A Thesis for the Degree of Ph.D. in Science

Small-time Existence of a Strong Solution of Primitive Equations for the Ocean and the Atmosphere

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Contents

Chapter 1

Introduction

The idea of weather forecast was conceived by Bjerknes [3] in 1904, and then numerical weather forecast was executed by Richardson in 1920's ([33]). He derived a system of equations describing the motion of the atmosphere, which is similar to the Navier–Stokes equations. His attempt unfortunately failed because of mainly the lack of stability of the calculations, however many attempts have carried on him.

Since around 1940–50's, digital computers made possible automatic calculations, so that the weather forecast with numerical calculation became practical; the first success was done by Charney, Fjörtoft and von Neumann [9] (see also [32]). In that period, many simplified models such as geostrophic and quasi-geostrophic models were proposed in order to lessen the amount of numerical calculations. For example, Charney [8] proposed the atmospheric circulation modeling with the quasi-geostrophic equations, which describes the conservation of absolute vorticity under the approximation of small Rossby number. The serious defficiency of this model is that it works well only over the limited regions, where the Coriolis force is dominant.

The first success in applying the primitive equations to modeling the atmosphere was achieved by Smagorinsky [35]. The main difference between his model and Richardson's was the presence of the frictional force, which was not taken into account (although Richardson recognized the necessity of it) in the Richardson's model. Other features of Smagorinsky's model are that the atmosphere is incompressible, its pressure is taken as a vertical coordinate (called

"p-coordinate system", which was found in Richardson's work [33]), and that the vertical wind velocity vanishes on the ocean surface. Smagorinsky also alloted the rigid lid hypothesis, which means that the ocean surface is fixed and flat. As for the ocean, in 1969, Bryan [6] formulated the ocean circulation with the application of the hydrostatic and Boussinesq approximations, and turbulent viscosity terms, which are anisotropic in the horizontal and the vertical directions. Nowadays his model equations are called primitive equations for the ocean [45].

Following Bryan's formulation Semtner [34] proposed the general circulation model and studied it in detail numerically. Although almost all works for the primitive equations were done under the rigid lid hypothesis, Crowley [10] regarded the ocean surface as a free surface, and considered the numerical model of the ocean.

Phillips [32] later reviewed various models used for the weather prediction, and revealed the cause of Richardson's error in 1970. Nowadays, the primitive equations can be solved numerically since the power of the computers intensively increases.

Mathematical arguments of primitive equations were begun in 1990's. The followings are the momentum and continuity equations of the primitive equations for the atmosphere (cf. Section 3):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + w\frac{\partial \mathbf{v}}{\partial x_3} - \frac{1}{\varrho}\left[\mu_1 \triangle \mathbf{v} + \mu_2 \frac{\partial^2 \mathbf{v}}{\partial x_3^2}\right] + f\mathbf{A}\mathbf{v} = -\frac{1}{\varrho}\nabla p + \mathbf{F}_1,$$

$$\frac{\partial p}{\partial x_3} = -\varrho g,$$

$$\frac{\partial \varrho}{\partial t} + \nabla\cdot(\varrho \mathbf{v}) + \frac{\partial}{\partial x_3}(\varrho w) = 0.$$

The fourth term in the left-hand side of the momentum equation is the effect of the frictional force, which is called as turbulent viscosity, and the fifth the Coriolis force. The second equation is derived from the hydrostatic approximation. Note that the atmosphere is compressible, and the continuity equation takes the form of the third equation. The equation of state for the ideal gas, $p = \varrho R\theta$ is also utilized. They are also described in the spherical coordinate system. The corresponding parts of primitive equations for the ocean are formulated as follows:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + w\frac{\partial \mathbf{v}}{\partial x_3} - \left[\mu_1\Delta\mathbf{v} + \mu_2\frac{\partial^2\mathbf{v}}{\partial x_3^2}\right] + f\mathbf{A}\mathbf{v} = -\frac{1}{\varrho}\nabla p + \mathbf{F}_1,$$

$$\frac{\partial p}{\partial x_3} = -\varrho g,$$

$$\nabla\cdot\mathbf{v} + \frac{\partial w}{\partial x_3} = 0.$$

Here ϱ may be taken to be a positive constant, or the Boussinesq approximation with $\varrho = \varrho(p,\theta,S)$ is applied. One of the main features of primitive equations is the fact that the vertical velocity is determined by the horizontal velocities via the continuity equation, since the vertical velocity disappears in the vertical component of equations of motion due to the hydrostatic approximation.

