Preface

A normalized positive linear functional φ on a unital $*$-algebra $\mathcal{A}$ is called a state. A fundamental property of states is that the set of states forms a convex set which is called a state space, particularly, a extreme point of the state space is called a pure state. For example, A state φ on the n-by-n matrix algebra $M_n(\mathbb{C})$ is given by $\varphi(a) = tr(\rho a)$, $a \in M_n(\mathbb{C})$ by using a density matrix $\rho = \{x \in M_+(n, \mathbb{C}) \mid tr(x) = 1\}$, where $M_+(n, \mathbb{C})$ is the set of positive semi-definite matrices, i.e.

$$M_+(n, \mathbb{C}) = \{a \in M_n(\mathbb{C}) \mid \ \langle \omega, a\omega \rangle \geq 0, \ \forall \omega \in \mathbb{C}^n\}.$$

The correspondence $\varphi \mapsto \rho$ is one to one affine. Thus the state space of $M_n(\mathbb{C})$ is equivalent to the set of density matrices as a convex set. A pure state φ_p of $M_n(\mathbb{C})$ given by $\varphi(a) = \langle y, ay \rangle$, $a \in M_n(\mathbb{C})$, $\|y\| = 1$, is called a vector state.

A pair $(\mathcal{A}, \varphi)$ is called an algebraic probability space (cf. [1],[11]), or sometimes a quantum probability space (cf. [24]) or a $*$-probability space (cf. [30]). Algebraic probability spaces are generalizations of algebras of random variables, and their study leads to generalizations of various probabilistic notions, particularly through applications of powerful techniques from noncommutative geometry.

A state φ on $\mathcal{A}$ is called a tracial state if $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$. The set of tracial states on a $*$-algebra $\mathcal{A}$ forms a convex set, called the tracial state space. Determining extreme points of this convex set help us understand the tracial state space.

In this thesis, we give a survey of extreme points of the tracial state space for several coordinate algebras (polynomial algebras) of noncommutative spaces. We study extreme points of the tracial state space of noncommutative algebras in order to understand properties of noncommutative algebras through comparing the tracial state space of commutative algebras to the tracial state space of algebras deformed by complex parameters. In this thesis, we focus in particular on θ-deformed $2m$-planes $C^{alg}(\mathbb{R}^{2m}_{\theta})$, $m \in \mathbb{N}$, and noncommutative 3-spheres $C^{alg}(S^3_{\theta})$.

Ordinarily, the investigation of these problems uses operator algebra techniques for C^*-algebras or von Neumann algebras. We deliberately avoid this approach, since many useful applications of free probability theory to combinatorics and random matrix theory use purely algebraically techniques. Thus we construct and determine extreme points of tracial state spaces purely algebraically.

In Chapter 1, we introduce fundamental notions of algebraic probability theory which are used in subsequent chapters. First, we introduce $*$-algebras and their states and describe their fundamental properties. We describe tracial states and extreme points of the tracial state space, and we give a Jensen-type inequality for states on $*$-algebras as follows:

I. *(Proposition 1.31) Let $\mathcal{A}$ be a unital $*$-algebra and φ be a state on $\mathcal{A}$. For all $a \in \mathcal{A}$, we have $\varphi((a^*a)^n) \geq \varphi(a^*a)^n$, $\forall n \in \mathbb{N}$.*

This inequality will help us evaluate the moment sequence of $\alpha^*\alpha$, where $\alpha^*\alpha$ is a self-adjoint element of the noncommutative 3-sphere $C^{alg}(S^3_\theta)$.

In Chapter 2, we survey a construction of nontrivial tracial states of the even-dimensional θ-deformed plane $C^{alg}(\mathbb{R}^{2m}_\theta)$, $m \in \mathbb{N}$, $\theta = (\theta_{ij})$. The $*$-algebra $C^{alg}(\mathbb{R}^{2m}_\theta)$ corresponds to the unital $*$-algebra of complex polynomial functions on the θ-deformed $2m$-plane $\mathbb{R}^{2m}_\theta$. θ-deformation is a deformation of coordinate algebras using elements θ_{ij}, $i, j = 1, \cdots, m$, of an m-by-m skew symmetric matrix $\theta = (\theta_{ij})$ as deformation parameters, in contrast to the deformation parameter $q \in \mathbb{C} - \{0\}$ ordinarily used for quantum groups or quantum enveloping algebras. $C^{alg}(\mathbb{R}^{2m}_\theta)$ is a fundamental example of θ-deformations.

We first introduce a tracial state Ψ^0 on $C^{alg}(\mathbb{R}^{2m}_\theta)$ as a trivial extreme point of the tracial state space of $C^{alg}(\mathbb{R}^{2m}_\theta)$. We give a definition of Ψ^0 using the notion of regular monomials, and we construct nontrivial tracial states $\Psi^{i(1)\cdots i(t)}$, $1 \le i(1) < \cdots < i(t) \le m$, $1 \le t \le m$ from the viewpoint of the notion. Moreover, we see that $\Psi^{i(1)\cdots i(t)}$ generalizes naturally to a tracial state denoted $\Psi^{i(1)\cdots i(t)}_{x_1\cdots x_t}$, $x_1, \cdots, x_t > 0$.

Set $\Psi^{(2)} = \Psi^{i(1)i(2)}_{x_1x_2}$, where $1 \le i(1) < i(2) \le m$, $x_1, x_2 > 0$. Then we show the following:

II. *(Proposition 2.30) If $\theta_{i(1)i(2)}$ is irrational, then $\Psi^{(2)}$ is an extreme point of the tracial state space of $C^{alg}(\mathbb{R}^{2m}_\theta)$.*

The following is a direct consequence of Proposition 2.30.

III. *(Proposition 2.34) $\Psi^{i(1)\cdots i(t)}_{x_1\cdots x_t}$, $t \ge 2$, is an extreme point of the tracial state space of $C^{alg}(\mathbb{R}^{2m}_\theta)$ if the deformation parameters θ_{ij}, $i, j = 1, \cdots, m$, are irrational numbers satisfying $\sum_{i<j} k_{ij}\theta_{ij} \notin \mathbb{Z}_{\ge 0}$ for all integers k_{ij} such that $\sum_{i<j} |k_{ij}| \ne 0$.*

Chapter 3 is the core of the thesis. We apply the method of construction of tracial states for $C^{alg}(\mathbb{R}^{2m}_\theta)$ to noncommutative 3-spheres $C^{alg}(S^3_\theta)$, and we determine the extreme points of the tracial state space of $C^{alg}(S^3_\theta)$. We first introduce the noncommutative 3-sphere $C^{alg}(S^3_\theta)$ as a quotient algebra of the θ-deformed 4-plane $C^{alg}(\mathbb{R}^4_\theta)$. The $*$-algebra $C^{alg}(S^3_\theta)$ corresponds to the unital $*$-algebra of complex polynomials on the noncommutative 3-sphere S^3_θ. We construct a class of extreme points $\Psi^\alpha_t, \Psi^\beta_t, \Psi_x$, $t \in \mathbb{C}$, $|t| = 1$, $0 < x < 1$, of the tracial state space of $C^{alg}(S^3_\theta)$, where α and β are generators of $C^{alg}(S^3_\theta)$. Then $\Psi^\alpha_t, \Psi^\beta_t$ are characters of $C^{alg}(S^3_\theta)$.

For an algebraic probability space $(\mathcal{A}, \varphi)$, the moment sequence of a self-adjoint element $a \in \mathcal{A}$ is defined by $\varphi(a^k)$, $k = 1, 2, \cdots$. We study the moment sequence of $\alpha^*\alpha$, as the moment sequence play a significant role in the determination of the extreme points of the tracial state space of $C^{alg}(S^3_\theta)$. Specifically, we show that for an extreme points f of the tracial state space of $C^{alg}(S^3_\theta)$ with $f(\alpha^*\alpha) = x$, $x \in (0,1)$, $f(X)$ is determined by the moment sequence of $\alpha^*\alpha$ for all $X \in C^{alg}(S^3_\theta)$. Moreover, for a state Φ on $C^{alg}(S^3_\theta)$ satisfying $\Phi(\alpha^*\alpha) = 1$, the moment sequence of $\alpha^*\alpha$ are $\Phi((\alpha^*\alpha)^k) = 1$, $k \in \mathbb{N}$. In addition, $\Phi(X)$ is determined by $\{\Phi(\alpha^k) : k \in \mathbb{N}\}$ for all $X \in C^{alg}(S^3_\theta)$.

Finally, we completely determine the extreme points of the tracial state of $C^{alg}(S^3_\theta)$.

IV. *(Theorem 3.31) The set of all extreme points of the tracial state space of $C^{alg}(S^3_\theta)$ equals the set $\{\Psi_x, \Psi^\alpha_t, \Psi^\beta_t : x \in (0,1), t \in \mathbb{C}, |t| = 1\}$.*

More precisely, the extreme points of the tracial state space of $C^{alg}(S^3_\theta)$ are given by three families of states, one parametrized by the interval $(0,1)$, and the other two by the circle.

Acknowledgements

I would like to express my deep gratitude to my supervisor Professor Yoshiaki Maeda for his intensive and pleasant guidance, and I would like to express hearty gratitude to members of doctoral committee. I would also like to thank Professor Giuseppe Dito and Professor Steve Rosenberg for many discussions and valuable advices.

Contents

Chapter 1

Algebraic Probability Theory

In this chapter, we prepare some fundamental notions of algebraic probability theory so as to state the results of the thesis.

1.1 $*$-algebras

First, we define algebras over $\mathbb{C}$.

DEFINITION 1.1. *An algebra $\mathcal{A}$ over $\mathbb{C}$ is a complex vector space $\mathcal{A}$ with a $\mathbb{C}$-bilinear mapping $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying*

$$a(bc) = (ab)c, \qquad \lambda(ab) = (\lambda a)b = a(\lambda b), \qquad \forall \lambda \in \mathbb{C},\ a, b \in \mathcal{A}.$$

If $\mathcal{A}$ has a multiplicative identity element (which is called a unit) $1_{\mathcal{A}} \in \mathcal{A}$, i.e. $1_{\mathcal{A}} a = a 1_{\mathcal{A}} = a$, $\forall a \in \mathcal{A}$, then we say that $\mathcal{A}$ is a unital algebra.

In this thesis, we basically deal with unital algebras.

DEFINITION 1.2. *Let $\mathcal{A}$ be an algebra. We say that $\mathcal{A}$ is commutative if $ab = ba$, $\forall a, b \in \mathcal{A}$, otherwise $\mathcal{A}$ is noncommutative.*

LEMMA 1.3. *A unital algebra $\mathcal{A}$ has a unique unit.*

Proof. If $\mathcal{A}$ has units $1_{\mathcal{A}}, 1'_{\mathcal{A}}$, then we have

$$1_{\mathcal{A}} = 1_{\mathcal{A}}1'_{\mathcal{A}} = 1'_{\mathcal{A}}1_{\mathcal{A}} = 1'_{\mathcal{A}}.$$

$\square$

DEFINITION 1.4. *Let $\mathcal{A}, \mathcal{B}$ are unital algebras. A homomorphism between $\mathcal{A}$ and $\mathcal{B}$ is a linear map $f : \mathcal{A} \to \mathcal{B}$ satisfying*

1. $f(ab) = f(a)f(b)$, $\forall a, b \in \mathcal{A}$,
2. $f(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

If f is bijective, then f is called an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. And if there exists an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then we call $\mathcal{A}$ and $\mathcal{B}$ isomorphic.

DEFINITION 1.5. *$*$-algebra $\mathcal{A}$ is an algebra, with a map $* : \mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$ satisfying*

1. $(a^*)^* = a$,

2. $(ab)^* = b^*a^*$,

3. $(\lambda a + \mu b)^* = \overline{\lambda}a^* + \overline{\mu}b^*$, $\forall \lambda, \mu \in \mathbb{C}$, $\forall a, b \in \mathcal{A}$.

The map $*$ is called the involution, and a^* is called an adjoint element of a. If $a^* = a$, then we say that a is a self-adjoint element.

DEFINITION 1.6. *Let $\mathcal{A}$ be a $*$-algebra. An element $a \in \mathcal{A}$ is called a projection if $a = a^* = a^2$.*

DEFINITION 1.7. *Let $\mathcal{A}$ be a $*$-algebra and let $\mathcal{B}$ be a subset of $\mathcal{A}$. We say that $\mathcal{B}$ is a $*$-subalgebra of $\mathcal{A}$, if the algebraic operations are closed when algebraic operations of $\mathcal{A}$ are restricted to $\mathcal{B}$.*

DEFINITION 1.8. *Let $\mathcal{A}, \mathcal{B}$ are $*$-algebras and let f be a homomorphism between $\mathcal{A}$ and $\mathcal{B}$. If $f(a^*) = f(a)^*$, $\forall a \in \mathcal{A}$, then f is called a $*$-homomorphism. For a $*$-homomorphism f between $\mathcal{A}$ and $\mathcal{B}$, if f is bijection, then f is called a $*$-isomorphism. If there exists a $*$-isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then we call $\mathcal{A}$ and $\mathcal{B}$ $*$-isomorphic.*

DEFINITION 1.9. *Let $\mathcal{A}$ be a $*$-algebra and I be an ideal of $\mathcal{A}$. We say that I is a $*$-ideal, if I satisfies $a^* \in I$, $\forall a \in I$.*

DEFINITION 1.10. *Let $\mathcal{A}$ be a $*$-algebra and I be a $*$-ideal. The quotient*

$*$-*algebra* $\mathcal{A}/I = \{a + I : a \in \mathcal{A}\} = \{[a] : a \in \mathcal{A}\}$ *is defined by*

$$\lambda[a] = [\lambda a],$$

$$[a]^* = [a^*],$$

$$[a] + [b] = [a + b],$$

$$[a][b] = [ab], \qquad \forall \lambda \in \mathbb{C}, \quad \forall a, b \in \mathcal{A}.$$

1.2 States

Next, we define states on $*$-algebras.

DEFINITION 1.11. *Let* $\mathcal{A}$ *be a unital* $*$-*algebra and* φ *be a functional* $\varphi : \mathcal{A} \to \mathbb{C}$. *We say that* φ *is a state on* $\mathcal{A}$ *if* φ *satisfies*

1. $\varphi(\lambda a + \mu b) = \lambda\varphi(a) + \mu\varphi(b), \quad \forall \lambda, \mu \in \mathbb{C}, \forall a, b \in \mathcal{A},$

2. $\varphi(a^* a) \geq 0, \quad \forall a \in \mathcal{A},$

3. $\varphi(1_{\mathcal{A}}) = 1,$

where $1_{\mathcal{A}}$ *is a unit element of* $\mathcal{A}$.

DEFINITION 1.12. *For linear functionals* φ_1, φ_2 *on* $\mathcal{A}$ *and* $\lambda \in \mathbb{C}, a \in \mathcal{A}$, *we define addition and scalar multiplication as follows:*

$$(\varphi_1 + \varphi_2)(a) = \varphi_1(a) + \varphi_2(a),$$

$$(\lambda\varphi_1)(a) = \lambda\varphi_1(a).$$

Under the above operations, the set of all linear functionals on $\mathcal{A}$ becomes a vector space. The vector space is called the dual space, which is usually denoted as $\mathcal{A}^*$.

DEFINITION 1.13. *We denote the set of all states on $\mathcal{A}$ by $\mathfrak{S}(\mathcal{A})$. $\mathfrak{S}(\mathcal{A})$ is called the state space of $\mathcal{A}$. Clearly it holds $\mathfrak{S}(\mathcal{A}) \subset \mathcal{A}^*$.*

LEMMA 1.14. *$\mathfrak{S}(\mathcal{A})$ is a convex set. Hence*

$$\lambda\varphi_1 + (1-\lambda)\varphi_2 \in \mathfrak{S}(\mathcal{A})$$

for $\varphi_1, \varphi_2 \in \mathfrak{S}(\mathcal{A})$, $0 \leq \lambda \leq 1$.

Proof. We will prove in three steps.

First step: linearlity.

$$\begin{aligned}
&(\lambda\varphi_1 + (1-\lambda)\varphi_2)(\mu a + \nu b) = \lambda\varphi_1(\mu a + \nu b) + (1-\lambda)\varphi_2(\mu a + \nu b) \\
&= \mu\lambda\varphi_1(a) + \nu\lambda\varphi_1(b) + \mu(1-\lambda)\varphi_2(a) + \nu(1-\lambda)\varphi_2(b) \\
&= \mu(\lambda\varphi_1 + (1-\lambda)\varphi_2)(a) + \nu(\lambda\varphi_1 + (1-\lambda)\varphi_2)(b)
\end{aligned}$$

for $\mu, \nu \in \mathbb{C}$, $a, b \in \mathcal{A}$.

Second step: positivity.

$$\begin{aligned}
&(\lambda\varphi_1 + (1-\lambda)\varphi_2)(a^*a) = \lambda\varphi_1(a^*a) + (1-\lambda)\varphi_2(a^*a), \qquad \forall a \in \mathcal{A}, \\
&= \lambda\varphi_1(a^*a) + \varphi_2(a^*a) - \lambda\varphi_2(a^*a).
\end{aligned}$$

Since $0 \leq \lambda \leq 1$,

$$\lambda\varphi_1(a^*a) + \varphi_2(a^*a) - \lambda\varphi_2(a^*a) \geq 0.$$

Third step: unity.

$$\begin{aligned}(\lambda\varphi_1 + (1-\lambda)\varphi_2)(1_{\mathcal{A}}) &= \lambda\varphi_1(1_{\mathcal{A}}) + (1-\lambda)\varphi_2(1_{\mathcal{A}}) \\ &= \lambda + (1-\lambda) = 1.\end{aligned}$$

□

DEFINITION 1.15. *An extreme point of state space is called a pure state.*

The following relation is fundamental.

