $\mathbf{b} = (b_1, b_2, b_3)(X,t)$ the external body forces and $(\operatorname{div}\mathbb{T})_i = \sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial X_j}$.

In this study we shall investigate the model equations (1.3.1) mathematically, *viz.* prove the existence of a solution of (1.3.1) with certain initial and boundary conditions.

1.3.2 Initial and boundary conditions

We assign the initial conditions

$$(\varrho, \mathbf{v})|_{t=0} = (\varrho_0, \mathbf{v}_0) \quad \text{for } X \in \Omega. \tag{1.3.2}$$

Concerning the boundary conditions, several types of them can be applied to the motion of inhomogeneous incompressible fluid-like bodies in a fixed container.

The simplest condition is the so-called adherence condition

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma, \tag{1.3.3}$$

where Γ is the boundary of Ω. It represents the situation that the bodies adhere on the boundary.

The adherence condition is a standard condition for Newtonian fluids, however it seems to be less suitable for motion of granular materials. Not only for granular materials and also for non-Newtonian fluids in a container like a pipe, one should consider the slip phenomena on the wall of the container. One of the reasonable conditions taking into account the slip effect is the generalized Navier's slip condition given by

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \mathbf{v} + K\Pi\mathbb{T}\mathbf{n} = \mathbf{0} \quad \text{ on } \Gamma. \tag{1.3.4}$$

Here, $\mathbf{n}$ is the unit outward normal vector to Γ, Π the projection onto the tangential plane given by $\Pi\mathbf{f} = \mathbf{f} - (\mathbf{f} \cdot \mathbf{n})\mathbf{n}$ and $K = K(X,t)(\geq 0)$ the slip rate. This condition means that the tangential velocity is in proportion to the normal stress with the proportion coefficient K. In general K is likely to be a function of the shear and the normal stresses, nevertheless we assume that it is be predictable *a priori* for the sake of simplicity. The basic form of (1.3.4) was first considered by Navier [26] at the dawn of fluid mechanics.

We should remark that when $K \equiv 0$, (1.3.4) obviously becomes (1.3.3). Moreover, when $K \equiv +\infty$, (1.3.4) becomes

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \Pi\mathbb{T}\mathbf{n} = \mathbf{0} \quad \text{ on } \Gamma,$$

which represents the perfect-slip case on the boundary. The general slip condition (1.3.4) may formally connect the adherence to the perfect-slip cases through K.

In this paper we shall consider the solvability of the initial-boundary value problem (1.3.1)-(1.3.2)-(1.3.4).

1.4 Reformulation in the Lagrangian coordinate system

The problem formulated above is written in the Eulerian coordinates system X. Now, we rewrite it in the Lagrangian coordinates system x. Let $\mathbf{u}(x,t)$ and $q(x,t)$ be the velocity vector field and pressure, respectively, expressed as functions of the Lagrangian coordinates. The relationship between Lagrangian and Eulerian coordinates is given by

$$X = x + \int_0^t \mathbf{u}(x,\tau)\,\mathrm{d}\tau \equiv X_{\mathbf{u}}(x,t), \quad \mathbf{u}(x,t) = \mathbf{v}(X_{\mathbf{u}}(x,t),t). \tag{1.4.1}$$

From $(1.3.1)_1$ it is easy to derive

$$\frac{\partial \varrho_{\mathbf{u}}}{\partial t}(x,t) = 0 \tag{1.4.2}$$

for $\varrho_{\mathbf{u}}(x,t) := \varrho(X_{\mathbf{u}}(x,t),t)$. Integrating (1.4.2) over $(0,t)$ yields

$$\varrho_{\mathbf{u}}(x,t) = \varrho_{\mathbf{u}}(x,0) = \varrho(X_{\mathbf{u}}(x,0),0) = \varrho(x,0) = \varrho_0(x). \tag{1.4.3}$$

This means that the density at each point in the Lagrangian coordinates does not vary in time.

Moreover, we denote the Jacobian matrix of the transformation $X_{\mathbf{u}}$ by $\mathbb{A}(x,t) = (a_{ij}(x,t))_{i,j=1,2,3}$ with elements $a_{ij}(x,t) = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial x_j}(x,\tau)\,\mathrm{d}\tau$ and its adjugate matrix by $\mathcal{A} = (A_{ij}(x,t))_{i,j=1,2,3} = \det\mathbb{A}\cdot\mathbb{A}^{-1}$. Jacobian $J_{\mathbf{u}}(x,t) = \det\mathbb{A}(x,t)$ satisfies the equality

$$\begin{aligned}\frac{\partial J_{\mathbf{u}}(x,t)}{\partial t} &= \sum_{i,j=1}^{3}\frac{\partial a_{ij}}{\partial t}A_{ji} = \sum_{i,j=1}^{3}A_{ji}\frac{\partial u_i}{\partial x_j} = \sum_{i,j=1}^{3}A_{ji}\sum_{k=1}^{3}\frac{\partial v_i}{\partial X_k}(X_{\mathbf{u}}(x,t),t)a_{kj}\\ &= J_{\mathbf{u}}(x,t)(\mathrm{div}\mathbf{v})(X_{\mathbf{u}}(x,t),t) = 0\end{aligned}$$

according to $(1.3.1)_2$. Since $J_{\mathbf{u}}(x,0)=1$, we have $J_{\mathbf{u}}(x,t)\equiv 1$. In general,

$$\nabla_x\{F(X_{\mathbf{u}}(x,t),t)\} = \mathbb{A}^{\mathrm{T}}\nabla_X F(X,t),$$

so that

$$\nabla_X F(X,t) = \mathcal{A}^{\mathrm{T}} \nabla_x F_{\mathbf{u}}(x,t) \equiv \nabla_{\mathbf{u}} F_{\mathbf{u}}(x,t), \quad F_{\mathbf{u}}(x,t) := F(X_{\mathbf{u}}(x,t),t),$$

since $\mathbb{A}^{-\mathrm{T}} = (\mathbb{A}^{-1})^{\mathrm{T}} = \mathcal{A}^{\mathrm{T}}$.

In the same way as (1.4.3), we have $\mathbf{u}(x,0) = \mathbf{v}_0(x)$. Thus the problem (1.3.1)-(1.3.2)-(1.3.4) becomes

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{u}}{\partial t} = \nabla_{\mathbf{u}} \cdot \mathbb{T}_{\mathbf{u}} + \varrho_0 \mathbf{b}_{\mathbf{u}}, \quad \nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \quad \text{for } x \in \Omega,\ t > 0, \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{for } x \in \Omega, \\ \mathbf{u} \cdot \mathbf{n}_{\mathbf{u}} = 0, \quad \mathbf{u} + K_{\mathbf{u}} \Pi_{\mathbf{u}} \mathbb{T}_{\mathbf{u}} \mathbf{n}_{\mathbf{u}} = \mathbf{0} \quad \text{for } x \in \Gamma,\ t > 0. \end{cases} \tag{1.4.4}$$

Here,

$$\begin{aligned} &\mathbb{T}_{\mathbf{u}} = -q\,\mathbb{I} + 2\nu(\varrho_0)\mathbb{D}_{\mathbf{u}}(\mathbf{u}) - \beta\Big(\nabla_{\mathbf{u}}\varrho_0 \otimes \nabla_{\mathbf{u}}\varrho_0 - \frac{1}{3}|\nabla_{\mathbf{u}}\varrho_0|^2\,\mathbb{I}\Big), \\ &\mathbb{D}_{\mathbf{u}}(\mathbf{w}) = \frac{1}{2}(\nabla_{\mathbf{u}}\mathbf{w} + [\nabla_{\mathbf{u}}\mathbf{w}]^{\mathrm{T}}), \quad \mathbf{b}_{\mathbf{u}}(x,t) = \mathbf{b}(X_{\mathbf{u}}(x,t),t), \\ &\mathbf{n}_{\mathbf{u}}(x,t) = \mathbf{n}(X_{\mathbf{u}}(x,t)), \quad K_{\mathbf{u}}(x,t) = K(X_{\mathbf{u}}(x,t),t), \quad \Pi_{\mathbf{u}}\mathbf{f} = \mathbf{f} - (\mathbf{f}\cdot\mathbf{n}_{\mathbf{u}})\mathbf{n}_{\mathbf{u}}, \\ &\Pi_{\mathbf{u}}\mathbb{T}_{\mathbf{u}}\mathbf{n}_{\mathbf{u}} = 2\nu(\varrho_0)\Pi_{\mathbf{u}}\mathbb{D}_{\mathbf{u}}(\mathbf{u})\mathbf{n}_{\mathbf{u}} - \beta\Pi_{\mathbf{u}}(\nabla_{\mathbf{u}}\varrho_0 \otimes \nabla_{\mathbf{u}}\varrho_0)\mathbf{n}_{\mathbf{u}}. \end{aligned}$$

It should be noted the following fact. This transformation into the Lagrangian coordinate system has been used for free boundary problems in order to transform them into the fixed domain problems (for example [39, 40]). Though problem (1.3.1)-(1.3.2)-(1.3.4) is posed in the fixed domain from the beginning, we apply this transformation to our problem.

The most important advantage is to hold (1.4.3). Even for equations (1.3.1) we still meet the difficulties similar to (1.2.1). Nevertheless, they can be removed by the use of (1.4.3) by virtue of the incompressible constraint and the Lagrangian coordinates system.

On the other hand, applying the transformation may cause some disadvantages. The functions and spatial derivatives in the equations are all transformed into non-linear terms, for example, $\nabla_{\mathbf{u}}$, $\mathbf{b}_{\mathbf{u}}$, *etc.* Moreover, the boundary conditions also become quasi-linear unlike the original problem. These terms seem to cause another difficulties, however (1.4.4) is much easier to handle when we proceed the “time-local” analysis. For any mathematical works like the time-global solvability or stability, we must need the existence of the time-local solution to the problem as the first step.

The aim of this dissertation is to prove a theorem on time-local solvability of problem (1.4.4) in Sobolev–Slobodetskiĭ spaces.

The plan of the rest of this paper is as follows. First, in Chapter 2 time-local existence of a unique solution of problem (1.4.4) with the adherence boundary condition ($K \equiv 0$) is proved. Then, in Chapter 3 the general slip case ($\inf K > 0$) is considered.

Chapter 2

Initial-boundary value problem under the adherence condition

2.1 Introduction

In this chapter we consider the initial-boundary value problem whose boundary condition is the so-called adherence condition. This means that the velocity vector field vanishes at the boundary, thus the particles of the continuum are fixed at the wall of its container. Certainly it is an ideal situation for the model problem of the flow of granular matter, however it is worth while considering the problem with the stress of new type as the first step.

Here we have the initial-boundary value problem (1.4.4) with $K \equiv 0$ as follows:

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{u}}{\partial t} = \nabla_{\mathbf{u}} \cdot \mathbb{T}_{\mathbf{u}} + \varrho_0 \mathbf{b}_{\mathbf{u}}, \quad \nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \quad \text{for } x \in \Omega,\ t > 0, \\ \mathbf{u}\big|_{t=0} = \mathbf{v}_0 \quad \text{for } x \in \Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{for } x \in \Gamma,\ t > 0. \end{cases} \tag{2.1.1}$$

The aim of this chapter is to prove a theorem on local in time solvability of problem (2.1.1) in Sobolev–Slobodetskiĭ spaces. The author refers the readers to Apendix for the definition and the properties of Sobolev–Slobodetskiĭ spaces.

Furthermore, we consider the following linear problem

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{u}}{\partial t} = -\nabla q + \nu_1(x)\Delta \mathbf{u} + \varrho_0 \mathbf{f}, \quad \nabla \cdot \mathbf{u} = g \quad \text{for } x \in \Omega,\ t > 0, \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{for } x \in \Omega, \\ \mathbf{u} = \mathbf{d} \quad \text{for } x \in \Gamma,\ t > 0, \end{cases} \tag{2.1.2}$$

where $\nu_1(x)$ is a given positive function defined in Ω, $\mathbf{f}$ and g given functions defined for $x \in \Omega,\ t > 0$ and $\mathbf{d}$ a given function for $x \in \Gamma,\ t > 0$.

2.2 Mathematical Results

Let us describe the results in this chapter. First of all, we consider the problem (2.1.2) in the spaces $H_h^{2+l,1+l/2}(Q_T)$ and $H_h^{l,l/2}(Q_T)$ for $Q_T = \Omega \times (0, T)$. The following lemma will be proved in §2.3.

Lemma 2.2.1 *Let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary $\Gamma \in W_2^{3/2+l}$, $l \in (1/2, 1)$, $Q_T = \Omega \times (0, T)$ and $G_T = \Gamma \times (0, T)$, $0 < T < +\infty$. Suppose that $\mathbf{v}_0 \equiv \mathbf{0}$, $\varrho_0 \in W_2^{2+l}(\Omega)$, $\varrho_0(x) \geq R_0 > 0$, $\nu_1 \in W_2^{2+l}(\Omega)$, $\inf \nu_1 > 0$. For arbitrary $\mathbf{f} \in H_h^{l,l/2}(Q_T)$, $g = \nabla \cdot \mathbf{G}$, $\mathbf{G} \in H_h^{2+l,1+l/2}(Q_T)$, $\mathbf{d} \in H_h^{3/2+l,3/4+l/2}(G_T)$, $\mathbf{G}|_\Gamma = \mathbf{d}$ on Γ, the problem (2.1.2) has a unique solution $\mathbf{u} \in H_h^{2+l,1+l/2}(Q_T)$, $\nabla q \in H_h^{l,l/2}(Q_T)$, provided that h is sufficiently large. And the solution satisfies the following estimate*

$$\|\mathbf{u}\|_{H_h^{2+l,1+l/2}(Q_T)} + \|\nabla q\|_{H_h^{l,l/2}(Q_T)} \leq c \Big(\|\mathbf{f}\|_{H_h^{l,l/2}(Q_T)} + \|g\|_{H_h^{1+l,1/2+l/2}(Q_T)} + \|\mathbf{G}\|_{H_h^{0,1/2+l/2}(Q_T)} + \|\mathbf{d}\|_{H_h^{3/2+l,3/4+l/2}(G_T)} \Big) \tag{2.2.1}$$

for some constant c independent of T.

This lemma is proved in the same way as that in [38]. First, we consider the problem with constant coefficients in the half-space and in the whole-space. Using those results, we prove Lemma 2.2.1 in a bounded domain. In the case of the half-space and the whole-space, we give an explicit formula for the solution, and in a bounded domain we prove a priori estimates and establish the solvability of the problem (2.1.2) by the construction of a regularizer. This method was used in the theory of general parabolic initial-boundary value problems [36].

Next, the problem (2.1.2) is considered in the spaces $W_2^{2+l,1+l/2}(Q_T)$ and $W_2^{l,l/2}(Q_T)$.

Lemma 2.2.2 *Let Ω, Γ, l, T, ϱ_0, ν_1, be the same as in Lemma* 2.2.1. *For arbitrary $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, $\mathbf{f} \in W_2^{l,l/2}(Q_T)$, $g = \nabla \cdot \mathbf{G}$, $\mathbf{G} \in W_2^{2+l,1+l/2}(Q_T)$ and $\mathbf{d} \in W_2^{3/2+l,3/4+l/2}\ (G_T)$ satisfying the compatibility conditions*

$$\nabla \cdot \mathbf{v}_0 = g(\,\cdot\,,0) \ \textit{in}\ \Omega, \quad \mathbf{v}_0 = \mathbf{d}(\,\cdot,0) \ \textit{on}\ \Gamma, \quad \mathbf{G}|_\Gamma = \mathbf{d} \ \textit{on}\ \Gamma,$$

problem (2.1.2) *has a unique solution $(\mathbf{u}, \nabla q)$ in $W_2^{2+l,1+l/2}(Q_T) \times W_2^{l,l/2}(Q_T)$ and*

$$\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla q\|_{Q_T}^{(l,l/2)} \le c(T) \Big(\|\mathbf{f}\|_{Q_T}^{(l,l/2)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}$$
$$+ \|g\|_{W_2^{1+l,1/2+l/2}(Q_T)} + \|\mathbf{G}\|_{Q_T}^{(0,1+l/2)} + \|\mathbf{d}\|_{W_2^{3/2+l,3/4+l/2}(G_T)} \Big), \tag{2.2.2}$$

where $c(T)$ is a non-decreasing function of T.

Finally, we consider the problem (2.1.1), and the following theorem on temporally local solvability is proved in § 2.5.

Theorem 2.2.1 *Let Ω be a bounded domain with a boundary $\Gamma \in W^{3/2+l}$, $l \in (1/2, 1)$, $\varrho_0 \in W_2^{2+l}(\Omega)$, $\varrho_0(x) \ge R_0 > 0$, $\nu \in C^2(\mathbb{R}_+)$, $\nu > 0$. Assume that $\mathbf{b}$ has continuous derivatives up to order two and that $\mathbf{b}$, $\nabla_X \mathbf{b}$ satisfy the Lipschitz condition in X and the Hölder condition with the exponent $1/2$ in t, and that $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$ satisfies the compatibility conditions*

$$\nabla \cdot \mathbf{v}_0 = 0 \ \textit{in}\ \Omega, \quad \mathbf{v}_0 = \mathbf{0} \ \textit{on}\ \Gamma.$$

Then the problem (2.1.1) *has a unique solution $(\mathbf{u}, \nabla q) \in W_2^{2+l,1+l/2}(Q_{T'}) \times W_2^{l,l/2}(Q_{T'})$ on a finite interval $(0, T')$ whose magnitude T' depends on the data, i.e., on the norms of $\mathbf{b}$, $\mathbf{v}_0$ and ϱ_0 (see the condition* (2.5.7) *below).*

2.3 Proofs of Lemmata 2.2.1 and 2.2.2

2.3.1 Problem in the half-space

First of all, in order to discuss the problem (2.1.2) in the space $H_h^{l,l/2}(Q_T)$, let $\varrho_0 \equiv 1$ and $\nu_1 \equiv \text{const} > 0,$ and we consider the initial-boundary value problem for the homogeneous Stokes system in the half-space $D_{+\infty} \equiv \mathbb{R}^3_+ \times (0, \infty)$, $(x_3 > 0,\ t > 0)$:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu_1 \Delta \mathbf{u} + \nabla q = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } D_{+\infty}, \\ \mathbf{u}|_{t=0} = \mathbf{0} \quad \text{on } \mathbb{R}^3_+, \quad \mathbf{u}|_{x_3=0} = (d_1, d_2, 0) \quad \text{on } D_\infty \equiv \mathbb{R}^2 \times (0, \infty). \end{cases} \tag{2.3.1}$$

Extend $\mathbf{u} = (u_1, u_2, u_3)$, q, $\mathbf{d}' = (d_1, d_2)$ to the half-space $t < 0$ by zero and make the Fourier transformation with respect to $x' = (x_1, x_2)$ and the Laplace transformation with respect to t:

$$\hat{f}(\xi', x_3, s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{d}t \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}x'\cdot\xi'} f(x', x_3, t)\mathrm{d}x'.$$

Then we have the following system of ordinary differential equations:

$$\begin{cases} \nu_1 \left(r^2 - \dfrac{\mathrm{d}^2}{\mathrm{d}x_3^2} \right) \hat{u}_j + \mathrm{i}\xi_j \hat{q} = 0 \quad (j = 1, 2), \\ \nu_1 \left(r^2 - \dfrac{\mathrm{d}^2}{\mathrm{d}x_3^2} \right) \hat{u}_3 + \dfrac{\mathrm{d}\hat{q}}{\mathrm{d}x_3} = 0, \\ \mathrm{i}\xi_1 \hat{u}_1 + \mathrm{i}\xi_2 \hat{u}_2 + \dfrac{\mathrm{d}\hat{u}_3}{\mathrm{d}x_3} = 0, \\ \hat{\mathbf{u}}|_{x_3=0} = (\hat{d}_1, \hat{d}_2, 0), \quad (\hat{\mathbf{u}}, \hat{q}) \longrightarrow 0 \ (x_3 \to +\infty), \end{cases} \tag{2.3.2}$$

where

$$r^2 = \frac{s}{\nu_1} + |\xi'|^2, \quad |\xi'|^2 = \xi_1^2 + \xi_2^2, \quad \arg r \in \left(-\frac{\pi}{4}, \frac{\pi}{4} \right).$$

This problem is easily solved by the same way as in [38] as the second order ordinary differential equation for $\hat{u}_j$ explicitly, i.e.,

$$\begin{cases} \hat{u}_j = \dfrac{-\mathrm{i}\xi_j e_1(x_3)}{|\xi'|} \displaystyle\sum_{k=1}^{2} \mathrm{i}\xi_k \hat{d}_k + \hat{d}_j e_0(x_3) \quad (j = 1, 2), \\ \hat{u}_3 = e_1(x_3) \displaystyle\sum_{k=1}^{2} \mathrm{i}\xi_k \hat{d}_k, \\ \hat{q} = \dfrac{-\nu_1(r + |\xi'|)e_2(x_3)}{|\xi'|} \displaystyle\sum_{k=1}^{2} \mathrm{i}\xi_k \hat{d}_k, \end{cases} \tag{2.3.3}$$

where

$$e_0(x_3) = \mathrm{e}^{-rx_3}, \qquad e_1(x_3) = \frac{\mathrm{e}^{-rx_3} - \mathrm{e}^{-|\xi'|x_3}}{r - |\xi'|}, \qquad e_2(x_3) = \mathrm{e}^{-|\xi'|x_3}.$$

In estimating the solution, it is convinient to use the new norms $\|\cdot\|^2_{\gamma,h,D_\infty}$ and $\|\cdot\|^2_{\gamma,h,D_{+\infty}}$ for $\gamma \geq 0$ (see Appendix A.2). They are equivalent to the norms in $H_h^{\gamma,\gamma/2}(D_\infty)$ and $H_h^{\gamma,\gamma/2}(D_{+\infty})$, respectively.

According to [38], for the functions $e_j(x_3)$, $j = 0, 1$, we have

Lemma 2.3.1 *Let $s = h + i\xi_0$, $h > 0$, $j = 0, 1, 2, \dots$, and $\alpha \in (0, 1)$. Then there exists a positve constant c independent of r and $|\xi'|$ such that*

(i) $$\int_0^\infty \Big| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_0(x_3) \Big|^2 \mathrm{d}x_3 \leq c|r|^{2j-1},$$

(ii) $$\int_0^\infty \int_0^\infty \Big| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_0(x_3 + z) - \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_0(x_3) \Big|^2 \frac{\mathrm{d}x_3 \mathrm{d}z}{z^{1+2\alpha}} \leq c|r|^{2(j+\alpha)-1},$$

(iii) $$\int_0^\infty \Big| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_1(x_3) \Big|^2 \mathrm{d}x_3 \leq c\frac{|r|^{2j-1} + |\xi'|^{2j-1}}{|r|^2},$$

(iv) $$\int_0^\infty \int_0^\infty \Big| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_1(x_3 + z) - \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_1(x_3) \Big|^2 \frac{\mathrm{d}x_3 \mathrm{d}z}{z^{1+2\alpha}}$$
$$\leq c\frac{|r|^{2(j+\alpha)-1} + |\xi'|^{2(j+\alpha)-1}}{|r|^2},$$

for all $\xi' \in \mathbb{R}^2$.