In [23] and [24], Lions, Temam and Wang formulated the evolution problem of primitive equations for the atmosphere and the ocean, respectively, and showed the existence of a weak solution in $L_2(0,T;H^1(\Omega))\bigcap L_\infty(0,T;L_2(\Omega))$ by the Galerkin method. In [24], the region is $\Omega = \bigcup_s M_s$, each M_s is a connected domain with both horizontally and vertically flat boundaries, while in [23], $\Omega = S^2\times[0,1]$. They also investigated the Hausdorff dimension of the attractor to the slightly modified equations in these papers. In a series of papers [25], [26], [28]–[30], they studied the evolutionary 3-dimensional problem of the primitive equations for the coupled atmosphere and ocean model under the rigid lid hypothesis. In [30] they showed the well-posedness of the model formulated in [28] in the same function space as above. In [27] they derived the following quasi-geostrophic equations from primitive equations by the asymptotic expansion with respect to the Rossby number:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right]\cdot\left[\Delta p + \frac{\partial}{\partial z}\left(\sigma^{-1}\frac{\partial p}{\partial z}\right) + f^1\right] - \left[\frac{1}{Re_1}\Delta + \frac{1}{\delta^2 Re_2}\frac{\partial^2}{\partial z^2}\right]\Delta p$$
$$-\frac{\partial}{\partial z}\left[\sigma^{-1}\left(\frac{1}{Rt_1}\Delta + \frac{1}{\delta^2 Rt_2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial p}{\partial z}\right)\right] = -\frac{\partial}{\partial z}\left(-\sigma^{-1}Q\right),$$

and showed the existence and uniqueness of a weak solution, global in time, in the similar function spaces as above. However, their approaches were based mainly on the Galerkin method, which are not applicable to the free boundary problem. Besides, from the viewpoint of the phenomena, the boundary conditions imposed in [28]–[30] was not realistic, since the interaction of the atmosphere and the ocean was not taken into account.

The stationary problem of the linear primitive equations was discussed by Ziane [47], [48]. In [47], he showed the existence of a weak solution to the stationary linear problem in $S^2 \times [0, 1]$ under the boundary conditions similar to Lions [23] et al. However, this result is very restrictive, in the sense that it make little contributions to the evolutionary problems. In [48], he showed the H^2 regularity of the solution to the linear stationary problem in the region with sidewalls (the vertically flat lateral boundary) perpendicular to the ocean surface (ocean surface is also fixed and flat). This result itself is restrictive, but some of the following results concerning the strong solution of the evolutionary primitive equations referred to it.

Guillén-González, Masmoudi and Rodríguez-Bellido in [14], [15] discussed the initial boundary value problem of the primitive equations for the ocean in the domain surrounded by the rigid lid, sidewalls and the bottom. They showed the existence of a global strong solution with small data and a local strong solution with any data in $L^\infty(0,T; H^1(\Omega)) \bigcap W_2^{2,1}(\Omega_T)$. In the similar situation as that in [15], Temam and Ziane [44] verified the existence and uniqueness of a strong local in time solution of primitive equations for the ocean in $C(0,T; L_2(\Omega)) \bigcap L_2(0,T; H^2(\Omega))$. Their results are based on [48]. Later, Cao and Titi [7] showed the existence and uniqueness of a global solution in $C(0,T; H^1(\Omega)) \bigcap L_2(0,T; H^2(\Omega)) \bigcap W_1^1(0,T; L_2(\Omega))$.