LEMMA 1.16. *Let $\mathcal{A}$ be a unital $*$-algebra and φ be a state on $\mathcal{A}$. Then we have*

$$\varphi(a^*) = \overline{\varphi(a)}, \qquad \forall a \in \mathcal{A}.$$

Proof. By the positivity of states,

$$\varphi\left((a+\lambda 1_{\mathcal{A}})^*(a+\lambda 1_{\mathcal{A}})\right) \geq 0, \qquad a \in \mathcal{A}, \lambda \in \mathbb{C}.$$

For

$$\begin{aligned}(a+\lambda 1_{\mathcal{A}})^*(a+\lambda 1_{\mathcal{A}}) &= \left(a^* + \overline{\lambda} 1_{\mathcal{A}}\right)(a+\lambda 1_{\mathcal{A}}) \\ &= a^*a + \lambda a^* + \overline{\lambda} a + |\lambda|^2 1_{\mathcal{A}},\end{aligned}$$

since $\varphi(a^*a) \geq 0$, $\varphi(|\lambda|^2 1_{\mathcal{A}}) \geq 0$,

$$\lambda\varphi(a^*) + \overline{\lambda}\varphi(a) \in \mathbb{R}.$$

Then we have

$$\lambda\varphi(a^*) + \overline{\lambda}\varphi(a) = \overline{\lambda\varphi(a^*) + \overline{\lambda}\varphi(a)} = \overline{\lambda\varphi(a^*)} + \lambda\overline{\varphi(a)}. \tag{1.1}$$

By taking $\lambda = 1$,

$$\varphi(a^*) + \varphi(a) = \overline{\varphi(a^*)} + \overline{\varphi(a)}. \tag{1.2}$$

And by taking $\lambda = i$,

$$\varphi(a^*) - \varphi(a) = -\overline{\varphi(a^*)} + \overline{\varphi(a)}. \tag{1.3}$$

Eventually, by taking (1.2)+(1.3), we get $\varphi(a) = \overline{\varphi(a)}$. $\square$

The following corollary holds by the proof of Lemma 1.16.

COROLLARY 1.17. *Let $\mathcal{A}$ be a unital $*$-algebra and φ be a positive linear functional on $\mathcal{A}$. Then we have*

$$\varphi(a^*) = \overline{\varphi(a)}, \qquad \forall a \in \mathcal{A}.$$

REMARK 1.18. *Corollary 1.17 means that the relation $\varphi(a^*) = \overline{\varphi(a)}$, $\forall a \in \mathcal{A}$ holds without the normalized property; $\varphi(1_{\mathcal{A}}) = 1$.*

LEMMA 1.19. *Let $\mathcal{A}$ be a unital $*$-algebra and φ be a state on $\mathcal{A}$. For $a \in \mathcal{A}$, if $\varphi(a^*a) = 0$, then $\varphi(a) = 0$.*

Proof. Suppose $\varphi(a^*a) = 0$. Since $\varphi(a^*) = \overline{\varphi(a)}$, we have

$$\begin{aligned}\varphi((\mathbf{1} + ra)^*(\mathbf{1} + ra)) &= 1 + \bar{r}\varphi(a^*) + r\varphi(a) \\ &= 1 + 2\mathrm{Re}(r\varphi(a)).\end{aligned}$$

If $\varphi(a) \neq 0$, then clearly there exists $r \in \mathbb{C}$ such that

$$1 + 2\mathrm{Re}(r\varphi(a)) < 0.$$

This contradicts the positivity of φ. □

DEFINITION 1.20. *Let $\mathcal{A}$ be a $*$-algebra and φ be a positive linear functional on $\mathcal{A}$. We say that φ is faithful if φ satisfies:*

$$\varphi(a^*a) = 0, \ \forall a \in \mathcal{A} \ \Rightarrow \ a = 0.$$

DEFINITION 1.21. *A pair $(\mathcal{A}, \varphi)$ is called an algebraic probability space. We say that $a \in \mathcal{A}$ is an algebraic random variable, and $\varphi(a)$ is the mean of a.*

DEFINITION 1.22. *Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. For $a \in \mathcal{A}$, if $a^* = a$, then a is called a real random variable or a self-adjoint random variable.*

DEFINITION 1.23. *Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. We say that $a \in \mathcal{A}$ is a unitary random variable if $a^*a = aa^* = 1_{\mathcal{A}}$, and $a \in \mathcal{A}$ is a normal random variable if $a^*a = aa^*$.*

1.3 Tracial states

DEFINITION 1.24. *Let $\mathcal{A}$ be a unital algebra. A state φ is called a tracial state if φ has the property: $\varphi(ab) = \varphi(ba)$, $\forall a, b \in \mathcal{A}$. The set of tracial states of $\mathcal{A}$ is called the tracial state space of $\mathcal{A}$.*

The following holds as with Lemma 1.14.

LEMMA 1.25. *The tracial state space of $\mathcal{A}$ forms a convex set.*

DEFINITION 1.26. *Let $\mathcal{A}$ be a unital algebra and let τ, ρ be positive linear functionals on $\mathcal{A}$. Define $\tau \leq \rho$ if $\rho - \tau$ is a positive linear functional on $\mathcal{A}$.*

The following proposition is Lemma 3.4.6 in [19].

PROPOSITION 1.27. *Let $\mathcal{A}$ be a unital $*$-algebra, let τ be a positive linear functional on $\mathcal{A}$ satisfying $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$ and ρ be an extreme point of the tracial state space of $\mathcal{A}$. If $\tau \leq \rho$, then τ is a scalar multiple of ρ.*

Proof. Since $\tau \leq \rho$,

$$0 \leq \tau(1_{\mathcal{A}}) \leq 1.$$

For $\forall a \in \mathcal{A},\ a \neq 1_{\mathcal{A}}$, when $\tau(1_{\mathcal{A}}) = 0$,

$$\begin{aligned}\tau((a + r_1 1_{\mathcal{A}})^*(a + r_1 1_{\mathcal{A}})) &= \tau\left(a^*a + |r_1|^2 1_{\mathcal{A}} + \overline{r_1}a + r_1 a^*\right) \\ &= \tau(a^*a) + \overline{r_1}\tau(a) + r_1\tau(a^*),\end{aligned}$$

by Corollary 1.17,

$$\tau((a + r_1 1_{\mathcal{A}})^*(a + r_1 1_{\mathcal{A}})) = \tau(a^*a) + 2\mathrm{Re}(\overline{r_1}\tau(a)),$$

if $\tau(a) \neq 0$, then there exists $r_1 \in \mathbb{C}$ such that

$$\tau((a + r_1 1_{\mathcal{A}})^*(a + r_1 1_{\mathcal{A}})) < 0. \tag{1.4}$$

(1.4) is contradictory to the positivity of τ, so $\tau = 0$. i.e. We have $\tau(a) = 0$ for all $a \in \mathcal{A}$. If $\tau(1_{\mathcal{A}}) = 1$, then we have

$$(\rho - \tau)(1_{\mathcal{A}}) = 0.$$

Then we get $\rho - \tau = 0$ similar to the case of $\tau(1_{\mathcal{A}}) = 0$. Thus $\rho = \tau$. If $0 < \tau(1_{\mathcal{A}}) < 1$, then we have

$$\rho = (1 - r_2)\rho_1 + r_2\rho_2,$$

but

$$r_2 = \tau(1_{\mathcal{A}}), \qquad \rho_1 = (1 - r_2)^{-1}(\rho - \tau), \qquad \rho_2 = r_2^{-1}\tau.$$

In fact,

$$\begin{aligned}\rho &= (1 - r_2)\rho_1 + r_2\rho_2 \\ &= (1 - r_2)(1 - r_2)^{-1}(\rho - \tau) + r_2 r_2^{-1}\tau \\ &= \rho.\end{aligned}$$

Note that ρ_1, ρ_2 are tracial states. ρ is an extreme point of the tracial state space of $\mathcal{A}$, so $\rho_2 = \rho$, namely, we obtain $\tau = r_2\rho$.

This completes the proof of Proposition 1.27. $\square$

The primary description of lemma in [19] is as follows:

LEMMA 1.28. *Let $\mathcal{A}$ be a unital $*$-algebra, let τ be a positive linear functional on $\mathcal{A}$ and ρ be a pure state of $\mathcal{A}$. If $\tau \leq \rho$, then τ is a scalar multiple of ρ.*

1.4 A Jensen-type inequality for states

Jensen 's inequality is known as an useful tool in many fields of mathematical sciences, and various generalizations are discussed. Particularly, from the analytic point of view, there are many research results.

Meanwhile, our main research interests are in pure algebraic aspects of noncommutative probability theory. And so, in this section, we introduce a Jensen-type inequality for states on unital $*$-algebras in an abstract algebraic setting. The inequality will be useful in the evaluation of the moment sequence of a self-adjoint element of the noncommutative 3-sphere $C^{alg}(S^3_\theta)$.

First, we recall a finite form of Jensen's inequality.

LEMMA 1.29. *([12]) Let $\{\lambda_i\}_{i=1}^n$ be positive numbers such that $\sum_{i=1}^n \lambda_i =$ 1. For a convex function f, $\{x_i\}_{i=1}^n$ are in its domain. Then we have the* *following:*

$$\sum_{i=1}^{n} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{n} \lambda_i x_i\right). \tag{1.5}$$

REMARK 1.30. *If* $n = 2$ *for* (1.5)*, we get the definition of convex function.*

In this section we give an algebraic Jensen-type inequality as follows:

PROPOSITION 1.31. *([27]) Let* $\mathcal{A}$ *be a unital* $*$*-algebra and* φ *a state on* $\mathcal{A}$*. For all* $a \in \mathcal{A}$*, we have*

$$\varphi((a^*a)^n) \geq \varphi(a^*a)^n, \qquad \forall n \in \mathbb{N}. \tag{1.6}$$

Proof. We will prove the statement by mathematical induction. If $\varphi(a^*a) = 0$, then clearly the statement holds. i.e. we consider in case of $\varphi(a^*a) > 0$. If $n = 1$, then (1.6) is trivially true. Let k be any natural number. We assume inequalities: $\varphi((a^*a)^s) \geq \varphi(a^*a)^s$ for all $s \leq k$.

First step.

When $k = 2p - 1$, $\forall p \in \mathbb{N}$, we have

$$\begin{aligned} &\varphi(((a^*a)^p - \varphi(a^*a)^p 1_{\mathcal{A}})^*((a^*a)^p - \varphi(a^*a)^p 1_{\mathcal{A}})) \\ &= \varphi((a^*a)^{2p}) + \varphi(a^*a)^{2p} - 2\varphi((a^*a)^p)\varphi(a^*a)^p \geq 0 \end{aligned} \tag{1.7}$$

by the positivity of states. However since $p \leq k$, it holds

$$\varphi((a^*a)^p) \geq \varphi(a^*a)^p.$$

Namely

$$\varphi(a^*a)^{2p} - 2\varphi((a^*a)^p)\varphi(a^*a)^p \leq -\varphi(a^*a)^{2p}. \tag{1.8}$$

It follows from (1.7) and (1.8) that

$$\varphi((a^*a)^{2p}) \geq \varphi(a^*a)^{2p}.$$

Since $2p = k + 1$, we get

$$\varphi((a^*a)^{k+1}) \geq \varphi(a^*a)^{k+1}.$$

Second step.

When $k = 2p$, $\forall p \in \mathbb{N}$, we have

$$\begin{aligned} &\varphi((a(a^*a)^p - a\varphi(a^*a)^p)^*(a(a^*a)^p - a\varphi(a^*a)^p)) \\ &= \varphi((a^*a)^{2p+1}) + \varphi(a^*a)^{2p+1} - 2\varphi((a^*a)^{p+1})\varphi(a^*a)^p \geq 0 \end{aligned} \tag{1.9}$$

by the property of the positivity of states. However since $p + 1 \leq k$, it holds

$$\varphi((a^*a)^{p+1}) \geq \varphi(a^*a)^{P+1}.$$

Namely

$$\varphi(a^*a)^{2p+1} - 2\varphi((a^*a)^{p+1})\varphi(a^*a)^p \leq -\varphi(a^*a)^{2p+1}. \tag{1.10}$$

It follows from (1.9) and (1.10) that

$$\varphi((a^*a)^{2p+1}) \geq \varphi(a^*a)^{2p+1}.$$

Since $2p+1 = k+1$, we get

$$\varphi((a^*a)^{k+1}) \geq \varphi(a^*a)^{k+1}. \tag{1.11}$$

Eventually, the statement holds for all $n \in \mathbb{N}$. $\square$

Let us recall that the set of states of an $*$-algebra $\mathcal{A}$ forms a convex set. So, the functional $\sum_{i=1}^n \lambda_i \varphi_i$ is also a state when $\{\varphi_i\}_i$ are states on $\mathcal{A}$ and $\{\lambda_i\}_i$ are real numbers such that $\sum_{i=1}^n \lambda_i = 1,\ \lambda_i \geq 0$. Then we have the following inequality for $\forall a \in \mathcal{A}$ from (1.6)

$$\sum_{i=1}^n \lambda_i \varphi_i\left((a^*a)^m\right) \geq \left(\sum_{i=1}^n \lambda_i \varphi_i(a^*a)\right)^m, \qquad \forall m \in \mathbb{N}. \tag{1.12}$$

If each φ_i is multiplicative and putting $\varphi_i(a^*a) = x_i$, then (1.12) is represented as follows.

$$\sum_{i=1}^n \lambda_i x_i^m \geq \left(\sum_{i=1}^n \lambda_i x_i\right)^m, \qquad x_i \geq 0. \tag{1.13}$$

The inequality (1.13) is a spacial case[1] of a finite form of Jensen's inequality as described previously. In this meaning, (1.6) can be interpreted as a Jensen-type.

1.5 A generalized Cauchy-Schwarz inequality

In this section, we show that (1.6) gives a generalization of the Cauchy-Schwarz inequality for functionals. First, we recall the standard Cauthy-

[1] $f(x) = x^m$ in (1.5).

Schwarz inequality for functionals.

PROPOSITION 1.32. *Let $\mathcal{A}$ be a unital $*$-algebra and φ be a state on $\mathcal{A}$. For $\forall a, b \in \mathcal{A}$ we have*

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b). \tag{1.14}$$

Proof. Suppose that $\lambda = xe^{i\theta}$, $\forall x \in \mathbb{R} - \{0\}$ with the property:

$$e^{i\theta}\varphi(b^*a) = |\varphi(b^*a)|\,.$$

We have

$$\begin{aligned}\varphi\left((\lambda a + b)^*(\lambda a + b)\right) &= |\lambda|^2 \varphi(a^*a) + \lambda\varphi(b^*a) + \overline{\lambda}\varphi(a^*b) + \varphi(b^*b) \\ &= x^2\varphi(a^*a) + 2x\,|\varphi(b^*a)| + \varphi(b^*b) \geq 0.\end{aligned}$$

If $\varphi(a^*a) \neq 0$, then we have

$$|\varphi(b^*a)|^2 - \varphi(a^*a)\varphi(b^*b) \leq 0,$$

from the discriminant of quadratic polynomial. So (1.14) holds. Meanwhile, if $\varphi(a^*a) = 0$, we have

$$2x\,|\varphi(b^*a)| + \varphi(b^*b) \geq 0,$$

but x is arbitrary, so

$$|\varphi(b^*a)| = 0.$$

Hence (1.14) holds. This completes the proof of Proposition 1.32. $\square$

It follows from (1.6), (1.14) that the following generalized Cauchy-Schwarz inequality:

PROPOSITION 1.33.

$$|\varphi(a^*b)|^{2n} \leq \varphi((a^*a)^n)\varphi((b^*b)^n), \qquad \forall n \in \mathbb{N}. \tag{1.15}$$

The case when $n = 1$, the Cauchy-Schwarz inequality recovers from (1.15).

COROLLARY 1.34. *If the case of $b = 1_{\mathcal{A}}$ in (1.15), then we have*

$$|\varphi(a)|^{2n} \leq \varphi((a^*a)^n), \qquad \forall n \in \mathbb{N}.$$

DEFINITION 1.35. *The case when $n = 1$ in $\varphi((a^*a)^n) - |\varphi(a)|^{2n}$ is called the variance of a.*

COROLLARY 1.36. *If a is a self-adjoint element, i.e. $a^* = a$, then we have*

$$\varphi(a)^{2n} \leq \varphi(a^{2n}), \qquad \forall n \in \mathbb{N}.$$

PROPOSITION 1.37. *Let $\forall(z_1, \cdots, z_n), (w_1, \cdots, w_n) \in \mathbb{C}^n$ and $\forall m \in \mathbb{N}$. Then we have*

$$\left|\sum_{i=1}^{n} \alpha_i \overline{z}_i w_i\right|^{2m} \leq \sum_{i=1}^{n} \alpha_i |z_i|^{2m} \sum_{i=1}^{n} \alpha_i |w_i|^{2m} \tag{1.16}$$

where $\sum_{i=1}^{n} \alpha_i = 1, \ \alpha_i \geq 0$.