Therefore, from (2.3.3) and Lemma 2.3.1 it follows that

Lemma 2.3.2 *Let* $h > 0$ *and* $l \in (1/2, 1)$. *Then the solution* $(\mathbf{u}, q)$ *of the problem* (2.3.1) *satisfies the estimate*

$$\|\mathbf{u}\|^2_{2+l,h,D_{+\infty}} + \|\nabla q\|^2_{l,h,D_{+\infty}} \leq c\,\|\mathbf{d}'\|^2_{3/2+l,h,D_\infty}, \tag{2.3.4}$$

where c *is a constant independent of* h.

2.3.2 Non-homogeneous Stokes system in the half-space and in the whole space

In this subsection we shall generalize Lemma 2.3.2 to that for the Stokes system with the non-homogeneous terms:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu_1 \Delta \mathbf{u} + \nabla q = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = g \quad \text{in } D_{+T}, \\ \mathbf{u}|_{t=0} = \mathbf{0} \quad \text{on } \mathbb{R}^3_+, \quad \mathbf{u}|_{x_3=0} = \mathbf{d} = (d_1, d_2, d_3) \quad \text{on } D_T, \end{cases} \tag{2.3.5}$$

where $D_{+T} \equiv \mathbb{R}^3_+ \times (0,T)$ and $D_T \equiv \mathbb{R}^2 \times (0,T)$. We prove

Lemma 2.3.3 *Let* h, l, ν, d_1, d_2 *be as in Lemma* 2.3.2 (D_∞ *should be replaced by* D_T). *Suppose that* $\mathbf{f} \in H_h^{l,l/2}(D_{+T})$, $g \in H_h^{1+l,1/2+l/2}(D_{+T})$, $g = \nabla \cdot \mathbf{G}$ *with* $\mathbf{G} = (G_1, G_2, G_3)$, $\mathbf{G} \in H_h^{0,1+l/2}(D_{+T})$, $d_3 \in H_h^{3/2+l,3/4+l/2}(D_T)$ *and the condition* $G_3|_{x_3=0} = d_3$ *is satisfied. Then there exists a unique solution* $(\mathbf{u}, \nabla q)$ *of* (2.3.5) *such that* $\mathbf{u} \in H_h^{2+l,1+l/2}(D_{+T})$, $\nabla q \in H_h^{l,l/2}(D_{+T})$ *satisfying the estimate*

$$\begin{aligned} \|\mathbf{u}\|_{H_h^{2+l,1+l/2}(D_{+T})} + \|\nabla q\|_{H_h^{l,l/2}(D_{+T})} \leq c \Big(&\|\mathbf{f}\|_{H_h^{l,l/2}(D_{+T})} + \|g\|_{H_h^{1+l,1/2+l/2}(D_{+T})} \\ &+ \|\mathbf{G}\|_{H_h^{0,1+l/2}(D_{+T})} + \|\mathbf{d}\|_{H_h^{3/2+l,3/4+l/2}(D_T)} \Big). \end{aligned} \tag{2.3.6}$$

Proof. We can assume $T = \infty$ after an appropriate extension of $(\mathbf{f}, g, \mathbf{G}, \mathbf{d})$. We seek a solution of (2.3.5) in the form

$$(\mathbf{u}, q) = (\mathbf{w}^{(1)} + \nabla\phi + \mathbf{w}^{(2)},\ \pi - \phi_t + \nu_1 g'). \tag{2.3.7}$$

Here $\mathbf{w}^{(1)}$ is a solution of the Dirichlet problem for the heat equation:

$$\begin{cases} \dfrac{\partial \mathbf{w}^{(1)}}{\partial t} - \nu_1 \Delta \mathbf{w}^{(1)} = \mathbf{f} \quad \text{in } D_{+\infty}, \\ \mathbf{w}^{(1)}|_{t=0} = \mathbf{0} \quad \text{on } \mathbb{R}^3_+, \qquad \mathbf{w}^{(1)}|_{x_3=0} = \mathbf{0} \quad \text{on } D_\infty. \end{cases} \tag{2.3.8}$$

Next, ϕ is a solution of the Neumann problem:

$$\Delta\phi = g - \nabla \cdot \mathbf{w}^{(1)} \equiv g' \quad \text{in } \mathbb{R}^3_+, \qquad \frac{\partial \phi}{\partial x_3}\Big|_{x_3=0} = d_3 \quad \text{on } \mathbb{R}^2. \tag{2.3.9}$$

Then $(\mathbf{w}^{(2)}, \pi)$ is a solution of the problem similar to (2.3.1):

$$\begin{cases} \dfrac{\partial \mathbf{w}^{(2)}}{\partial t} - \nu_1 \Delta \mathbf{w}^{(2)} + \nabla\pi = 0, \quad \nabla \cdot \mathbf{w}^{(2)} = 0 \quad \text{in } D_{+\infty}, \\ \mathbf{w}^{(2)}|_{t=0} = \mathbf{0} \quad \text{on } \mathbb{R}^3_+, \quad \mathbf{w}^{(2)}|_{x_3=0} = (\widetilde{d}_1, \widetilde{d}_2, 0) \quad \text{on } D_\infty, \end{cases} \tag{2.3.10}$$

where $\widetilde{d}_j = d_j - \frac{\partial\phi}{\partial x_j}|_{x_3=0}$ $(j = 1, 2)$. It is well known that a solution of (2.3.8) satisfies the estimate

$$\|\mathbf{w}^{(1)}\|_{H_h^{2+l,1+l/2}(D_{+\infty})} \le c\|\mathbf{f}\|_{H_h^{l,l/2}(D_{+\infty})}. \tag{2.3.11}$$

Next, from the classical result of the Neumann problem [1], we have the following estimate of the solution to (2.3.9):

$$\|\nabla\phi\|_{\dot{W}_2^{2+l}(\mathbb{R}^3_+)} \le c\left(\|g\|_{\dot{W}_2^{1+l}(\mathbb{R}^3_+)} + \|\mathbf{w}^{(1)}\|_{\dot{W}_2^{2+l}(\mathbb{R}^3_+)} + \|d_3\|_{\dot{W}_2^{3/2+l}(\mathbb{R}^2)}\right). \tag{2.3.12}$$

In order to estimate $\|\nabla\phi\|_{H_h^{0,1+l/2}(D_{+\infty})}$, we express the solution of (2.3.9) by virtue of the Neumann function N (see [41]) as

$$\begin{aligned} \phi &= \int_{\mathbb{R}^3_+} N(x,y) g'(y,t)\mathrm{d}y + \int_{\mathbb{R}^2} N(x,y') d_3(y',t)\mathrm{d}y' \\ &= -\int_{\mathbb{R}^3_+} \nabla_y N(x,y) \cdot (\mathbf{G} - \mathbf{w}^{(1)})\mathrm{d}y. \end{aligned}$$

Calderón-Zygmund theorem implies the estimate

$$\|\nabla\phi\|_{H_h^{0,1+l/2}(D_{+\infty})} \le c\left(\|\mathbf{G}\|_{H_h^{0,1+l/2}(D_{+\infty})} + \|\mathbf{w}^{(1)}\|_{H_h^{0,1+l/2}(D_{+\infty})}\right). \tag{2.3.13}$$

Finally, applying Lemma 2.3.2 to $(\mathbf{w}^{(2)}, \pi)$, one can obtain

$$\begin{aligned}
&\|\mathbf{w}^{(2)}\|_{H_h^{2+l,1+l/2}(D_{+\infty})} + \|\nabla\pi\|_{H_h^{l,l/2}(D_{+\infty})} \le c\sum_{j=1}^{2}\|\widetilde{d_j}\|_{H_h^{3/2+l,3/4+l/2}(D_\infty)} \\
&\le c\left(\sum_{j=1}^{2}\|d_j\|_{H_h^{3/2+l,3/4+l/2}(D_\infty)} + \|\nabla\phi\|_{H_h^{2+l,1+l/2}(D_{+\infty})}\right).
\end{aligned} \tag{2.3.14}$$

Combining (2.3.11)-(2.3.14), one can obtain the estimate (2.3.6).

Uniqueness of the solution follows from the standard energy method. Indeed, let $(\mathbf{u}', q')$ and $(\mathbf{u}'', q'')$ be solutions of (2.3.5), then $(\mathbf{u}, q) := (\mathbf{u}' - \mathbf{u}'', q' - q'')$ satisfies the same equations as (2.3.5) with $\mathbf{f} = \mathbf{0}$, $g = 0$, $\mathbf{d} = \mathbf{0}$, *viz.*, (2.3.1) with $\mathbf{d}' = \mathbf{0}$. Then, Lemma 2.3.2 leads to $(\mathbf{u}, \nabla q) = (\mathbf{0}, \mathbf{0})$. Consequently, the solution of (2.3.5) is unique and given by (2.3.7). ∎

We can prove a similar result for the Cauchy problem.

Lemma 2.3.4 *Let h, l, ν_1 be as in Lemma 2.3.2 (D_∞ should be replaced by D_T) and $\mathbb{R}^3_T \equiv \mathbb{R}^3 \times (0, T)$. Suppose that $\mathbf{f} \in H_h^{l,l/2}(\mathbb{R}^3_T)$, $g \in H_h^{1+l,1/2+l/2}(\mathbb{R}^3_T)$, $g = \nabla\cdot\mathbf{G}$, $\mathbf{G} \in H_h^{0,1+l/2}(\mathbb{R}^3_T)$. Then the Cauchy problem*

$$\frac{\partial\mathbf{u}}{\partial t} - \nu_1\Delta\mathbf{u} + \nabla q = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = g \quad \text{in } \mathbb{R}^3_T, \qquad \mathbf{u}|_{t=0} = \mathbf{0} \quad \text{on } \mathbb{R}^3$$

has a unique solution $(\mathbf{u}, \nabla q) \in H_h^{2+l,1+l/2}(\mathbb{R}^3_T) \times H_h^{l,l/2}(\mathbb{R}^3_T)$, which satisfies the estimate

$$\begin{aligned}
&\|\mathbf{u}\|_{H_h^{2+l,1+l/2}(\mathbb{R}^3_T)} + \|\nabla q\|_{H_h^{l,l/2}(\mathbb{R}^3_T)} \\
&\qquad \le c\left(\|\mathbf{f}\|_{H_h^{l,l/2}(\mathbb{R}^3_T)} + \|g\|_{H_h^{1+l,1/2+l/2}(\mathbb{R}^3_T)} + \|\mathbf{G}\|_{H_h^{0,1+l/2}(\mathbb{R}^3_T)}\right).
\end{aligned} \tag{2.3.15}$$

2.3.3 Proof of Lemma 2.2.1

Proof of Lemma 2.2.1. Here, according to [40], we present some preliminaries. Because of the condition of Ω and Γ, in the neighbourhood of an arbitrary point $\xi \in \Gamma$, the surface Γ is determined by the equation

$$y_3 = \varphi(y'), \quad y' = (y_1, y_2) \in K_d$$

in Cartesian coordinates system (y_1, y_2, y_3) with the origin at ξ and with y_3-axis directed along $-\mathbf{n}(\xi)$, $\mathbf{n}$ being the outward normal to Γ. The function φ is defined in a disc $K_d : |y'| < d$, and it satisfies the condition $\varphi(0) = 0$, $\nabla'\varphi(0) = \mathbf{0}$ (∇' is the gradient with respect to y') and $\|\varphi\|_{W^{3/2+l}}(K_d) \leq M$. The constants d and M are indepenent of ξ.

It can be assumed that φ is extended into $\mathbb{R}^3_+$ (see [38, 40]), belongs to $W_2^{2+l}(\mathbb{R}^3_+)$, and satisfies $\varphi(0) = 0$, $\nabla\varphi(0) = \mathbf{0}$,

$$\sup_{|y|\leq\lambda} |\varphi(y)| \leq cM\lambda, \qquad \sup_{|y|\leq\lambda} |\nabla\varphi(y)| \leq cM\lambda^{1/2}. \tag{2.3.16}$$

The transformation $y = Y(z)$:

$$y_1 = z_1, \ y_2 = z_2, \ y_3 = z_3 + \varphi(z) \tag{2.3.17}$$

is invertible if $|\varphi_{z_3}| < 1$ and maps $\mathbb{R}^3_+$ onto the domain $\{y_3 > \varphi(y')\}$.

The solvability of (2.1.2) will be proved by the regularizer $\mathcal{R}$ (see for instance [36]), which is a linear continuous operator from the data $\mathbf{F} = (\mathbf{f}, g, \mathbf{d}) \in \mathcal{H}_{h,l} \equiv H_h^{l,l/2}(Q_T) \times H_h^{1+l,1/2+l/2}(Q_T) \times H_h^{3/2+l,3/4+l/2}(G_T)$ to the solution ($\boldsymbol{\omega}$, $\nabla\pi) \in H_h^{2+l,1+l/2}(Q_T) \times H_h^{l,l/2}(Q_T)$ of

$$\begin{cases} \dfrac{\partial\boldsymbol{\omega}}{\partial t} - \dfrac{\nu_1(x)}{\varrho_0(x)}\Delta\boldsymbol{\omega} + \dfrac{1}{\varrho_0(x)}\nabla\pi = \mathbf{f} + \mathcal{M}_1\mathbf{F}, \quad \nabla\cdot\boldsymbol{\omega} = g + \mathcal{M}_2\mathbf{F} \quad \text{in } Q_T, \\ \boldsymbol{\omega}|_{t=0} = \mathbf{0} \quad \text{in } \Omega, \quad \boldsymbol{\omega}|_{\Gamma} = \mathbf{d} + \mathcal{M}_3\mathbf{F} \quad \text{on } G_T, \end{cases} \tag{2.3.18}$$

where $\mathcal{M}\mathbf{F} = (\mathcal{M}_1\mathbf{F}, \mathcal{M}_2\mathbf{F}, \mathcal{M}_3\mathbf{F})$ is a contraction operator on $\mathcal{H}_{h,l}$ for sufficiently large h. The solution of (2.1.2) can be expressed in terms of the regularizer as $(\mathbf{w}, q) = \mathcal{R}(I + \mathcal{M})^{-1}(\mathbf{f}, g, \mathbf{d})$.

In order to establish the existence of a solution of (2.1.2), let $\{\Omega^{(k)}\}$ and $\{\omega^{(k)}\}$ be the coverings of $\bar{\Omega}$ for an arbitrary small number λ, as follows:

$$\Omega^{(k)} = \{|x - \xi^{(k)}| \leq \beta_k \lambda\}, \quad \omega^{(k)} = \{|x - \xi^{(k)}| \leq \frac{\beta_k \lambda}{2}\}.$$

It is convenient to assume that $\xi^{(k)} \in \Gamma$, $\beta_k = 2$ for $k = 1, 2, \ldots, M_\lambda$, and $\xi^{(k)} \in \Omega$, $\operatorname{dist}(\xi^{(k)}, \Gamma) \geq \frac{5\lambda}{4}$, $\beta_k = 1$ for $k = M_\lambda + 1, M_\lambda + 2, \ldots, N_\lambda$. Now we take two families of smooth functions $\{\zeta^{(k)}(x)\}$ and $\{\eta^{(k)}(x)\}$ associated with the coverings $\{\Omega^{(k)}\}$ and $\{\omega^{(k)}\}$ satisfying $\zeta^{(k)}(x) = 1$ for $x \in \omega^{(k)}$, $\zeta^{(k)}(x) = 0$ for $x \notin \overline{\Omega^{(k)}}$, $0 \leq \zeta^{(k)}(x) \leq 1$, $|D_x^\alpha \zeta^{(k)}(x)| \leq c\lambda^{-|\alpha|}$, $\eta^{(k)}(x) = \frac{\zeta^{(k)}(x)}{\sum_j (\zeta^{(j)}(x))^2}$. Obviously, $\{\eta^{(k)}(x)\}$ is a family of smooth functions satisfying $\eta^{(k)}(x) = 0$ for $x \notin \overline{\Omega^{(k)}}$, $\sum_k \eta^{(k)}(x)\zeta^{(k)}(x) = 1$ and $|D_x^\alpha \eta^{(k)}(x)| \leq c\lambda^{-|\alpha|}$, where c is independent of λ and k.

We define $(\bar{\mathbf{w}}, \bar{\pi}) = \mathcal{R}\mathbf{F}$ by the formula

$$(\bar{\mathbf{w}}, \bar{\pi})(x, t) = \sum_{k=1}^{N_\lambda} \eta^{(k)}(x)(\bar{\mathbf{w}}^{(k)}, \bar{\pi}^{(k)})(x, t),$$

where $(\bar{\mathbf{w}}^{(k)}, \bar{\pi}^{(k)})$ $(k = 1, 2, \ldots, N_\lambda)$ are given in the following way.

For $k = 1, 2, \ldots, M_\lambda$, let $\{y\}$ be local Cartesian coordinates in the neighbourhood of the point $\xi^{(k)} : y = \mathcal{C}_k(x - \xi^{(k)})$ with $\mathcal{C}_k$ being an orthogonal matrix satisfying $\mathcal{C}_k \mathbf{n}(\xi^{(k)}) = (0, 0, -1)^{\mathrm{T}}$, $\varphi^{(k)}(y')$ be the function defining Γ in the neighbourhood of $\xi^{(k)}$ and let Y_k be the corresponding transformation (2.3.17). The transformation $z = Z_k(x) = Y_k^{-1}\mathcal{C}_k(x - \xi^{(k)})$ maps the domain $\Omega^{(k)} \cap \Omega$ into the half space $\mathbb{R}^3_{+,k} = \{z \in \mathbb{R}^3 \,|z_3 > 0\}$ and its Jacobian matrix is the identity $\mathbb{I}$ at $\xi^{(k)}$. Set

$$\begin{aligned}
\mathbf{f}^{(k)}(z, t) &= \zeta^{(k)}(Z_k^{-1}(z))\mathcal{C}_k \mathbf{f}(Z_k^{-1}(z), t), \\
\mathbf{G}^{(k)}(z, t) &= \zeta^{(k)}(Z_k^{-1}(z))\mathcal{C}_k \mathbf{G}(Z_k^{-1}(z), t), \\
\mathbf{d}^{(k)}(z, t) &= \zeta^{(k)}(Z_k^{-1}(z))\mathcal{C}_k \mathbf{d}(Z_k^{-1}(z), t),
\end{aligned}$$

and extend them to the domain $\mathbb{R}^3_{+,k} \setminus Z_k^{-1}(\Omega^{(k)} \cap \Omega)$ by 0, which are denoted by the same symbols again.

Now let $(\mathbf{w}^{(k)}, \pi^{(k)})(z,t)$, $k = 1, 2, \ldots, M_\lambda$, be a solution of the half-space problem

$$
\begin{cases}
\dfrac{\partial \mathbf{w}^{(k)}}{\partial t}(z,t) - \dfrac{\nu_1(\xi^{(k)})}{\varrho_0(\xi^{(k)})}\Delta_z \mathbf{w}^{(k)}(z,t) + \dfrac{\nabla_z \pi^{(k)}(z,t)}{\varrho_0(\xi^{(k)})} = \mathbf{f}^{(k)}(z,t) \quad \text{in } D_{+T}^{(k)}, \\
\nabla_z \cdot \mathbf{w}^{(k)}(z,t) = \nabla_z \cdot \mathbf{G}^{(k)}(z,t) \quad \text{in } D_{+T}^{(k)}, \\
\mathbf{w}^{(k)}(z,t)\big|_{t=0} = \mathbf{0} \quad \text{in } \mathbb{R}^3_{+,k}, \quad \mathbf{w}^{(k)}(z,t)\big|_{z_3=0} = \mathbf{d}^{(k)}(z,t) \quad \text{on } D_T^{(k)},
\end{cases}
\tag{2.3.19}
$$

where $D_{+T}^{(k)} \equiv \mathbb{R}^3_{+,k} \times (0,T)$ and $D_T^{(k)} \equiv \partial\mathbb{R}^3_{+,k} \times (0,T)$. According to (2.3.6), we have

$$
\begin{aligned}
&\|\mathbf{w}^{(k)}\|_{H_h^{2+l,1+l/2}(D_{+T}^{(k)})} + \|\nabla \pi^{(k)}\|_{H_h^{l,l/2}(D_{+T}^{(k)})} \le c\left(\|\mathbf{f}^{(k)}\|_{H_h^{l,l/2}(D_{+T}^{(k)})}\right. \\
&\left.+\|\nabla \cdot \mathbf{G}^{(k)}\|_{H_h^{1+l,1/2+l/2}(D_{+T}^{(k)})} + \|\mathbf{G}^{(k)}\|_{H_h^{0,1+l/2}(D_{+T}^{(k)})} + \|\mathbf{d}^{(k)}\|_{H_h^{3/2+l,3/4+l/2}(D_T^{(k)})}\right).
\end{aligned}
\tag{2.3.20}
$$

Then we define for $k = 1, 2, \ldots, M_\lambda$

$$
\bar{\mathbf{w}}^{(k)}(x,t) = \mathcal{C}_k^{-1}\mathbf{w}^{(k)}(Z_k(x),t), \quad \bar{\pi}^{(k)}(x,t) = \pi^{(k)}(Z_k(x),t). \tag{2.3.21}
$$

Next, $(\mathbf{w}^{(k)}, \pi^{(k)})(x,t)$, $k = M_\lambda + 1, M_\lambda + 2, \ldots, N_\lambda$, is a solution of the Cauchy problem

$$
\begin{cases}
\dfrac{\partial \mathbf{w}^{(k)}}{\partial t}(x,t) - \dfrac{\nu_1(\xi^{(k)})}{\varrho_0(\xi^{(k)})}\Delta \mathbf{w}^{(k)}(x,t) + \dfrac{1}{\varrho_0(\xi^{(k)})}\nabla \pi^{(k)}(x,t) = \zeta^{(k)}\mathbf{f}(x,t), \\
\nabla \cdot \mathbf{w}^{(k)}(x,t) = \nabla \cdot (\zeta^{(k)}\mathbf{G})(x,t), \\
\mathbf{w}^{(k)}(x,t)\big|_{t=0} = \mathbf{0}.
\end{cases}
\tag{2.3.22}
$$

Then we define

$$
\bar{\mathbf{w}}^{(k)}(x,t) = \mathbf{w}^{(k)}(x,t), \quad \bar{\pi}^{(k)}(x,t) = \pi^{(k)}(x,t) \tag{2.3.23}
$$

for $k = M_\lambda + 1, M_\lambda + 2, \ldots, N_\lambda$.

Finally, we restrict $\bar{\pi}^{(k)}$ by

$$\int_{\Omega^{(k)}\cap\Omega} \bar{\pi}^{(k)}(x,t)\mathrm{d}x = 0. \tag{2.3.24}$$

For such $\bar{\pi}^{(k)}$ we have

$$\|\bar{\pi}^{(k)}\|_{L_2(Q_T^{(k)})} \leq c\lambda\|\nabla\bar{\pi}^{(k)}\|_{L_2(Q_T^{(k)})} \leq c\lambda\|\nabla\pi^{(k)}\|_{L_2(D_{+T}^{(k)})}, \tag{2.3.25}$$

where $Q_T^{(k)} = (\Omega^{(k)}\cap\Omega)\times(0,T)$. Consequently, $\eta^{(k)}\bar{\pi}^{(k)}$ is uniquely determined in Q_T, and $\mathcal{R}$ is well-defined.