Now, we review the results in two-dimensional case. In this case, we have a few results concerning the strong global in time solution. Guillén-González and Rodríguez-Bellido [13] showed the existence and uniqueness of the strong solution to the primitive equations for the ocean in the two-dimensional region with a sidewall under the smallness condition of data. Bresch et al. [5] also showed the existence and uniqueness of the strong solution with large data, and also showed the uniqueness of the weak solution in the similar region as [13]. However, in these cases, Coriolis force does not appear, and are not interesting from the viewpoint of the phenomena. Hu et al. [19] proved the existence and uniqueness of the strong solution of the primitive equations on thin domains with a non-flat bottom. Owing to this, Hu [20] discussed the primitive equations for the ocean under the small depth assumption, and proved that the solution asymptotically becomes a barotropic flow both in the horizontal and vertical directions as the depth of the ocean goes to zero. For the weak solutions in the domain without sidewalls, as far as the present author knows, we have had only two results so far, Azerad and Guillén-González [1] for non-stationary case, and Besson and Laydi [4] for the stationary case. In [1], they showed the existence of the weak solution of the Navier–Stokes equations with anisotropic viscosity terms and its convergence to a weak solution of primitive equations as the aspect ratio of depth to width of the domain tends to zero. However, the uiqueness of weak solution is remained open. Besides, as far as the present author knows, we have no results concerning the strong solutions for such a problem.

Until now, comparing to the mathematical results of the primitive equations for the ocean, we have had fewer results for the atmosphere. One of the reasons why we have fewer results concerning the atmosphere model than those of the ocean is that the atmosphere should be considered to be compressible fluid. In order to avoid the difficulty, the p-coordinate system is used to make the compressible framework to the incompressible one. Thereby, in the atmosphere model equation, the unknown temperature appears in the turbulent viscosity term, which gives another difficulty. In a series of works by Lions, Temam and Wang and Ewald and Temam [11], the temperature in the viscosity term was replaced by the average temperature along the isobar of the pressure, which makes all the situations of the problem easier to analyze. Besides, they added the known functions in the equations after the coordinates transform, which seems to be unrealistic from the original phenomena. In [11], they considered the potential temperature of the atmosphere model following Lions et al. [23], and showed the positivity of the temperature by using the maximum principle. However, their discussion is based on a weak solution, and it seems incredible to us whether the results holds true in the pointwise sense.

Now we state the mathematical issues concerning the present prolems and the features of our works [17], [18].

First, almost all the results cited above were obtained under the rigid lid hypothesis, and the addition of the turbulent viscosity terms as an empirical claim. In this thesis, we take into account the effect of the surface movement following Crowley [10], and model the ocean surface as a free surface. For Navier–Stokes equations, we have many important results concerning the free boundary problems and two phase problems due to Solonnikov and Tani ([36]–[38], for example) and Tanaka and Tani ([22], [39]–[43], for example). In these papers, the existence and uniqueness of strong solutions to the free boundary problem was established. As far as the present author knows, however, unlike the Navier–Stokes equations, the primitive equations have not been investigated its derivatives and formulations rigorously from mathematics, geophysics and oceanography. It seems important to study the primitive equations from the viewpoint of the free boundary problem. In the present author's works [17] and [18], in addition to the standard terms in the primitive equations, there appear the nonstandard terms in the boundary conditions such as heat flux due to the movement and the surface tension, which is similar to those in [43], for example.

Second, it is to be noted that in the papers cited above, ocean and atmosphere models are described in Cartesian and p-coordinates, respectively. Lions *et al.* [25], [26], [28]-[30] used these coordinates for the coupled ocean and atmosphere model in each layer, and made a physically unrealistic assumption that the height of the pressure isobar coincides with that of the ocean surface. Aiming the study of the coupled model in future [16], we use p-coordinates both for the ocean and the atmosphere models in this thesis, which is the first study on the primitive equations for the ocean.