Proof. An n-dimensional complex Euclidean space $\mathbb{C}^n$ becomes $*$-algebra define by

$$\begin{aligned} z+w &= (z_1+w_1,\cdots,z_n+w_n), \\ \lambda z &= (\lambda z_1,\cdots,\lambda z_n), \qquad\qquad \forall\lambda\in\mathbb{C}, \\ zw &= (z_1w_1,\cdots,z_nw_n), \\ z^* &= (\overline{z}_1,\cdots,\overline{z}_n). \end{aligned}$$

Then φ on $*$-algebra $\mathbb{C}^n$ is described as follows:

$$\varphi(z)=\sum_{i=1}^n \alpha_i z_i, \qquad\qquad \sum_{i=1}^n \alpha_i = 1,\ \alpha_i\geq 0. \tag{1.17}$$

Substituting z, w, φ into (1.15), we obtain (1.16). □

If the case of $\alpha_i = 1/n$ $(i = 1,\cdots,n)$ is considered, then the following inequality holds.

COROLLARY 1.38.

$$\frac{1}{n^{2m-2}}\left|\sum_{i=1}^n \overline{z}_i w_i\right|^{2m} \leq \sum_{i=1}^n |z_i|^{2m}\sum_{i=1}^n |w_i|^{2m}. \tag{1.18}$$

The case when $m = 1$ in (1.18), the Cauchy-Schwarz inequality in $\mathbb{C}^n$ with the standard inner product recovers.

An interesting problem will be to generalize other inequalies relate to probability theory, for example, the Minkowski inequality.

1.6 A generalized Jensen-type inequality

In this section, we describe a generalized Jensen-type inequality [28].

PROPOSITION 1.39. *Let $\mathcal{A}$ be a $*$-algebra and let φ be a state on $\mathcal{A}$. Then, we have*

$$\varphi((a^*a+b^*b)^n) \geq \varphi(a^*a+b^*b)^n \tag{1.19}$$

for $\forall a, b \in \mathcal{A}$, $\forall n \in \mathbb{N}$.

Proof. We prove the statement by induction on n. If $n=1$, (1.19) trivially holds. Let k be any natural number. We assume that the following inequality holds for any natural number $s \leq k$.

$$\varphi((a^*a+b^*b)^s) \geq \varphi(a^*a+b^*b)^s.$$

We consider two cases whether k is odd or even.

Case 1.

If $k=2p-1$, $p \in \mathbb{N}$, then we have

$$\begin{aligned}&\varphi(((a^*a+b^*b)^p-\varphi(a^*a+b^*b)^p1_{\mathcal{A}})^*((a^*a+b^*b)^p-\varphi(a^*a+b^*b)^p1_{\mathcal{A}}))\\ &\quad=\varphi((a^*a+b^*b)^{2p})+\varphi(a^*a+b^*b)^{2p}-2\varphi((a^*a+b^*b)^p)\varphi(a^*a+b^*b)^p \geq 0\end{aligned} \tag{1.20}$$

by the positivity of states. Since $p \leq k$,

$$\varphi((a^*a + b^*b)^p) \geq \varphi(a^*a + b^*b)^p. \tag{1.21}$$

(1.21) implies

$$\varphi((a^*a + b^*b)^p)\varphi(a^*a + b^*b)^p \geq \varphi(a^*a + b^*b)^{2p}. \tag{1.22}$$

So, it follows from (1.20),(1.22) that

$$\varphi((a^*a + b^*b)^{2p}) \geq \varphi(a^*a + b^*b)^{2p}.$$

Since $k = 2p - 1$, we get

$$\varphi((a^*a + b^*b)^{k+1}) \geq \varphi(a^*a + b^*b)^{k+1}.$$

Case 2.

If $k = 2p$, $p \in \mathbb{N}$, then we have

$$\begin{aligned}
&\varphi((a(a^*a + b^*b)^p - a\varphi(a^*a + b^*b)^p)^*(a(a^*a + b^*b)^p - a\varphi(a^*a + b^*b)^p)) \\
&\quad = \varphi((a^*a + b^*b)^p a^*a(a^*a + b^*b)^p) + \varphi(a^*a)\varphi(a^*a + b^*b)^{2p} \\
&\quad - \varphi((a^*a + b^*b)^p a^*a)\varphi(a^*a + b^*b)^p - \varphi(a^*a(a^*a + b^*b)^p)\varphi(a^*a + b^*b)^p \geq 0,
\end{aligned} \tag{1.23}$$

and

$$\begin{aligned}
&\varphi((b(a^*a+b^*b)^p-b\varphi(a^*a+b^*b)^p)^*(b(a^*a+b^*b)^p-b\varphi(a^*a+b^*b)^p)) \\
&\quad=\varphi((a^*a+b^*b)^pb^*b(a^*a+b^*b)^p)+\varphi(b^*b)\varphi(a^*a+b^*b)^{2p} \\
&\quad-\varphi((a^*a+b^*b)^pb^*b)\varphi(a^*a+b^*b)^p-\varphi(b^*b(a^*a+b^*b)^p)\varphi(a^*a+b^*b)^p\geq 0
\end{aligned} \tag{1.24}$$

by the positivity of states. Then it follows from (1.23),(1.24) and the linearity of states that

$$\varphi((a^*a+b^*b)^{2p+1})+\varphi(a^*a+b^*b)^{2p+1}-2\varphi((a^*a+b^*b)^{p+1})\varphi(a^*a+b^*b)^p\geq 0. \tag{1.25}$$

Since $p+1\leq k$,

$$\varphi((a^*a+b^*b)^{p+1})\geq\varphi(a^*a+b^*b)^{p+1}.$$

So

$$\varphi((a^*a+b^*b)^{p+1})\varphi(a^*a+b^*b)^p\geq\varphi(a^*a+b^*b)^{2p+1}. \tag{1.26}$$

Thus, by (1.25),(1.26),

$$\varphi((a^*a+b^*b)^{2p+1})\geq\varphi(a^*a+b^*b)^{2p+1}.$$

Since $k=2p$, we get

$$\varphi((a^*a+b^*b)^{k+1})\geq\varphi(a^*a+b^*b)^{k+1}.$$

By Case 1 and Case 2, we obtain (1.19) for $\forall n\in\mathbb{N}$. $\square$

In (1.19), if $b^*b = 0$, then we get a Jensen type inequality $\varphi((a^*a)^n) \geq \varphi(a^*a)^n$, $\forall n \in \mathbb{N}$. So, the inequality (1.19) can be interpreted as a generalized Jensen-type.

Moreover, Proposition 1.39 can be more generalized straightforwardly as follows:

PROPOSITION 1.40. *Let $\mathcal{A}$ be a unital $*$-algebra and φ be a state on $\mathcal{A}$. Then we have*

$$\varphi((a_1^*a_1 + \cdots + a_m^*a_m)^n) \geq \varphi(a_1^*a_1 + \cdots + a_m^*a_m)^n,$$
$$\forall a_i \in \mathcal{A},\ \ i = 1, \cdots, m,\ \ \forall m, n \in \mathbb{N}.$$

Proof. The proof is same as Proposition 1.39. □

Chapter 2

θ-deformed $2m$-planes

A well-known deformation method for function algebras is q-deformation using one parameter $q \in \mathbb{C} - \{0\}$. Quantum groups (cf. [9], [13], [14]) or quantum enveloping algebras are fundamental examples of q-deformations.

In contrast, θ-deformation is a deformation of coordinate algebras using elements θ_{ij} of an anti-symmetric real-valued matrix $\theta = (\theta_{ij})$ as deformation parameters, which was studied by Connes and Dubois-Violette in [4]. The C^*-algebra $C(T^m_\theta)$, $\forall m \in \mathbb{N}$ (cf. [32]) corresponds to the algebra of continuous functions on the noncommutative torus T^m_θ, which is well-known as a fundamental example of θ-deformations.

In this chapter, we restrict our attention to the θ-defomed $2m$-planes $C^{alg}(\mathbb{R}^{2m}_\theta)$, $\forall m \in \mathbb{N}$. The $*$-algebra $C^{alg}(\mathbb{R}^{2m}_\theta)$ corresponds to the unital $*$-algebra of polynomial functions on the θ-deformed $2m$-plane $\mathbb{R}^{2m}_\theta$. The purpose of this chapter is to construct nontrivial tracial states on θ-defomed $2m$-planes $C^{alg}(\mathbb{R}^{2m}_\theta)$.

NOTE 2.1. *In Chapter 2, we use* $^-$ *instead of* $*$*-operation in consideration of the simplification of the description.*

2.1 θ-deformed $2m$-planes

We begin by recalling the definition of the θ-deformed $2m$-plane $C^{alg}(\mathbb{R}^{2m}_\theta)$.

DEFINITION 2.2. *Let* $C^{alg}(\mathbb{R}^{2m}_\theta)$ *be the unital* $*$*-algebra generated by* m *elements* z^i, $i = 1, \cdots, m$, *with relations:*

$$\begin{aligned} z^i z^j &= \lambda^{ij} z^j z^i, \\ \bar{z}^i \bar{z}^j &= \lambda^{ij} \bar{z}^j \bar{z}^i, \\ z^i \bar{z}^j &= \lambda^{ji} \bar{z}^j z^i, \qquad\qquad 1 \leq i, j \leq m. \end{aligned} \tag{2.1}$$

Here λ^{ij} *is defined as* $\lambda^{ij} = e^{2\pi i \theta_{ij}} = \overline{\lambda^{ji}}$, *where* $\theta = (\theta_{ij})$ *is an anti-symmetric real-valued matrix of degree* m.

We denote the unit element of $C^{alg}(\mathbb{R}^{2m}_\theta)$ by $\mathbf{1}$. The noncommutative m-torus $C^{alg}(T^m_\theta)$, $m \in \mathbb{N}$, is defined as a quotient $*$-algebra of $C^{alg}(\mathbb{R}^{2m}_\theta)$.

DEFINITION 2.3. $C^{alg}(T^m_\theta)$ *is a quotient algebra of* $C^{alg}(\mathbb{R}^{2m}_\theta)$ *such that*

$$C^{alg}(T^m_\theta) = C^{alg}(\mathbb{R}^{2m}_\theta) / \bar{z}^1 z^1 = \mathbf{1}, \cdots, \bar{z}^m z^m = \mathbf{1}.$$

The $*$-algebra $C^{alg}(T^m_\theta)$ corresponds to the algebra of complex polynomial functions on T^m_θ.

2.2 Trivial state Ψ^0 on $C^{alg}(\mathbb{R}^{2m}_\theta)$

We define a trivial state Ψ^0 on $C^{alg}(\mathbb{R}^{2m}_\theta)$.

DEFINITION 2.4. *Let $n_1, n_1^{'}, \cdots, n_m, n_m^{'}$ be in $\mathbb{Z}_{\geq 0}$, and consider the monomial $X = (z^1)^{n_1}(\bar{z}^1)^{n_1^{'}} \cdots (z^m)^{n_m}(\bar{z}^m)^{n_m^{'}} \in C^{alg}(\mathbb{R}^{2m}_\theta)$. Ψ^0 is a linear functional defined by*

$$\Psi^0(X) = \begin{cases} 1, & \text{if } n_1 = n_1^{'} = \cdots = n_m = n_m^{'} = 0, \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

for X.

LEMMA 2.5. *The functional Ψ^0 is a pure state on $C^{alg}(\mathbb{R}^{2m}_\theta)$.*

Proof. Since it is trivial that Ψ^0 is a state, we only prove that Ψ^0 is a pure state.

We assume that there exists states φ_1, φ_2 on $C^{alg}(\mathbb{R}^{2m}_\theta)$ such that

$$\Psi^0 = \lambda\varphi_1 + (1-\lambda)\varphi_2, \qquad 0 < \lambda < 1.$$

Let a be a monomial of $C^{alg}(\mathbb{R}^{2m}_\theta)$ such that $a \neq \mathbf{1}$. It suffices to prove that $\varphi_1(a) = \varphi_2(a) = 0$ for that Ψ^0 is a pure state. We have

$$0 = \lambda\varphi_1(\overline{a}a) + (1-\lambda)\varphi_2(\overline{a}a),$$

but φ_1, φ_2 are states (i.e. $\varphi_1(\overline{a}a), \varphi_2(\overline{a}a) \geq 0$), so

$$\varphi_1(\overline{a}a) = \varphi_2(\overline{a}a) = 0.$$

By Lemma 1.19, we get

$$\varphi_1(a) = \varphi_2(a) = 0.$$

Thus, we see that Ψ^0 is a pure state on $C^{alg}(\mathbb{R}^{2m}_\theta)$. $\square$

We prepare some notions. Let $t, i(1), \cdots, i(t)$ be in $\mathbb{N}$ such that $1 \leq i(1) < \cdots < i(t) \leq m$, $1 \leq t \leq m$.

DEFINITION 2.6. *Let $T^{i(1)\cdots i(t)}$ be the set of monomials formed by generators $\mathbf{1}, z^{i(1)}, \bar{z}^{i(1)}, \cdots, z^{i(t)}, \bar{z}^{i(t)} \in C^{alg}(\mathbb{R}^{2m}_\theta)$. Particularly, we denote the set $\{\mathbf{1}\}$ by T^0.*

EXAMPLE 2.7. $\mathbf{1}, \bar{z}^1\bar{z}^3, z^1z^2 \in T^{1\cdot 2\cdot 3}$.

DEFINITION 2.8. *We say that X is regular in $T^{i(1)\cdots i(t)}$ or simply we say that X is regular, if there exists a monomial $Y \in T^{i(1)\cdots i(t)}$ such that $X = \lambda\overline{Y}Y$, $\lambda \in \mathbb{C} - \{0\}$.*

EXAMPLE 2.9. *A monomial $X = \bar{z}^2z^1z^2\bar{z}^1$ is regular in $T^{1\cdot 2}$. In fact, if we set $Y = z^2\bar{z}^1 \in T^{1\cdot 2}$, then it holds $X = \lambda^{21}\overline{Y}Y$.*

We characterize this trivial functional Ψ^0 from a little general viewpoints. The functional Ψ^0 can be expressed as follows by using the above terms.

LEMMA 2.10.

$$\Psi^0(X) = \begin{cases} \phi(X), & \text{if } X \text{ is regular in } T^0, \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

for any monomial $X \in C^{alg}(\mathbb{R}^{2m}_\theta)$.

2.3 Generalization of Ψ^0

In this section, we generalize the funtional Ψ^0 from a viewpoint of (2.3). We consider a map $\phi : C^{alg}(\mathbb{R}^{2m}_\theta) \to \mathbb{C}$ satisfying

$$\begin{array}{lll} \phi(z^i z^j) = \lambda^{ij}, & \phi(z^j \bar{z}^i) = \lambda^{ij}, & \phi(z^i \bar{z}^j) = \lambda^{ji}, \\ \phi(\bar{z}^j \bar{z}^i) = \lambda^{ji}, & \phi(z^j z^i) = 1, & \phi(\bar{z}^j z^i) = 1, \\ \phi(\bar{z}^i \bar{z}^j) = 1, & \phi(\bar{z}^i z^j) = 1, & \phi(z^i) = 0, \\ \phi(\bar{z}^i) = 0, & \phi(\mathbf{1}) = 1, & 1 \le i \le j \le m. \end{array} \tag{2.4}$$

We put $z^1 = w_1, \bar{z}^1 = w_2, \cdots, z^m = w_{2m-1}, \bar{z}^m = w_{2m}$, and let $\iota(1), \cdots, \iota(n)$ be in $\mathbb{N}$ such that $1 \le \iota(1), \cdots, \iota(n) \le 2m$, where $\iota(1), \cdots, \iota(n)$ are allowed overlapping. Furthermore, we require that

$$\phi(w_{\iota(1)} \cdots w_{\iota(n)}) = \prod_{k<l}^{n} \phi(w_{\iota(k)} w_{\iota(l)}) \tag{2.5}$$

for the monomial $A = w_{\iota(1)} \cdots w_{\iota(n)}$ of degree more than 2, and linearlity such that

$$\phi(\lambda X + \mu Y) = \lambda\phi(X) + \mu\phi(Y), \qquad \lambda, \mu \in \mathbb{C},\ X, Y \in C^{alg}(\mathbb{R}^{2m}_\theta).$$

LEMMA 2.11. *ϕ is well-defined uniquely by (2.4) and (2.5) as a linear functional.*

Proof. It suffices to show following equalities based on (2.5).

$$\phi(w_{\iota(1)}\cdots w_{\iota(k)}(w_{2p-1}w_{2q-1}-\lambda^{pq}w_{2q-1}w_{2p-1})w_{\iota(k+1)}\cdots w_{\iota(n)})=0,$$

$$\phi(w_{\iota(1)}\cdots w_{\iota(k)}(w_{2p}w_{2q}-\lambda^{pq}w_{2q}w_{2p})w_{\iota(k+1)}\cdots w_{\iota(n)})=0,$$

$$\phi(w_{\iota(1)}\cdots w_{\iota(k)}(w_{2p}w_{2q-1}-\lambda^{qp}w_{2q-1}w_{2p})w_{\iota(k+1)}\cdots w_{\iota(n)})=0,$$

$$p,q,k\in\mathbb{N},\ 1\le p,q\le m,\ 1\le k\le n-1. \quad (2.6)$$

Note that $w_{2p-1}=z^p, w_{2p}=\bar{z}^p, w_{2q-1}=z^q, w_{2q}=\bar{z}^q$.