Clearly, $\mathcal{R}$ is a linear operator on $\mathcal{H}_{h,l}$. To calculate the norm of $\mathcal{M}\mathbf{F}$, we rewrite the problem (2.3.19) in coordinates $\{x\}$ in the neighbourhood $\Omega^{(k)}\cap\Omega$ of $\xi^{(k)}$. Then,

$$\begin{cases} \dfrac{\partial\bar{\mathbf{w}}^{(k)}}{\partial t}(x,t) - \dfrac{\nu_1(\xi^{(k)})}{\varrho_0(\xi^{(k)})}\bar{\Delta}^{(k)}\bar{\mathbf{w}}^{(k)}(x,t) + \dfrac{1}{\varrho_0(\xi^{(k)})}\bar{\nabla}^{(k)}\bar{\pi}^{(k)}(x,t) = \zeta^{(k)}\mathbf{f}(x,t), \\ \bar{\nabla}^{(k)}\cdot\bar{\mathbf{w}}^{(k)}(x,t) = \bar{\nabla}^{(k)}\cdot(\zeta^{(k)}\mathbf{G}(x,t)) \quad \text{in } Q_T^{(k)}, \\ \bar{\mathbf{w}}^{(k)}|_{t=0} = \mathbf{0} \quad \text{in } \Omega^{(k)}\cap\Omega, \quad \bar{\mathbf{w}}^{(k)} = \zeta^{(k)}\mathbf{d}(x,t) \quad \text{on } G_T^{(k)}, \end{cases} \tag{2.3.26}$$

where $Q_T^{(k)} = (\Omega^{(k)}\cap\Omega)\times(0,T)$, $G_T^{(k)} = (\Omega^{(k)}\cap\Gamma)\times(0,T)$, $\bar{\nabla}^{(k)} = \mathcal{C}_k^{-1}\mathcal{Z}_k^{-\mathrm{T}}\nabla$, $\bar{\Delta}^{(k)} = \bar{\nabla}^{(k)}\cdot\bar{\nabla}^{(k)}$, and $\mathcal{Z}_k$ is the Jacobian matrix of the transformation Z_k.

Thus one can obtain

$$\begin{aligned} \mathcal{M}_1\mathbf{F} \;=\; & \sum_{k=M_\lambda+1}^{N_\lambda}\eta^{(k)}\left(\frac{\nu_1(\xi^{(k)})}{\varrho_0(\xi^{(k)})} - \frac{\nu_1(x)}{\varrho_0(x)}\right)\Delta\bar{\mathbf{w}}^{(k)} \\ & + \sum_{k=1}^{M_\lambda}\eta^{(k)}\left(\frac{\nu_1(\xi^{(k)})}{\varrho_0(\xi^{(k)})}\bar{\Delta}^{(k)}\bar{\mathbf{w}}^{(k)} - \frac{\nu_1(x)}{\varrho_0(x)}\Delta\bar{\mathbf{w}}^{(k)}\right) \\ & + \sum_{k=1}^{N_\lambda}\frac{\nu_1(x)}{\varrho_0(x)}\left(\eta^{(k)}\Delta\bar{\mathbf{w}}^{(k)} - \Delta(\eta^{(k)}\bar{\mathbf{w}}^{(k)})\right) \\ & - \sum_{k=M_\lambda+1}^{N_\lambda}\eta^{(k)}\left(\frac{1}{\varrho_0(\xi^{(k)})} - \frac{1}{\varrho_0(x)}\right)\nabla\bar{\pi}^{(k)} \\ & - \sum_{k=1}^{M_\lambda}\eta^{(k)}\left(\frac{1}{\varrho_0(\xi^{(k)})}\bar{\nabla}^{(k)}\bar{\pi}^{(k)} - \frac{1}{\varrho_0(x)}\nabla\bar{\pi}^{(k)}\right) \end{aligned}$$

$$
\begin{aligned}
& - \sum_{k=1}^{N_\lambda} \frac{1}{\varrho_0(x)} \left(\eta^{(k)}\nabla\bar{\pi}^{(k)} - \nabla(\eta^{(k)}\bar{\pi}^{(k)})\right), && (2.3.27) \\
\mathcal{M}_2\mathbf{F} &= -\sum_{k=1}^{M_\lambda} \left\{\eta^{(k)}\nabla\cdot\bar{\mathbf{w}}^{(k)} - \nabla\cdot(\eta^{(k)}\bar{\mathbf{w}}^{(k)})\right\} \\
& - \sum_{k=M_\lambda+1}^{N_\lambda} \left\{\eta^{(k)}\bar{\nabla}^{(k)}\cdot\bar{\mathbf{w}}^{(k)} - \nabla\cdot(\eta^{(k)}\bar{\mathbf{w}}^{(k)})\right\}, && (2.3.28) \\
\mathcal{M}_3\mathbf{F} &= \mathbf{0}.
\end{aligned}
$$

In the same scheme as [40, 41], we can show that

$$
\|\mathcal{M}\mathbf{F}\|_{\mathcal{H}_{h,l}} \le (c\lambda^\beta + c'(\lambda)h^{-1/2})\|\mathbf{F}\|_{\mathcal{H}_{h,l}} \tag{2.3.29}
$$

with c independent of λ. Hence, for small λ and large h, $\mathcal{M}$ is a contraction operator, so that the solvability of (2.1.2) is proved.

Furthermore, from the same way as that of (2.3.29) (see [40, 41]), it holds that

$$
\|\mathbf{w}\|_{H_h^{2+l,1+l/2}(Q_T)} + \|\nabla q\|_{H_h^{l,l/2}(Q_T)} \le c\|\mathbf{F}\|_{\mathcal{H}_{h,l}}. \tag{2.3.30}
$$

Thus the estimate (2.2.1) follows from (2.3.30).

2.3.4 Proof of Lemma 2.2.2

Proof of Lemma 2.2.2. The trace theorem implies that there exists the vector field $\mathbf{u}^* \in W_2^{2+l,1+l/2}(Q_T)$ satisfying the initial condition $\mathbf{u}^*|_{t=0} = \mathbf{v}_0$ and the inequality

$$
\|\mathbf{u}^*\|_{W_2^{2+l,1+l/2}(Q_T)} \le c\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}. \tag{2.3.31}
$$

For the difference $\mathbf{U} = \mathbf{u} - \mathbf{u}^*$ we get the problem (2.1.2) with homogeneous initial condition, i.e.,

$$
\begin{cases}
\dfrac{\partial\mathbf{U}}{\partial t} - \dfrac{\nu_1(y)}{\varrho_0(y)}\Delta\mathbf{U} + \dfrac{1}{\varrho_0(y)}\nabla q = \mathbf{f} - \dfrac{\partial\mathbf{u}^*}{\partial t} + \dfrac{\nu_1(y)}{\varrho_0(y)}\Delta\mathbf{u}^* =: \mathbf{f}^*, \\
\nabla\cdot\mathbf{U} = \nabla\cdot\mathbf{G} - \nabla\cdot\mathbf{u}^* =: \nabla\cdot\mathbf{G}^* =: g^* \\
\mathbf{U}|_{t=0} = \mathbf{0}, \quad \mathbf{U}|_\Gamma = \mathbf{d} - \mathbf{u}^*|_\Gamma = \mathbf{d}^*,
\end{cases} \tag{2.3.32}
$$

and the compatibility conditions reduce to

$$g^*(\,\cdot\,,0)=0, \quad \mathbf{d}^*(\,\cdot,0)=\mathbf{0}, \quad \mathbf{G}^*|_\Gamma=\mathbf{d}^*.$$

Hence from the conditions above, we have $\mathbf{f}^*\in H_h^{l,l/2}(Q_T)$, $g^*\in H_h^{1+l,1/2+l/2}(Q_T)$, $\mathbf{G}^*\in H_h^{2+l,1+l/2}(Q_T)$, $\mathbf{d}^*\in H_h^{3/2+l,3/4+l/2}(G_T)$. By applying Lemma 2.2.1 to (2.3.32) and taking into account (2.3.31) and Remark A.2.1, the assertion of Lemma 2.2.2 immediately follows.

2.4 Auxiliary estimates

Before proving Theorem 2.2.1, we begin with auxiliary propositions.

In this section we assume that $\mathbf{u}\in W_2^{2+l,1+l/2}(Q_T)$ satisfies

$$T^{1/2}\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)}\le\delta \tag{2.4.1}$$

with sufficiently small $\delta>0$.

The problem (2.1.1) is rewritten in the form

$$\begin{cases}\varrho_0\dfrac{\partial\mathbf{u}}{\partial t}-\nu(\varrho_0)\Delta\mathbf{u}+\nabla q=\mathbf{l}_1^{(\mathbf{u})}(\mathbf{u},q)+2\nu'(\varrho_0)\mathbb{D}_{\mathbf{u}}(\mathbf{u})\nabla_{\mathbf{u}}\varrho_0\\ \qquad\qquad\qquad -\dfrac{\beta}{3}(\nabla_{\mathbf{u}}^{(j)}\nabla_{\mathbf{u}}^{(i)}\varrho_0)\nabla_{\mathbf{u}}\varrho_0-\beta\Delta_{\mathbf{u}}\varrho_0\nabla_{\mathbf{u}}\varrho_0+\varrho_0\mathbf{b}_{\mathbf{u}},\\ \nabla\cdot\mathbf{u}=l_2^{(\mathbf{u})}(\mathbf{u}),\quad \mathbf{u}|_{t=0}=\mathbf{v}_0,\quad \mathbf{u}|_\Gamma=\mathbf{0},\end{cases} \tag{2.4.2}$$

where $(\nabla_{\mathbf{u}}^{(j)}\nabla_{\mathbf{u}}^{(i)}\varrho_0)$ is a 3×3 matrix whose (i,j) element is given by $\nabla_{\mathbf{u}}^{(j)}\nabla_{\mathbf{u}}^{(i)}\varrho_0$,

$$\begin{aligned}\mathbf{l}_1^{(\mathbf{u})}(\mathbf{w},s)&=\nu(\varrho_0)(\Delta_{\mathbf{u}}-\Delta)\mathbf{w}-(\nabla_{\mathbf{u}}-\nabla)s,\\ l_2^{(\mathbf{u})}(\mathbf{w})&=(\nabla-\nabla_{\mathbf{u}})\cdot\mathbf{w}=\nabla\cdot\mathcal{L}^{(\mathbf{u})}(\mathbf{w}).\end{aligned} \tag{2.4.3}$$

Hereafter we estimate the right-hand side of (2.4.2), which is neccesary to prove the solvability of the problem (2.1.1). Let us introduce the following notation:

$$a_{ij}=\delta_{ij}+b_{ij}, \quad b_{ij}=\int_0^t\frac{\partial u_i}{\partial x_j}\mathrm{d}\tau, \quad A_{ij}=\delta_{ij}+B_{ij},$$

where $\mathcal{A} = (A_{ij})$ (see p. 14). For $i \neq j$, $j \neq k$, $k \neq i$ from

$$A_{ii} = a_{jj}a_{kk} - a_{jk}a_{kj}, \quad A_{ij} = a_{ki}a_{jk} - a_{ji}a_{kk}$$

it follows that

$$B_{ii} = b_{jj} + b_{kk} + b_{jj}b_{kk} - b_{jk}b_{kj}, \quad B_{ij} = -b_{ji} + b_{ki}b_{jk} - b_{ji}b_{kk}. \tag{2.4.4}$$

Consequently, we have

$$l_2^{(\mathbf{u})}(\mathbf{w}) = -(\mathcal{B}^{\mathrm{T}}\nabla)\cdot\mathbf{w} = -\sum_{i,j=1}^{3} B_{ji}\frac{\partial w_i}{\partial x_j} = -\nabla\cdot(\mathcal{B}\mathbf{w}),$$

since $\sum_{j=1}^{3}\frac{\partial B_{ji}}{\partial x_j} = 0$ for $i = 1,2,3$. This yields that

$$\mathcal{L}^{(\mathbf{u})}(\mathbf{w}) = -\mathcal{B}\mathbf{w}. \tag{2.4.5}$$

We denote by a'_{ij}, b'_{ij}, A'_{ij}, B'_{ij} the same functions corresponding to another vector field $\mathbf{u}'(x,t)$, and set $\tilde{b}_{ij} = b_{ij} - b'_{ij}$, $\widetilde{B}_{ij} = B_{ij} - B'_{ij}$, etc. We have

$$\begin{aligned}
\widetilde{B}_{ii} &= \tilde{b}_{jj}(1+b_{kk}) + \tilde{b}_{kk}(1+b'_{jj}) - b_{kj}\tilde{b}_{jk} - b'_{jk}\tilde{b}_{kj},\\
\widetilde{B}_{ij} &= -\tilde{b}_{ji}(1+b_{kk}) - \tilde{b}_{kk}b'_{ji} + \tilde{b}_{jk}b_{ki} + b'_{jk}\tilde{b}_{ki}.
\end{aligned} \tag{2.4.6}$$

Finally, set

$$D\mathbf{u} = \left\{\frac{\partial u_i}{\partial x_j}\right\}_{i,j=1,2,3}, \; D^2\mathbf{u} = \left\{\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right\}_{i,j,k=1,2,3}, \; |D\mathbf{u}|_\Omega = \max_{i,j}\sup_{x\in\Omega}\left|\frac{\partial u_i}{\partial x_j}\right|,$$

$$|D^2\mathbf{u}|_\Omega = \max_{i,j,k}\sup_{x\in\Omega}\left|\frac{\partial^2 u_i}{\partial x_j\partial x_k}\right|, \quad \|D\mathbf{u}\|_{W_2^r(\Omega)} = \left(\sum_{j=1}^{3}\left\|\frac{\partial \mathbf{u}}{\partial x_j}\right\|^2_{W_2^r(\Omega)}\right)^{1/2}, \; \textit{etc.}$$

We proceed to estimates of the functions (2.4.4) and (2.4.6). All lemmata stated below were proved mainly in [39].

Lemma 2.4.1 *If* $\mathbf{u}$, $\mathbf{u}' \in W_2^{2+l,1+l/2}(Q_T)$, *then*

$$|\widetilde{B}_{ij}(x,t)| \leq 2\int_0^t |D(\mathbf{u}-\mathbf{u}')|\mathrm{d}\tau$$

$$\times\left(1+\int_0^t |D\mathbf{u}|_\Omega \mathrm{d}\tau+\int_0^t |D\mathbf{u}'|_\Omega \mathrm{d}\tau\right), \tag{2.4.7}$$

$$\|\widetilde{B}_{ij}(\cdot,t)\|_{W_2^{1+l}(\Omega)} \le c\int_0^t \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^{1+l}(\Omega)}\mathrm{d}\tau$$

$$\times\left(1+\int_0^t \|D\mathbf{u}\|_{W_2^{1+l}(\Omega)}\mathrm{d}\tau \int_0^t \|D\mathbf{u}'\|_{W_2^{1+l}(\Omega)}\mathrm{d}\tau\right), \tag{2.4.8}$$

$$\|\widetilde{B}_{ij}(\cdot,t)-\widetilde{B}_{ij}(\cdot,t-\tau)\|_{L_q(\Omega)}$$

$$\le 2\int_{t-\tau}^t \|D(\mathbf{u}-\mathbf{u}')\|_{L_q(\Omega)}\mathrm{d}\tau'\left(1+\int_0^t |D\mathbf{u}|_\Omega \mathrm{d}\tau'+\int_0^t |D\mathbf{u}'|_\Omega \mathrm{d}\tau'\right)$$

$$+2\int_0^t |D(\mathbf{u}-\mathbf{u}')|_\Omega \mathrm{d}\tau'\int_{t-\tau}^t (\|D\mathbf{u}\|_{L_q(\Omega)}+\|D\mathbf{u}'\|_{L_q(\Omega)})\mathrm{d}\tau', \tag{2.4.9}$$

$$\|\nabla\widetilde{B}_{ij}(\cdot,t)-\nabla\widetilde{B}_{ij}(\cdot,t-\tau)\|_{L_2(\Omega)}$$

$$\le 2\int_{t-\tau}^t \|D^2(\mathbf{u}-\mathbf{u}')\|_{L_2(\Omega)}\mathrm{d}\tau'\left(1+\int_0^t |D\mathbf{u}|_\Omega \mathrm{d}\tau'+\int_0^t |D\mathbf{u}'|_\Omega \mathrm{d}\tau'\right)$$

$$+2\int_0^t \|D^2(\mathbf{u}-\mathbf{u}')\|_{L_3(\Omega)}\mathrm{d}\tau'\int_{t-\tau}^t (\|D\mathbf{u}\|_{L_6(\Omega)}+\|D\mathbf{u}'\|_{L_6(\Omega)})\mathrm{d}\tau'$$

$$+2\int_{t-\tau}^t \|D(\mathbf{u}-\mathbf{u}')\|_{L_6(\Omega)}\mathrm{d}\tau'\int_0^t (\|D^2\mathbf{u}\|_{L_3(\Omega)}+\|D^2\mathbf{u}'\|_{L_3(\Omega)})\mathrm{d}\tau'$$

$$+2\int_0^t |D(\mathbf{u}-\mathbf{u}')|_\Omega \mathrm{d}\tau'\int_{t-\tau}^t (\|D^2\mathbf{u}\|_{L_2(\Omega)}+\|D^2\mathbf{u}'\|_{L_2(\Omega)})\mathrm{d}\tau', \tag{2.4.10}$$

where $\tau\in(0,t)$. *Such estimates* (*with* $\mathbf{u}'=\mathbf{0}$ *on the right hand side*) *also hold for the functions* B_{ij}.

Inequalities (2.4.7)–(2.4.10) can be obtained directly from formulae (2.4.6). In the proof of (2.4.10) we used the Hölder inequality

$$\|fg\|_{L_2(\Omega)} \le \|f\|_{L_3(\Omega)}\|g\|_{L_6(\Omega)}.$$

We note that

$$\int_0^t \|D\mathbf{u}\|_{W_2^{1+l}(\Omega)} \mathrm{d}\tau \le \sqrt{t}\ \|\mathbf{u}\|_{W_2^{2+l,0}(Q_T)} \le \delta, \tag{2.4.11}$$

$$\int_0^t \|D\mathbf{u}'\|_{W_2^{1+l}(\Omega)} \mathrm{d}\tau \le \sqrt{t}\, \|\mathbf{u}'\|_{W_2^{2+l,0}(Q_T)} \le \delta, \tag{2.4.12}$$

$$\begin{aligned}\int_0^t \|D\mathbf{u}\|_{W_2^l(\Omega)} \frac{\mathrm{d}\tau}{(t-\tau)^{1/2}} &\le \frac{t^{1/2-l/2}}{\sqrt{1-l}} \left(\int_0^t \|D\mathbf{u}\|_{W_2^l(\Omega)} \mathrm{d}\tau\right)^{1/2} \\ &\le \frac{T^{1/2}}{\sqrt{1-l}} \|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} \le \frac{\delta}{\sqrt{1-l}}\end{aligned} \tag{2.4.13}$$

hold.

Lemma 2.4.2 *If* $\mathbf{u},\ \mathbf{u}' \in W_2^{2+l,1+l/2}(Q_T)$ *satisfy condition* (2.4.1), *then for* $t \le T$

$$\|\widetilde{B}_{ij}\|_{W_2^{1+l}(\Omega)} \le c \int_0^t \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^{1+l}(\Omega)} \mathrm{d}\tau, \tag{2.4.14}$$

$$\begin{aligned}&\left(\int_0^t \|\widetilde{B}_{ij}(\cdot,t) - \widetilde{B}_{ij}(\cdot,t-\tau)\|^2_{W_2^l(\Omega)} \frac{\mathrm{d}\tau}{\tau^{1+l}}\right)^{1/2} \\ &\le c\left(\int_0^t \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^{1+l}(\Omega)} \mathrm{d}\tau + \int_0^t \frac{\|D(\mathbf{u}-\mathbf{u}')\|_{W_2^l(\Omega)}}{(t-\tau)^{l/2}} \mathrm{d}\tau\right).\end{aligned} \tag{2.4.15}$$

Such inequalities (*with* $\mathbf{u}' = \mathbf{0}$ *on the right side*) *hold also for* B_{ij}.

To derive (2.4.15) the fact that $W_2^{1+l}(\Omega)$ is embedded in $C(\overline{\Omega})$ (and also in $L_6(\Omega)$) and $W_2^l(\Omega)$ is embedded in $L_3(\Omega)$ is used.

Lemma 2.4.3 *If* $\mathbf{u},\ \mathbf{u}' \in W_2^{2+l,1+l/2}(Q_T)$ *satisfy condition* (2.4.1), *then for any* $f \in W_2^{l,l/2}(Q_T)$ *and* $h \in W_2^{1+l,1/2+l/2}(Q_T)$

$$\left\|\widetilde{B}_{ij} f\right\|_{Q_T}^{(l,l/2)} \le c\sqrt{T} \|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)} \|f\|_{Q_T}^{(l,l/2)}, \tag{2.4.16}$$

$$\|\widetilde{B}_{ij}h\|_{W_2^{1+l,1/2+l/2}(Q_T)} \le c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)}$$
$$\times(\|h\|_{W_2^{1+l,1/2+l/2}(Q_T)} + \|\nabla h\|_{Q_T}^{(0,l/2)} + \|h\|_{Q_T}^{(0,l/2)}). \quad (2.4.17)$$

Setting $\mathbf{u}' = \mathbf{0}$ in (2.4.16) and (2.4.17) and noting (2.4.11) and (2.4.13), we arrive at the following proposition.

Lemma 2.4.4 *If* $\mathbf{u}$ *satisfies* (2.4.1), *then*

$$\|B_{ij}f\|_{Q_T}^{(l,l/2)} \le c\delta\,\|f\|_{Q_T}^{(l,l/2)}, \quad (2.4.18)$$

$$\|B_{ij}h\|_{W_2^{1+l,1/2+l/2}(Q_T)}$$
$$\le c\delta(\|h\|_{W_2^{1+l,1/2+l/2}(Q_T)} + \|\nabla h\|_{Q_T}^{(0,l/2)} + \|h\|_{Q_T}^{(0,l/2)}). \quad (2.4.19)$$

Lemma 2.4.5 *Let* $\mathbf{u} \in W_2^{2+l,1+l/2}(Q_T)$, $T_0 > 0$. *It holds that*

$$\|D\mathbf{u}\|_{Q_T}^{(l,l/2)} \le c(T_0)\left(T^{1/2}\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} + T^{1/2-l/2}\|\mathbf{u}(\cdot,0)\|_{W_2^l(\Omega)}\right) \quad (2.4.20)$$

for any $T \le T_0$.