For the atmosphere model, since the temperature appears in the hydrostatic equation through the equation of state $p = \varrho R\theta$, two unknown functions, temperature θ and pressure at the upper boundary h, appear in all the transformed equations. Especially, θ in the transformed equations is determined by an implicit form, as is seen just after (3.1.11). This makes it difficult in estimating the coordinates transformed functions since we need to estimate θ in the Cartesian coordinates by the coordinate transformed one. Due to the same reason, the spatial and temporal derivatives of the ocean surface Ψ are provided by more complicated calculations than in the case of the ocean model. Moreover, the additional regularity for θ is required due to the form of the hydrostatic equation. In this thesis, we are concerned with the free boundary problems of the primitive equations for the ocean and the atmosphere. Here, we enumerate the major features of this thesis:

1. The surface of the ocean is free, not the rigid lid.
2. The boundary conditions on the free surface are described by the stress tensor and the effect of vapour evaporation.
3. p-coordinate system is applied both to the ocean and the atmosphere models.
4. We consider the problem in the 3-dimensional strip without sidewalls.
5. The existence of the strong solution is proved in the Sobolev–Slobodetskiĭ spaces.

This thesis is organized as follows. In Chapter 2 and Chapter 3, we study the initial boundary value problem of the primitive equations for the ocean and the atmosphere in 3-dimensional strip, respectively. Definitions of the function spaces and proofs of some lemmas are provided in Appendices A and B, respectively.

Chapter 2

Primitive Equations for the Ocean

In this chapter, we study the initial boundary value problem of the primitive equations for the ocean. First, we formulate the problem in the orthogonal coordinate system, and then rewrite it in the p-coordinate system. Furthermore, another transform of the coordinate system is introduced in order to make the free boundary fix. Then we study the transformed problem in Sobolev–Slobodetskiĭ spaces by an iteration method.

2.1 Formulation of the Problem

By adopting f-plane approximation, our problem can be formulated in the strip-like region. By $x = (x_1, x_2, x_3)$, we denote an orthogonal Cartesian coordinate system with x_3 being the vertical direction. Let the unknown free surface and the known bottom of the ocean be represented by the equations $x_3 = d(x', t)$ and $x_3 = b(x')$ $(x' = (x_1, x_2))$, respectively, where $d(x', t)$ is assumed to be a function satisfying $d(x', t) > b(x')$ for any $x' \in \mathbf{R}^2$ and $t \geq 0$. Then the domain $\Omega(t)$ of the ocean at time t is represented as $\{(x', x_3)|x' \in \mathbf{R}^2, b(x') < x_3 < d(x', t)\}$. The equations that we consider in this chapter are as follows:

$$
\begin{cases}
\dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + w\dfrac{\partial \mathbf{v}}{\partial x_3} - \left[\mu_1\Delta\mathbf{v} + \mu_2\dfrac{\partial^2\mathbf{v}}{\partial x_3^2}\right] + f\mathbf{A}\mathbf{v} = -\dfrac{1}{\varrho}\nabla p + \mathbf{F}_1, \\
\dfrac{\partial p}{\partial x_3} = F_{13} - \varrho g =: \tilde{F}_{13}, \\
\nabla\cdot\mathbf{v} + \dfrac{\partial w}{\partial x_3} = 0, \\
\dfrac{\partial \theta}{\partial t} + (\mathbf{v}\cdot\nabla)\theta + w\dfrac{\partial \theta}{\partial x_3} - \left[\mu_3\Delta\theta + \mu_4\dfrac{\partial^2\theta}{\partial x_3^2}\right] = F_2, \\
\dfrac{\partial S}{\partial t} + (\mathbf{v}\cdot\nabla)S + w\dfrac{\partial S}{\partial x_3} - \left[\mu_5\Delta S + \mu_6\dfrac{\partial^2 S}{\partial x_3^2}\right] = F_3, \quad x\in\Omega(t),\ t>0.
\end{cases}
\tag{2.1.1}
$$