We show the first equation of (2.6).

$$\begin{aligned}&\phi(w_{\iota(1)}\cdots w_{\iota(k)}(w_{2p-1}w_{2q-1}-\lambda^{pq}w_{2q-1}w_{2p-1})w_{\iota(k+1)}\cdots w_{\iota(n)})\\ &=\phi(w_{\iota(1)}\cdots w_{\iota(k)}w_{2p-1}w_{2q-1}w_{\iota(k+1)}\cdots w_{\iota(n)})\\ &-\lambda^{pq}\phi(w_{\iota(1)}\cdots w_{\iota(k)}w_{2q-1}w_{2p-1}w_{\iota(k+1)}\cdots w_{\iota(n)}).\end{aligned}$$

By (2.5),

$$\begin{aligned}&=\phi(w_{2p-1}w_{2q-1})\prod_{e=1}^{k}\phi(w_{\iota(e)}w_{2p-1})\phi(w_{\iota(e)}w_{2q-1})\\ &\times\prod_{f=k+1}^{n}\phi(w_{2p-1}w_{\iota(f)})\phi(w_{2q-1}w_{\iota(f)})\times\prod_{1\le\iota(g)<\iota(g')\le n}w_{\iota(g)}w_{\iota(g')}\\ &-\lambda^{pq}\phi(w_{2q-1}w_{2p-1})\prod_{g=1}^{k}\phi(w_{\iota(g)}w_{2p-1})\phi(w_{\iota(g)}w_{2q-1})\\ &\times\prod_{h=k+1}^{n}\phi(w_{2p-1}w_{\iota(h)})\phi(w_{2q-1}w_{\iota(h)})\times\prod_{1\le\iota(g)<\iota(g')\le n}w_{\iota(g)}w_{\iota(g')},\end{aligned}$$

so, to sum up,

$$= (\phi(w_{2p-1}w_{2q-1}) - \lambda^{pq}\phi(w_{2q-1}w_{2p-1}))\prod_{e=1}^{k}\phi(w_{\iota(e)}w_{2p-1})\phi(w_{\iota(e)}w_{2q-1})$$

$$\times \prod_{f=k+1}^{n}\phi(w_{2p-1}w_{\iota(f)})\phi(w_{2q-1}w_{\iota(f)}) \times \prod_{1\le\iota(g)<\iota(g')\le n} w_{\iota(g)}w_{\iota(g')}. \qquad (2.7)$$

Since $\phi(w_{2p-1}w_{2q-1}) = \lambda^{pq}\phi(w_{2q-1}w_{2p-1})$, it holds (2.7)$= 0$.

Hence the first equation is proved. The remaining are proved similarly. So, it is clear that any such ϕ is uniquely determined as a linear functional.

$\square$

Our intention is to generalize Ψ^0 in accordance with the form of (2.3). We define a functional $\Psi^{i(1)\cdots i(t)}$ on $C^{alg}(\mathbb{R}^{2m}_\theta)$. Suppose that $n_1, n_1', \cdots, n_m, n_m' \in \mathbb{Z}_{\ge 0}$.

DEFINITION 2.12. *Let $\Psi^{i(1)\cdots i(t)}$ be the linear functional defined by*

$$\Psi^{i(1)\cdots i(t)}(X) = \begin{cases} \phi(X), & \text{if } X \text{ is regular in } T^{i(1)\cdots i(t)}, \\ 0, & \text{otherwise} \end{cases}$$

for the monomial $X = (z^1)^{n_1}(\bar{z}^1)^{n_1'}\cdots(z^m)^{n_m}(\bar{z}^m)^{n_m'} \in C^{alg}(\mathbb{R}^{2m}_\theta)$.

In the following, we denote $\Psi^{i(1)\cdots i(t)}$ simply by Ψ^i.

REMARK 2.13. *In definition 2.12, if $T^{i(1)\cdots i(t)} = T^0$, then $\Psi^{i(1)\cdots i(t)} = \Psi^0$.*

2.4 Tracial states Ψ^i

We now prove that Ψ^i is a tracial state on $C^{alg}(\mathbb{R}^{2m}_\theta)$. The following lemma is fundamental.

LEMMA 2.14. *We put $z^1 = w_1, \bar{z}^1 = w_2, \cdots, z^m = w_{2m-1}, \bar{z}^m = w_{2m}$. Suppose that $X = w_{j(1)} \cdots w_{j(k)} \in T^{i(1)\cdots i(t)}$ for $j(1), \cdots, j(k) \in \{1, \cdots, 2m\}, k \in \mathbb{N}$. Then we have*

$$\Psi^i(\overline{X}X) = 1. \tag{2.8}$$

Proof. If $X = \mathbf{1}$, then (2.8) is obvious. Therefore, we assume that $X \neq \mathbf{1}$. Let p, q be in $\mathbb{N}$ such that $1 \leq q < p \leq k$. Then we get

$$\begin{aligned}
\Psi^i(\overline{X}X) &= \Psi^i(\bar{w}_{j(k)} \cdots \bar{w}_{j(1)} w_{j(1)} \cdots w_{j(k)}) \\
&= \phi(\bar{w}_{j(k)} \cdots \bar{w}_{j(1)} w_{j(1)} \cdots w_{j(k)}) \\
&= \prod_{p \neq q} \underbrace{\phi(\bar{w}_{j(p)}\bar{w}_{j(q)})\phi(w_{j(q)}w_{j(p)})}_{1} \underbrace{\phi(\bar{w}_{j(p)}w_{j(q)})\phi(\bar{w}_{j(q)}w_{j(p)})}_{1} \\
&= 1.
\end{aligned}$$

$\square$

LEMMA 2.15. *If X and Y are regular, then $XY, \overline{X}Y$ are regular.*

Note that the converse of Lemma 2.15 is not true in general.

PROPOSITION 2.16. *If X and Y are regular, then we have*

$$\Psi^i(XY) = \Psi^i(X)\Psi^i(Y). \tag{2.9}$$

Proof. If X or Y is a scalar multiplication of $\mathbf{1}$, then (2.9) is obvious. Therefore, we assume $X, Y \neq \lambda\mathbf{1}, \lambda \in \mathbb{C} - \{0\}$. Let z^j be one of elements which forms X. Then $\bar{z}^j$ is also one of elements which forms X. Similarly, let $z^k, \bar{z}^k$ be elements which form Y. Then we obtain the following equality.

$$\phi(z^j z^k)\phi(z^j \bar{z}^k)\phi(\bar{z}^j z^k)\phi(\bar{z}^j \bar{z}^k) = 1.$$

Since the elements z^j and z^k are arbitrary, the result is given. □

In relation to Proposition 2.16, we have the following:

LEMMA 2.17. *If X is regular and Y is not regular, then*

$$\Psi^i(XY) = \Psi^i(\overline{X}Y) = 0. \tag{2.10}$$

Proof. If X is regular and Y is not regular, then $XY, \overline{X}Y$ are not regular. Namely, (2.10) is proved by Definition 2.12. □

Ψ^i has the following property.

PROPOSITION 2.18. $\Psi^i(xy) = \Psi^i(yx)$ *for* $\forall x, y \in C^{alg}(\mathbb{R}^{2m}_\theta)$.

Proof. By definition of Ψ^i, it suffices to consider the case that xy is regualr. We put $z^j = e_1, \bar{z}^j = e_2, z^k = e_3, \bar{z}^k = e_4,\ j, k = 1, \cdots, m$. Suppose that $k_1, k_2, k_3, k_4 \in \{1, \cdots, 4\}$, however $i \neq j \Rightarrow k_i \neq k_j,\ i, j = 1, \cdots, 4$. Then it is easy to see that

$$e_{k_1}e_{k_2}e_{k_3}e_{k_4} = e_{k_2}e_{k_3}e_{k_4}e_{k_1} = e_{k_3}e_{k_4}e_{k_1}e_{k_2} = e_{k_4}e_{k_1}e_{k_2}e_{k_3}.$$

Hence if $X = xy$ is regular, then it holds $xy = yx$. □

By relations of $C^{alg}(\mathbb{R}^{2m}_\theta)$, it holds the following property concerned with Ψ^i.

LEMMA 2.19. *Let x be regular and let y be a monomial in $C^{alg}(\mathbb{R}^{2m}_\theta)$. Then it holds*

$$\Psi^i(xy) = \Psi^i(yx).$$

Proof. For $z^i, \bar{z}^i, z^j$, we have

$$z^i\bar{z}^i z^j = z^j z^i \bar{z}^i.$$

Thus we get the conclusition. $\square$

DEFINITION 2.20. *Let X, Y be in $T^{i(1)\cdots i(t)}$. We denote by $X \sim Y$ if $\overline{X}Y$ is regular.*

The relation $\sim$ is an equivalence relation. We denote the equivalence class of A by $[A]$ for $A \in T^{i(1)\cdots i(t)}$. Let $\deg(A)$ denote the degree of a monomial A, and let $[A]_{\min}$ be the subset of $[A]$ such that

$$[A]_{\min} = \{x \in [A] \mid \deg(x) \leq \deg(y),\ \forall y \in [A]\}.$$

LEMMA 2.21. *Let X, Y be in $T^{i(1)\cdots i(t)}$. If $X \nsim Y$, then*

$$\Psi^i(\overline{X}Y) = 0.$$

Proof. If $X \nsim Y$, then $\overline{X}Y$ is not regular. i.e. $\Psi^i(\overline{X}Y) = 0$. $\square$

The following proposition is the core result in this section.

PROPOSITION 2.22. Ψ^i *is a positive functional.*

Proof. By Lemma 2.21 and definition of Ψ^i, it suffices to prove $\Psi^i(\overline{X}X) \geq 0$ for $X = \sum_{t=1}^k r_t x_t,\ x_p \sim x_q,\ p, q = 1, \cdots, k,\ r_1, \cdots, r_k \in \mathbb{C}$ in order for Ψ^i to be a positive functional. We can denote X by $\sum_{t=1}^k r_t' u y_t$, where $u \in [x_t]_{\min}$, $y_1, \cdots, y_k$ are regular, and $r_1', \cdots, r_k' \in \mathbb{C}$. Then it follows from Proposition 2.16 and Lemma 2.19 that

$$\begin{aligned}
\Psi^i(\overline{X}X) &= \Psi^i((\sum_{t=1}^k \bar{r}_t' \bar{y}_t \bar{u})(\sum_{t=1}^k r_t' u y_t)) \\
&= \Psi^i(\bar{u}u)(\sum_{t=1}^k \bar{r}_t' \Psi^i(\bar{y}_t))(\sum_{t=1}^k r_t' \Psi^i(y_t)) \\
&= \Psi^i(\bar{u}u)\overline{(\sum_{t=1}^k r_t' \Psi^i(y_t))}(\sum_{t=1}^k r_t' \Psi^i(y_t)) \geq 0.
\end{aligned}$$

Thus it is proved that Ψ^i is a state on $C^{alg}(\mathbb{R}_\theta^{2m})$. □

It follows from Proposition 2.18 and Proposition 2.22 that the following proposition.

PROPOSITION 2.23. Ψ^i *is a tracial state on* $C^{alg}(\mathbb{R}_\theta^{2m})$.

2.5 Generalization of Ψ^i

Tracial state Ψ^i is generalized naturally as following. We define a linear functional $\Psi_{x_1 \cdots x_t}^{i(1) \cdots i(t)}$ on $C^{alg}(\mathbb{R}_\theta^{2m})$. Suppose that $x_1, \cdots, x_t > 0,\ n_1, n_1' \cdots, n_m, n_m' \in$

$\mathbb{Z}_{\geq 0}$.

DEFINITION 2.24. *Let $\Psi^{i(1)\cdots i(t)}_{x_1\cdots x_t}$ be the linear functional defined by*

$$\Psi^{i(1)\cdots i(t)}_{x_1\cdots x_t}(X) = \begin{cases} x_1^{n_{i(1)}} \cdots x_t^{n_{i(t)}} \phi(X), & \text{if } X \text{ is regular in } T^{i(1)\cdots i(t)}, \\ 0, & \text{otherwise} \end{cases}$$

for the monomial $X = (z^1)^{n_1}(\bar{z}^1)^{n_1'} \cdots (z^m)^{n_m}(\bar{z}^m)^{n_m'} \in C^{alg}(\mathbb{R}^{2m}_\theta)$.

In the following, we denote $\Psi^{i(1)\cdots i(t)}_{x_1\cdots x_t}$ simply by Ψ^i_x. The functional Ψ^i is restored from Ψ^i_x as follows:

REMARK 2.25. *If $x_1 = \cdots = x_t = 1$, then Ψ^i_x=Ψ^i.*

We have the following:

PROPOSITION 2.26. *If X and Y are regular, then*

$$\Psi^i_x(XY) = \Psi^i_x(X)\Psi^i_x(Y).$$

Proof. Let X and Y be regular in $T^{i(1)\cdots i(t)}$. Then X, Y can be represented as follows:

$$\begin{aligned} X =& c(z^1)^{n_1}(\bar{z}^1)^{n_1} \cdots (z^m)^{n_m}(\bar{z}^m)^{n_m}, \\ Y =& c^{'}(z^1)^{k_1}(\bar{z}^1)^{k_1} \cdots (z^m)^{k_m}(\bar{z}^m)^{k_m}, \\ & \qquad c, c' \in \mathbb{C} - \{0\},\ n_1, \cdots, n_m, k_1, \cdots, k_m \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

By Definition 2.24,

$$\Psi_x^i(XY) = x_1^{n_{i(1)}} \cdots x_t^{n_{i(t)}} \Psi^i(X) \times x_1^{k_{i(1)}} \cdots x_t^{k_{i(t)}} \Psi^i(Y)$$
$$= \Psi_x^i(X)\Psi_x^i(Y).$$

□

PROPOSITION 2.27. $\Psi_x^i(xy) = \Psi_x^i(yx)$, $\forall x, y \in C^{alg}(\mathbb{R}_\theta^{2m})$.

Proof. Same as Proposition 2.16. □

Thus, we have the following:

PROPOSITION 2.28. Ψ_x^i *is a tracial state on* $C^{alg}(\mathbb{R}_\theta^{2m})$.

REMARK 2.29. *The unital* $*$*-algebra* $C^{alg}(\mathbb{R}_\theta^{2m+1})$ *is defined by adding a self-adjoint generator* z^{m+1} *to* $C^{alg}(\mathbb{R}_\theta^{2m})$ *with relations* $z^i z^{m+1} = z^{m+1} z^i$, $1 \leq i \leq m$. *We can construct tracial states on* $C^{alg}(\mathbb{R}_\theta^{2m+1})$ *as with the case of* $C^{alg}(\mathbb{R}_\theta^{2m})$.

2.6 Extreme points of the tracial state space

We would introduce extreme points of the tracial state space of $C^{alg}(\mathbb{R}_\theta^{2m})$, if deformation parameter θ_{ij}, $1 \leq i < j \leq m$, are irrational numbers. Let $\Psi^{(2)}$ be the tracial state, which is assumed to be $t = 2$ in $\Psi_{x_1 \cdots x_t}^{i(1)\cdots i(t)}$. Namely it holds $\Psi^{(2)} = \Psi_{x_1 x_2}^{i(1)i(2)}$, where $1 \leq i(1) < i(2) \leq m$, $x_1, x_2 > 0$.

PROPOSITION 2.30. *If $\theta_{i(1)i(2)}$ is an irrational number, then $\Psi^{(2)}$ is an extreme point of the tracial state space of $C^{alg}(\mathbb{R}^{2m}_\theta)$.*

Proof. We assume that there exist tracial states Ψ_1, Ψ_2 on $C^{alg}(\mathbb{R}^{2m}_\theta)$ such that

$$\Psi^{(2)} = (1-s)\Psi_1 + s\Psi_2, \qquad 0 < s < 1. \tag{2.11}$$

We prove Proposition 2.30 in three steps.

First step.

Let K be a monomial formed from the set of generators

$$\{z^1, \bar{z}^1, \cdots, z^m, \bar{z}^m\} - \{z^{i(1)}, \bar{z}^{i(1)}, z^{i(2)}, \bar{z}^{i(2)}\}.$$

Suppose that $L = \overline{K}K$. By definition of $\Psi^{(2)}$ and (2.11), we have

$$(1-s)\Psi_1(L) + s\Psi_2(L) = 0.$$

Since a state is positive,

$$\Psi_1(L) \geq 0, \qquad \Psi_2(L) \geq 0.$$

Hence

$$\Psi_1(L) = \Psi_2(L) = 0.$$

Let $\lambda_1, \lambda_2 \in \mathbb{C} - \{0\}$, but assume that $\lambda_1, \lambda_2 \neq 0$. We assume $\Psi_1(K) \neq 0$. Then we have

$$\begin{aligned}\Psi_1(\overline{(\lambda_1 \mathbf{1} + \lambda_2 K)}(\lambda_1 \mathbf{1} + \lambda_2 K)) &= |\lambda_1|^2 + |\lambda_2|^2 \Psi_1(\overline{K}K) + 2\mathrm{Re}(\overline{\lambda_1}\lambda_2 \Psi_1(K)) \\ &= |\lambda_1|^2 + |\lambda_2|^2 \Psi_1(L) + 2\mathrm{Re}(\overline{\lambda_1}\lambda_2 \Psi_1(K)) \\ &= |\lambda_1|^2 + 2\mathrm{Re}(\overline{\lambda_1}\lambda_2 \Psi_1(K)) \geq 0. \qquad (2.12)\end{aligned}$$

However, if we take

$$\lambda_2 = -\lambda_1 \frac{1}{\Psi_1(K)},$$

then we have

$$|\lambda_1|^2 + 2\mathrm{Re}(\overline{\lambda_1}\lambda_2 \Psi_1(K)) = -|\lambda_1|^2 < 0. \qquad (2.13)$$

(2.13) contradicts (2.12). Hence we get $\Psi_1(K) = 0$. As with Ψ_1, we get $\Psi_2(K) = 0$.