(2.4.20) is derived from the interpolation inequality

$$\|Df\|_{L_2(\Omega)} \le c(\varepsilon\|D^2 f\|_{L_2(\Omega)} + \varepsilon^{-1}\|f\|_{L_2(\Omega)}).$$

We proceed to estimates of $\mathbf{l}_1^{(\mathbf{u})}(\mathbf{w},s) - \mathbf{l}_1^{(\mathbf{u}')}(\mathbf{w},s)$, $l_2^{(\mathbf{u})}(\mathbf{w}) - l_2^{(\mathbf{u}')}(\mathbf{w})$ and $\mathcal{L}^{(\mathbf{u})}(\mathbf{w}) - \mathcal{L}^{(\mathbf{u}')}(\mathbf{w})$, where $\mathbf{l}_1^{(\mathbf{u})}$, $\mathbf{l}_1^{(\mathbf{u}')}$, *etc.*, are determined by formulae (2.4.3) on the basis of the vector fields $\mathbf{u}$ and $\mathbf{u}'$.

From (A.1.7) for $\varrho_0 \in W_2^{1+l}$ satisfying $\varrho_0(x) \ge R_0 > 0$ we have

$$\|\nu(\varrho_0)f\|_{Q_T}^{(l,l/2)} \le c\|\nu(\varrho_0)\|_{W_2^{1+l}(\Omega)}\,\|f\|_{Q_T}^{(l,l/2)} \le c(\varrho_0)\,\|f\|_{Q_T}^{(l,l/2)}, \quad (2.4.21)$$

where

$$c(\varrho_0) = c\left\{\sup_{R_0\le\varrho\le R_1}|\nu(\varrho)||\Omega|^{\frac{1}{2}} + \left(\sup_{R_0\le\varrho\le R_1}|\nu'(\varrho)| + \|\nabla\varrho_0\|_{W_2^l(\Omega)}\right)\|\nabla\varrho_0\|_{W_2^l(\Omega)}\right\},$$

and $R_1 = \sup_{x\in\Omega}\varrho_0(x) \le c\|\varrho_0\|_{W_2^{1+l}(\Omega)} < +\infty$.

Then we obtain the following estimates:

Lemma 2.4.6 *Let* $\mathbf{u}$ *and* $\mathbf{u}'$ *satisfy condition* (2.4.1). *For arbitrary* $\mathbf{w} \in W_2^{2+l,1+l/2}$ *it holds that* (Q_T), $\nabla s \in W_2^{l,l/2}(Q_T)$

$$\left\|\mathbf{l}_1^{(\mathbf{u})}(\mathbf{w},s) - \mathbf{l}_1^{(\mathbf{u}')}(\mathbf{w},s)\right\|_{Q_T}^{(l,l/2)}$$
$$\leq c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)}(\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla s\|_{Q_T}^{(l,l/2)}), \quad (2.4.22)$$

$$\|l_2^{(\mathbf{u})}(\mathbf{w}) - l_2^{(\mathbf{u}')}(\mathbf{w})\|_{W_2^{1+l,1/2+l/2}(Q_T)}$$
$$\leq c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)}\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)}, \quad (2.4.23)$$

$$\left\|\frac{\partial}{\partial t}(\mathcal{L}^{(\mathbf{u})}(\mathbf{w}) - \mathcal{L}^{(\mathbf{u}')}(\mathbf{w}))\right\|_{Q_T}^{(0,l/2)} \leq c\Big(\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)}$$
$$+T^{1/2-l/2}\|\mathbf{u}(\cdot,0)-\mathbf{u}'(\cdot,0)\|_{W_2^l(\Omega)}\Big)\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)}. \quad (2.4.24)$$

If $\mathbf{w}|_{t=0} = \mathbf{0}$, *then* (2.4.24) *is valid also without the second term in the parenthesis of the right hand side.*

Setting $\mathbf{u}' = \mathbf{0}$ in (2.4.22)–(2.4.24), we obtain that

Lemma 2.4.7 *If* $\mathbf{u}$ *satisfies condition* (2.4.1), *then*

$$\left\|\mathbf{l}_1^{(\mathbf{u})}(\mathbf{w},s)\right\|_{Q_T}^{(l,l/2)} \leq c\,\delta\left(\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla s\|_{Q_T}^{(l,l/2)}\right), \quad (2.4.25)$$

$$\|l_2^{(\mathbf{u})}(\mathbf{w})\|_{W_2^{1+l,1/2+l/2}(Q_T)} \leq c\,\delta\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)}, \quad (2.4.26)$$

$$\left\|\frac{\partial}{\partial t}\mathcal{L}^{(\mathbf{u})}(\mathbf{w})\right\|_{Q_T}^{(0,l/2)} \leq c\Big(\delta + T^{1/2-l/2}\|\mathbf{u}(\cdot,0)\|_{W_2^l(\Omega)}\Big)\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)}. \quad (2.4.27)$$

In the case $\mathbf{w}|_{t=0} = \mathbf{0}$ *the second term in the parenthesis of the right hand side of* (2.4.27) *can be dropped.*

The next auxiliary proposition concerns the difference

$$\mathbf{b}_{\mathbf{u}}(x,t) - \mathbf{b}_{\mathbf{u}'}(x,t) = \mathbf{b}(X_{\mathbf{u}},t) - \mathbf{b}(X_{\mathbf{u}'},t)$$
$$= \sum_{k=1}^{3}\int_0^1 \mathbf{b}_{X_k}(X_{\mathbf{u}_\theta},t)\mathrm{d}\theta\int_0^t (u_k - u_k')\mathrm{d}\tau, \quad (2.4.28)$$

where $\mathbf{u}-\mathbf{u}' = \tilde{\mathbf{u}}$, $\mathbf{u}_\theta = \mathbf{u}'+\theta\tilde{\mathbf{u}}$ $(\theta \in (0,1))$, $X_{\mathbf{u}} = x+\int_0^t \mathbf{u}\mathrm{d}\tau$, $X_{\mathbf{u}'} = x+\int_0^t \mathbf{u}'\mathrm{d}\tau$ and $X_{\mathbf{u}_\theta} = x + \int_0^t \mathbf{u}_\theta \mathrm{d}\tau$.

Lemma 2.4.8 *If* $\mathbf{b}$ *satisfies the conditions of Theorem* 2.2.1 *and condition* (2.4.1) *is satisfied, then*

$$\|\mathbf{b}_{\mathbf{u}} - \mathbf{b}_{\mathbf{u}'}\|_{Q_T}^{(l,l/2)} \le c(T) \int_0^T \|\mathbf{u}-\mathbf{u}'\|_{W_2^l(\Omega)} \mathrm{d}t, \tag{2.4.29}$$

where $c(T)$ *is a nondecreasing (power) function of* T.

Finally, we remark that by elementary calculation we have

$$\left\|\varrho_0^{-1} f\right\|_{Q_T}^{(l,l/2)} \le \frac{1}{R_0}\|f\|_{Q_T}^{(l,l/2)} + \frac{c}{R_0^2}\|\varrho_0\|_{W_2^{2+l}(\Omega)}\|f\|_{L_2(\Omega)}. \tag{2.4.30}$$

2.5 Proof of Theorem 2.2.1

Proof of Theorem 2.2.1. We solve the problem (2.4.2) by the method of successive approximations, setting $\mathbf{u}_0 = \mathbf{0}$, $q_0 = 0$ and determining $(\mathbf{u}_{m+1}, q_{m+1})$ $(m = 0,1,2,\ldots)$ as a solution of the problem

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{u}_{m+1}}{\partial t} - \nu(\varrho_0)\Delta \mathbf{u}_{m+1} + \nabla q_{m+1} \\ \qquad\qquad = \mathbf{l}_1^{(m)}(\mathbf{u}_m, q_m) + 2\nu'(\varrho_0)\mathbb{D}_m(\mathbf{u}_m)\nabla_m \varrho_0 \\ \qquad\qquad\quad -\dfrac{\beta}{3}(\nabla_m^{(j)}\nabla_m^{(i)}\varrho_0)\nabla_m\varrho_0 - \beta\Delta_m\varrho_0\nabla_m\varrho_0 + \varrho_0\mathbf{b}_m, \\ \nabla\cdot\mathbf{u}_{m+1} = l_2^{(m)}(\mathbf{u}_m), \quad \mathbf{u}_{m+1}|_{t=0} = \mathbf{v}_0, \quad \mathbf{u}_{m+1}|_\Gamma = \mathbf{0}. \end{cases} \tag{2.5.1}$$

Here $\nabla_m = \nabla_{\mathbf{u}_m}$, $\Delta_m = \Delta_{\mathbf{u}_m}$, $\mathbf{l}_1^{(m)} = \mathbf{l}_1^{(\mathbf{u}_m)}$, $l_2^{(m)} = l_2^{(\mathbf{u}_m)}$, $\mathbb{D}_m(\mathbf{w}) = \mathbb{D}_{\mathbf{u}_m}(\mathbf{w})$, $\mathbf{b}_m = \mathbf{b}_{\mathbf{u}_m}$. From Lemma 2.2.2 and the estimates in §§ 2.3 and 2.4 it follows that $(\mathbf{u}_{m+1}, \nabla q_{m+1})$ is uniquely determined, and $(\mathbf{u}_1, q_1)$ is a solution of problem (2.5.1), *i.e.*,

$$\begin{cases} \dfrac{\partial \mathbf{u}_1}{\partial t} - \dfrac{\nu(\varrho_0)}{\varrho_0}\Delta\mathbf{u}_1 + \dfrac{1}{\varrho_0}\nabla q_1 = -\dfrac{\beta}{3\varrho_0}(\nabla^{(j)}\nabla^{(i)}\varrho_0)\nabla\varrho_0 - \dfrac{\beta}{\varrho_0}\Delta\varrho_0\nabla\varrho_0 + \mathbf{b}, \\ \nabla\cdot\mathbf{u}_1 = 0, \quad \mathbf{u}_1|_{t=0} = \mathbf{v}_0, \quad \mathbf{u}_1|_\Gamma = \mathbf{0} \end{cases} \tag{2.5.2}$$

with the estimates

$$N[\mathbf{u}_1, q_1] := \|\mathbf{u}_1\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla q_1\|_{Q_T}^{(l,l/2)} \le c\left(\frac{\beta}{3}\left\|\frac{1}{\varrho_0}(\nabla^{(j)}\nabla^{(i)}\varrho_0)\nabla\varrho_0\right\|_{Q_T}^{(l,l/2)}\right.$$

$$\left. + \beta\left\|\frac{1}{\varrho_0}\Delta\varrho_0\nabla\varrho_0\right\|_{Q_T}^{(l,l/2)} + \|\mathbf{b}\|_{Q_T}^{(l,l/2)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right)$$

$$\le c_1\left((T^{1/2} + T^{1/2-l/2})\|\varrho_0\|^3_{W_2^{2+l}(\Omega)} + \|\mathbf{b}\|_{Q_T}^{(l,l/2)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right), \quad (2.5.3)$$

where c_1 is a nondecreasing function of T.

For the differences $\mathbf{Z}_{m+1} := \mathbf{u}_{m+1} - \mathbf{u}_m$, $P_{m+1} := q_{m+1} - q_m$ $(m = 1, 2, 3\ldots)$, we have

$$\begin{cases} \varrho_0\dfrac{\partial\mathbf{Z}_{m+1}}{\partial t} - \nu(\varrho_0)\Delta\mathbf{Z}_{m+1} + \nabla P_{m+1} \\ \quad = \mathbf{l}_1^{(m)}(\mathbf{Z}_m, P_m) + \mathbf{l}_1^{(m)}(\mathbf{u}_{m-1}, q_{m-1}) - \mathbf{l}_1^{(m-1)}(\mathbf{u}_{m-1}, q_{m-1}) \\ \qquad +2\nu'(\varrho_0)\left(\mathbb{D}_m(\mathbf{u}_m)\nabla_m\varrho_0 - \mathbb{D}_{m-1}(\mathbf{u}_{m-1})\nabla_{m-1}\varrho_0\right) \\ \qquad -\dfrac{\beta}{3}\left\{(\nabla_m^{(j)}\nabla_m^{(i)}\varrho_0)\nabla_m\varrho_0 - (\nabla_{m-1}^{(j)}\nabla_{m-1}^{(i)}\varrho_0)\nabla_{m-1}\varrho_0\right\} \\ \qquad -\beta\left(\Delta_m\varrho_0\nabla_m\varrho_0 - \Delta_{m-1}\varrho_0\nabla_{m-1}\varrho_0\right) + \varrho_0(\mathbf{b}_m - \mathbf{b}_{m-1}), \\ \nabla\cdot\mathbf{Z}_{m+1} = l_2^{(m)}(\mathbf{Z}_m) + l_2^{(m)}(\mathbf{u}_{m-1}) - l_2^{(m-1)}(\mathbf{u}_{m-1}), \\ \mathbf{Z}_{m+1}|_{t=0} = 0, \quad \mathbf{Z}_{m+1}|_{\Gamma} = \mathbf{0}. \end{cases}$$

We suppose that the condition (2.4.1) is satisfied for $\mathbf{u}_n$ $(n \le m)$. Then the lemmata in § 2.4 yield

$$\left\|\mathbf{l}_1^{(m)}(\mathbf{Z}_m, P_m)\right\|_{Q_T}^{(l,l/2)} + \left\|\mathbf{l}_1^{(m)}(\mathbf{u}_{m-1}, q_{m-1}) - \mathbf{l}_1^{(m-1)}(\mathbf{u}_{m-1}, q_{m-1})\right\|_{Q_T}^{(l,l/2)}$$

$$\le c\delta\left(\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla P_m\|_{Q_T}^{(l,l/2)}\right),$$

$$\|\mathbb{D}_m(\mathbf{u}_m)\nabla_m\varrho_0 - \mathbb{D}_{m-1}(\mathbf{u}_{m-1})\nabla_{m-1}\varrho_0\|_{Q_T}^{(l,l/2)}$$

$$\le c\|\varrho_0\|_{W_2^{2+l}(\Omega)}\left(1 + T^{1/2-l/2}\|\mathbf{v}_0\|_{W_2^l(\Omega)}\right)T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)};$$

$$\left\|(\nabla_m^{(j)}\nabla_m^{(i)}\varrho_0)\nabla_m\varrho_0 - (\nabla_{m-1}^{(j)}\nabla_{m-1}^{(i)}\varrho_0)\nabla_{m-1}\varrho_0\right\|_{Q_T}^{(l,l/2)}$$
$$\leq c\|\varrho_0\|^2_{W_2^{2+l}(\Omega)}(T^{1/2}+T^{1/2-l/2})T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$
$$\|\Delta_m\varrho_0\nabla_m\varrho_0 - \Delta_{m-1}\varrho_0\nabla_{m-1}\varrho_0\|_{Q_T}^{(l,l/2)}$$
$$\leq c\|\varrho_0\|^2_{W_2^{2+l}(\Omega)}(T^{1/2}+T^{1/2-l/2})T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$
$$\|\mathbf{b}_m - \mathbf{b}_{m-1}\|_{Q_T}^{(l,l/2)} \leq c\,T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$
$$\|l_2^{(m)}(\mathbf{Z}_m)\|_{W_2^{1+l,1/2+l/2}(Q_T)} + \|l_2^{(m)}(\mathbf{u}_{m-1}) - l_2^{(m-1)}(\mathbf{u}_{m-1})\|_{W_2^{1+l,1/2+l/2}(Q_T)}$$
$$\leq c\delta\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$
$$\left\|\frac{\partial}{\partial t}\mathcal{L}^{(m)}(\mathbf{Z}_m)\right\|_{Q_T}^{(0,l/2)} + \left\|\frac{\partial}{\partial t}\left(\mathcal{L}^{(m)}(\mathbf{u}_{m-1}) - \mathcal{L}^{(m-1)}(\mathbf{u}_{m-1})\right)\right\|_{Q_T}^{(0,l/2)}$$
$$\leq c\delta\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)}.$$

Then, we obtain that

$$N[\mathbf{Z}_{m+1},P_{m+1}] \equiv \|\mathbf{Z}_{m+1}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla P_{m+1}\|_{Q_T}^{(l,l/2)}$$
$$\leq C\left(\delta N[\mathbf{Z}_m,P_m] + T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)}\right), \tag{2.5.4}$$

where $C = C(T;\mathbf{v}_0,\varrho_0)$ is a nondecreasing function with respect to T. Taking into account the condition (2.4.1) for $\mathbf{u}_n$ $(n \leq m)$, we also have (2.5.4) for $m = 0,1,\ldots,m-1$. If we choose δ satisfying $C\delta < 1/4$, we obtain

$$N[\mathbf{Z}_{n+1},P_{n+1}] \leq \frac{1}{4}N[\mathbf{Z}_n,P_n] + CT^{1/2}\|\mathbf{Z}_n\|_{Q_T}^{(2+l,1+l/2)}$$
$$\leq \left(\frac{1}{4}+CT^{1/2}\right)N[\mathbf{Z}_n,P_n] \leq \ldots \leq \left(\frac{1}{4}+CT^{1/2}\right)^n N[\mathbf{Z}_1,P_1] \tag{2.5.5}$$

for $n = 0,1,\ldots,m$. We set $\Sigma_{m+1} = \sum_{n=0}^m N[\mathbf{Z}_{n+1},P_{n+1}]$. Since

$$\sum\nolimits_{m+1} \leq N[\mathbf{u}_1,q_1]\sum_{n=0}^m\left(\frac{1}{4}+CT^{1/2}\right)^n \leq c_1\Big\{(T^{1/2}+T^{1/2-l/2})\|\varrho_0\|^3_{W_2^{2+l}(\Omega)}$$
$$+\|\mathbf{b}\|_{Q_T}^{(l,l/2)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\Big\}\sum_{n=0}^m\left(\frac{1}{4}+CT^{1/2}\right)^n,$$

we obtain

$$N[\mathbf{u}_{m+1},q_{m+1}]\le \sum\nolimits_{m+1}+N[\mathbf{u}_1,q_1]\le c_1\Big\{(T^{1/2}+T^{1/2-l/2})\|\varrho_0\|^3_{W_2^{2+l}(\Omega)}$$
$$+\|\mathbf{b}\|^{(l,l/2)}_{Q_T}+\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\Big\}\left\{1+\sum_{n=0}^{m}\left(\frac{1}{4}+CT^{1/2}\right)^n\right\}. \quad (2.5.6)$$

Note that c_1 and C are nondecreasing functions of T, then condition (2.4.1) for $\mathbf{u}_{m+1}$ is satisfied if $CT^{1/2}\le 1/4$ and

$$3T^{1/2}c_1\left((T^{1/2}+T^{1/2-l/2})\|\varrho_0\|^3_{W_2^{2+l}(\Omega)}+\|\mathbf{b}\|^{(l,l/2)}_{Q_T}+\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right)\le\delta. \quad (2.5.7)$$

The left-hand side does not depend on m. Thus, $N[\mathbf{u}_m,q_m]$ is uniformly bounded, the sequence $\{\mathbf{u}_m,q_m\}$ converges in the norm $N[\,\cdot\,,\,\cdot\,]$, and the limit is a solution of the problem (2.4.2).

The solution is unique, since the difference of two solutions $\mathbf{w}=\mathbf{u}-\mathbf{u}'$, $s=q-q'$ satisfies the relations

$$\begin{cases}\varrho_0\dfrac{\partial\mathbf{w}}{\partial t}-\nu(\varrho_0)\Delta\mathbf{w}+\nabla s\\ \qquad=\mathbf{l}_1^{(\mathbf{u})}(\mathbf{u},q)-\mathbf{l}_1^{(\mathbf{u}')}(\mathbf{u}',q')+2\nu'(\varrho_0)\left(\mathbb{D}_{\mathbf{u}}(\mathbf{u})\nabla_{\mathbf{u}}\varrho_0-\mathbb{D}_{\mathbf{u}'}(\mathbf{u}')\nabla_{\mathbf{u}'}\varrho_0\right)\\ \qquad\quad-\dfrac{\beta}{3}\left\{(\nabla^{(j)}_{\mathbf{u}}\nabla^{(i)}_{\mathbf{u}}\varrho_0)\nabla_{\mathbf{u}}\varrho_0-(\nabla^{(j)}_{\mathbf{u}'}\nabla^{(i)}_{\mathbf{u}'}\varrho_0)\nabla_{\mathbf{u}'}\varrho_0\right\}\\ \qquad\quad-\beta\left(\Delta_{\mathbf{u}}\varrho_0\nabla_{\mathbf{u}}\varrho_0-\Delta_{\mathbf{u}'}\varrho_0\nabla_{\mathbf{u}'}\varrho_0\right)+\varrho_0(\mathbf{b}_{\mathbf{u}}-\mathbf{b}_{\mathbf{u}'}),\\ \nabla\cdot\mathbf{w}=l_2^{(\mathbf{u})}(\mathbf{w})+l_2^{(\mathbf{u})}(\mathbf{u}')-l_2^{(\mathbf{u}')}(\mathbf{u}'),\\ \mathbf{Z}_{m+1}|_{t=0}=\mathbf{0},\quad \mathbf{Z}_{m+1}|_{\Gamma}=\mathbf{0}.\end{cases}$$

Applying to this problem the estimate (2.2.2) and repeating the arguments carried out, we arrive at inequality

$$N[\mathbf{w},s]\le c(\delta+T^{1/2})N[\mathbf{w},s].$$

This implies $(\mathbf{w},\nabla s)=(\mathbf{0},\mathbf{0})$, and Theorem 2.2.1 is proved.

Chapter 3

Initial-boundary value problem under the generalized Navier's slip condition

3.1 Introduction

The motion of inhomogeneous incompressible fluid-like bodies under the generalized Navier's slip condition is studied in this chapter. We should pay attention to the slip phenomena of the granular body at the boundary. Unlike the adhering behaviour of Newtonian fluids at the boundary, non-Newtonian fluids including granular materials may slip in general at the surface of the solid in contact with the fluids. Moreover, this slip effect may cause the significant consequence for motion. Thus, taking into account this slip phenomena, we analyse the motion of inhomogenous incompressible fluid-like bodies.

The system of equations of interest is (1.4.4), namely

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{u}}{\partial t} = \nabla_{\mathbf{u}} \cdot \mathbb{T}_{\mathbf{u}} + \varrho_0 \mathbf{b}_{\mathbf{u}}, \quad \nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \quad \text{for } x \in \Omega,\ t > 0, \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{for } x \in \Omega, \\ \mathbf{u} \cdot \mathbf{n}_{\mathbf{u}} = 0, \quad \mathbf{u} + K_{\mathbf{u}} \Pi_{\mathbf{u}} \mathbb{T}_{\mathbf{u}} \mathbf{n}_{\mathbf{u}} = \mathbf{0} \quad \text{for } x \in \Gamma,\ t > 0. \end{cases} \tag{3.1.1}$$

The aim of this chapter is to prove a theorem on local in time solvability of problem (3.1.1) in Sobolev–Slobodetskiĭ spaces.