Here, $f\mathbf{A}\mathbf{v}$ is a Coriolis force with $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the Coriolis parameter f (a positive constant); ∇ and Δ are 2 dimensional gradient and Laplacian, respectively; $\mathbf{F}_1'$ and $\tilde{F}_{13}$ are the horizontal and vertical components of external forces given in $\mathbf{R}^3\times[0,\infty)$. The horizontal component of the velocity is represented by $\mathbf{v}$ and the vertical component w; p is the pressure, ϱ is the density (a positive constant), g is gravity force (a positve constant), θ is the temperature, S is the salinity; F_2 and F_3 are the sources of heat and salinity, respectively; μ_1 and μ_2 are the coefficients of turbulent viscosity; (μ_3,μ_4) and (μ_5,μ_6) are, respectively, given by scaling sum of turbulent and molecular diffusivities of heat and salinity.

The conditions on the free surface $\Gamma_s(t) = \{x\in\mathbf{R}^3 | x_3 = d(x',t), t>0\}$ are as follows:

$$
\begin{cases}
\mathbf{T}(\mathbf{v})\mathbf{n} - (\mathbf{T}(\mathbf{v})\mathbf{n}\cdot\mathbf{n}')\mathbf{n}' = |\mathbf{v}|^\alpha\mathbf{v}, \\
-\left(\mu_3\nabla\theta\cdot\mathbf{n}' + \mu_4\dfrac{\partial\theta}{\partial x_3}n_3\right) = -la(\theta_e)\mathcal{V} + g_1|\mathbf{v}|^\alpha\theta + \sigma LK, \\
(\theta, S, p) = (\theta_e, S_e, p_0),
\end{cases}
\tag{2.1.2}
$$

where

$$
\mathbf{T}(\mathbf{v}) = \begin{pmatrix} \mu_1\dfrac{\partial v_1}{\partial x_1} & \mu_1\dfrac{\partial v_1}{\partial x_2} & \mu_2\dfrac{\partial v_1}{\partial x_3} \\ \mu_1\dfrac{\partial v_2}{\partial x_1} & \mu_1\dfrac{\partial v_2}{\partial x_2} & \mu_2\dfrac{\partial v_2}{\partial x_3} \end{pmatrix}
\tag{2.1.3}
$$

is a part of the stress tensor, $\mathcal{V}$ is the normal velocity of the free surface, $la(\theta_e)$ is a latent heat with saturation temperature θ_e, S_e is the salinity on the surface of the ocean, σ is the surface tension coefficient (a positive constant), K is twice mean curvature, $\mathbf{n} = (n_1, n_2, n_3)^{\mathrm{T}} = \left(\mathbf{n}'^{\mathrm{T}},\ n_3\right)^{\mathrm{T}}$ is the unit normal vector to $\Gamma_s(t)$ at time t pointing to the atmospheric region, L is the heat capacity, g_1 is a given function representing the turbulent transition on the free surface including the albedo of the earth, and p_0 is atmospheric pressure at the ocean surface (positive constant). The conditions of the form (2.1.2) are called bulk formulae ([12], [23], [31], [45]).

Since $\mathcal{V} = \frac{\partial d}{\partial t}/\sqrt{1+|\nabla d|^2}$, the condition $(2.1.2)_2$ can be written in the explicit form

$$\frac{\partial d}{\partial t} = L_{4,d} d + G_{6,d}(\mathbf{v}, \theta), \quad x' \in \mathbf{R}^2,\ t > 0, \tag{2.1.4}$$

where

$$L_{4,d}\tilde{d} := \frac{\sigma L}{la(\theta_e)(1+|\nabla d|^2)} \left\{ \left(1 + \left(\frac{\partial d}{\partial x_2}\right)^2\right) \frac{\partial^2 \tilde{d}}{\partial x_1^2} - 2 \frac{\partial d}{\partial x_1} \frac{\partial d}{\partial x_2} \frac{\partial^2 \tilde{d}}{\partial x_1 \partial x_2} + \left(1 + \left(\frac{\partial d}{\partial x_1}\right)^2\right) \frac{\partial^2 \tilde{d}}{\partial x_2^2} \right\},$$

$$G_{6,d}(\mathbf{v}, \theta) := \frac{1}{la(\theta_e)} \left[\left(-\mu_3 \nabla\theta\big|_{\Gamma_s(t)} \cdot \nabla d + \mu_4 \frac{\partial \theta}{\partial x_3}\Big|_{\Gamma_s(t)} \right) + g_1 \sqrt{1+|\nabla d|^2} |\mathbf{v}|^{\alpha} \theta\big|_{\Gamma_s(t)} \right].$$