Second step.

Let M be regular in $T^{i(1)i(2)}$. Then we have

$$|\Psi^{(2)}(M)|^2 = \Psi^{(2)}(\overline{M}M)$$

by definition of $\Psi^{(2)}$. Therefore, there exists ω, $0 \leq \omega < 2\pi$, that satisfies

$$\begin{aligned}
&\Psi^{(2)}(\overline{(|\Psi^{(2)}(M)|\mathbf{1} + e^{i\omega}M)}(|\Psi^{(2)}(M)|\mathbf{1} + e^{i\omega}M)) \\
&\quad = \left|\Psi^{(2)}(M)\right|^2 + \Psi^{(2)}(\overline{M}M) + 2\mathrm{Re}\left(\left|\Psi^{(2)}(M)\right| \Psi^{(2)}(M)\right) \\
&\quad = 2\left|\Psi^{(2)}(M)\right|^2 + 2\mathrm{Re}\left(e^{i\omega}\left|\Psi^{(2)}(M)\right| \Psi^{(2)}(M)\right) \\
&\quad = 0. \qquad (2.14)
\end{aligned}$$

(i.e. $e^{i\omega}\Psi^{(2)}(M) = -|\Psi^{(2)}(M)|$)

We denote $|\Psi^{(2)}(M)|\mathbf{1} + e^{i\omega}M$ by S. By (2.11) and (2.14) we have

$$(1-s)\Psi_1(\overline{S}S) + s\Psi_2(\overline{S}S) = 0.$$

However, since a state is positive

$$\Psi_1(\overline{S}S) \geq 0, \qquad \Psi_2(\overline{S}S) \geq 0.$$

Hence

$$\Psi_1(\overline{S}S) = \Psi_2(\overline{S}S) = 0.$$

Let $r_1, r_2 \in \mathbb{C} - \{0\}$. Assuming that $\Psi^{(2)}(S) \neq 0$, then we have

$$\begin{aligned}
\Psi^{(2)}(\overline{(r_1\mathbf{1} + r_2S)}(r_1\mathbf{1} + r_2S)) &= |r_1|^2 + |r_2|^2\,\Psi^{(2)}(\overline{S}S) + 2\mathrm{Re}(\overline{r_1}r_2\Psi^{(2)}(S)) \\
&= |r_1|^2 + 2\mathrm{Re}(\overline{r_1}r_2\Psi^{(2)}(S)) \geq 0. \qquad (2.15)
\end{aligned}$$

However, if we take

$$r_2 = -\frac{r_1}{\Psi^{(2)}(S)},$$

then

$$|r_1|^2 + 2\mathrm{Re}(\overline{r_1} r_2 \Psi^{(2)}(S)) = -|r_1|^2 < 0. \tag{2.16}$$

(2.16) contradicts (2.15). Hence, we get

$$\Psi^{(2)}(S) = 0.$$

In the similar way, we get

$$\Psi_1(S) = \Psi_2(S) = 0.$$

Consequently, we obtain

$$\Psi^{(2)}(M) = \Psi_1(M) = \Psi_2(M).$$

Third step.

Suppose that $P, Q \in T^{i(1)i(2)}$ and P is not regular in $T^{i(1)i(2)}$. Then we have

$$\begin{aligned}
&\Psi_1((\overline{P\Psi_1(\overline{Q}Q) - P\overline{Q}Q})(P\Psi_1(\overline{Q}Q) - P\overline{Q}Q)) \\
&= \Psi_1(\overline{P}P)\Psi_1(\overline{Q}Q)^2 + \Psi_1(\overline{Q}Q\overline{P}P\overline{Q}Q) - 2\Psi_1(\overline{P}Q\overline{Q}P)\Psi_1(\overline{Q}Q) \\
&= 0.
\end{aligned} \tag{2.17}$$

By Lemma 1.19, we get the following as with the second step.

$$\Psi_1(P\overline{Q}Q) = \Psi_1(P)\Psi_1(\overline{Q}Q). \tag{2.18}$$

Considering (2.18), We see that $\Psi_1(P) = 0$ in order for Ψ_1 to be a tracial state in the case that $\theta_{i(1)i(2)}$ is an irrational number. Let m, n be in natural numbers. In fact, if $\Psi_1((z^{i(1)})^m(z^{i(2)})^n) \neq 0$, then we get

$$\Psi_1((z^{i(1)})^m(z^{i(2)})^n) \neq \Psi_1((z^{i(2)})^n(z^{i(1)})^m) \tag{2.19}$$

since $\theta_{i(1)i(2)}$ is an irrational number. On the other hand, if $\Psi_1((z^{i(1)})^m) \neq 0$, then (2.18) shows that

$$\Psi_1((z^{i(1)})^m)\Psi_1((\bar{z}^{i(2)})^n(z^{i(2)})^n) = \Psi_1((z^{i(1)})^m(\bar{z}^{i(2)})^n(z^{i(2)})^n) \neq 0.$$

Then we obtain

$$\Psi_1((z^{i(1)})^m(\bar{z}^{i(2)})^n(z^{i(2)})^n) \neq \Psi_1((z^{i(2)})^n(z^{i(1)})^m(\bar{z}^{i(2)})^n)$$

as well as (2.19). Eventually, it turns out in these cases that Ψ_1 is not tracial. This contradicts to that Ψ_1 is a tracial state. Hence we see that $\Psi_1(P) = 0$ generally. As well as Ψ_1, we see that $\Psi_2(P) = 0$.

Eventually we obtain

$$\Psi^{(2)} = \Psi_1 = \Psi_2$$

from three steps. This completes the proof. $\square$

In relation to Proposition 2.30, we have the following:

COROLLARY 2.31. *Let $\mathcal{A}^{ab}$ be the quotient of $C^{alg}(\mathbb{R}^4_\theta)$ by the two-sided ideal generated by $\bar{z}^1z^1 - a\mathbf{1}$ and $\bar{z}^2z^2 - b\mathbf{1}$, $a, b > 0$. If θ_{12} is an irrational number, then $\mathcal{A}^{ab}$ has the unique tracial state.*

REMARK 2.32. *For $\mathcal{A}^{ab}$, if $a = b = 1$, then $\mathcal{A}^{ab}$ is a noncommutative 2-torus $C^{alg}(\mathbb{T}^2_\theta)$.*

Suppose that deformation parameters θ_{ij}, $i, j = 1, \cdots, m$, of $C^{alg}(\mathbb{R}^{2m}_\theta)$ are irrational numbers and satisfy the following condition:

For any integers k_{ij}, $i, j = 1, \cdots, m$, such that $\sum_{i<j} |k_{ij}| \neq 0$, θ_{ij} satisfy $\sum_{i<j} k_{ij}\theta_{ij} \notin \mathbb{Z}_{\geq 0}$.

Then we have the following:

LEMMA 2.33. *Let x, y be monomials of $C^{alg}(\mathbb{R}^{2m}_\theta)$ which both are not scalar multiple of $\mathbf{1}$. If a monomial xy is not regular and xy is formed by two or more different generators, then $xy \neq yx$.*

Based on Lemma 2.33, the following holds.

PROPOSITION 2.34. *If $t \geq 2$ for a tracial state $\Psi^i_x = \Psi^{i(1)\cdots i(t)}_{x_1 \cdots x_t}$, then Ψ^i_x is an extreme point of the tracial state space of $C^{alg}(\mathbb{R}^{2m}_\theta)$.*

Proof. Same as Proposition 2.30. □

2.7 Pure states

We give non-trivial pure states on $C^{alg}(\mathbb{R}^{2m}_\theta)$. Suppose that $n_1, n_1' \cdots, n_m, n_m' \in \mathbb{Z}_{\geq 0}$, $t \in \mathbb{C} - \{0\}$, $k = 1, \cdots, m$.

DEFINITION 2.35. *Let Φ_t^k be the linear functional defined by*

$$\Phi_t^k(X) = \begin{cases} t^{n_k}\bar{t}^{n_k'}, & \text{if } X \in T^k, \\ 0, & \text{otherwise} \end{cases}$$

for the monomial $X = (z^1)^{n_1}(\bar{z}^1)^{n_1'} \cdots (z^m)^{n_m}(\bar{z}^m)^{n_m'} \in C^{alg}(\mathbb{R}_\theta^{2m})$

We have the following:

PROPOSITION 2.36. *Φ_t^k is a pure state on $C^{alg}(\mathbb{R}_\theta^{2m})$.*

Proof. Suppose that $\Phi_t^k = (1-s)\Phi_1 + s\Phi_2,\ 0 \le s \le 1$, where Φ_1, Φ_2 are states on $C^{alg}(\mathbb{R}_\theta^{2m})$. If $X = (z^1)^{n_1}(\bar{z}^1)^{n_1'} \cdots (z^m)^{n_m}(\bar{z}^m)^{n_m'} \in T^k$, then we have

$$\Phi_t^k((\overline{X - t^{n_k}\bar{t}^{n_k'}})(X - t^{n_k}\bar{t}^{n_k'})) = 0.$$

So, by the positivity of states and the equality $\Phi_t^k = (1-s)\Phi_1 + s\Phi_2$,

$$\Phi_1((\overline{X - t^{n_k}\bar{t}^{n_k'}})(X - t^{n_k}\bar{t}^{n_k'})) = \Phi_2((\overline{X - t^{n_k}\bar{t}^{n_k'}})(X - t^{n_k}\bar{t}^{n_k'})) = 0. \quad (2.20)$$

It follows from Lemma 1.19 and (2.20) that

$$\Phi_1(X) = \Phi_2(X) = t^{n_k}\bar{t}^{n_k'}.$$

On the other hand, if $X \notin T^k$, then $\Phi_t^k(\overline{X}X) = 0$. So we get $\Phi_1(\overline{X}X) = \Phi_2(\overline{X}X) = 0$. By Lemma 1.19, $\Phi_t^k(X) = \Phi_1(X) = \Phi_2(X) = 0$. Thus we obtain $\Phi_t^k = \Phi_1 = \Phi_2$. $\square$

Chapter 3

Noncommutative 3-spheres

In chapter 2, we studied tracial states on the θ-deformed $2m$-plane $C^{alg}(\mathbb{R}_\theta^{2m})$. In this chapter, we restrict our attention to noncommutative 3-spheres $C^{alg}(S_\theta^3)$ (cf. [4]) with irrational deformation parameters. The algebra $C^{alg}(S_\theta^3)$ corresponds to the unital $*$-algebra of polynomial functions on the noncommutative 3-sphere S_θ^3. The main result of this chapter is to determine the extreme points of the tracial state space of $C^{alg}(S_\theta^3)$ completely.

3.1 Noncommutative 3-spheres

We consider the θ-defomed $2m$-plane of dimension 4. Here we rewrite the definition.

DEFINITION 3.1. *Let $C^{alg}(\mathbb{R}_\theta^4)$ be the unital $*$-algebra generated by two* *elements α, β with relations:*

$$\alpha\beta = \lambda\beta\alpha, \qquad \alpha\beta^* = \overline{\lambda}\beta^*\alpha, \qquad \alpha^*\alpha = \alpha\alpha^*, \qquad \beta^*\beta = \beta\beta^*.$$

Here $\lambda = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} - \mathbb{Q}$.

Let I be the two sided ideal of $C^{alg}(\mathbb{R}^4_\theta)$ generated by the element $\alpha^*\alpha + \beta^*\beta - \mathbf{1}$, where $\mathbf{1}$ is the unit element in $C^{alg}(\mathbb{R}^4_\theta)$. Let π be the natural projection from $C^{alg}(\mathbb{R}^4_\theta)$ to $C^{alg}(\mathbb{R}^4_\theta)/I$:

$$\pi : \ C^{alg}(\mathbb{R}^4_\theta) \to C^{alg}(\mathbb{R}^4_\theta)/I,$$
$$X \mapsto \tilde{X}.$$

Then $C^{alg}(\mathbb{R}^4_\theta)/I$ is a unital $*$-algebra with the well-defined product $\tilde{X}\tilde{Y} := \widetilde{XY}$.

DEFINITION 3.2. *The noncommutative* 3*-sphere is* $C^{alg}(S^3_\theta) := C^{alg}(\mathbb{R}^4_\theta)/I$*.*

Thus $C^{alg}(S^3_\theta)$ is a unital $*$-algebra generated by two elements $\tilde{\alpha}, \tilde{\beta}$ with the relations

$$\tilde{\alpha}\tilde{\beta} = \lambda\tilde{\beta}\tilde{\alpha}, \qquad \tilde{\alpha}\tilde{\beta}^* = \overline{\lambda}\tilde{\beta}^*\tilde{\alpha}, \qquad \tilde{\alpha}^*\tilde{\alpha} = \tilde{\alpha}\tilde{\alpha}^*, \qquad \tilde{\beta}^*\tilde{\beta} = \tilde{\beta}\tilde{\beta}^*,$$
$$\tilde{\alpha}^*\tilde{\alpha} + \tilde{\beta}^*\tilde{\beta} = \tilde{\mathbf{1}}.$$

3.2 Extreme points of the tracial state space

In this section, we construct a class of extreme points $\Psi_x, \ 0 < x < 1$, of the tracial state space of $C^{alg}(S^3_\theta)$. First we consider a map $\phi : C^{alg}(\mathbb{R}^4_\theta) \to \mathbb{C}$

satisfying

$$
\begin{array}{lll}
\phi(\alpha\beta) = \lambda, & \phi(\beta\alpha^*) = \lambda, & \phi(\alpha\beta^*) = \overline{\lambda}, \\
\phi(\beta^*\alpha^*) = \overline{\lambda}, & \phi(\beta\alpha) = 1, & \phi(\beta^*\alpha) = 1, \\
\phi(\alpha^*\beta^*) = 1, & \phi(\alpha^*\beta) = 1, & \phi(\alpha^*\alpha) = 1, \\
\phi(\alpha\alpha^*) = 1, & \phi(\beta^*\beta) = 1, & \phi(\beta\beta^*) = 1, \\
\phi(\mathbf{1}) = 1, & \phi(\alpha) = \phi(\alpha^*) = 0, & \phi(\beta) = \phi(\beta^*) = 0.
\end{array}
\tag{3.1}
$$

We put $\alpha = w_1, \alpha^* = w_2, \beta = w_3, \beta^* = w_4$. For $i(1), \cdots, i(n) \in \mathbb{N}$ with $1 \leq i(1), \cdots, i(n) \leq 4$, we demand that ϕ satisfies

$$
\phi(w_{i(1)} \cdots w_{i(n)}) = \prod_{1 \leq k < l \leq n} \phi(w_{i(k)} w_{i(l)}) \tag{3.2}
$$

for any monomial $w_{i(1)} \cdots w_{i(n)}$ of degree more than two, and that ϕ be linear:

$$
\phi(\lambda X + \mu Y) = \lambda\phi(X) + \mu\phi(Y), \qquad \lambda, \mu \in \mathbb{C}, \qquad X, Y \in C^{alg}(\mathbb{R}^4_\theta). \tag{3.3}
$$

LEMMA 3.3. *ϕ is defined uniquely by* (3.1)*,* (3.2)*, and* (3.3) *as a linear functional on $C^{alg}(\mathbb{R}^4_\theta)$.*

Proof. We must check that $\phi\left(X(\alpha\beta - \lambda\beta\alpha)Y\right) = 0$ for all monomials $X, Y \in C^{alg}(\mathbb{R}^4_\theta)$, as well as the corresponding equations for the other relations in Definition 3.1. By (3.3), it suffices to show that

$$\phi(w_{i(1)} \cdots w_{i(k)}(\alpha\beta - \lambda\beta\alpha)w_{i(k+1)} \cdots w_{i(n)}) = 0,$$

$$\phi(w_{i(1)} \cdots w_{i(k)}(\alpha^*\beta - \overline{\lambda}\beta\alpha^*)w_{i(k+1)} \cdots w_{i(n)}) = 0,$$

$$k, n \in \mathbb{N},\ 1 \leq k \leq n-1. \tag{3.4}$$

For the first equation of (3.4),

$$\begin{aligned}
&\phi(w_{i(1)} \cdots w_{i(k)}(\alpha\beta - \lambda\beta\alpha)w_{i(k+1)} \cdots w_{i(n)}) \\
&= \phi(w_{i(1)} \cdots w_{i(k)}\alpha\beta w_{i(k+1)} \cdots w_{i(n)}) - \lambda\phi(w_{i(1)} \cdots w_{i(k)}\beta\alpha w_{i(k+1)} \cdots w_{i(n)}) \\
&= (\phi(\alpha\beta) - \lambda\phi(\beta\alpha)) \prod_{e=1}^{k} \phi(w_{i(e)}\alpha)\phi(w_{i(e)}\beta) \prod_{f=k+1}^{n} \phi(\alpha w_{i(f)})\phi(\beta w_{i(f)}) \\
&\times \prod_{1 \leq g < g' \leq n} \phi(w_{i(g)} w_{i(g')}) = 0
\end{aligned}$$

by (3.1) and (3.2). The second equation is proved similarly. It is clear by (3.2) that any such ϕ is unique. □

Fix $x \in (0,1)$ and let $n_1, n_1', n_2, n_2' \in \mathbb{Z}_{\geq 0} = \{0\} \cup \mathbb{N}$.