Furthermore, we consider the following linear problem:

$$\begin{cases} \varrho_0(x)\dfrac{\partial \mathbf{u}}{\partial t} = -\nabla q + \nu_1(x)\Delta\mathbf{u} + \varrho_0(x)\mathbf{f} \quad \text{for } x \in \Omega,\ t > 0, \\ \nabla \cdot \mathbf{u} = g \quad \text{for } x \in \Omega,\ t > 0, \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{for } x \in \Omega, \quad \mathbf{u}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} \quad \text{for } x \in \Gamma,\ t > 0, \\ \Pi\mathbf{u} + 2\nu_1(x)K(x,t)\Pi\mathbb{D}(\mathbf{u})\mathbf{n} = K(x,t)\mathbf{d} \quad \text{for } x \in \Gamma,\ t > 0, \end{cases} \tag{3.1.2}$$

where $\nu_1(x)$ is a given positive function defined in Ω, $(\mathbf{f}, g)$ and $(\mathbf{b}, \mathbf{d})$ are given functions defined on $\Omega \times (0, +\infty)$ and on $\Gamma \times (0, +\infty)$, respectively.

3.2 Mathematical Results

Let us describe the results in this chapter. First of all, we consider the problem (3.1.2) in the spaces $H_h^{2+l,1+l/2}(Q_T)$ and $H_h^{l,l/2}(Q_T)$. The following lemma is proved in § 3.3. Note that Q_T and G_T are the same as those in Chapter 2, respectively.

Lemma 3.2.1 *Let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary $\Gamma \in W_2^{5/2+l}$, $l \in (1/2, 1)$, $0 < T < +\infty$, $\mathbf{v}_0 \equiv \mathbf{0}$, $\varrho_0 \in W_2^{2+l}(\Omega)$, $\varrho_0(x) \geq R_0 > 0$, $\nu_1 \in W_2^{2+l}(\Omega)$, $\inf \nu_1(x) > 0$, $K \in W_2^{3/2+l,3/4+l/2}(G_T)$ and $\inf K > 0$. For arbitrary $\mathbf{f} \in H_h^{l,l/2}(Q_T)$, $g = \nabla \cdot \mathbf{G}$, $\mathbf{G} \in H_h^{2+l,1+l/2}(Q_T)$, $\mathbf{b} \in H_h^{3/2+l,3/4+l}(G_T)$, $\mathbf{b} = \mathbf{G}|_\Gamma$, $\mathbf{d} \in H_h^{1/2+l,1/4+l}(G_T)$, and $\mathbf{d}\cdot\mathbf{n} = 0$, problem (3.1.2) has a unique solution $\mathbf{u} \in H_h^{2+l,1+l/2}(Q_T)$, $\nabla q \in H_h^{l,l/2}(Q_T)$, provided h is sufficiently large. And this solution satisfies the following estimate:*

$$\begin{aligned} \|\mathbf{u}\|_{H_h^{2+l,1+l/2}(Q_T)} + \|\nabla q\|_{H_h^{l,l/2}(Q_T)} \leq c(T) \Big(&\|\mathbf{f}\|_{H_h^{l,l/2}(Q_T)} + \|g\|_{H_h^{1+l,1/2+l/2}(Q_T)} \\ &+ \|\mathbf{G}\|_{H_h^{0,1+l/2}(Q_T)} + \|\mathbf{b}\|_{H_h^{3/2+l,3/4+l/2}(G_T)} + \|\mathbf{d}\|_{H_h^{1/2+l,1/4+l/2}(G_T)} \Big), \end{aligned} \tag{3.2.1}$$

where $c(T)$ is a non-decreasing function of T.

This lemma is proved in the same way as that in Chapter 2; we first consider the constant coefficient case in the half-space, then the function coefficient case in a bounded domain. In the case of the half-space we give an explicit formula for the solution, and in a bounded domain we prove a priori estimates and establish the solvability of the problem (3.1.2) by the construction of a regularizer (see, for example, [24, 36]).

Next, problem (3.1.2) is considered in the Sobolev–Slobodetskiĭ spaces such as $W_2^{2+l,1+l/2}(Q_T)$ and $W_2^{l,l/2}(Q_T)$.

Lemma 3.2.2 *Let* Ω, Γ, T, l, ϱ_0, R_0, ν_1 *and* K *be the same as in Lemma* 3.2.1. *For arbitrary* $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, $\mathbf{f} \in W_2^{l,l/2}(Q_T)$, $g = \nabla \cdot \mathbf{G}$, $\mathbf{G} \in W_2^{2+l,1+l/2}(Q_T)$, $\mathbf{d} \in W_2^{3/2+l,3/4+l/2}$ *and* $\mathbf{d} \in W_2^{1/2+l,1/4+l/2}$ (G_T) *satisfying the compatibility conditions*

$$\nabla \cdot \mathbf{v}_0 = \nabla \cdot \mathbf{G}(\,\cdot\,,0) \ \text{in } \Omega, \quad \mathbf{b} = \mathbf{G}|_\Gamma \ \text{on } \Gamma, \quad \mathbf{d} \cdot \mathbf{n} = 0 \ \text{on } \Gamma,$$

$$\Pi\mathbf{v}_0 + 2\nu_1 K(\cdot,0)\Pi\mathbb{D}(\mathbf{v}_0)\mathbf{n} = K(\cdot,0)\mathbf{d}(\cdot,0) \ \text{on } \Gamma,$$

problem (3.1.2) *has a unique solution* $(\mathbf{u}, \nabla q)$ *in* $W_2^{2+l,1+l/2}(Q_T) \times W_2^{l,l/2}(Q_T)$ *such that*

$$\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla q\|_{Q_T}^{(l,l/2)} \le c(T) \Big(\|\mathbf{f}\|_{Q_T}^{(l,l/2)} + \|g\|_{W_2^{1+l,1/2+l/2}(Q_T)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}$$
$$+ \|\mathbf{G}\|_{Q_T}^{(0,1+l/2)} + \|\mathbf{b}\|_{W_2^{3/2+l,3/4+l/2}(G_T)} + \|\mathbf{d}\|_{W_2^{1/2+l,1/4+l/2}(G_T)} \Big), \qquad (3.2.2)$$

where $c(T)$ *is a non-decreasing function of* T.

Finally, we consider the quasi-linear problem (3.1.1), and the following theorem on time-local solvability is proved in § 3.5.

Theorem 3.2.1 *Let* Ω *be a bounded domain in* $\mathbb{R}^3$, $\Gamma \in W_2^{7/2+l}$, $l \in (1/2, 1)$, $\varrho_0 \in W_2^{2+l}(\Omega)$, $\varrho_0(x) \ge R_0 > 0$, $\nu \in C^2(\mathbb{R}_+)$, $\nu > 0$, $0 < T < +\infty$, $\mathbf{b} \in W_2^{l,l/2}(Q_T)$. *Assume that* $\mathbf{b}(X,t)$ *has continuous derivatives with respect to* x *and* $\mathbf{b}$, $\mathbf{b}_{X_k}$ *satisfy the Lipschitz condition in* x *and the Hölder condition with exponent* $1/2$ *in* t, *that* $K(X,t)$ *has continuous derivatives up to order* 3 *with* *respect to* X, $\inf K > 0$ *and* $D_X^\alpha K$ $(|\alpha| \le 3)$ *satisfy the Lipschitz condition in* x *and* t, *and that* $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$ *satisfies the compatibility conditions*

$$\nabla \cdot \mathbf{v}_0 = 0 \ \textit{in} \ \Omega, \quad \mathbf{v}_0 \cdot \mathbf{n} = 0 \ \textit{on} \ \Gamma,$$

$$\mathbf{v}_0 + K(\cdot, 0)\Pi\left\{2\nu(\varrho_0)\mathbb{D}(\mathbf{v}_0)\mathbf{n} - \beta(\nabla\varrho_0 \otimes \nabla\varrho_0)\mathbf{n}\right\} = \mathbf{0} \ \textit{on} \ \Gamma.$$

Then problem (3.1.1) *has a unique solution* $(\mathbf{u}, \nabla q) \in W_2^{2+l,1+l/2}(Q_{T'}) \times W_2^{l,l/2}$ $(Q_{T'})$ *on some interval* $(0, T^{'})$ $(0 < T' \le T)$, *whose magnitude* $T^{'}$ *depends on the data* (*see condition* (3.5.8) *below*).

3.3 Proofs of Lemmata 3.2.1 and 3.2.2

3.3.1 Problem in the half-space

In this subsection we shall consider the initial-boundary value problem for the homogeneous Stokes system in a half-space $D_{+T} \equiv \mathbb{R}^3_+ \times (0,T) = \{x_3 > 0,\ 0 < t < T\}$ $(0 < T < +\infty)$:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu_0 \Delta \mathbf{u} + \nabla q = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } D_{+T}, \\ \mathbf{u}|_{t=0} = \mathbf{0} \quad \text{on } \mathbb{R}^3_+, \quad u_3|_{x_3=0} = 0 \quad \text{on } D_T \equiv \mathbb{R}^2 \times (0,T), \\ u_j - \nu_0 K_0 \left(\dfrac{\partial u_j}{\partial x_3} + \dfrac{\partial u_3}{\partial x_j} \right)\Bigg|_{x_3=0} = d_j \quad \text{on } D_T \ (j = 1,2), \end{cases} \tag{3.3.1}$$

where ν_0 and K_0 are positive constants, and $d_j \in H_h^{1/2+l,1/4+l/2}(D_T)$ $(j = 1,2)$ with $l \in (1/2, 1)$.

In considering the problem (3.3.1), we extend d_j from D_T to D_∞ such that $d_j \in H_h^{1/2+l,1/4+l/2}(D_\infty)$ (denoted by the same symbol) and

$$\|d_j\|_{H_h^{1/2+l,1/4+l/2}(D_\infty)} \le c\|d_j\|_{H_h^{1/2+l,1/4+l/2}(D_T)}, \tag{3.3.2}$$

where c is independent of h and T (see [38], §2).

Next, extending $\mathbf{u} = (u_1, u_2, u_3)$, q, $\mathbf{d}' = (d_1, d_2)$ to the half-space $t < 0$ by 0 and make the Fourier transformation with respect to $x' = (x_1, x_2)$ and the

Laplace transformation with respect to t in the same way as in Chapter 2, we have the following system of ordinary differential equations:

$$\begin{cases} \nu_0\left(r^2-\dfrac{\mathrm{d}^2}{\mathrm{d}x_3^2}\right)\hat{u}_j+\mathrm{i}\xi_j\hat{q}=0 \quad (j=1,2), \\ \nu_0\left(r^2-\dfrac{\mathrm{d}^2}{\mathrm{d}x_3^2}\right)\hat{u}_3+\dfrac{\mathrm{d}\hat{q}}{\mathrm{d}x_3}=0, \quad \mathrm{i}\xi_1\hat{u}_1+\mathrm{i}\xi_2\hat{u}_2+\dfrac{\mathrm{d}\hat{u}_3}{\mathrm{d}x_3}=0, \\ \hat{u_3}\big|_{x_3=0}=0, \quad \hat{u}_j-\nu_0K_0\left(\dfrac{\mathrm{d}\hat{u}_j}{\mathrm{d}x_3}+\mathrm{i}\xi_j\hat{u}_3\right)\Big|_{x_3=0}=\hat{d}_j, \\ (\hat{\mathbf{u}},\hat{q})\longrightarrow(\mathbf{0},0)\ (x_3\to+\infty), \end{cases} \tag{3.3.3}$$

where

$$r^2=\frac{s}{\nu_0}+|\xi'|^2, \quad |\xi'|^2=\xi_1^2+\xi_2^2, \quad \arg r\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right).$$

This problem is easily solved by the same way as in Chapter 2, whose solution is given explicitly by

$$\begin{cases} \hat{u}_j=\dfrac{\hat{d}_j}{1+\nu_0K_0r}e_0(x_3)+\dfrac{\mathrm{i}\xi_j\nu_0K_0\sum_{k=1}^2\mathrm{i}\xi_k\hat{d}_k}{|\xi'|(1+\nu_0K_0r)\{\nu_0K_0(r+|\xi'|)+1\}}e_0(x_3) \\ \qquad +\dfrac{-\mathrm{i}\xi_j\sum_{k=1}^2\mathrm{i}\xi_k\hat{d}_k}{|\xi'|\{\nu_0K_0(r+|\xi'|)+1\}}e_1(x_3) \quad (j=1,2), \\ \hat{u}_3=\dfrac{\sum_{k=1}^2\mathrm{i}\xi_k\hat{d}_k}{\nu_0K_0(r+|\xi'|)+1}e_1(x_3), \\ \hat{q}\ =\dfrac{-\nu_0(r+|\xi'|)\sum_{k=1}^2\mathrm{i}\xi_k\hat{d}_k}{|\xi'|\{\nu_0K_0(r+|\xi'|)+1\}}e_2(x_3), \end{cases} \tag{3.3.4}$$

where

$$e_0(x_3)=\mathrm{e}^{-rx_3}, \qquad e_1(x_3)=\frac{\mathrm{e}^{-rx_3}-\mathrm{e}^{-|\xi'|x_3}}{r-|\xi'|}, \qquad e_2(x_3)=\mathrm{e}^{-|\xi'|x_3}.$$

In estimating this solution, it is also convinient to use the equivalent norms $\|\cdot\|_{\gamma,h,D_\infty}$ and $\|\cdot\|_{\gamma,h,D_{+\infty}}$ in $H_h^{\gamma,\gamma/2}(D_\infty)$ and $H_h^{\gamma,\gamma/2}(D_{+\infty})$, respectively for $\gamma\ge 0$. Then, the formula (3.3.4) and Lemma 2.3.1 in Chapter 2 yield that for $h>0$ the solution $(\mathbf{u},q)$ of the problem (3.3.1) with $T=\infty$ satisfies the estimate

$$\|\mathbf{u}\|^2_{2+l,h,D_{+\infty}}+\|\nabla q\|^2_{l,h,D_{+\infty}}\le\frac{c}{K_0^2}\|\mathbf{d}'\|^2_{1/2+l,h,D_\infty}, \tag{3.3.5}$$

where c is a constant independent of h and K_0.

Consequently, taking into account (3.3.5) and restricting the domain of $\mathbf{u}$ and q, we have

Lemma 3.3.1 *Let $h > 0$ and $l \in (1/2, 1)$. Then the solution $(\mathbf{u}, q)$ of the problem* (3.3.1) *satisfies the estimate*

$$\|\mathbf{u}\|_{H_h^{2+l,1+l/2}(D_{+T})} + \|\nabla q\|_{H_h^{l,l/2}(D_{+T})} \leq \frac{c}{K_0} \|\mathbf{d}'\|_{H_h^{1/2+l,1/4+l/2}(D_T)}, \tag{3.3.6}$$

where c is a constant independent of h and K_0.

3.3.2 Proof of Lemma 3.2.1

We shall use the same framework as in Chapter 2. Because of the condition of Ω and Γ, in the neighbourhood of an arbitrary point $\xi \in \Gamma$, the surface Γ is represented by the equation

$$y_3 = \varphi(y'), \quad y' = (y_1, y_2) \in K_d \quad (K_d = \{y' : |y'| < d\})$$

in a Cartesian local coordinates system (y_1, y_2, y_3) with the origin at ξ and with y_3-axis directed along $-\mathbf{n}(\xi)$, $\mathbf{n}(\xi)$ being the unit outward normal vector to Γ at ξ. The function φ may be considered to be defined on $\mathbb{R}^2$ such that its support is included in a disc K_{2d} and $\varphi(0) = 0$, $\nabla'\varphi(0) = \mathbf{0}$ (∇' is the gradient with respect to y') and $\|\varphi\|_{W_2^{5/2+l}(\mathbb{R}^2)} \leq M$ ($M > 0$) hold. It is to be noted that the constants d and M are taken indepenently of ξ. Furthermore, φ can be extended into $\mathbb{R}^3_+$ (see [38, 40]) so that it belongs to $W_2^{3+l}(\mathbb{R}^3_+)$, and $\varphi(0) = 0$, $\nabla\varphi(0) = \mathbf{0}$ and

$$\sup_{|y| \leq \lambda} |\varphi(y)| \leq cM\lambda, \qquad \sup_{|y| \leq \lambda} |\nabla\varphi(y)| \leq cM\lambda.$$

Then the transformation $y = Y(z)$:

$$y_1 = z_1, \; y_2 = z_2, \; y_3 = z_3 + \varphi(z) \tag{3.3.7}$$

is invertible if $|\varphi_{z_3}| < 1$ and maps $\mathbb{R}^3_+$ onto the domain $\{y_3 > \varphi(y')\}$.

The solvability of (3.1.2) will be proved by using the regularizer $\mathcal{R}$ (see, for instance, [36, 41]), which is a linear continuous operator from the data

$$\begin{aligned}\mathbf{F} = (\mathbf{f}, g, \mathbf{b}, \mathbf{d}) \in \mathcal{H}_{h,l} = H_h^{l,l/2}(Q_T) \times H_h^{1+l,1/2+l/2}(Q_T)\\ \times H_h^{3/2+l,3/4+l/2}(G_T) \times H_h^{1/2+l,1/4+l/2}(G_T)\end{aligned}$$

to the solution $(\bar{\mathbf{w}}, \nabla\bar{\pi}) \in H_h^{2+l,1+l/2}(Q_T) \times H_h^{l,l/2}(Q_T)$ of

$$\begin{cases} \dfrac{\partial \bar{\mathbf{w}}}{\partial t} - \dfrac{\nu_1(x)}{\varrho_0(x)}\Delta\bar{\mathbf{w}} + \dfrac{1}{\varrho_0(x)}\nabla\bar{\pi} = \mathbf{f} + \mathcal{M}_1\mathbf{F} \quad \text{in } Q_T, \\ \nabla\cdot\bar{\mathbf{w}} = g + \mathcal{M}_2\mathbf{F} \quad \text{in } Q_T, \\ \bar{\mathbf{w}}|_{t=0} = \mathbf{0} \quad \text{in } \Omega, \quad \bar{\mathbf{w}}\cdot\mathbf{n} = (\mathbf{b} + \mathcal{M}_3\mathbf{F})\cdot\mathbf{n} \quad \text{on } G_T, \\ \Pi\bar{\mathbf{w}} + 2\nu_1(x)K(x,t)\Pi\mathbb{D}(\bar{\mathbf{w}})\mathbf{n} = K(x,t)(\mathbf{d} + \mathcal{M}_4\mathbf{F}) \quad \text{on } G_T \end{cases} \tag{3.3.8}$$

with $\mathcal{M}\mathbf{F} = (\mathcal{M}_1\mathbf{F}, \mathcal{M}_2\mathbf{F}, \mathcal{M}_3\mathbf{F}, \mathcal{M}_4\mathbf{F})$ being a contraction operator on $\mathcal{H}_{h,l}$ for sufficiently large h and small T. The solution of (3.1.2) can be obtained in terms of the regularizer as $(\mathbf{w}, \pi) = \mathcal{R}(I + \mathcal{M})^{-1}(\mathbf{f}, \mathbf{G}, \mathbf{b}, \mathbf{d})$. Note that $(I + \mathcal{M})^{-1}$ can be represented by the Neumann series if $\mathcal{M}$ is a contraction operator, *i.e.*, $(I + \mathcal{M})^{-1} = \sum_{j=0}^{\infty}(-\mathcal{M})^j$.

In order to establish the existence of a solution of problem (3.1.2), we use the same coverings $\{\Omega^{(k)}\}$ and $\{\omega^{(k)}\}$, and the same functions $\{\zeta^{(k)}(x)\}$ and $\{\eta^{(k)}(x)\}$ as in Chapter 2 (see §2.3.3).

We define $(\bar{\mathbf{w}}, \bar{\pi}) = \mathcal{R}\mathbf{F}$ by the formula

$$(\bar{\mathbf{w}}, \bar{\pi})(x,t) = \sum_{k=1}^{N_\lambda} \eta^{(k)}(x)(\bar{\mathbf{w}}^{(k)}, \bar{\pi}^{(k)})(x,t),$$

where $(\bar{\mathbf{w}}^{(k)}, \bar{\pi}^{(k)})$ $(k = 1, 2, \ldots, M_\lambda)$ are given in the following way.

For $k = 1, 2, \ldots, M_\lambda$, let $\{y\}$ be local Cartesian coordinates in the neighbourhood of the point $\xi^{(k)} : y = \mathcal{C}_k(x - \xi^{(k)})$ with $\mathcal{C}_k$ being an orthogonal matrix satisfying $\mathcal{C}_k\mathbf{n}(\xi^{(k)}) = (0, 0, -1)^{\mathrm{T}}$, $\varphi^{(k)}(y')$ be the function defining Γ in the neighbourhood of $\xi^{(k)}$ and let Y_k be the corresponding transformation (3.3.7). The transformation $z = Z_k(x) = Y_k^{-1}\mathcal{C}_k(x - \xi^{(k)})$ maps the domain

$\Omega^{(k)} \cap \Omega$ into the half space $\mathbb{R}^3_{+,k} = \{z \in \mathbb{R}^3 \, | z_3 > 0\}$ and its Jacobian matrix is $\mathbb{I}$ at $\xi^{(k)}$. Set

$$\begin{aligned}
&\mathbf{b}^{(k)}(z,t) = \zeta^{(k)}(Z_k^{-1}(z))\mathcal{C}_k \mathbf{b}(Z_k^{-1}(z),t),\\
&\mathbf{d}^{(k)}(z,t) = \zeta^{(k)}(Z_k^{-1}(z))K(Z_k^{-1}(z),t)\mathcal{C}_k \mathbf{d}(Z_k^{-1}(z),t),
\end{aligned}$$

and let $\mathbf{f}^{(k)}$ and $\mathbf{G}^{(k)}$ are the same as those in Chapter 2. Then extend them to the domain $\mathbb{R}^3_{+,k} \setminus Z_k^{-1}(\Omega^{(k)} \cap \Omega)$ by zero, which are denoted by the same symbols again.

Let $(\mathbf{w}^{(k)}, \pi^{(k)})(z,t)$, $k = 1, 2, \ldots, M_\lambda$, be a solution of the half-space problem

$$\begin{cases}
\dfrac{\partial \mathbf{w}^{(k)}}{\partial t}(z,t) - \dfrac{\nu_1(\xi^{(k)})}{\varrho_0(\xi^{(k)})}\Delta_z \mathbf{w}^{(k)}(z,t) + \dfrac{1}{\varrho_0(\xi^{(k)})}\nabla_z \pi^{(k)}(z,t) = \mathbf{f}^{(k)}(z,t),\\
\nabla_z \cdot \mathbf{w}^{(k)}(z,t) = \nabla_z \cdot \mathbf{G}^{(k)}(z,t) \quad \text{in } D^{(k)}_{+T},\\
\mathbf{w}^{(k)}(z,t)|_{t=0} = \mathbf{0} \quad \text{in } \mathbb{R}^3_{+,k}, \quad w_3^{(k)}(z,t)|_{z_3=0} = b_3^{(k)}(z,t)|_{z_3=0} \quad \text{on } D^{(k)}_T,\\
w_j^{(k)} - \nu_1(\xi^{(k)})K(\xi^{(k)},0)\left(\dfrac{\partial w_j^{(k)}}{\partial z_3} + \dfrac{\partial w_3^{(k)}}{\partial z_j}\right)\Bigg|_{z_3=0} = b_j^{(k)} + d_j^{(k)}\Big|_{z_3=0} \text{ on } D^{(k)}_T\\
\hfill (j = 1, 2),
\end{cases} \tag{3.3.9}$$

where $D^{(k)}_{+T} \equiv \mathbb{R}^3_{+,k} \times (0,T)$ and $D^{(k)}_T \equiv \partial\mathbb{R}^3_{+,k} \times (0,T)$. According to (3.3.6) and Lemma 2.3.3 in Chapter 2, we have

$$\begin{aligned}
&\|\mathbf{w}^{(k)}\|_{H_h^{2+l,1+l/2}(D^{(k)}_{+T})} + \|\nabla\pi^{(k)}\|_{H_h^{l,l/2}(D^{(k)}_{+T})}\\
&\le c\Bigg(\|\mathbf{f}^{(k)}\|_{H_h^{l,l/2}(D^{(k)}_{+T})} + \|\nabla \cdot \mathbf{G}^{(k)}\|_{H_h^{1+l,1/2+l/2}(D^{(k)}_{+T})}\\
&\qquad +\|\mathbf{G}^{(k)}\|_{H_h^{0,1+l/2}(D^{(k)}_{+T})} + \frac{1}{\inf K}\|\mathbf{d}^{(k)}\|_{H_h^{1/2+l,1/4+l/2}(D^{(k)}_{T})}\Bigg)
\end{aligned} \tag{3.3.10}$$

with a constant c independent of h and K.