The conditions on the bottom $\Gamma_b = \{(x', b(x')) | x' \in \mathbf{R}^2\}$ are

$$(\mathbf{v}, w, \theta, S)(x', b(x'), t) = (\mathbf{0}, 0, \theta_b, S_b)(x', t), \quad x' \in \mathbf{R}^2,\ t > 0. \tag{2.1.5}$$

Initial conditions are

$$(\mathbf{v}, \theta, S)(x, 0) = (\mathbf{v}_0, \theta_0, S_0)(x), \quad x \in \Omega := \Omega(0), \tag{2.1.6}$$

$$d(x', 0) = d_0(x'), \quad x' \in \mathbf{R}^2. \tag{2.1.7}$$

Let us introduce the "p-coordinate system". From $(2.1.1)_2$ and $(2.1.2)_3$, p can be represented as

$$p = p_0 + \int_d^{x_3} \frac{\partial p}{\partial x_3}\, \mathrm{d}x_3 = p_0 + \int_d^{x_3} \tilde{F}_{13}\, \mathrm{d}x_3. \tag{2.1.8}$$

We assume that

$$|F_{13}| < \varrho g \quad \text{in} \quad \mathbf{R}^3 \times [0, \infty),$$

which means the gravity force is dominant in the vertical direction. Now we denote the pressure at the bottom of the ocean by

$$h(x', t) := p_0 + \int_d^b \tilde{F}_{13} \, \mathrm{d}x_3,$$

and

$$h_0 := p_0 + \int_{d_0}^b \tilde{F}_{13}|_{t=0} \, \mathrm{d}x_3.$$

We assume $d_0(x') - b(x') > c_0$ on $\mathbf{R}^2$ with a positive constant c_0. Since $\partial p / \partial x_3 = \tilde{F}_{13} < 0$, we can define a map

$$y_3 \longmapsto p_0 + \int_{y_3}^b \tilde{F}_{13} \, \mathrm{d}x_3 =: \Phi_1(y_3),$$

for which there exists an inverse function

$$d = \Psi(x', t; h) := \Phi_1^{-1}(h)$$

first, and a map

$$y_3 \longmapsto p_0 + \int_{\Psi(y';h)}^{y_3} \tilde{F}_{13} \, \mathrm{d}x_3 =: \Phi_2(y_3; h),$$

for which there exists an inverse function

$$x_3 = X_3(x', p, t; h) := \Phi_2^{-1}(p; h).$$

From (2.1.8), we get

$$\nabla p = -\tilde{F}_{13}(x', d, t) \nabla d + \int_d^{x_3} \nabla \tilde{F}_{13} \, \mathrm{d}x_3 =: \mathbf{F}_5(x', x_3, t), \tag{2.1.9}$$

$$\frac{\partial p}{\partial t} = -\tilde{F}_{13}(x', d, t) \frac{\partial d}{\partial t} + \int_d^{x_3} \frac{\partial \tilde{F}_{13}}{\partial t} \, \mathrm{d}x_3 =: F_6(x', x_3, t). \tag{2.1.10}$$