DEFINITION 3.4. *Let Ψ_x be the linear functional on $C^{alg}(S^3_\theta)$ defined by*

$$\Psi_x(\tilde{X}) = \begin{cases} x^{n_1}(1-x)^{n_2}\phi(X), & \textit{if } n_1 = n_1',\ n_2 = n_2', \\ 0, & \textit{otherwise} \end{cases}$$

for a monomial $\tilde{X} = \tilde{\alpha}^{*n_1}\tilde{\alpha}^{n_1'}\tilde{\beta}^{*n_2}\tilde{\beta}^{n_2'} \in C^{alg}(S^3_\theta)$.

To justify the definition, we check that Ψ_x is well-defined by proving

$$\Psi_x(\tilde{X}(\tilde{\alpha}^*\tilde{\alpha} + \tilde{\beta}^*\tilde{\beta} - \tilde{\mathbf{1}})\tilde{Y}) = 0$$

for $\tilde{X} = \tilde{\alpha}^{*n_1}\tilde{\alpha}^{n_1'}\tilde{\beta}^{*n_2}\tilde{\beta}^{n_2'},\ \tilde{Y} = \tilde{\alpha}^{*n_3}\tilde{\alpha}^{n_3'}\tilde{\beta}^{*n_4}\tilde{\beta}^{n_4'} \in C^{alg}(S^3_\theta)$. From Definition 3.4, it suffices to consider the case $n_1 = n_1',\ n_2 = n_2',\ n_3 = n_3',\ n_4 = n_4'$.

Since $\beta^*\beta\alpha^*\alpha = \alpha^*\alpha\beta^*\beta$, we have

$$\begin{aligned}\Psi_x(\tilde{X}(\tilde{\alpha}^*\tilde{\alpha} + \tilde{\beta}^*\tilde{\beta} - \tilde{\mathbf{1}})\tilde{Y}) =&\Psi_x(\tilde{\alpha}^{*n_1+n_3+1}\tilde{\alpha}^{n_1+n_3+1}\tilde{\beta}^{*n_2+n_4}\tilde{\beta}^{n_2+n_4})\\ &+\Psi_x(\tilde{\alpha}^{*n_1+n_3}\tilde{\alpha}^{n_1+n_3}\tilde{\beta}^{*n_2+n_4+1}\tilde{\beta}^{n_2+n_4+1})\\ &-\Psi_x(\tilde{\alpha}^{*n_1+n_3}\tilde{\alpha}^{n_1+n_3}\tilde{\beta}^{*n_2+n_4}\tilde{\beta}^{n_2+n_4})\end{aligned}$$

$$\begin{aligned}&=x^{n_1+n_3+1}(1-x)^{n_2+n_4}\phi(\alpha^{*n_1+n_3+1}\alpha^{n_1+n_3+1}\beta^{*n_2+n_4}\beta^{n_2+n_4})\\ &+x^{n_1+n_3}(1-x)^{n_2+n_4+1}\phi(\alpha^{*n_1+n_3}\alpha^{n_1+n_3}\beta^{*n_2+n_4+1}\beta^{n_2+n_4+1})\\ &-x^{n_1+n_3}(1-x)^{n_2+n_4}\phi(\alpha^{*n_1+n_3}\alpha^{n_1+n_3}\beta^{*n_2+n_4}\beta^{n_2+n_4}),\end{aligned}$$

since $\phi(\alpha^*\beta^*)\phi(\alpha^*\beta)\phi(\alpha\beta^*)\phi(\alpha\beta) = 1$, we get

$$\begin{aligned}&\Psi_x(\tilde{X}(\tilde{\alpha}^*\tilde{\alpha} + \tilde{\beta}^*\tilde{\beta} - \tilde{\mathbf{1}})\tilde{Y})\\ &\quad = x^{n_1+n_3+1}(1-x)^{n_2+n_4}\phi(\alpha^{*n_1+n_3}\alpha^{n_1+n_3}\beta^{*n_2+n_4}\beta^{n_2+n_4})\\ &\quad + x^{n_1+n_3}(1-x)^{n_2+n_4+1}\phi(\alpha^{*n_1+n_3}\alpha^{n_1+n_3}\beta^{*n_2+n_4}\beta^{n_2+n_4})\end{aligned}$$

$$\begin{aligned}&- x^{n_1+n_3}(1-x)^{n_2+n_4}\phi(\alpha^{*n_1+n_3}\alpha^{n_1+n_3}\beta^{*n_2+n_4}\beta^{n_2+n_4})\\ &= (x + (1-x) - 1)\, x^{n_1+n_3}(1-x)^{n_2+n_4}\phi(\alpha^{*n_1+n_3}\alpha^{n_1+n_3}\beta^{*n_2+n_4}\beta^{n_2+n_4})\\ &= 0\end{aligned}$$

by Lemma 3.3.

LEMMA 3.5. $\Psi_x(\tilde{\mathbf{1}}) = 1.$

Proof.

$$\begin{aligned}\Psi_x(\tilde{\mathbf{1}}) &= \Psi_x(\tilde{\alpha}^*\tilde{\alpha} + \tilde{\beta}^*\tilde{\beta}) \\ &= \Psi_x(\tilde{\alpha}^*\tilde{\alpha}) + \Psi_x(\tilde{\beta}^*\tilde{\beta}) \\ &= x\phi(\alpha^*\alpha) + (1-x)\phi(\beta^*\beta) \\ &= x + (1-x) = 1\end{aligned}$$

□

NOTE 3.6. *Henceforth, we denote* $\tilde{X} \in C^{alg}(S^3_\theta)$ *just by* X *for concise description.*

In chapter 2, we found extreme points of the tracial state space of θ-deformed $2m$-planes. The proof carries over to noncommutative 3-spheres.

PROPOSITION 3.7. *For* $x \in (0,1), \Psi_x$ *is an extreme point of the tracial state space of* $C^{alg}(S^3_\theta)$.

3.3 Pure states

Next we find pure states $\Psi^\alpha_t, \Psi^\beta_t$ on $C^{alg}(S^3_\theta)$, for $t \in \mathbb{C},\ |t| = 1$ and α, β are generators of $C^{alg}(S^3_\theta)$. Suppose that $n_1, n'_1, n_2, n'_2 \in \mathbb{Z}_{\geq 0}$.

DEFINITION 3.8. *Let Ψ_t^α be the linear functional on $C^{alg}(S_\theta^3)$ defined by*

$$\Psi_t^\alpha(X) = \begin{cases} \overline{t^{n_1}} t^{n_1'}, & \text{if } n_2 = n_2' = 0, \\ 0, & \text{otherwise} \end{cases}$$

for a monomial $X = \alpha^{*n_1}\alpha^{n_1'}\beta^{*n_2}\beta^{n_2'} \in C^{alg}(S_\theta^3)$.

Let Ψ_t^β be the linear functional on $C^{alg}(S_\theta^3)$ defined by

$$\Psi_t^\beta(X) = \begin{cases} \overline{t^{n_2}} t^{n_2'}, & \text{if } n_1 = n_1' = 0, \\ 0, & \text{otherwise} \end{cases}$$

for a monomial $X = \alpha^{*n_1}\alpha^{n_1'}\beta^{*n_2}\beta^{n_2'} \in C^{alg}(S_\theta^3)$.

PROPOSITION 3.9. *$\Psi_t^\alpha, \Psi_t^\beta$ are pure states on $C^{alg}(S_\theta^3)$. Moreover, $\Psi_t^\alpha, \Psi_t^\beta$ are extreme points of the tracial state space of $C^{alg}(S_\theta^3)$.*

Proof. Proof of the first statement is same as Proposition 2.36.

We show the second statement. Since the tracial state space is a convex subset of the state space, pure states in the tracial state space are extreme points of the tracial state space. However, $\Psi_t^\alpha, \Psi_t^\beta$ are clearly tracial states.

□

3.4 Moment sequences

In this section, we study the moment sequence of $\alpha^*\alpha$. We would find the moment sequence plays a significant role for characterizing the tracial state space of $C^{alg}(S^3_\theta)$.

DEFINITION 3.10. *[1] Let* $(\mathcal{A}, \varphi)$ *be an algebraic probability space. We say that* $\varphi(a^k)$*,* $k = 1, 2, \cdots$ *, is the moment sequence of* a*, if* a *is a self-adjoint element*[1] *of* $\mathcal{A}$*. Particularly,* $\varphi(a^k)$ *is called a m-th moment of* a*.*

DEFINITION 3.11. *We say that* X *is a regular monomial in* $C^{alg}(S^3_\theta)$ *or simply we say that* X *is regular, if there exists a monomial* $Y \in C^{alg}(S^3_\theta)$ *such that* $X = \gamma Y^* Y$*,* $\gamma \in \mathbb{C} - \{0\}$*.*

PROPOSITION 3.12. *Let* Φ *be a state on* $C^{alg}(S^3_\theta)$*. For every regular monomial* $b \in C^{alg}(S^3_\theta)$*,* $\Phi(b)$ *is determined by the moment sequence of* $\alpha\alpha^*$*.*

Proof. It follows from the spherical relation $\alpha^*\alpha + \beta^*\beta = \mathbf{1}$ and the binomial theorem that

$$(\beta^*\beta)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-\alpha^*\alpha)^k.$$

There exists $r \in \mathbb{C} - \{0\}$, $n_1, n_2 \in \mathbb{Z}_{\geq 0}$ such that

$$b = r(\alpha^*\alpha)^{n_1}(\beta^*\beta)^{n_2}.$$

Then

$$\begin{aligned} \Phi(b) &= \Phi(r(\alpha^*\alpha)^{n_1}(\beta^*\beta)^{n_2}) \\ &= r\Phi((\alpha^*\alpha)^{n_1} \sum_{k=0}^{n_2} \begin{pmatrix} n_2 \\ k \end{pmatrix} (-\alpha^*\alpha)^k). \end{aligned} \tag{3.5}$$

[1] i.e. $a^* = a$.

Obviously, (3.5) is determined by the moment sequence of $\alpha^*\alpha$. $\square$

LEMMA 3.13. *For Ψ_x, the moment sequence of $\alpha^*\alpha$ is $\Psi_x((\alpha^*\alpha)^k) = x^k$. For Ψ_t^α, the moment sequence of $\alpha^*\alpha$ is $\Psi_t^\alpha((\alpha^*\alpha)^k) = 1$, and for Ψ_t^β, the moment sequence of $\beta^*\beta$ is $\Psi_t^\beta((\beta^*\beta)^k) = 1$, $k = 1, 2, \cdots$.*

Proof. The proof is immediate. $\square$

Let Φ be a state on $C^{alg}(S^3_\theta)$. It follows from (1.6) that

$$\begin{aligned}
&\Phi((\alpha^*\alpha)^k) \geq \Phi(\alpha^*\alpha)^k, \\
&\Phi((\beta^*\beta)^k) \geq \Phi(\beta^*\beta)^k, \qquad\qquad \forall k \in \mathbb{N}.
\end{aligned} \tag{3.6}$$

COROLLARY 3.14. *Let Φ be a state on $C^{alg}(S^3_\theta)$. Then*

$$\Phi(\alpha^*\alpha) \geq \Phi((\alpha^*\alpha)^2) \geq \Phi((\alpha^*\alpha)^3) \geq \cdots, \tag{3.7}$$

$$\Phi(\beta^*\beta) \geq \Phi((\beta^*\beta)^2) \geq \Phi((\beta^*\beta)^3) \geq \cdots. \tag{3.8}$$

Proof. For $k \in \mathbb{N}$, $k \geq 2$, we have

$$\begin{aligned}
\Phi((\alpha^*\alpha)^k)) &= \Phi((\alpha^*\alpha)^{k-1}\alpha^*\alpha) \\
&= \Phi((\alpha^*\alpha)^{k-1}(\mathbf{1} - \beta^*\beta)) \\
&= \Phi((\alpha^*\alpha)^{k-1}) - \Phi((\alpha^*\alpha)^{k-1}\beta^*\beta) \\
&= \Phi((\alpha^*\alpha)^{k-1}) - \Phi(\beta^*(\alpha^*\alpha)^{k-1}\beta).
\end{aligned}$$

Since $\Phi(\beta^*(\alpha^*\alpha)^{k-1}\beta) \geq 0$, we get

$$\Phi((\alpha^*\alpha)^{k-1}) \geq \Phi((\alpha^*\alpha)^k)).$$

□

The following proposition follows from (3.6),(3.7),(3.8).

PROPOSITION 3.15. *Let Φ be a state on $C^{alg}(S^3_\theta)$. If $\Phi(\alpha^*\alpha) = 1$, then the moment sequence of $\alpha^*\alpha$ is $\Phi((\alpha^*\alpha)^k) = 1$, and the moment sequence of $\beta^*\beta$ is $\Phi((\beta^*\beta)^k) = 0$, $k = 1, 2, \cdots$.*

COROLLARY 3.16. *Let Φ be a state on $C^{alg}(S^3_\theta)$. If $\Phi(\beta^*\beta) = 1$, then the moment sequence of $\beta^*\beta$ are $\Phi((\beta^*\beta)^k) = 1$, and the moment sequence of $\alpha^*\alpha$ is $\Phi((\alpha^*\alpha)^k) = 0$, $k = 1, 2, \cdots$.*

3.5 The extreme points

In this section, we determine the extreme points of the tracial state space of $C^{alg}(S^3_\theta)$.

LEMMA 3.17. *Let f be an extreme point of the tracial state space of $C^{alg}(S^3_\theta)$. Then we have*

$$f((\alpha^*\alpha)^k) = f(\alpha^*\alpha)^k, \qquad \forall k \in \mathbb{N}.$$

Proof. Let f' be the linear functional on $C^{alg}(S^3_\theta)$ defined by

$$f'(X) := f(X\alpha^*\alpha), \qquad \forall X \in C^{alg}(S^3_\theta).$$

By the spherical relation $\alpha^*\alpha + \beta^*\beta = \mathbf{1}$ and the positivity of states, $0 \leq f(\alpha^*\alpha) \leq 1$. Thus we get $0 \leq f'(\mathbf{1}) \leq 1$. In addition, since it holds $X\alpha^*\alpha = \alpha^*\alpha X$ for all $X \in C^{alg}(S^3_\theta)$,

$$f'(X^*X) = f(X^*X\alpha^*\alpha) = f(X^*\alpha^*\alpha X) \geq 0.$$

Hence f' is a positive linear functional on $C^{alg}(S^3_\theta)$. We then have

$$\begin{aligned} f(X^*X) - f'(X^*X) &= f(X^*X) - f(X^*X\alpha^*\alpha) \\ &= f(X^*X(\mathbf{1} - \alpha^*\alpha)) \\ &= f(X^*X\beta^*\beta) \\ &= f(X^*\beta^*\beta X) \geq 0. \end{aligned}$$

Thus $f \geq f'$. It follows from Proposition 1.27 that there exists $r \in [0, 1]$ such that $f' = rf$. Therefore

$$f(X\alpha^*\alpha) = f'(X) = rf(X) = rf(\mathbf{1})f(X) = f'(\mathbf{1})f(X) = f(\alpha^*\alpha)f(X). \tag{3.9}$$

For $k \in \mathbb{N}$, it follows from (3.9) that

$$f((\alpha^*\alpha)^k) = f(\alpha^*\alpha)f((\alpha^*\alpha)^{k-1}) = f(\alpha^*\alpha)^2 f((\alpha^*\alpha)^{k-2}) = \cdots = f(\alpha^*\alpha)^k.$$

□

COROLLARY 3.18. *Let f be an extreme point of the tracial state space of $C^{alg}(S^3_\theta)$. Then we have*

$$f((\beta^*\beta)^k) = f(\beta^*\beta)^k, \qquad \forall k \in \mathbb{N}.$$

The next proposition will be used in the proof of the main result[2].