Then for $k = 1, 2, \ldots, M_\lambda$ we define

$$\bar{\mathbf{w}}^{(k)}(x,t) = \mathcal{C}_k^{-1}\mathbf{w}^{(k)}(Z_k(x),t), \quad \bar{\pi}^{(k)}(x,t) = \pi^{(k)}(Z_k(x),t). \tag{3.3.11}$$

For $k = M_\lambda + 1, M_\lambda + 2, \ldots, N_\lambda$ $(\mathbf{w}^{(k)}, \pi^{(k)})$ is a solution of the Cauchy problem (2.3.22) in Chapter 2, and let $(\bar{\mathbf{w}}^{(k)}, \bar{\pi}^{(k)}) = (\mathbf{w}^{(k)}, \pi^{(k)})$.

Again we restrict $\bar{\pi}^{(k)}$ by

$$\int_{\Omega^{(k)}\cap\Omega} \bar{\pi}^{(k)}(x,t)\mathrm{d}x = 0. \tag{3.3.12}$$

For such $\bar{\pi}^{(k)}$ we have

$$\|\bar{\pi}^{(k)}\|_{L_2(Q_T^{(k)})} \leq c\lambda\|\nabla\bar{\pi}^{(k)}\|_{L_2(Q_T^{(k)})} \leq c\lambda\|\nabla\pi^{(k)}\|_{L_2(D_{+T}^{(k)})}, \tag{3.3.13}$$

where $Q_T^{(k)} = (\Omega^{(k)} \cap \Omega) \times (0,T)$. Consequently, $\eta^{(k)}\bar{\pi}^{(k)}$ is uniquely determined in Q_T, and $\mathcal{R}$ is well-defined.

Clearly, $\mathcal{R}$ is a linear operator on $\mathcal{H}_{h,l}$. To calculate the norm of $\mathcal{M}\mathbf{F}$, we rewrite the problem (3.3.9) in coordinates $\{x\}$ in the neighbourhood $\Omega^{(k)} \cap \Omega$ of $\xi^{(k)}$. Then,

$$\begin{cases} \dfrac{\partial\bar{\mathbf{w}}^{(k)}}{\partial t}(x,t) - \dfrac{\nu_1(\xi^{(k)})}{\varrho_0(\xi^{(k)})}\bar{\Delta}^{(k)}\bar{\mathbf{w}}^{(k)}(x,t) + \dfrac{1}{\varrho_0(\xi^{(k)})}\bar{\nabla}^{(k)}\bar{\pi}^{(k)}(x,t) = \zeta^{(k)}\mathbf{f}, \\ \bar{\nabla}^{(k)} \cdot \bar{\mathbf{w}}^{(k)}(x,t) = \bar{\nabla}^{(k)} \cdot (\zeta^{(k)}\mathbf{G}) \quad \text{in } Q_T^{(k)}, \\ \bar{\mathbf{w}}^{(k)}|_{t=0} = \mathbf{0} \quad \text{in } \Omega^{(k)} \cap \Omega, \quad \bar{\mathbf{w}}^{(k)} \cdot \mathbf{n}(\xi^{(k)}) = \zeta^{(k)}\mathbf{b} \cdot \mathbf{n}(\xi^{(k)}) \quad \text{on } G_T^{(k)}, \\ \bar{\Pi}^{(k)}\bar{\mathbf{w}}^{(k)} + 2\nu_1(\xi^{(k)})K(\xi^{(k)},0)\bar{\Pi}^{(k)}\bar{\mathbb{D}}^{(k)}(\bar{\mathbf{w}}^{(k)})\mathbf{n}(\xi^{(k)}) \\ \qquad\qquad = \zeta^{(k)}(x)K(x,t)\bar{\Pi}^{(k)}\mathbf{d} \quad \text{on } G_T^{(k)}, \end{cases} \tag{3.3.14}$$

where $Q_T^{(k)} = (\Omega^{(k)} \cap \Omega) \times (0,T)$, $G_T^{(k)} = (\Omega^{(k)} \cap \Gamma) \times (0,T)$, $\bar{\Pi}^{(k)}\mathbf{f} = \mathbf{f} - (\mathbf{f} \cdot \mathbf{n}(\xi^{(k)}))\mathbf{n}(\xi^{(k)})$, $\bar{\mathbb{D}}^{(k)}(\mathbf{f}) = \frac{1}{2}(\bar{\nabla}^{(k)}\mathbf{f} + [\bar{\nabla}^{(k)}\mathbf{f}]^{\mathrm{T}})$, $\bar{\nabla}^{(k)} = \mathcal{C}_k^{-1}\mathcal{Z}_k^{-\mathrm{T}}\nabla$, $\bar{\Delta}^{(k)} = \bar{\nabla}^{(k)} \cdot \bar{\nabla}^{(k)}$, and $\mathcal{Z}_k$ is the Jacobian matrix of the transformation Z_k.

Thus one can obtain $\mathcal{M}_i\mathbf{F}$ in problem (3.3.8) as follows:

$$\begin{aligned} \mathcal{M}_3\mathbf{F} &= \sum_{k=1}^{M_\lambda} \eta^{(k)} \left\{\bar{\mathbf{w}}^{(k)} \cdot \left(\mathbf{n}(x) - \mathbf{n}(\xi^{(k)})\right)\right\} \mathbf{n}, \\ \mathcal{M}_4\mathbf{F} &= \sum_{k=1}^{M_\lambda} \eta^{(k)}\zeta^{(k)}\Pi\left(\bar{\Pi}^{(k)} - \Pi\right)\mathbf{d} \end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=1}^{M_\lambda} 2\eta^{(k)}\Pi \Bigg(\nu_1(\xi^{(k)}) \frac{K(\xi^{(k)},0)}{K(x,t)} \bar{\Pi}^{(k)}\bar{\mathbb{D}}^{(k)}(\bar{\mathbf{w}}^{(k)})\mathbf{n}(\xi^{(k)}) \\
&\qquad\qquad - \nu_1(x)\Pi\mathbb{D}(\bar{\mathbf{w}}^{(k)})\mathbf{n}(x) \Bigg) \\
&- \sum_{k=1}^{M_\lambda} \nu_1(x)\Pi \left(\bar{\mathbf{w}}^{(k)} \otimes \nabla\eta^{(k)} + \eta^{(k)} \otimes \bar{\mathbf{w}}^{(k)} \right) \mathbf{n}(x),
\end{aligned}
$$

and $\mathcal{M}_i\mathbf{F}$ $(i = 1, 2)$ are the same as in (2.3.27) and (2.3.28).

In the same way as [24, 40, 41], we can show that

$$
\|\mathcal{M}\mathbf{F}\|_{\mathcal{H}_{h,l}} \le \left\{ c(\lambda + T^{1/2}) + c'(\lambda)h^{-l/2} \right\} \|\mathbf{F}\|_{\mathcal{H}_{h,l}} \tag{3.3.15}
$$

with a constant c independent of λ and T. Hence, for small λ, small T' and large h, $\mathcal{M}$ becomes a contraction operator on $\mathcal{H}_{l,T'}$, so that the solvability of (3.1.2) is proved on the interval $(0, T')$.

Furthermore, by the same way as that of (3.3.15) (see [40, 41]), it holds that

$$
\|\mathbf{w}\|_{H_h^{2+l,1+l/2}(Q_{T'})} + \|\nabla\pi\|_{H_h^{l,l/2}(Q_{T'})} \le c\|\mathbf{F}\|_{\mathcal{H}_{h,l}}. \tag{3.3.16}
$$

From this estimate and (3.3.10) the estimate (3.2.1) on $(0, T')$ follows. Then it is easy to prove the uniqueness of the solution of problem (3.1.2) on $(0, T')$.

If $T' < T$, we next consider the similar problem to problem (3.1.2) on $(T'/2, T)$:

$$
\begin{cases}
\dfrac{\partial \mathbf{u}^*}{\partial t} - \dfrac{\nu_1(x)}{\varrho_0(x)}\Delta\mathbf{u}^* + \dfrac{1}{\varrho_0(x)}\nabla q^* = \mathbf{f} \quad \text{in } Q_{T'/2,3T'/2}, \\
\nabla \cdot \mathbf{u}^* = g \quad \text{in } Q_{T'/2,3T'/2}, \\
\mathbf{u}^*|_{t=T'/2} = \mathbf{u}|_{t=T'/2} \quad \text{in } \Omega, \quad \mathbf{u}^* \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{in } G_{T'/2,3T'/2}, \\
\Pi\mathbf{u}^* + 2\nu_1(x)K(x,t)\Pi\mathbb{D}(\mathbf{u}^*)\mathbf{n} = K(x,t)\mathbf{d} \quad \text{in } G_{T'/2,3T'/2},
\end{cases} \tag{3.3.17}
$$

where $Q_{T_1,T_2} = \Omega \times (T_1, T_2)$, $G_{T_1,T_2} = \Gamma \times (T_1, T_2)$. To establish the solvability theorem of problem (3.3.17) we shall rewrite it.

First, extend $\mathbf{u} \in H_h^{2+l,1+l/2}(Q_{T'})$ to Q_∞ so that $\mathbf{u} \in H_h^{2+l,1+l/2}(Q_\infty)$ (denoted by the same letter) and

$$\|\mathbf{u}\|_{H_h^{2+l,1+l/2}(Q_\infty)} \leq c\|\mathbf{u}\|_{H_h^{2+l,1+l/2}(Q_{T'})}, \tag{3.3.18}$$

where c is independent of T'.

Let $\mathbf{u}^\#(x,t) = \mathbf{u}^*(x,t+T'/2) - \mathbf{u}(x,t+T'/2)$ and $q^\#(x,t) = q(x,t+T'/2)$, then $(\mathbf{u}^\#, q^\#)$ satisfies the following equations:

$$\begin{cases} \dfrac{\partial \mathbf{u}^\#}{\partial t} - \dfrac{\nu_1(x)}{\varrho_0(x)}\Delta \mathbf{u}^\# + \dfrac{1}{\varrho_0(x)}\nabla q^\# = \mathbf{f}^\# \quad \text{in } Q_{T'}, \\ \nabla \cdot \mathbf{u}^\# = g^\# \quad \text{in } Q_{T'}, \\ \mathbf{u}^\#|_{t=0} = \mathbf{0} \quad \text{in } \Omega, \quad \mathbf{u}^\# \cdot \mathbf{n} = \mathbf{b}^\# \cdot \mathbf{n} \quad \text{in } G_{T'}, \\ \Pi \mathbf{u}^\# + 2\nu_1(x) K^\#(x,t) \Pi \mathbb{D}(\mathbf{u}^\#)\mathbf{n} = K^\#(x,t)\mathbf{d}^\# \quad \text{in } G_{T'}, \end{cases} \tag{3.3.19}$$

where

$$\begin{cases} \mathbf{f}^\#(x,t) = \mathbf{f}(x,t+T'/2) - \dfrac{\partial \mathbf{u}}{\partial t}(x,t+T'/2) + \dfrac{\nu_1(x)}{\varrho_0(x)}\Delta \mathbf{u}(x,t+T'/2), \\ g^\#(x,t) = g(x,t+T'/2) - \nabla \cdot \mathbf{u}(x,t+T'/2), \\ \mathbf{G}^\#(x,t) = \mathbf{G}(x,t+T'/2) - \mathbf{u}(x,t+T'/2), \\ \mathbf{b}^\#(x,t) = \mathbf{b}(x,t+T'/2) - (\mathbf{u}(x,t+T'/2)\cdot \mathbf{n})\mathbf{n}|_\Gamma, \\ K^\#(x,t) = K(x,t+T'/2), \\ \mathbf{d}^\#(x,t) = \mathbf{d}(x,t+T'/2) - \dfrac{1}{K^\#(x,t)}\Pi \mathbf{u}(x,t+T'/2)|_\Gamma \\ \qquad\qquad -2\nu_1(x)\Pi\mathbb{D}(\mathbf{u}(x,t+T'/2))\mathbf{n}|_\Gamma. \end{cases} \tag{3.3.20}$$

We can see that, for example,

$$g^\#|_{t=0} = (g - \nabla \cdot \mathbf{u})|_{t=T'/2} = 0,$$

since $\mathbf{u}$ is a solution of problem (3.1.2) on $(0,T')$, and also we have

$$\mathbf{G}^\#|_{t=0} = \mathbf{0}, \quad \mathbf{d}^\#|_{t=0} = \mathbf{0}, \quad \mathbf{b}^\#|_{t=0} = 0.$$

Thus $\mathbf{f}^\#$, $g^\#$, $\mathbf{G}^\#$, $\mathbf{b}^\#$ and $\mathbf{d}^\#$ satisfy the same conditions as those of Lemma 3.2.1. Repeating the same argument carried out above, one can also obtain the solution $(\mathbf{u}^{\#}, q^{\#})$ of problem (3.3.19) on the same interval $(0, T')$ for the same h and the estimate

$$\|\mathbf{u}^{\#}\|_{H_h^{2+l,1+l/2}(Q_{T'})} + \|\nabla q^{\#}\|_{H_h^{l,l/2}(Q_{T'})} \leq c \Big(\|\mathbf{f}^{\#}\|_{H_h^{l,l/2}(Q_{T'})} + \|g^{\#}\|_{H_h^{1+l,1/2+l/2}(Q_{T'})}$$

$$+ \|\mathbf{G}^{\#}\|_{H_h^{0,1+l/2}(Q_{T'})} + \|\mathbf{b}^{\#}\|_{H_h^{3/2+l,3/4+l/2}(G_{T'})} + \|\mathbf{d}^{\#}\|_{H_h^{1/2+l,1/4+l/2}(G_{T'})} \Big). \quad (3.3.21)$$

Consequently, we obtain a unique solution $(\mathbf{u}^*, \nabla q^*)$ of problem (3.3.17) on $(T'/2, 3T'/2)$.

Due to the unique existence of the solution of problem (3.1.2) on $(T'/2, T')$,

$$\mathbf{u} \equiv \mathbf{u}^*, \quad \nabla q \equiv \nabla q^* \text{ on } (T'/2, T') \quad (3.3.22)$$

holds. Thus, let

$$\mathbf{u}^{**} = \begin{cases} \mathbf{u} & \text{on } (0, T'), \\ \mathbf{u}^* & \text{on } (T', 3T'/2), \end{cases} \quad \nabla q^{**} = \begin{cases} \nabla q & \text{on } (0, T'), \\ \nabla q^* & \text{on } (T', 3T'/2), \end{cases}$$

then it holds $\mathbf{u}^{**} \in H_h^{2+l,1+l/2}(Q_{3T'/2})$ and $\nabla q^{**} \in H_h^{l,l/2}(Q_{3T'/2})$ because of (3.3.22). Moreover, it is obvious that $(\mathbf{u}^{**}, q^{**})$ is a solution of problem (3.1.2) on the extended interval $(0, 3T'/2)$ and satisfies the estimate (3.2.1) on $(0, 3T'/2)$ by virtue of the estimates (3.3.18), (3.3.21).

This means that we can extend the length of the interval by $T'/2$, on which the solution of problem (3.1.2) exists. Repeating this argument up to T, we can conclude that a unique solution $(\mathbf{u}, q)$ of problem (3.1.2) exists on $(0, T)$ and the estimate (3.2.1) holds.

3.3.3 Proof of Lemma 3.2.2

The trace theorem implies that for any $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$ there exists $\mathbf{u}^\bullet \in W_2^{2+l,1+l/2}(Q_\infty)$ satisfying the initial condition $\mathbf{u}^\bullet|_{t=0} = \mathbf{v}_0$ and the inequality

$$\|\mathbf{u}^\bullet\|_{W_2^{2+l,1+l/2}(Q_\infty)} \leq c \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}. \quad (3.3.23)$$

The difference $\mathbf{U} = \mathbf{u} - \mathbf{u}^\bullet$ satisfies the following equations with the homogeneous initial data:

$$
\begin{cases}
\dfrac{\partial \mathbf{U}}{\partial t} - \dfrac{\nu_1(x)}{\varrho_0(x)}\Delta \mathbf{U} + \dfrac{1}{\varrho_0(x)}\nabla q = \mathbf{f} - \dfrac{\partial \mathbf{u}^\bullet}{\partial t} + \dfrac{\nu_1(x)}{\varrho_0(x)}\Delta \mathbf{u}^\bullet =: \mathbf{f}^\bullet \quad \text{in } Q_T, \\
\nabla \cdot \mathbf{U} = g - \nabla \cdot \mathbf{u}^\bullet =: g^\bullet \quad \text{in } Q_T, \\
\mathbf{U}|_{t=0} = \mathbf{0} \quad \text{in } \Omega, \quad \mathbf{U}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} - \mathbf{u}^\bullet\cdot\mathbf{n} =: \mathbf{b}^\bullet\cdot\mathbf{n} \quad \text{in } G_T, \\
\Pi\mathbf{U} + 2\nu_1(x)K(x,t)\Pi\mathbb{D}(\mathbf{U})\mathbf{n} = K(x,t)\mathbf{d} - \Pi\mathbf{u}^\bullet - 2\nu_1(x)K(x,t)\Pi\mathbb{D}(\mathbf{u}^\bullet)\mathbf{n} \\
\qquad\qquad =: K(x,t)\mathbf{d}^\bullet \quad \text{in } G_T,
\end{cases}
\tag{3.3.24}
$$

and the compatibility conditions reduce to

$$
g^\bullet(\,\cdot\,,0) = 0, \quad \mathbf{b}^\bullet(\,\cdot\,,0) = \mathbf{d}^\bullet(\,\cdot,0) = \mathbf{0}, \quad \mathbf{G}^\bullet|_\Gamma = \mathbf{b}^\bullet, \quad \mathbf{d}^\bullet\cdot\mathbf{n} = 0,
$$

where $\mathbf{G}^\bullet = \mathbf{G} - \mathbf{u}^\bullet$. It is easily seen that the conditions above imply $\mathbf{f}^\bullet \in H_h^{l,l/2}(Q_T)$, $g^\bullet \in H_h^{1+l,1/2+l/2}(Q_T)$, $g^\bullet = \nabla\cdot\mathbf{G}^\bullet$, $\mathbf{G}^\bullet \in H_h^{0,1+l/2}(Q_T)$, $\mathbf{b}^\bullet \in H_h^{3/2+l,3/4+l/2}(G_T)$, $\mathbf{d}^\bullet \in H_h^{1/2+l,1/4+l/2}(G_T)$. By applying Lemma 3.2.1 to problem (3.3.24) and taking into account the estimate (3.3.23) and Remark A.2.1, the assertion of Lemma 3.2.2 immediately follows.

3.4 Auxiliary estimates

Before proving Theorem 3.2.1, we begin with auxiliary propositions.

We assume here and in what follows that $\mathbf{u} \in W_2^{2+l,1+l/2}(Q_T)$ satisfies

$$
T^{1/2}\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} \le \delta \tag{3.4.1}
$$

with sufficiently small $\delta > 0$.

The problem (3.1.1) is rewritten in the form

$$
\begin{cases}
\varrho_0 \dfrac{\partial \mathbf{u}}{\partial t} - \nu(\varrho_0)\Delta \mathbf{u} + \nabla q = \mathbf{l}_1^{(\mathbf{u})}(\mathbf{u}, q) + 2\nu'(\varrho_0)\mathbb{D}_{\mathbf{u}}(\mathbf{u})\nabla_{\mathbf{u}}\varrho_0 \\
\qquad\qquad -\dfrac{\beta}{3}(\nabla_{\mathbf{u}}^{(j)}\nabla_{\mathbf{u}}^{(i)}\varrho_0)\nabla_{\mathbf{u}}\varrho_0 - \beta\Delta_{\mathbf{u}}\varrho_0\nabla_{\mathbf{u}}\varrho_0 + \varrho_0\mathbf{b}_{\mathbf{u}}, \\
\nabla \cdot \mathbf{u} = l_2^{(\mathbf{u})}(\mathbf{u}), \quad \mathbf{u}|_{t=0} = \mathbf{v}_0, \quad \mathbf{u}\cdot\mathbf{n}|_\Gamma = l_3^{(\mathbf{u})}(\mathbf{u})|_\Gamma, \\
\mathbf{u} + 2\nu(\varrho_0)K\Pi\mathbb{D}(\mathbf{u})\mathbf{n}|_\Gamma = \mathbf{l}_4^{(\mathbf{u})}(\mathbf{u}) + \beta K_{\mathbf{u}}\Pi_{\mathbf{u}}(\nabla_{\mathbf{u}}\varrho_0 \otimes \nabla_{\mathbf{u}}\varrho_0)\mathbf{n}_{\mathbf{u}}|_\Gamma,
\end{cases}
\tag{3.4.2}
$$

where

$$
\begin{cases}
\nabla_{\mathbf{u}} = (\nabla_{\mathbf{u}}^{(1)}, \nabla_{\mathbf{u}}^{(2)}, \nabla_{\mathbf{u}}^{(3)}), \quad \Delta_{\mathbf{u}} = \nabla_{\mathbf{u}} \cdot \nabla_{\mathbf{u}}, \\
\mathbf{l}_1^{(\mathbf{u})}(\mathbf{w}, s) = \nu(\varrho_0)(\Delta_{\mathbf{u}} - \Delta)\mathbf{w} - (\nabla_{\mathbf{u}} - \nabla)s, \\
l_2^{(\mathbf{u})}(\mathbf{w}) = (\nabla - \nabla_{\mathbf{u}})\cdot\mathbf{w} = \nabla\cdot\mathcal{L}^{(\mathbf{u})}(\mathbf{w}), \quad l_3^{(\mathbf{u})}(\mathbf{w}) = \mathbf{w}\cdot(\mathbf{n} - \mathbf{n}_{\mathbf{u}}), \\
\mathbf{l}_4^{(\mathbf{u})}(\mathbf{w}) = 2\nu(\varrho_0)\left(K\Pi\mathbb{D}(\mathbf{w})\mathbf{n} - K_{\mathbf{u}}\Pi_{\mathbf{u}}\mathbb{D}_{\mathbf{u}}(\mathbf{w})\mathbf{n}_{\mathbf{u}}\right).
\end{cases}
\tag{3.4.3}
$$

In the rest of this section we estimate the right-hand side of (3.4.2), which is neccesary to prove the solvability of problem (3.1.1).