Note that after introducing p-coordinates, the ocean surface becomes flat and is represented by the equation $p = p_0$. Hereafter, we denote a function $f(x', x_3, t)$ after this coordinate transform by

$$f^{(h)}(x', p, t) = f(x', X_3(x', p, t; h), t).$$

Moreover, we introduce another mapping:

$$y' = x', \quad y_3 = (p_0 - h_0(x'))\frac{p - h(x',t)}{p_0 - h(x',t)} + h_0(x'), \tag{2.1.11}$$

which is similar to that used in [2]. For simplicity, instead of $\tilde{F}_{13}w$ we use $\bar{w}$. By composing these transformations, it is clear that the regions

$$\bigcup_{0\le t\le T} (\Omega(t) \times \{t\}), \quad \bigcup_{0\le t\le T} (\Gamma_b \times \{t\}), \quad \bigcup_{0\le t\le T} (\Gamma_s(t) \times \{t\})$$

are transformed onto the regions

$$\tilde{\Omega}_T := \tilde{\Omega} \times [0,T],\ \tilde{\Gamma}_{bT} := \tilde{\Gamma}_b \times [0,T],\ \tilde{\Gamma}_{sT} := \tilde{\Gamma}_s \times [0,T],$$

respectively, where

$$\tilde{\Omega} = \{(y',y_3) | y' \in \mathbf{R}^2,\ p_0 < y_3 < h_0(y')\},$$
$$\tilde{\Gamma}_b = \{(y',y_3) | y' \in \mathbf{R}^2,\ y_3 = h_0(y')\},$$
$$\tilde{\Gamma}_s = \{(y',y_3) | y' \in \mathbf{R}^2,\ y_3 = p_0\},$$

where $\mathbf{R}^2_T := \mathbf{R}^2 \times [0,T]$.

We denote the inverse of transposed matrix of the Jacobian matrix by

$$\left(J[(x',p)/(y',y_3)]^{\mathrm{T}}\right)^{-1} = (a^{ij}) = (a^{ij}(h)) \quad (i,j = 1,2,3).$$

Then one can easily derive

$$\begin{aligned} \mathbf{a}^3 := \left(a^{13},\ a^{23}\right)^{\mathrm{T}} &= \frac{(p_0 - h_0)(p(y,t) - p_0)}{(p_0 - h)^2}\nabla h + \frac{p_0 - p(y,t)}{p_0 - h}\nabla h_0 \\ &=: A_1(y,t)\nabla h + \mathbf{B}_1(y,t), \end{aligned}$$

$$p(y,t) = \frac{(p_0 - h)(y_3 - h_0)}{p_0 - h_0} + h,$$

$$a^{33} = \frac{p_0 - h_0}{p_0 - h}, \quad a^{ij} = \delta_{ij}\ \ (i = 1,2,\ j = 1,2,3). \tag{2.1.12}$$

In the following, we use the notation

$$\begin{pmatrix} \nabla_h \\ \nabla_{h,3} \end{pmatrix} = J\left[\frac{(x',p)}{(y',y_3)}\right]^{-\mathrm{T}} \begin{pmatrix} \nabla_{y'} \\ \dfrac{\partial}{\partial y_3} \end{pmatrix};$$

$$\tilde{X}_3(y',y_3,t;h) := X_3(x',p,t;h)\big|_{x'=y',\ p=p(y,t)};$$

$$f^{(h)*}(y',y_3,t) := f^{(h)}(x',p,t)\big|_{x'=y',\ p=p(y,t)}.$$

Here $\nabla_{y'}$ is the derivative with rispect to y'. Now let us derive the explicit representation of $\mathbf{F}_5^{(h)*}$ and $F_6^{(h)*}$. Representing the integral term in (2.1.8) by p-coordinate system, we have

$$\mathbf{F}_5^{(h)}(x',p,t) = -\tilde{F}_{13}\big|_{x_3=\Psi}\nabla\Psi + \int_{p_0}^{p}\frac{1}{\tilde{F}_{13}^{(h)}}\left(\nabla\tilde{F}_{13}^{(h)} + \mathbf{F}_5^{(h)}\frac{\partial\tilde{F}_{13}^{(h)}}{\partial p}\right)\,\mathrm{d}p.$$

We have the following boundary condition from this integral equation

$$\mathbf{F}_5^{(h)}|_{p=p_0} = -\tilde{F}_{13}|_{x_3=\Psi(x',t)}\nabla\Psi(x',t).$$