PROPOSITION 3.19. *Let f be an extreme point of the tracial state space of $C^{alg}(S^3_\theta)$. If $f(\alpha^*\alpha) = x$ for some $x \in (0,1)$, then $f = \Psi_x$.*

Proof. By Lemma 3.17, we have

$$f((\alpha^*\alpha)^k) = f(\alpha^*\alpha)^k = x^k$$

for $k \in \mathbb{N}$. Let b be a regular monomial in $C^{alg}(S^3_\theta)$. From Proposition 3.12, it follows that

$$f(b) = \Psi_x(b).$$

Let P, Q be monomials of $C^{alg}(S^3_\theta)$, but assume that P is not regular. We prove that $f(P) = 0$. We have

$$\begin{aligned}
&f((Pf(Q^*Q) - PQ^*Q)^*(Pf(Q^*Q) - PQ^*Q)) \\
&= f(P^*P)f(Q^*Q)^2 + f(Q^*QP^*PQ^*Q) - 2f(P^*QQ^*P)f(Q^*Q) \\
&= 0.
\end{aligned}$$

It then follows from Lemma 1.19 that

$$f(PQ^*Q) = f(P)f(Q^*Q). \tag{3.10}$$

Let $m, n \in \mathbb{N}$. Suppose that $f(\alpha^m\beta^n) \neq 0$. We have

$$f(\alpha^m\beta^n) = \lambda^{mn} f(\beta^n\alpha^m) = e^{2mn\pi i\theta} f(\beta^n\alpha^m) \tag{3.11}$$

[2]Theorem 3.31.

by the commutation relations $\alpha\beta = \lambda\beta\alpha$, where $\lambda = e^{2\pi i\theta}$. Since θ is irrational, $e^{2mn\pi i\theta} \neq 1$, so we get $f(\alpha^m\beta^n) \neq f(\beta^n\alpha^m)$. This contradicts the tracial property of f. Hence, we get $f(\alpha^m\beta^n) = 0$. On the other hand, if $f(\alpha^m) \neq 0$, then (3.10) shows that

$$f(\alpha^m)f(\beta^{*n}\beta^n) = f(\alpha^m\beta^{*n}\beta^n) \neq 0,$$

so arguing as above, we obtain $f(\alpha^m) = 0$. Thus we see in general that $f(P) \neq 0$ contradicts the tracial property of f. Hence

$$f(P) = \Psi_x(P) = 0.$$

i.e. $f = \Psi_x$. This completes the proof of Proposition 3.19. □

COROLLARY 3.20. *Let f be an extreme point of the tracial state space of $C^{alg}(S^3_\theta)$ with $f(\alpha^*\alpha) = x$, $x \in (0,1)$. It follows from the proof of Proposition 3.19 that for all $X \in C^{alg}(S^3_\theta)$, $f(X)$ is determined by the moment sequence of $\alpha^*\alpha$.*

The next two lemmas are needed for the proof of Proposition 3.25.

LEMMA 3.21. *Let Φ be a state on $C^{alg}(S^3_\theta)$ and let $n_1, n_1', n_2, n_2' \in \mathbb{Z}_{\geq 0}$ with either $n_2 \geq 1$ or $n_2' \geq 1$. If $\Phi(\alpha^*\alpha) = 1$, then*

$$\Phi(\alpha^{*n_1}\alpha^{n_1'}\beta^{*n_2}\beta^{n_2'}) = 0.$$

Proof. It follows from Proposition 3.15 that

$$\Phi((\beta^{*n_2}\beta^{n_2'})^*(\beta^{*n_2}\beta^{n_2'})) = 0.$$

First, if $n_1 = n_1' = 0$, then $\Phi(\beta^{*n_2}\beta^{n_2'}) = 0$ by Lemma 1.19.

Next we consider the case $n_1 \neq 0$ or $n_1' \neq 0$. By Proposition 3.15,

$$\Phi((\alpha^{*n_1'}\alpha^{n_1})^*(\alpha^{*n_1'}\alpha^{n_1})) = 1.$$

It follows that if $\Phi(\alpha^{*n_1}\alpha^{n_1'}\beta^{*n_2}\beta^{n_2'}) \neq 0$, there exists $r_2 \in \mathbb{C}$ such that

$$\begin{aligned}
&\Phi((\alpha^{*n_1'}\alpha^{n_1} + r_2\beta^{*n_2}\beta^{n_2'})^*(\alpha^{*n_1'}\alpha^{n_1} + r_2\beta^{*n_2}\beta^{n_2'})) \\
&\quad = \Phi((\alpha^{*n_1'}\alpha^{n_1})^*(\alpha^{*n_1'}\alpha^{n_1})) + |r_2|^2\,\Phi((\beta^{*n_2}\beta^{n_2'})^*(\beta^{*n_2}\beta^{n_2'})) \\
&\quad + 2\mathrm{Re}\left(r_2\Phi(\alpha^{*n_1}\alpha^{n_1'}\beta^{*n_2}\beta^{n_2'})\right) \\
&\quad = 1 + 2\mathrm{Re}\left(r_2\Phi(\alpha^{*n_1}\alpha^{n_1'}\beta^{*n_2}\beta^{n_2'})\right) < 0, \qquad (3.12)
\end{aligned}$$

which contradicts the positivity of Φ. Thus we obtain

$$\Phi(\alpha^{*n_1}\alpha^{n_1'}\beta^{*n_2}\beta^{n_2'}) = 0.$$

This completes the proof of Lemma 3.21. $\square$

COROLLARY 3.22. *Let Φ be a state on $C^{alg}(S^3_\theta)$ and let $n_1, n_1', n_2, n_2' \in \mathbb{Z}_{\geq 0}$ with either $n_1 \geq 1$ or $n_1' \geq 1$. If $\Phi(\beta^*\beta) = 1$, then*

$$\Phi(\alpha^{*n_1}\alpha^{n_1'}\beta^{*n_2}\beta^{n_2'}) = 0.$$

LEMMA 3.23. *Let Φ be a state on $C^{alg}(S^3_\theta)$. If $|\Phi(\alpha^{*n_1}\alpha^{n_1'})| = 1$ for all $n_1, n_1' \in \mathbb{Z}_{\geq 0}$, then $\Phi = \Psi^\alpha_t$ where $t = \Phi(\alpha)$.*

Proof. Since $|\Phi(\alpha^{*n_1}\alpha^{n'_1})| = 1$ for all $n_1, n'_1 \in \mathbb{Z}_{\geq 0}$, there exists ξ_1, $0 \leq \xi_1 < 2\pi$, such that

$$\begin{aligned} \Phi((\alpha^* + e^{i\xi_1}\alpha)^*(\alpha^* + e^{i\xi_1}\alpha)) &= 2\Phi(\alpha^*\alpha) + 2\mathrm{Re}(e^{i\xi_1}\Phi(\alpha^2)) \\ &= 2 + 2\mathrm{Re}(e^{i\xi_1}\Phi(\alpha^2)) = 0, \end{aligned} \tag{3.13}$$

so, we have $\mathrm{Re}(e^{i\xi_1}\Phi(\alpha^2)) = -1$. Since $|\Phi(\alpha^{*n_1}\alpha^{n'_1})| = 1$ for all $n_1, n'_1 \in \mathbb{Z}_{\geq 0}$, we get

$$e^{i\xi_1}\Phi(\alpha^2) = -1. \tag{3.14}$$

By (3.13) and Lemma 1.19,

$$\Phi(\alpha^*) + e^{i\xi_1}\Phi(\alpha) = 0. \tag{3.15}$$

It follows from (3.14) that

$$e^{i\xi_1} = -\frac{1}{\Phi(\alpha^2)}. \tag{3.16}$$

By (3.15) and (3.16),

$$\Phi(\alpha^*) - \frac{1}{\Phi(\alpha^2)}\Phi(\alpha) = 0,$$

so

$$\Phi(\alpha^2) = \frac{\Phi(\alpha)}{\Phi(\alpha^*)}. \tag{3.17}$$

Since $|\Phi(\alpha^{*n_1}\alpha^{n_1'})| = 1$ for all $n_1, n_1' \in \mathbb{Z}_{\geq 0}$, we have $|\Phi(\alpha^*)| = 1$. Thus we have

$$\Phi(\alpha^2) = \Phi(\alpha)\overline{\Phi(\alpha^*)} \tag{3.18}$$

from (3.17). Hence, it follows from Lemma 1.16 and (3.18) that

$$\Phi(\alpha^2) = \Phi(\alpha)^2.$$

Let $n \in \mathbb{N},\ n \geq 2$. Assume that $\Phi(\alpha^n) = \Phi(\alpha)^n$. Since $|\Phi(\alpha^{*n_1}\alpha^{n_1'})| = 1$ for all $n_1, n_1' \in \mathbb{Z}_{\geq 0}$, there exists $\xi_2,\ 0 \leq \xi_2 < 2\pi$, such that

$$\begin{aligned}\Phi((\alpha^* + e^{i\xi_2}\alpha^n)^*(\alpha^* + e^{i\xi_2}\alpha^n)) &= \Phi(\alpha^*\alpha) + \Phi(\alpha^{*n}\alpha^n) + 2\mathrm{Re}(e^{i\xi_2}\Phi(\alpha^{n+1})) \\ &= 2 + 2\mathrm{Re}(e^{i\xi_2}\Phi(\alpha^{n+1})) = 0.\end{aligned}$$

As in (3.14) and (3.15), we have

$$e^{i\xi_2}\Phi(\alpha^{n+1}) = -1,$$
$$\Phi(\alpha^*) + e^{i\xi_2}\Phi(\alpha^n) = 0. \tag{3.19}$$

Since assumed $\Phi(\alpha^n) = \Phi(\alpha)^n$, (3.19) implies

$$e^{i\xi_2}\Phi(\alpha^{n+1}) = -1,$$
$$\Phi(\alpha^*) + e^{i\xi_2}\Phi(\alpha)^n = 0. \tag{3.20}$$

By (3.20),

$$\Phi(\alpha^{n+1}) = \frac{\Phi(\alpha)^n}{\Phi(\alpha^*)}, \tag{3.21}$$

as in (3.18), we have

$$\Phi(\alpha^{n+1}) = \Phi(\alpha)^n \overline{\Phi(\alpha^*)}. \tag{3.22}$$

From Lemma 1.16 and (3.22), we get

$$\Phi(\alpha^{n+1}) = \Phi(\alpha)^{n+1}.$$

Hence we obtain $\Phi(\alpha^k) = \Phi(\alpha)^k$ for all $k \in \mathbb{N}$. By Lemma 1.16, we also get $\Phi(\alpha^{*k}) = \Phi(\alpha^*)^k$ for all $k \in \mathbb{N}$.

Now we prove that

$$\Phi(\alpha^{*n_2}\alpha^{n_2'}) = \Psi_t^\alpha(\alpha^{*n_2}\alpha^{n_2'}) \tag{3.23}$$

for all $n_2, n_2' \in \mathbb{N}$. It suffices to consider the case $n_2 \geq n_2'$. Since $|\Phi(\alpha^{*n_1}\alpha^{n_1'})| = 1$ for all n_1, n_1', we have

$$\begin{aligned}
&\Phi((\alpha^{*n_2}\alpha^{n_2'} - \alpha^{*(n_2-n_2')})^*(\alpha^{*n_2}\alpha^{n_2'} - \alpha^{*(n_2-n_2')}))\\
&= \Phi(\alpha^{*(n_2+n_2')}\alpha^{n_2+n_2'}) + \Phi(\alpha^{*(n_2-n_2')}\alpha^{n_2-n_2'}) - 2\mathrm{Re}(\Phi(\alpha^{*n_2'}\alpha^{n_2})\Phi(\alpha^{*(n_2-n_2')}))\\
&= 2 - 2\mathrm{Re}(\Phi(\alpha^{n_2}\alpha^{*n_2})) = 0.
\end{aligned}$$

Thus implies

$$\Phi(\alpha^{*n_2}\alpha^{n_2'}) = \Phi(\alpha^{*(n_2-n_2')}) = \overline{t^{(n_2-n_2')}} = \Psi_t^\alpha(\alpha^{*n_2}\alpha^{n_2'}) \tag{3.24}$$

by Lemma 1.19, which proves (3.23). It now follows from (3.24) and Lemma 3.21 that $\Phi = \Psi_t^\alpha$. □

COROLLARY 3.24. *Let Φ be a state on $C^{alg}(S^3_\theta)$. If $|\Phi(\beta^{*n_2}\beta^{n_2'})| = 1$ for all $n_2, n_2' \in \mathbb{Z}_{\geq 0}$, then $\Phi = \Psi^\beta_t$ where $t = \Phi(\beta)$.*

PROPOSITION 3.25. *If f is an extreme point of the tracial state space of $C^{alg}(S^3_\theta)$ satisfying $f(\alpha^*\alpha) = 1$, then there exists $t \in \mathbb{C},\ |t| = 1$, such that $f = \Psi^\alpha_t$.*

Proof. If $|f(\alpha)| > 0$, there exits $l,\ 0 \leq l \leq 2\pi$, such that

$$f((\alpha + e^{il}\mathbf{1})^*(\alpha + e^{il}\mathbf{1})) = 2 + 2\mathrm{Re}(e^{-il}f(\alpha)) < 0.$$

This contradicts the positivity of states. Thus we have

$$0 \leq |f(\alpha)| \leq 1. \tag{3.25}$$

Then we have

$$\begin{aligned} f\left((\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha)^*(\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha)\right) &= \frac{1}{4} + \frac{1}{4}f(\alpha^*\alpha) + \frac{1}{2}\mathrm{Re}(f(\alpha)) \\ &= \frac{1}{2} + \frac{1}{2}\mathrm{Re}(f(\alpha)). \end{aligned} \tag{3.26}$$

By (3.25),(3.26), we obtain

$$0 \leq f\left((\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha)^*(\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha)\right) \leq 1. \tag{3.27}$$

We denote $\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha$ by P. Let f' be the linear functional defined by

$$f'(X) := f(XP^*P), \qquad \forall X \in C^{alg}(S^3_\theta).$$

By (3.27),

$$0 \leq f'(\mathbf{1}) \leq 1.$$

It follows from Lemma 3.21 that

$$f'(X^*X) = f(X^*XP^*P) = f(P^*X^*XP) \geq 0$$

for all $X \in C^{alg}(S^3_\theta)$, so f' is a positive linear functional on $C^{alg}(S^3_\theta)$. For $X \in C^{alg}(S^3_\theta)$, we have

$$\begin{aligned}
(f - f')(X^*X) &= f(X^*X) - f'(X^*X) \\
&= f(X^*X) - f(X^*XP^*P) \\
&= f(X^*X) - f\left(X^*X\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha^*\right)\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha\right)\right) \\
&= f(X^*X) - f\left(X^*X\left(\frac{1}{4}\mathbf{1} + \frac{1}{4}\alpha^*\alpha + \frac{1}{4}\alpha^* + \frac{1}{4}\alpha\right)\right) \\
&= \frac{3}{4}f(X^*X) - \frac{1}{4}f(X^*X\alpha^*\alpha) - \frac{1}{4}f(X^*X\alpha^*) - \frac{1}{4}f(X^*X\alpha).
\end{aligned} \tag{3.28}$$

By (3.9) and an assumption of Proposition 3.25,

$$f(X^*X\alpha^*\alpha) = f(X^*X)f(\alpha^*\alpha) = 1. \tag{3.29}$$

In addition, by Lemma 3.21,

$$f(X^*X\alpha) = f(\alpha X^*X). \tag{3.30}$$

So we get

$$(f - f')(X^*X) = \frac{1}{2}f(X^*X) - \frac{1}{2}\mathrm{Re}(f(X^*X\alpha)) \tag{3.31}$$

from (3.28),(3.29),(3.30). We also have

$$\begin{aligned}
f((X - X\alpha)^*(X - X\alpha)) &= f((X^* - \alpha^*X^*)(X - X\alpha)) \\
&= f(X^*X + \alpha^*X^*X\alpha - \alpha^*X^*X - X^*X\alpha) \\
&= f(X^*X) + f(\alpha^*X^*X\alpha) - 2\mathrm{Re}(f(X^*X\alpha)).
\end{aligned}$$

By Lemma 3.21,

$$f(\alpha^*X^*X\alpha) = f(X^*X\alpha^*\alpha), \tag{3.32}$$

thus we have

$$\begin{aligned}
f((X - X\alpha)^*(X - X\alpha)) &= f(X^*X) + f(\alpha^*X^*X\alpha) - 2\mathrm{Re}(f(X^*X\alpha)) \\
&= f(X^*X) + f(X^*X\alpha^*\alpha) - 2\mathrm{Re}(f(X^*X\alpha)) \\
&= f(X^*X(\mathbf{1} + \alpha^*\alpha)) - 2\mathrm{Re}(f(X^*X\alpha)) \\
&= 2f(X^*X) - f(X^*X\beta^*\beta) - 2\mathrm{Re}(f(X^*X\alpha)) \\
&\geq 0. \qquad (3.33)
\end{aligned}$$

However, by Lemma 3.21,

$$f(X^*X\beta^*\beta) = 0, \tag{3.34}$$

thus

$$(f - f')(X^*X) = \frac{1}{2}f(X^*X) - \frac{1}{2}\mathrm{Re}(f(X^*X\alpha)) \geq 0$$

from (3.31),(3.33),(3.34). Hence it holds

$$f \geq f'. \tag{3.35}$$

It follows from (3.35) and Proposition 1.27 that there exists $r \in [0,1]$ such that $f' = rf$. Then

$$f(XP^*P) = f'(X) = rf(X) = rf(\mathbf{1})f(X) = f'(\mathbf{1})f(X) = f(P^*P)f(X). \tag{3.36}$$

Then we have

$$\begin{aligned} f(\alpha P^*P) &= f\left(\alpha\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha^*\right)\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha\right)\right) \\ &= f\left(\alpha\left(\frac{1}{4}\mathbf{1} + \frac{1}{4}\alpha^*\alpha + \frac{1}{4}\alpha^* + \frac{1}{4}\alpha\right)\right) \\ &= \frac{1}{4}f(\alpha) + \frac{1}{4}f(\alpha\alpha^*\alpha) + \frac{1}{4}f(\alpha^2) + \frac{1}{4}. \end{aligned}$$