We use the same notation as that in Chapter 2 §2.4:

$$
a_{ij} = \delta_{ij} + b_{ij}, \quad b_{ij} = \int_0^t \frac{\partial u_i}{\partial x_j}\,\mathrm{d}\tau, \quad A_{ij} = \delta_{ij} + B_{ij}, \quad \mathbb{A} = \mathbb{I} + \mathcal{B},
$$

where $\mathbb{A} = (A_{ij})$, $\mathcal{B} = (B_{ij})$,

$$
A_{ii} = a_{jj}a_{kk} - a_{jk}a_{kj}, \quad A_{ij} = a_{kj}a_{ik} - a_{ij}a_{kk},
$$

$$
B_{ii} = b_{jj} + b_{kk} + b_{jj}b_{kk} - b_{jk}b_{kj}, \quad B_{ij} = -b_{ij} + b_{kj}b_{ik} - b_{ij}b_{kk}
$$

for $i, j, k = 1, 2, 3$, $i \neq j$, $j \neq k$, $k \neq i$, and $\mathcal{L}^{(\mathbf{u})}(\mathbf{w}) = -\mathcal{B}\mathbf{w}$.

We denote by a'_{ij}, b'_{ij}, A'_{ij}, B'_{ij} the corresponding functions to another vector field $\mathbf{u}'(x, t)$, and set $\tilde{b}_{ij} = b_{ij} - b'_{ij}$, $\widetilde{B}_{ij} = B_{ij} - B'_{ij}$. Finally, set

$$
D\mathbf{u} = \left\{\frac{\partial u_i}{\partial x_j}\right\}_{i,j=1,2,3}, \quad D^2\mathbf{u} = \left\{\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right\}_{i,j,k=1,2,3},
$$

$$|D\mathbf{u}|_\Omega = \max_{i,j}\sup_{x\in\Omega}\left|\frac{\partial u_i}{\partial x_j}\right|, \qquad |D^2\mathbf{u}|_\Omega = \max_{i,j,k}\sup_{x\in\Omega}\left|\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right|,$$

$$\|D\mathbf{u}\|_{W_2^r(\Omega)} = \left(\sum_{j=1}^{3}\left\|\frac{\partial \mathbf{u}}{\partial x_j}\right\|^2_{W_2^r(\Omega)}\right)^{1/2}.$$

Since, $\mathbf{l}_1^{(\mathbf{u})}$ and $l_2^{(\mathbf{u})}$ are the same as those in Chapter 2, we only proceed to the estimates of $l_3^{(\mathbf{u})}(\mathbf{w}) - l_3^{(\mathbf{u}')}(\mathbf{w})$, $\mathbf{l}_4^{(\mathbf{u})}(\mathbf{w}) - \mathbf{l}_4^{(\mathbf{u}')}(\mathbf{w})$.

Due to $\Gamma \in W_2^{7/2+l}$, we have $\mathbf{n} \in W_2^{5/2+l}(\Gamma)$. It is well known that $W_2^{5/2+l}(\Gamma)$ is embedded in $C^{2+l'}(\Gamma) = \{f \in C^2(\Gamma)\,|\,D^\alpha f\ (|\alpha| = 2)$ is Hölder continuous with exponent $l'\}$ for $0 < l' < l - 1/2$. Thus one can easily see that

Lemma 3.4.1 *Let* $\mathbf{u}$ *and* $\mathbf{u}'$ *satisfy condition* (3.4.1)*, and let* K *satisfy the conditions of Theorem* 3.2.1*. Then for arbitrary* $\mathbf{w} \in W_2^{2+l,1+l/2}(Q_T)$ *it holds*

$$\|K_\mathbf{u} - K_{\mathbf{u}'}\|_{W_2^{3/2+l,3/4+l/2}(\Gamma_T)} \le c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|^{(2+l,1+l/2)}_{Q_T},$$

$$\|l_3^{(\mathbf{u})}(\mathbf{w}) - l_3^{(\mathbf{u}')}(\mathbf{w})\|_{W_2^{3/2+l,3/4+l/2}(\Gamma_T)} \le c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|^{(2+l,1+l/2)}_{Q_T}\|\mathbf{w}\|^{(2+l,1+l/2)}_{Q_T},$$

$$\|\mathbf{l}_4^{(\mathbf{u})}(\mathbf{w}) - \mathbf{l}_4^{(\mathbf{u}')}(\mathbf{w})\|_{W_2^{1/2+l,1/4+l/2}(G_T)} \le c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|^{(2+l,1+l/2)}_{Q_T}\|\mathbf{w}\|^{(2+l,1+l/2)}_{Q_T}.$$

3.5 Proof of Theorem 3.2.1

We shall solve the problem (3.4.2) by the method of successive approximations: set $(\mathbf{u}_0, q_0) = (\mathbf{0}, 0)$ and let $(\mathbf{u}_{m+1}, q_{m+1})$ $(m = 0, 1, 2, \ldots)$ be a solution of the problem

$$\begin{cases} \varrho_0\dfrac{\partial \mathbf{u}_{m+1}}{\partial t} - \nu(\varrho_0)\Delta\mathbf{u}_{m+1} + \nabla q_{m+1} \\ \qquad\qquad = \mathbf{l}_1^{(m)}(\mathbf{u}_m, q_m) + 2\nu'(\varrho_0)\mathbb{D}_m(\mathbf{u}_m)\nabla_m\varrho_0 \\ \qquad\qquad\quad -\dfrac{\beta}{3}(\nabla_m^{(j)}\nabla_m^{(i)}\varrho_0)\nabla_m\varrho_0 - \beta\Delta_m\varrho_0\nabla_m\varrho_0 + \varrho_0\mathbf{b}_m, \\ \nabla\cdot\mathbf{u}_{m+1} = l_2^{(m)}(\mathbf{u}_m), \quad \mathbf{u}_{m+1}|_{t=0} = \mathbf{v}_0, \quad \mathbf{u}_{m+1}\cdot\mathbf{n}|_\Gamma = l_3^{(m)}(\mathbf{u}_m)|_\Gamma, \\ \mathbf{u}_{m+1} + 2\nu(\varrho_0)K\Pi\mathbb{D}(\mathbf{u}_{m+1})\mathbf{n}|_\Gamma = \mathbf{l}_4^{(m)}(\mathbf{u}_m) + \beta K_m\Pi_m(\nabla_m\varrho_0\otimes\nabla_m\varrho_0)\mathbf{n}_m|_\Gamma, \end{cases} \tag{3.5.1}$$

provided that $(\mathbf{u}_n, q_n)$, $n = 1,2,\dots,m$, satisfy $(\mathbf{u}_n, \nabla q_n) \in W_2^{2+l,1+l/2}(Q_T) \times W_2^{l,l/2}\ (Q_T)$ and the condition (3.4.1). Here $\nabla_m = \nabla_{\mathbf{u}_m}$, $\Delta_m = \nabla_m \cdot \nabla_m$, $\mathbf{l}_j^{(m)} = \mathbf{l}_j^{(\mathbf{u}_m)}$ $(j = 1,4)$, $l_j^{(m)} = l_j^{(\mathbf{u}_m)}$ $(j = 2,3)$, $\mathbb{D}_m = \mathbb{D}_{\mathbf{u}_m}$, $\mathbf{b}_m = \mathbf{b}_{\mathbf{u}_m}$, $K_m = K_{\mathbf{u}_m}$, $\Pi_m = \Pi_{\mathbf{u}_m}$, $\mathbf{n}_m = \mathbf{n}_{\mathbf{u}_m}$. From Lemma 3.2.2 and the estimates in § 3.4 it follows that $(\mathbf{u}_{m+1}, \nabla q_{m+1})$ is uniquely determined. In particular, $(\mathbf{u}_1, q_1)$ is a solution of problem (3.5.1) with $m = 0$, *i.e.*,

$$\begin{cases} \dfrac{\partial \mathbf{u}_1}{\partial t} - \dfrac{\nu(\varrho_0)}{\varrho_0}\Delta \mathbf{u}_1 + \dfrac{1}{\varrho_0}\nabla q_1 = -\dfrac{\beta}{3\varrho_0}(\nabla^{(j)}\nabla^{(i)}\varrho_0)\nabla\varrho_0 - \dfrac{\beta}{\varrho_0}\Delta\varrho_0\nabla\varrho_0 + \mathbf{b}, \\ \nabla\cdot\mathbf{u}_1 = 0, \quad \mathbf{u}_1|_{t=0} = \mathbf{v}_0, \qquad \mathbf{u}_1\cdot\mathbf{n}|_\Gamma = 0, \\ \mathbf{u}_1 + 2\nu(\varrho_0)K\Pi\mathbb{D}(\mathbf{u}_1)\mathbf{n}|_\Gamma = \beta K\Pi(\nabla\varrho_0\otimes\nabla\varrho_0)\mathbf{n}|_\Gamma, \end{cases} \tag{3.5.2}$$

where $\nabla^{(i)} = \frac{\partial}{\partial x_i}$, and the estimate

$$\begin{aligned} N[\mathbf{u}_1, q_1] &:= \|\mathbf{u}_1\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla q_1\|_{Q_T}^{(l,l/2)} \\ &\le c\left(\frac{\beta}{3}\left\|\frac{1}{\varrho_0}(\nabla^{(j)}\nabla^{(i)}\varrho_0)\nabla\varrho_0\right\|_{Q_T}^{(l,l/2)} + \beta\left\|\frac{1}{\varrho_0}\nabla^2\varrho_0\nabla\varrho_0\right\|_{Q_T}^{(l,l/2)}\right. \\ &\quad \left. + \|\mathbf{b}\|_{Q_T}^{(l,l/2)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)} + \beta\|K\Pi(\nabla\varrho_0\otimes\nabla\varrho_0)\mathbf{n}\|_{W_2^{1/2+l,1/4+l/2}(G_T)}\right) \\ &\le c_1(T)\left(1 + \|\varrho_0\|_{W_2^{2+l}(\Omega)}^3 + \|\mathbf{b}\|_{Q_T}^{(l,l/2)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right) \end{aligned} \tag{3.5.3}$$

holds with $c_1(T)$ being a non-decreasing function of T. Thus, if T_1 $(0 < T_1 \le T)$ satisfies

$$T_1^{1/2}\cdot 2c_1(T)\left(1 + \|\varrho_0\|_{W_2^{2+l}(\Omega)}^3 + \|\mathbf{b}\|_{Q_T}^{(l,l/2)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right) < \delta, \tag{3.5.4}$$

$\mathbf{u}_1$ satisfies the condition (3.4.1) on the interval $(0, T_1)$.

Next, suppose that the condition (3.4.1) is satisfied for $\mathbf{u}_n$ $(n = 1,2,\dots,m)$. Then the differences $\mathbf{Z}_{m+1} := \mathbf{u}_{m+1} - \mathbf{u}_m$, $P_{m+1} := q_{m+1} - q_m$ $(m = 1,2,3\dots)$

satisfy

$$
\begin{cases}
\varrho_0 \dfrac{\partial \mathbf{Z}_{m+1}}{\partial t} - \nu(\varrho_0)\Delta \mathbf{Z}_{m+1} + \nabla P_{m+1} \\
\qquad = \mathbf{l}_1^{(m)}(\mathbf{Z}_m, P_m) + \left(\mathbf{l}_1^{(m)}(\mathbf{u}_{m-1}, q_{m-1}) - \mathbf{l}_1^{(m-1)}(\mathbf{u}_{m-1}, q_{m-1})\right) \\
\qquad\qquad +2\nu'(\varrho_0)\left(\mathbb{D}_m(\mathbf{u}_m)\nabla_m\varrho_0 - \mathbb{D}_{m-1}(\mathbf{u}_{m-1})\nabla_{m-1}\varrho_0\right) \\
\qquad\qquad -\dfrac{\beta}{3}\left\{(\nabla_m^{(j)}\nabla_m^{(i)}\varrho_0)\nabla_m\varrho_0 - (\nabla_{m-1}^{(j)}\nabla_{m-1}^{(i)}\varrho_0)\nabla_{m-1}\varrho_0\right\} \\
\qquad\qquad -\beta\left(\Delta_m\varrho_0\nabla_m\varrho_0 - \Delta_{m-1}\varrho_0\nabla_{m-1}\varrho_0\right) + \varrho_0(\mathbf{b}_m - \mathbf{b}_{m-1}), \\
\nabla\cdot\mathbf{Z}_{m+1} = l_2^{(m)}(\mathbf{Z}_m) + \left(l_2^{(m)}(\mathbf{u}_{m-1}) - l_2^{(m-1)}(\mathbf{u}_{m-1})\right), \\
\mathbf{Z}_{m+1}|_{t=0} = \mathbf{0}, \quad \mathbf{Z}_{m+1}\cdot\mathbf{n}|_\Gamma = l_3^{(m)}(\mathbf{Z}_m) + \left(l_3^{(m)}(\mathbf{u}_{m-1}) - l_3^{(m-1)}(\mathbf{u}_{m-1})\right)|_\Gamma, \\
\mathbf{Z}_{m+1} + 2\nu(\varrho_0)K\Pi\mathbb{D}(\mathbf{Z}_{m+1})\mathbf{n}|_\Gamma = \mathbf{l}_4^{(m)}(\mathbf{Z}_m) + \left(\mathbf{l}_4^{(m)}(\mathbf{u}_{m-1}) - \mathbf{l}_4^{(m-1)}(\mathbf{u}_{m-1})\right) \\
\qquad +\beta\left\{K_m\Pi_m(\nabla_m\varrho_0\otimes\nabla_m\varrho_0)\mathbf{n}_m - K_{m-1}\Pi_{m-1}(\nabla_{m-1}\varrho_0\otimes\nabla_{m-1}\varrho_0)\mathbf{n}_{m-1}\right\}|_\Gamma.
\end{cases}
$$

We suppose that the condition (2.4.1) is satisfied for $\mathbf{u}_n$ $(n \le m)$. Then the lemmata in §§ 2.4 and 3.4 yield

$$
\left\|\mathbf{l}_1^{(m)}(\mathbf{Z}_m, P_m)\right\|_{Q_T}^{(l,l/2)} \le c\delta\left(\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla P_m\|_{Q_T}^{(l,l/2)}\right),
$$

$$
\begin{aligned}
&\left\|\mathbf{l}_1^{(m)}(\mathbf{u}_{m-1}, q_{m-1}) - \mathbf{l}_1^{(m-1)}(\mathbf{u}_{m-1}, q_{m-1})\right\|_{Q_T}^{(l,l/2)} \\
&\qquad \le c\delta\left(\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla P_m\|_{Q_T}^{(l,l/2)}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathbb{D}_m(\mathbf{u}_m)\nabla_m\varrho_0 - \mathbb{D}_{m-1}(\mathbf{u}_{m-1})\nabla_{m-1}\varrho_0\|_{Q_T}^{(l,l/2)} \\
&\qquad \le c\|\varrho_0\|_{W_2^{2+l}(\Omega)}\left(1 + T^{1/2-l/2}\|\mathbf{v}_0\|_{W_2^l(\Omega)}\right)T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},
\end{aligned}
$$

$$
\begin{aligned}
&\left\|(\nabla_m^{(j)}\nabla_m^{(i)}\varrho_0)\nabla_m\varrho_0 - (\nabla_{m-1}^{(j)}\nabla_{m-1}^{(i)}\varrho_0)\nabla_{m-1}\varrho_0\right\|_{Q_T}^{(l,l/2)} \\
&\qquad \le c\|\varrho_0\|_{W_2^{2+l}(\Omega)}^2(T^{1/2} + T^{1/2-l/2})T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\nabla_m^2\varrho_0\nabla_m\varrho_0 - \nabla_{m-1}^2\varrho_0\nabla_{m-1}\varrho_0\right\|_{Q_T}^{(l,l/2)} \\
&\qquad \le c\|\varrho_0\|_{W_2^{2+l}(\Omega)}^2(T^{1/2} + T^{1/2-l/2})T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},
\end{aligned}
$$

$$\|\mathbf{b}_m - \mathbf{b}_{m-1}\|_{Q_T}^{(l,l/2)} \le c\,T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$

$$\|l_2^{(m)}(\mathbf{Z}_m)\|_{W_2^{1+l,1/2+l/2}(Q_T)} \le c\delta\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$

$$\|l_2^{(m)}(\mathbf{u}_{m-1}) - l_2^{(m-1)}(\mathbf{u}_{m-1})\|_{W_2^{1+l,1/2+l/2}(Q_T)} \le c\delta\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$

$$\left\|\frac{\partial}{\partial t}\mathcal{L}^{(m)}(\mathbf{Z}_m)\right\|_{Q_T}^{(0,l/2)} + \left\|\frac{\partial}{\partial t}\left(\mathcal{L}^{(m)}(\mathbf{u}_{m-1}) - \mathcal{L}^{(m-1)}(\mathbf{u}_{m-1})\right)\right\|_{Q_T}^{(0,l/2)}$$
$$\le c\delta\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$

$$\|l_3^{(m)}(\mathbf{Z}_m)\|_{W_2^{3/2+l,3/4+l/2}(\Gamma_T)} \le c\delta\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$

$$\|l_3^{(m)}(\mathbf{u}_{m-1}) - l_3^{(m-1)}(\mathbf{u}_{m-1})\|_{W_2^{3/2+l,3/4+l/2}(\Gamma_T)} \le c\delta\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$

$$\|\mathbf{l}_4^{(m)}(\mathbf{Z}_m)\|_{W_2^{1/2+l,1/4+l/2}(G_T)} \le c\delta\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$

$$\|\mathbf{l}_4^{(m)}(\mathbf{u}_{m-1}) - \mathbf{l}_4^{(m-1)}(\mathbf{u}_{m-1})\|_{W_2^{1/2+l,1/4+l/2}(G_T)} \le c\delta\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)},$$

$$\|K_m\Pi_m(\nabla_m\varrho_0\otimes\nabla_m\varrho_0)\mathbf{n}_m$$
$$-K_{m-1}\Pi_{m-1}(\nabla_{m-1}\varrho_0\otimes\nabla_{m-1}\varrho_0)\mathbf{n}_{m-1}\|_{W_2^{1/2+l,1/4+l/2}(G_T)}$$
$$\le c\,T^{1/2}\|\varrho_0\|^2_{W_2^{2+l}(\Omega)}T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)}.$$

Hence, we obtain

$$\begin{aligned} N[\mathbf{Z}_{m+1}, P_{m+1}] &\equiv \|\mathbf{Z}_{m+1}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla P_{m+1}\|_{Q_T}^{(l,l/2)} \\ &\le c_2(T)\left(\delta N[\mathbf{Z}_m, P_m] + T^{1/2}\|\mathbf{Z}_m\|_{Q_T}^{(2+l,1+l/2)}\right), \quad (3.5.5) \end{aligned}$$

where $c_2(T)$ is a non-decreasing function with respect to T. Taking into account the condition (3.4.1) for $\mathbf{u}_n$ $(n \le m)$, we also have (3.5.5) for $m = 0, 1, \ldots, m-1$. Choosing δ in such a way that $c_2(T)\delta < 1/4$, we obtain

$$N[\mathbf{Z}_{n+1}, P_{n+1}] \le \frac{1}{4}N[\mathbf{Z}_n, P_n] + c_2(T)T^{1/2}\|\mathbf{Z}_n\|_{Q_T}^{(2+l,1+l/2)}$$
$$\le \left(\frac{1}{4} + c_2(T)T^{1/2}\right)N[\mathbf{Z}_n, P_n] \le \ldots \le \left(\frac{1}{4} + c_2(T)T^{1/2}\right)^n N[\mathbf{Z}_1, P_1]. \quad (3.5.6)$$

Set $\Sigma_{m+1} = \sum_{n=0}^{m} N[\mathbf{Z}_{n+1}, P_{n+1}]$. Since

$$\sum\nolimits_{m+1} \leq N[\mathbf{u}_1, q_1] \sum_{n=0}^{m} \left(\frac{1}{4} + c_2(T)T^{1/2}\right)^n \leq c_1(T)\Big(1 + \|\varrho_0\|^3_{W_2^{2+l}(\Omega)} + \|\mathbf{b}\|^{(l,l/2)}_{Q_T} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\Big) \sum_{n=0}^{m} \left(\frac{1}{4} + c_2(T)T^{1/2}\right)^n,$$

it is easy to see that

$$\begin{aligned} N[\mathbf{u}_{m+1}, q_{m+1}] &\leq \sum\nolimits_{m+1} + N[\mathbf{u}_1, q_1] \\ &\leq c_1(T)\left(1 + \|\varrho_0\|^3_{W_2^{2+l}(\Omega)} + \|\mathbf{b}\|^{(l,l/2)}_{Q_T} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right) \\ &\quad \times \left\{1 + \sum_{n=0}^{m} \left(\frac{1}{4} + c_2(T)T^{1/2}\right)^n\right\} \end{aligned} \tag{3.5.7}$$

holds. Note that $c_j(T)$ $(j = 1, 2)$ are non-decreasing functions of T, and hence the condition (3.4.1) with T' $(0 < T' \leq T)$ for $\mathbf{u}_{m+1}$ is satisfied if $c_2(T){T'}^{1/2} \leq 1/4$ and $T' \leq T_1$ (see (3.5.4) for T_1), namely,

$${T'}^{1/2} \cdot 2c_1(T)\left(1 + \|\varrho_0\|^3_{W_2^{2+l}(\Omega)} + \|\mathbf{b}\|^{(l,l/2)}_{Q_T} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right) \leq \delta. \tag{3.5.8}$$

Since the left hand side of (3.5.8) does not depend on m, it follows from (3.5.6) and (3.5.7) that $N[\mathbf{u}_m, q_m]$ is uniformly bounded on $(0, T')$. Therefore the sequence $\{(\mathbf{u}_m, q_m)\}$ converges to the limit function $(\mathbf{u}, q)$ in the norm $N[\,\cdot\,,\,\cdot\,]$ and this limit is a solution of the problem (3.4.2).

The solution is unique. Indeed, the difference of two solutions $\mathbf{w} = \mathbf{u} - \mathbf{u}'$,

$\pi = q - q'$ satisfies the equations

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{w}}{\partial t} - \nu(\varrho_0)\Delta \mathbf{w} + \nabla \pi = \mathbf{l}_1^{(\mathbf{u})}(\mathbf{w}, \pi) + \Big(\mathbf{l}_1^{(\mathbf{u})}(\mathbf{u}', q') - \mathbf{l}_1^{(\mathbf{u}')}(\mathbf{u}', q')\Big) \\ \qquad\qquad +2\nu'(\varrho_0)\left(\mathbb{D}_{\mathbf{u}}(\mathbf{u})\nabla_{\mathbf{u}}\varrho_0 - \mathbb{D}_{\mathbf{u}'}(\mathbf{u}')\nabla_{\mathbf{u}'}\varrho_0\right) \\ \qquad\qquad -\dfrac{\beta}{3}\Big\{(\nabla_{\mathbf{u}}^{(i)}\nabla_{\mathbf{u}}^{(j)}\varrho_0)\nabla_{\mathbf{u}}\varrho_0 - (\nabla_{\mathbf{u}'}^{(i)}\nabla_{\mathbf{u}'}^{(j)}\varrho_0)\nabla_{\mathbf{u}'}\varrho_0\Big\} \\ \qquad\qquad -\beta\left(\Delta_{\mathbf{u}}\varrho_0\nabla_{\mathbf{u}}\varrho_0 - \Delta_{\mathbf{u}'}\varrho_0\nabla_{\mathbf{u}'}\varrho_0\right) + \varrho_0(\mathbf{b}_{\mathbf{u}} - \mathbf{b}_{\mathbf{u}'}), \\ \nabla \cdot \mathbf{w} = l_2^{(\mathbf{u})}(\mathbf{w}) + \Big(l_2^{(\mathbf{u})}(\mathbf{u}') - l_2^{(\mathbf{u}')}(\mathbf{u}')\Big), \\ \mathbf{w}|_{t=0} = \mathbf{0}, \quad \mathbf{w} \cdot \mathbf{n}|_{\Gamma} = l_3^{(\mathbf{u})}(\mathbf{w}) + \Big(l_3^{(\mathbf{u})}(\mathbf{u}') - l_3^{(\mathbf{u}')}(\mathbf{u}')\Big)|_{\Gamma}, \\ \mathbf{w} + 2\nu(\varrho_0)K\Pi\mathbb{D}(\mathbf{w})\mathbf{n}|_{\Gamma} = \mathbf{l}_4^{(\mathbf{u})}(\mathbf{w}) + \Big(\mathbf{l}_4^{(\mathbf{u})}(\mathbf{u}') - \mathbf{l}_4^{(\mathbf{u}')}(\mathbf{u}')\Big) \\ \qquad +\beta\left\{K_{\mathbf{u}}\Pi_{\mathbf{u}}(\nabla_{\mathbf{u}}\varrho_0 \otimes \nabla_{\mathbf{u}}\varrho_0)\mathbf{n}_{\mathbf{u}} - K_{\mathbf{u}'}\Pi_{\mathbf{u}'}(\nabla_{\mathbf{u}'}\varrho_0 \otimes \nabla_{\mathbf{u}'}\varrho_0)\mathbf{n}_{\mathbf{u}'}\right\}|_{\Gamma}. \end{cases}$$

Applying to this problem the estimate (3.2.2) and repeating the arguments carried out, we arrive at inequality

$$N[\mathbf{w}, \pi] \le c(\delta + T'^{1/2})N[\mathbf{w}, \pi].$$

This implies $(\mathbf{w}, \nabla\pi) = (\mathbf{0}, \mathbf{0})$, and the proof of Theorem 3.2.1 is completed.

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Bibliography

[1] S. Agmon, A. Douglis and L. Nirenberg: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I.* Comm. Pure Appl. Math. **12** (1959), 623-727.

[2] G. Ahmadi: *A generalized continuum theory for granular materials.* Int. J. Non-Linear Mech. **17** (1982), 21-33.

[3] R. A. Bagnold: The Physics of Blown Sand and Desert Dunes, *London, Methuen*, 1941.

[4] O. V. Besov, V. P. Il'in and S. M. Nikol'skiĭ: Integral representations of functions and imbedding theorems. *"Nauka", Moscow*, 1975; English transl., Vols. 1, 2, Wiley, 1979.

[5] E. J. Boyle and M. Massoudi: *A theory for granular materials exhibiting normal stress effects based on Enskog's dense gas theory.* Int. J. Eng. Sci. **28** (1990), 1261-1275.

[6] M. Bulíček, J. Málek and K. R. Rajagopal: *Navier's slip and evolutionary Navier–Stokes-like systems with pressure and shear-rate dependent viscosity.* Indiana. Univ. Math. J. **56** (2007), 51-85.

[7] J. E. Dunn and J. Serrin: *On the thermomechanics of interstitial working.* Arch. Rational Mech. Anal. **88** (1985), 95-133.

[8] J. Duran: Sands, Powders, and Grains: An Introduction to the Physics of Granular Materials. *Springer*, 1991.

[9] M. A. Goodman and S. C. Cowin: *Two problems in the gravity flow of granular materials.* J. Fluid. Mech. **45** (1971), 321-339.

[10] M. A. Goodman and S. C. Cowin: *A continuum theory for granular materials.* Arch. Rational Mech. Anal. **44** (1972), 249-266.

[11] H. Hayakawa, S. Nasuno: Funtaino Butsuri, *Kyōritsu Shuppan, Gendaibutsuri Saizensen* **1** (2000), 49-115 (Japanese).

[12] H. Hayakawa: *Sanitsu ryushikeino rikigaku. Iwanamishoten*, 2003 (Japanese).

[13] K. Hutter and K. R. Rajagopal: *On flows of granular materials.* Continuum. Mech. Thermodyn. **6** (1994), 81-139.

[14] S. Itoh, N. Tanaka, A. Tani: *The initial value problem for the Navier–Stokes equations with general slip boundary condition in Hölder space.* J. Math. Fluid Mech. **5** (2003), 275-301.

[15] S. Itoh, A. Tani: *The initial value problem for the non-homogeneous Navier–Stokes equations with general slip boundary condition.* Proc. Roy. Soc. Edinburgh Sect. A, **130** (2000), 827-835.

[16] J. T. Jenkins: *Static equilibrium of granular materials.* J. Appl. Mech. **42** (1975), 603-606.

[17] K. I. Kanatani: *A micropolar continuum theory for the flow of granular materials.* Int. J. Engng. Sci. **17** (1979), 419-432.

[18] D. J. Kortweg: *Sur la forme que prenent les équations du mouvements des fluides si l'on tient compte des forces capilaires causées par des variations de densité considérables mains continues et sur la théorie de la capollarité dans l'hypothése d'une varation continue de la densité.* Arch. Néerl. Sci. Exactes Nat. Ser. II, **6** (1901), 1-24 (French).

[19] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva: Linear and Quasi-linear Equations of Parabolic Type. *Amer. Math. Soc., Providence, R. I.,* 1968.

[20] J. Málek and K. R. Rajagopal: *Mathematical Issues Concerning the Navier–Stokes Equations and Some of Its Generalizations.* Handbook of Differential Equations: Evolutionary Equations. Vol. II, North-Holland, Amsterdam, 371-459 (2005).

[21] J. Málek and K. R. Rajagopal: *On the modeling of inhomogeneous incompressible fluid-like bodies.* Mech. Mat. **38** (2006), 233-242.

[22] J. Málek and K. R. Rajagopal: *Incompressible rate type fluids with pressure and shear-rate dependent material moduli.* Nonlinear Analysis: Real World Applications **8** (2007), 156-164.

[23] M. Massoudi and E. J. Boyle: *A review of theories for flowing granular materials with applications to fluidized beds and solids transport.* U. S. Department of energy Report, DOE/PETC/TR-91/8 (1991).

[24] N. Nakano and A. Tani: *An initial-boundary value problem for motion of inhomogeneous incompressible fluid-like bodies.* To appear in Adv. Math. Sci. Appl..

[25] N. Nakano and A. Tani: *Navier's slip problem for motion of inhomogeneous incompressible fluid-like bodies.* To appear in J. Math. Fluid Mech., published online: 19th Sep. 2009.

[26] C. L. M. H. Navier, *Mémoir sur les lois du mouvement des fluides*, Mem. Acad. R. Sci. Paris **6**, (1823), 389-416 (French).

[27] K. R. Rajagopal: *Multiple natural configurations in continuum mechanics.* Technical Report 6, Institute for Computational and Applied Mechanics, University of Pittsburgh, 1995.

[28] K. R. Rajagopal, W. C. Troy and M. Massoudi: *Existence of solutions to the equations governing the flow of granular materials*. Eur. J. Mech. B/Fluids **11** (1992), 265-276.

[29] K. R. Rajagopal and M. Massoudi: *A method for measuring material moduli of granular materials: flow in an orthogonal rheomer*. Topical Report U.S. Department of Energy DOE/PETC/TR-90/3 (1990).

[30] K. R. Rajagopal and A. R. Srinivasa: *A thermodynamic framework for rate type fluid models*. J. Non-Newtonian Fluid Mech. **88** (2000), 207-227.

[31] O. Reynolds: *On the dilatancy of media composed of rigid particles in contact*. Phil. Mag. Ser. 5, **50** (1885), 469.

[32] J. R. Royer, D. J. Evans, L. Oyarte, Q. Guo, E. Kapit, M. E. Möbius, S. R. Waitukaitis and H. M. Jaeger; *High-speed tracking of rupture and clustering in freely falling granular streams*. Nature **459** (2009), 1110-1113.

[33] S. Savage: *Gravity flow of cohesionless granular materials in chutes and channels*. J. Fluid Mech. **92** (1979), 53-96.

[34] L. N. Slobodetskiĭ: *Estimates in L_2 for solutions of linear elliptic and parabolic systems* I. Estimates of solutions of an elliptic system. Ser. Mat. Mekh. Astr. **7** (1960), 28-47 (Russian).

[35] V. A. Solonnikov: *A priori estimates for certain boundary value problems*. Dokl. Akad. Nauk SSSR **138** (1961), 781-784 (Russian); English transl. in Soviet Math. Dokl., **2** (1961).

[36] V. A. Solonnikov: *On general initial-boundary value problems for linear parabolic systems*. Proc. Steklov Math. Inst. **83** (1965), 1-184.

[37] V. A. Solonnikov: *Estimates for solutions of nonstationary Navier–Stokes equations.* Zap. Nauchn. Sem. LOMI **38** (1973), 153-231 (Russian); English transl. in J. Soviet Math. **8** (1977).

[38] V. A. Solonnikov: *On an initial-boundary value problem for the Stokes system arising in the study of a problem with a free boundary.* Trudy Mat. Inst. Steklov. **188** (1990), 150-188 (Russian); English transl. in Proc. Steklov Inst. Math. **1991**, Issue 3 (1991), 191-239.

[39] V. A. Solonnikov: *Solvability of the problem of evolution of a viscous incompressible fluid bounded by a free surface on a finite time interval.* St. Petersburg Math. J. **3** (1992), 189-220.

[40] V. A. Solonnikov and A. Tani: *Free boundary problem for a viscous compressible flow with a surface tension.* Carathéodory: An International Tribute, World Scientific Publ. Co., 1270-1303 (1991).

[41] A. Tani, S. Itoh and N. Tanaka: *The initial value problem for the Navier–Stokes equations with general slip boundary condtition.* Adv. Math. Sci. Appl. **4** (1994), 51-69.

[42] C. Truesdell and W. Noll: The non-linear field theories of mechanics. *Handbuch der Physik III/3*, Springer, Berlin 1965.

[43] R. C. Yalamanchili, R. Gudhe and K. R. Rajagopal: *Flow of granular materials in a vertical channel under the action of gravity.* Powder Tech. **81** (1994), 65-73.

Appendix A

Function Spaces

A.1 Anisotropic Sobolev–Slobodetskiĭ spaces

A.1.1 Definitions

We introduce the function spaces used in this paper. Let $\mathcal{G}$ be a domain in $\mathbb{R}^n$ and γ is a non-negative number. By $W_2^\gamma(\mathcal{G})$ we denote the space of functions equipped with the standard norm

$$\|u\|^2_{W_2^\gamma(\mathcal{G})} = \sum_{|\alpha|<\gamma} \|D^\alpha u\|^2_{L_2(\mathcal{G})} + \|u\|^2_{\dot{W}_2^\gamma(\mathcal{G})}, \tag{A.1.1}$$

where

$$\|u\|^2_{\dot{W}_2^\gamma(\mathcal{G})} = \sum_{|\alpha|=\gamma} \|D^\alpha u\|^2_{L_2(\mathcal{G})}$$

if γ is an integer, and

$$\|u\|^2_{\dot{W}_2^\gamma(\mathcal{G})} = \sum_{|\alpha|=[\gamma]} \int_{\mathcal{G}} \int_{\mathcal{G}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\{\gamma\}}} \mathrm{d}x\mathrm{d}y$$

if γ is not an integer. Here $[\gamma]$ is the integral part and $\{\gamma\}$ the fractional part of r, respectively. The norm in $L_2(\mathcal{G})$ is denoted by $\|f\|_{L_2(\mathcal{G})} = (\int_{\mathcal{G}} |f(x)|^2 \mathrm{d}x)^{1/2}$, $D^\alpha f = \partial^{|\alpha|} f/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$ is the generalized derivative of the function f in the distribution sense of order $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, and $\alpha =$

$(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_+^n$ is a multi-index.

The anisotropic space $W_2^{r,r/2}(\mathfrak{G}_T)$ in the cylindrical domain $\mathfrak{G}_T = \mathcal{G}\times(0,T)$ is defined by $L_2(0,T; W_2^r(\mathcal{G})) \cap L_2(\mathcal{G}; W_2^{r/2}(0,T))$, whose norm is introduced by the formula

$$\begin{aligned} \|u\|^2_{W_2^{r,r/2}(\mathfrak{G}_T)} &= \int_0^T \|u\|^2_{W_2^r(\mathcal{G})}\mathrm{d}t + \int_{\mathcal{G}} \|u\|^2_{W_2^{r/2}(0,T)}\mathrm{d}x \\ &\equiv \|u\|^2_{W_2^{r,0}(\mathfrak{G}_T)} + \|u\|^2_{W_2^{0,r/2}(\mathfrak{G}_T)}, \end{aligned}$$

where $W_2^{r,0}(\mathfrak{G}_T) = L_2(0,T; W_2^r(\mathcal{G}))$ and $W_2^{0,r/2}(\mathfrak{G}_T) = L_2(\mathcal{G}; W_2^{r/2}(0,T))$. Similarly, the norm in $W_2^{r/2}(0,T)$ (for non-integral $r/2$) is defined by

$$\begin{aligned} \|u\|^2_{W_2^{r/2}(0,T)} &= \sum_{j=0}^{[r/2]} \left\| \frac{\mathrm{d}^j u}{\mathrm{d}t^j} \right\|^2_{L_2(0,T)} \\ &\quad + \int_0^T \mathrm{d}t \int_0^t \left| \frac{\mathrm{d}^{[r/2]} u(t)}{\mathrm{d}t^{[r/2]}} - \frac{\mathrm{d}^{[r/2]} u(t-\tau)}{\mathrm{d}t^{[r/2]}} \right|^2 \frac{\mathrm{d}\tau}{\tau^{1+2\{r/2\}}}. \end{aligned}$$

Other equivalent norms of this space are possible. For $l \in (0,1)$ we set

$$\|f\|^{(l,l/2)}_{\mathfrak{G}_T} = \left\{ \|f\|^2_{W_2^{l,l/2}(\mathfrak{G}_T)} + \frac{1}{T^l}\|f\|^2_{L_2(\mathfrak{G}_T)} \right\}^{1/2},$$

$$\begin{aligned} \|f\|^{(2+l,1+l/2)}_{\mathfrak{G}_T} = \Bigg\{ &\|f\|^2_{W_2^{2+l,1+l/2}(\mathfrak{G}_T)} + \left(\|f_t\|^{(l,l/2)}_{\mathfrak{G}_T}\right)^2 \\ &+ \sum_{|\alpha|=2} \left(\|D_x^\alpha f\|^{(l,l/2)}_{\mathfrak{G}_T}\right)^2 + \sup_{t\in(0,T)} \|f\|^2_{W_2^{1+l}(\mathcal{G})} \Bigg\}^{1/2}. \end{aligned}$$

For any finite $T > 0$ these norms are equivalent to the norms in the spaces $W_2^{l,l/2}(\mathfrak{G}_T)$ and $W_2^{2+l,1+l/2}(\mathfrak{G}_T)$, respectively. Let also

$$\|f\|^{(0,l/2)}_{\mathfrak{G}_T} = \left\{ \|f\|^2_{W_2^{0,l/2}(\mathfrak{G}_T)} + \frac{1}{T^l}\|f\|^2_{L_2(\mathfrak{G}_T)} \right\}^{1/2}.$$

If $\mathcal{G}$ is a smooth manifold (in this paper the boundary of a domain in $\mathbb{R}^3$ may play this role), then the norm in $W_2^r(\mathcal{G})$ is defined by means of local charts, *i.e.*, a partition of $\mathcal{G}$ into subsets each of which is mapped into a domain of Euclidean space where the norms of W_2^r are defined by formula (A.1.1). After this the spaces $W_2^{r,r/2}(\mathfrak{G}_T)$ on $\mathfrak{G}_T = \mathcal{G} \times (0,T)$ are introduced as indicated above.

The same symbols $W_2^r(\mathcal{G})$, $W_2^{r,r/2}(\mathfrak{G}_T)$ are used for the spaces of vector fields $\mathbf{f} = (f_1, f_2, \dots, f_n)$, *etc.* Their norms are introduced in standard form; for example,

$$\|\mathbf{f}\|^2_{W_2^r(\mathcal{G})} = \sum_{i=1}^n \|f_i\|^2_{W_2^r(\mathcal{G})}.$$

A.1.2 Well-known properties

We describe the well-known inequalities of norms in Sobolev–Slobodetskiĭ spaces (see Lemma 4.1 of [38]).

Lemma A.1.1 *For any* $f \in W_2^l(\Omega)$, g, $h \in W_2^{1+l}(\Omega)$, $\Omega \subset \mathbb{R}^3$, $l \in (1/2, 1)$

$$\|fg\|_{W_2^l(\mathcal{G})} \le c\|f\|_{W_2^l(\mathcal{G})}\|g\|_{W_2^{1+l}(\mathcal{G})}, \tag{A.1.2}$$

$$\|gh\|_{W_2^{1+l}(\mathcal{G})} \le c\|g\|_{W_2^{1+l}(\mathcal{G})}\|h\|_{W_2^{1+l}(\mathcal{G})}. \tag{A.1.3}$$

These estimates also hold in the case $n = 2$, when the index l may be replaced by $l - 1/2$.

For functions f, g depending also on $t \in (0,T)$ we obtain the inequalities

$$\|fg\|_{W_2^{l,0}(\mathfrak{G}_T)} \le c \sup_{0\le t\le T} \|g\|_{W_2^{1+l}(\mathcal{G})}\|f\|_{W_2^{l,0}(\mathfrak{G}_T)}, \tag{A.1.4}$$

$$\|fg\|_{W_2^{l,0}(\mathfrak{G}_T)} \le c \sup_{0\le t\le T} \|f\|_{W_2^{l}(\mathcal{G})}\|g\|_{W_2^{1+l,0}(\mathfrak{G}_T)}, \tag{A.1.5}$$

$$\|gh\|_{W_2^{1+l,0}(\mathfrak{G}_T)} \le c \sup_{0\le t\le T} \|g\|_{W_2^{1+l}(\mathcal{G})}\|h\|_{W_2^{1+l,0}(\mathfrak{G}_T)}. \tag{A.1.6}$$

And also for $f \in W_2^{l,l/2}(\mathfrak{G}_T)$ and $g \in W_2^{1+l}(\mathcal{G})$

$$\|fg\|^{(l,l/2)}_{\mathfrak{G}_T} \le c\|f\|^{(l,l/2)}_{\mathfrak{G}_T}\|g\|_{W_2^{1+l}(\mathcal{G})} \tag{A.1.7}$$

holds.

A.2 Weighted Sobolev–Slobodetskiĭ spaces

A.2.1 Definitions

We denote by $H_h^{r,r/2}(\mathfrak{G}_T)$, $h>0$, the space of functions $u(x,t)$ with a finite form

$$\|u\|^2_{H_h^{r,r/2}(\mathfrak{G}_T)} = \|u\|^2_{H_h^{r,0}(\mathfrak{G}_T)} + \|u\|^2_{H_h^{0,r/2}(\mathfrak{G}_T)},$$

$$\|u\|^2_{H_h^{r,0}(\mathfrak{G}_T)} = \int_0^T \mathrm{e}^{-2ht}\|u\|^2_{\dot W_2^r(\mathcal{G})}\mathrm{d}t,$$

$$\|u\|^2_{H_h^{0,r/2}(\mathfrak{G}_T)} = h^r\int_0^T \mathrm{e}^{-2ht}\|u\|^2_{L_2(\mathcal{G})}\mathrm{d}t$$

$$+\int_0^T \mathrm{e}^{-2ht}\mathrm{d}t\int_0^\infty \left\|\frac{\partial^{[r/2]}u_0(\cdot,t)}{\partial t^{[r/2]}} - \frac{\partial^{[r/2]}u_0(\cdot,t-\tau)}{\partial t^{[r/2]}}\right\|^2_{L_2(\mathcal{G})}\frac{\mathrm{d}\tau}{\tau^{1+2\{r\}}},$$

if $r/2$ is not an integer. Here, $u_0(x,t)=u(x,t)$ for $t>0$, $u_0(x,t)=0$ for $t<0$.

Remark A.2.1 For $T<\infty$, the space $H_h^{r,r/2}(\mathfrak{G}_T)$ can be identified with the subspace of $W_2^{r,r/2}(\mathfrak{G}_T)$ consisting of functions $u(x,t)$ that can be extended by zero into the domain $t<0$ without loss of smoothness. In the case $r>1$ this implies that $u\in H_h^{r,r/2}(\mathfrak{G}_T)$ satisfies

$$\left.\frac{\partial^i u}{\partial t^i}\right|_{t=0} = 0, \quad \text{for } i=0,\ldots,\left[\frac{r-1}{2}\right].$$

Equivalent norms of these spaces are also possible. In the cases $\mathfrak{G}_T = D_{+\infty}\equiv\mathbb{R}^3_+\times(0,\infty)$ or $\mathfrak{G}_T = D_\infty\equiv\mathbb{R}^2\times(0,\infty)$, it is convenient to use the following spaces: for $\gamma>0$

$$\|f\|^2_{\gamma,h,D_\infty} \equiv \int_{\mathbb{R}^2}\mathrm{d}\xi'\int_{-\infty}^{+\infty}|\hat f(\xi',h+\mathrm{i}\xi_0)|^2|r|^{2\gamma}\mathrm{d}\xi_0$$

and

$$\|f\|^2_{\gamma,h,D_{+\infty}} \equiv \sum_{j<\gamma} \int_{\mathbb{R}^2} \mathrm{d}\xi' \int_{-\infty}^{+\infty} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j \hat{f}(\xi', \cdot, h + \mathrm{i}\xi_0) \right\|^2_{L_2(\mathbb{R}_+)} |r|^{2(\gamma-j)} \mathrm{d}\xi_0$$

$$+ \int_{\mathbb{R}^2} \mathrm{d}\xi' \int_{-\infty}^{+\infty} \left\| \hat{f}(\xi', \cdot, h + \mathrm{i}\xi_0) \right\|^2_{\dot{W}_2^{\gamma}(\mathbb{R}_+)} \mathrm{d}\xi_0.$$

These norms are equivalent to the norms in $H_h^{\gamma,\gamma/2}(D_\infty)$ and $H_h^{\gamma,\gamma/2}(D_{+\infty})$ defined above, respectively (see [38]).