By (3.9),

$$f(\alpha P^*P) = \frac{1}{4}f(\alpha) + \frac{1}{4}f(\alpha)f(\alpha^*\alpha) + \frac{1}{4}f(\alpha^2) + \frac{1}{4}.$$

In addition, by an assumption of Proposition 3.25,

$$f(\alpha P^*P) = \frac{1}{2}f(\alpha) + \frac{1}{4}f(\alpha^2) + \frac{1}{4}. \tag{3.37}$$

By (3.36),

$$\begin{aligned} f(\alpha P^*P) &= f(\alpha)f(P^*P) \\ &= f(\alpha)f\left(\left(\frac{1}{2}\mathbf{1}+\frac{1}{2}\alpha^*\right)\left(\frac{1}{2}\mathbf{1}+\frac{1}{2}\alpha\right)\right) \\ &= f(\alpha)f\left(\frac{1}{4}\mathbf{1}+\frac{1}{4}\alpha^*\alpha+\frac{1}{4}\alpha^*+\frac{1}{4}\alpha\right) \\ &= \frac{1}{2}f(\alpha)+\frac{1}{4}f(\alpha)^2+\frac{1}{4}|f(\alpha)|^2. \end{aligned} \tag{3.38}$$

It follows from (3.37), (3.38) that

$$\mathrm{Im}(f(\alpha^2)) = \mathrm{Im}(f(\alpha)^2). \tag{3.39}$$

We assume that $e^{i\xi}f(\alpha)$ $(0 \leq \xi < 2\pi)$ is in $\{yi : y \in \mathbb{R}, y \geq 0\}$. Then

$$\begin{aligned} f((\frac{1}{2}\mathbf{1}+\frac{1}{2}e^{i\xi}\alpha)^*(\frac{1}{2}\mathbf{1}+\frac{1}{2}e^{i\xi}\alpha)) &= f\left(\frac{1}{4}\mathbf{1}+\frac{1}{4}\alpha^*\alpha+\frac{1}{4}e^{i\xi}\alpha+\frac{1}{4}e^{-i\xi}\alpha^*\right) \\ &= \frac{1}{4}+\frac{1}{4}f(\alpha^*\alpha). \end{aligned}$$

Then, by an assumption of Proposition 3.25,

$$f((\frac{1}{2}\mathbf{1}+\frac{1}{2}e^{i\xi}\alpha)^*(\frac{1}{2}\mathbf{1}+\frac{1}{2}e^{i\xi}\alpha)) = \frac{1}{2}. \tag{3.40}$$

We denote $\frac{1}{2}\mathbf{1}+\frac{1}{2}e^{i\xi}\alpha$ by Q. Let f'' be the linear functional defined by

$$f''(X) := f(XQ^*Q), \qquad \forall X \in C^{alg}(S^3_\theta).$$

By (3.40),

$$f''(\mathbf{1}) = \frac{1}{2},$$

and it follows from Lemma 3.21 that

$$f^{''}(X^*X) = f(X^*XQ^*Q) = f(Q^*X^*XQ) \geq 0.$$

Hence $f^{''}$ is a positive linear functional on $C^{alg}(S^3_\theta)$. We have

$$\begin{aligned}(f - f^{''})(X^*X) =& f(X^*X) - f^{''}(X^*X) \\ =& f(X^*X) - f(X^*XQ^*Q) \\ =& f(X^*X) - f\left(X^*X\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}e^{-i\xi}\alpha^*\right)\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}e^{i\xi}\alpha\right)\right) \\ =& f(X^*X) - f\left(X^*X\left(\frac{1}{4}\mathbf{1} + \frac{1}{4}\alpha^*\alpha + \frac{1}{4}e^{-i\xi}\alpha^* + \frac{1}{4}e^{i\xi}\alpha\right)\right) \\ =& \frac{3}{4}f(X^*X) - \frac{1}{4}f(X^*X\alpha^*\alpha) \\ & - \frac{1}{4}e^{-i\xi}f(X^*X\alpha^*) - \frac{1}{4}e^{i\xi}f(X^*X\alpha).\end{aligned}$$

Then, by (3.29),(3.30),

$$(f - f^{''})(X^*X) = \frac{1}{2}f(X^*X) - \frac{1}{2}\mathrm{Re}(e^{i\xi}f(X^*X\alpha)).$$

In addition, we have

$$
\begin{aligned}
&f((X-e^{i\xi}X\alpha)^*(X-e^{i\xi}X\alpha)) \\
&\quad = f((X^*-e^{-i\xi}\alpha^*X^*)(X-e^{i\xi}X\alpha)) \\
&\quad = f(X^*X+\alpha^*X^*X\alpha-e^{i\xi}X^*X\alpha-e^{-i\xi}\alpha^*X^*X) \\
&\quad = f(X^*X)+f(\alpha^*X^*X\alpha)-2\mathrm{Re}(e^{i\xi}f(X^*X\alpha)) \\
&\quad = f(X^*X)+f(X^*X\alpha^*\alpha)-2\mathrm{Re}(e^{i\xi}f(X^*X\alpha)) \\
&\quad = f(X^*X(\mathbf{1}+\alpha^*\alpha))-2\mathrm{Re}(e^{i\xi}f(X^*X\alpha)) \\
&\quad = 2f(X^*X)-f(X^*X\beta^*\beta)-2\mathrm{Re}(e^{i\xi}f(X^*X\alpha)) \\
&\quad \geq 0 \qquad (3.41)
\end{aligned}
$$

from (3.32). By (3.34),(3.41), we get

$$\frac{1}{2}f(X^*X)-\frac{1}{2}\mathrm{Re}(e^{i\xi}f(X^*X\alpha))\geq 0,$$

so $(f-f'')(X^*X)\geq 0$, i.e. $f\geq f''$. As above, there exists $s\in[0,1]$ such that $f''=sf$. Then

$$f(XQ^*Q)=f''(X)=sf(X)=sf(\mathbf{1})f(X)=f''(\mathbf{1})f(X)=f(Q^*Q)f(X). \qquad (3.42)$$

We have

$$
\begin{aligned}
e^{i\xi} f(\alpha Q^* Q) &= e^{i\xi} f\left(\alpha\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}e^{-i\xi}\alpha^*\right)\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}e^{i\xi}\alpha\right)\right) \\
&= e^{i\xi} f\left(\alpha\left(\frac{1}{4}\mathbf{1} + \frac{1}{4}\alpha^*\alpha + \frac{1}{4}e^{i\xi}\alpha + \frac{1}{4}e^{-i\xi}\alpha^*\right)\right) \\
&= e^{i\xi} f\left(\frac{1}{4}\alpha + \frac{1}{4}\alpha\alpha^*\alpha + \frac{1}{4}e^{i\xi}\alpha^2 + \frac{1}{4}e^{-i\xi}\alpha^*\alpha\right) \\
&= \frac{1}{4}e^{i\xi} f(\alpha) + \frac{1}{4}e^{i\xi} f(\alpha\alpha^*\alpha) + \frac{1}{4}e^{2i\xi} f(\alpha^2) + \frac{1}{4} f(\alpha^*\alpha).
\end{aligned}
$$

Then, by (3.9), and an assumption of Proposition 3.25,

$$
e^{i\xi} f(\alpha Q^* Q) = \frac{1}{2}e^{i\xi} f(\alpha) + \frac{1}{4}e^{2i\xi} f(\alpha^2) + \frac{1}{4}. \tag{3.43}
$$

In addition, (3.42) implies

$$
\begin{aligned}
e^{i\xi} f(\alpha Q^* Q) &= e^{i\xi} f(\alpha) f(Q^* Q) \\
&= e^{i\xi} f(\alpha) f\left(\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}e^{-i\xi}\alpha^*\right)\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}e^{i\xi}\alpha\right)\right) \\
&= e^{i\xi} f(\alpha) f\left(\frac{1}{4}\mathbf{1} + \frac{1}{4}\alpha^*\alpha + \frac{1}{4}e^{i\xi}\alpha + \frac{1}{4}e^{-i\xi}\alpha^*\right) \\
&= e^{i\xi} f(\alpha) f\left(\frac{1}{2} + \frac{1}{4}e^{i\xi} f(\alpha) + \frac{1}{4}e^{-i\xi} f(\alpha^*)\right) \\
&= \frac{1}{2}e^{i\xi} f(\alpha) + \frac{1}{4}e^{2i\xi} f(\alpha)^2 + \frac{1}{4}|f(\alpha)|^2. \qquad (3.44)
\end{aligned}
$$

By (3.37),(3.38), if $f(\alpha) = 0$, then $f(\alpha^2) = -1$. It also follows from (3.43),(3.44) that if $f(\alpha) = 0$, then $e^{2i\xi} f(\alpha^2) = -1$ for every $\xi \in [0, 2\pi)$, because for every such ξ, $e^{i\xi} f(\alpha)$ is in $\{yi : y \in \mathbb{R}, y \geq 0\}$. This contradicts $f(\alpha^2) = -1$. Hence $f(\alpha) \neq 0$. Then for all ξ,

$$
e^{2i\xi} f(\alpha)^2 < 0, \tag{3.45}
$$

since $e^{i\xi}f(\alpha)$ is in $\{yi : y \in \mathbb{R}, y > 0\}$. From (3.43),(3.44) and $0 \leq |f(\alpha)|^2 \leq 1$, it follows that

$$e^{2i\xi}f(\alpha^2) < 0. \tag{3.46}$$

By (3.39),(3.45),(3.46),

$$f(\alpha^2) = f(\alpha)^2. \tag{3.47}$$

By (3.37),(3.38),(3.47), we obtain

$$|f(\alpha)| = 1. \tag{3.48}$$

Let $n \in \mathbb{N}$. We assume that $f(\alpha^s) = f(\alpha)^s$ for each natural number $s \leq n$. We have

$$\begin{aligned} f(\alpha^n P^* P) &= f\left(\alpha^n\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha^*\right)\left(\frac{1}{2}\mathbf{1} + \frac{1}{2}\alpha\right)\right) \\ &= f\left(\alpha^n\left(\frac{1}{4}\mathbf{1} + \frac{1}{4}\alpha^*\alpha + \frac{1}{2}\alpha^* + \frac{1}{2}\alpha\right)\right) \\ &= \frac{1}{4}f(\alpha^n) + \frac{1}{4}f(\alpha^n\alpha^*\alpha) + \frac{1}{2}f(\alpha^{n-1}\alpha^*\alpha) + \frac{1}{2}f(\alpha^{n+1}). \end{aligned}$$

By (3.9),

$$f(\alpha^n P^* P) = \frac{1}{2}f(\alpha^n) + \frac{1}{4}f(\alpha^{n+1}) + \frac{1}{4}f(\alpha^{n-1}). \tag{3.49}$$

In addition, it follows from (3.36) that

$$\begin{aligned} f(\alpha^n P^*P) &= f(\alpha^n)f(P^*P) \\ &= f(\alpha^n)f\left((\frac{1}{2}\mathbf{1}+\frac{1}{2}\alpha^*)\left(\frac{1}{2}\mathbf{1}+\frac{1}{2}\alpha\right)\right) \\ &= f(\alpha^n)f\left(\frac{1}{4}\mathbf{1}+\frac{1}{4}\alpha^*\alpha+\frac{1}{4}\alpha^*+\frac{1}{4}\alpha\right) \\ &= \frac{1}{2}f(\alpha^n)+\frac{1}{4}f(\alpha^n)f(\alpha^*)+\frac{1}{4}f(\alpha^n)f(\alpha). \end{aligned}$$

By an assumption $f(\alpha^s) = f(\alpha)^s$, $s \leq n$,

$$f(\alpha^n)f(P^*P) = \frac{1}{2}f(\alpha^n)+\frac{1}{4}f(\alpha)^{n+1}+\frac{1}{4}f(\alpha^{n-1})\left|f(\alpha)\right|^2.$$

By (3.48),

$$f(\alpha^n)f(P^*P) = \frac{1}{2}f(\alpha^n)+\frac{1}{4}f(\alpha)^{n+1}+\frac{1}{4}f(\alpha^{n-1}). \tag{3.50}$$

By (3.49),(3.50), we get $f(\alpha^{n+1}) = f(\alpha)^{n+1}$. Therefore,

$$f(\alpha^k) = f(\alpha)^k, \qquad k \in \mathbb{N}. \tag{3.51}$$

This implies $f(\alpha^{*k}) = f(\alpha^*)^k$, $k \in \mathbb{N}$, since $f(\alpha^*) = \overline{f(\alpha)}$. By (3.48),

$$|f(\alpha^k)| = |f(\alpha^{*k})| = 1, \qquad k \in \mathbb{N}.$$

Set $f(\alpha) = t$ for $t \in \mathbb{C}$, $|t| = 1$.

We now prove that

$$f(\alpha^{*n_1}\alpha^{n_1'}) = \Psi_t^\alpha(\alpha^{*n_1}\alpha^{n_1'}) \tag{3.52}$$

for $n_1, n_1' \in \mathbb{N}$. It suffices to consider the case $n_1 \geq n_1'$. It follows from Proposition 3.15 that

$$\begin{aligned} f(&(\alpha^{*n_1}\alpha^{n_1'} - \alpha^{*(n_1-n_1')})^*(\alpha^{*n_1}\alpha^{n_1'} - \alpha^{*(n_1-n_1')})) \\ &= f((\alpha^{*n_1^*}\alpha^{n_1} - \alpha^{(n_1-n_1')})(\alpha^{*n_1}\alpha^{n_1'} - \alpha^{*(n_1-n_1')})) \\ &= f(\alpha^{*(n_1+n_1')}\alpha^{(n_1+n_1')}) + f(\alpha^{*(n_1-n_1')}\alpha^{(n_1-n_1')}) - 2f(\alpha^{*n_1}\alpha^{n_1}) \\ &= 0 \end{aligned}$$

By Lemma 1.19,

$$f(\alpha^{*n_1}\alpha^{n_1'}) = f(\alpha^{*(n_1-n_1')}) = \overline{t^{(n_1-n_1')}} = \Psi_t^{\alpha}(\alpha^{*n_1}\alpha^{n_1'}),$$

which proves (3.52). By (3.52) and Lemma 3.23, $f = \Psi_t^{\alpha}$. □

COROLLARY 3.26. *Let f be an extreme point of the tracial state space of $C^{alg}(S_\theta^3)$ such that $f(\beta^*\beta) = 1$. Then there exists $t \in \mathbb{C}$, $|t| = 1$, such that $f = \Psi_t^{\beta}$.*

In fact, the following proposition holds.

PROPOSITION 3.27. *If $\mathcal{F}$ is a pure state of the state space of $C^{alg}(S_\theta^3)$ satisfying $\mathcal{F}(\alpha^*\alpha) = 1$, then there exists $t \in \mathbb{C}$, $|t| = 1$, such that $\mathcal{F} = \Psi_t^{\alpha}$.*

Proof. The Proof is same as Proposition 3.25. □

COROLLARY 3.28. *If $\mathcal{F}$ is a pure state of the state space of $C^{alg}(S_\theta^3)$ satisfying $\mathcal{F}(\beta^*\beta) = 1$, then there exists $t \in \mathbb{C}$, $|t| = 1$, such that $\mathcal{F} = \Psi_t^{\beta}$.*

It follows from Proposition 3.15 and Lemma 3.21 that the following proposition.

PROPOSITION 3.29. *Let Φ be a state on $C^{alg}(S^3_\theta)$ satisfying $\Phi(\alpha^*\alpha) = 1$. For $X \in C^{alg}(S^3_\theta)$, $\Phi(X)$ is determined by $\{\Phi(\alpha^k) : k \in \mathbb{N}\}$.*

COROLLARY 3.30. *Let Φ' be a state on $C^{alg}(S^3_\theta)$ satisfying $\Phi'(\beta^*\beta) = 1$. For $X \in C^{alg}(S^3_\theta)$, $\Phi'(X)$ is determined by $\{\Phi'(\beta^k) : k \in \mathbb{N}\}$.*

By Proposition 3.19, Proposition 3.25 and Corollary 3.26, we have the following result:

THEOREM 3.31. *The set of all extreme points of the tracial state space of $C^{alg}(S^3_\theta)$ equals the set*

$$\{\Psi_x, \Psi^\alpha_t, \Psi^\beta_t : x \in (0,1), t \in \mathbb{C}, |t| = 1\}.$$

For the noncommutative 3-sphere $C^{alg}(S^3_\theta)$, the extreme points of the tracial state space is given by three families of states, one parameterized by the interval $(0,1)$, and the other two by the circle. The two circles correspond to characters. An important problem considered immediatery is to study the extreme points of the tracial state space of noncommutative spheres in general dimension.

On the other hand, for the noncommutative 4-plane $C^{alg}(\mathbb{R}^4_\theta)$ [25], the extreme points of the tracial state space include at least four families of states.

Two of the families are parametrized by $\mathbb{C} - \{0\}$, one by $(0, \infty) \times (0, \infty)$, and one is just a point. Further detailed studies of the extreme points of the tracial state space of $C^{alg}(\mathbb{R}^4_\theta)$ are in progress.

An interesting problem is to consider the tracial state space of the noncommutative 3-plane given by a noncommutative analogue of stereographic projection from a 4-sphere as a natural intermediate case between $C^{alg}(S^3_\theta)$ and $C^{alg}(\mathbb{R}^4_\theta)$.

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