We derive the explicit representation

$$\mathbf{F}_5^{(h)}(x',p,t) = \tilde{F}_{13}^{(h)}\left(-\nabla\Psi(x',t) + \int_{p_0}^{p}\frac{\nabla\tilde{F}_{13}^{(h)}}{\tilde{F}_{13}^{(h)2}}\,\mathrm{d}p\right),$$

and hence

$$\begin{aligned}
&\mathbf{F}_5^{(h)*}(y',y_3,t)\\
&= \tilde{F}_{13}^{(h)*}\left\{-\nabla\Psi(y',t) + \int_{p_0}^{y_3}\frac{1}{\tilde{F}_{13}^{(h)*2}}\left(\nabla\tilde{F}_{13}^{(h)*} + \mathbf{a}^3(h)\frac{\partial\tilde{F}_{13}^{(h)*}}{\partial y_3}\right)\frac{p_0-h}{p_0-h_0}\,\mathrm{d}y_3\right\}\\
&=: -\tilde{F}_{13}^{(h)*}\nabla\Psi(y',t) + \mathbf{C}_1(y,t). \qquad (2.1.13)
\end{aligned}$$

Similarly, we obtain

$$F_6^{(h)}(x',p,t) = \tilde{F}_{13}^{(h)}\left(-\frac{\partial\Psi}{\partial t}(x',t) + \int_{p_0}^{p}\frac{1}{\tilde{F}_{13}^{(h)2}}\frac{\partial\tilde{F}_{13}^{(h)}}{\partial t}\,\mathrm{d}p\right),$$

$$\begin{aligned}
&F_6^{(h)*}(y',y_3,t)\\
&= \tilde{F}_{13}^{(h)*}\left\{-\frac{\partial\Psi}{\partial t}(y',t) + \int_{p_0}^{y_3}\frac{1}{\tilde{F}_{13}^{(h)*2}}\left(\frac{\partial\tilde{F}_{13}^{(h)*}}{\partial t} + \left(\frac{\partial y_3}{\partial t}\right)^*\frac{\partial\tilde{F}_{13}^{(h)*}}{\partial y_3}\right)\frac{p_0-h}{p_0-h_0}\,\mathrm{d}y_3\right\}\\
&=: -\tilde{F}_{13}^{(h)*}\frac{\partial\Psi}{\partial t}(y',t) + \tilde{C}_1(y,t), \quad \left(\frac{\partial y_3}{\partial t}\right)^* = A_1(y,t)\frac{\partial h}{\partial t}. \qquad (2.1.14)
\end{aligned}$$

Differentiating the relation

$$h(x',t) = p(x',b(x'),t) = p_0 + \int_{\Psi}^{b}\tilde{F}_{13}\,\mathrm{d}x_3$$

with respect to x' and t leads us to the following equalities, respectively:

$$\begin{aligned}\nabla\Psi &= \frac{1}{\tilde F_{13}|_{x_3=\Psi}}\left\{-\nabla h + \tilde F_{13}|_{x_3=b}\nabla b + \int_\Psi^b \nabla \tilde F_{13}\,\mathrm{d}x_3\right\}\\ &=: -\frac{\nabla h}{\tilde F_{13}|_{x_3=\Psi}} + \mathbf{D}_1, \quad \mathbf{D}_1 = (D_1,\ D_2)^{\mathrm{T}},\end{aligned} \tag{2.1.15}$$

$$\begin{aligned}\frac{\partial^2\Psi}{\partial y_i\partial y_j} &= -\frac{1}{\tilde F_{13}|_{x_3=\Psi}}\frac{\partial^2 h}{\partial y_i\partial y_j} - \frac{\partial}{\partial y_j}\left(\frac{1}{\tilde F_{13}|_{x_3=\Psi}}\right)\frac{\partial h}{\partial y_i} + \frac{\partial D_i}{\partial y_j}\\ &=: -\frac{1}{\tilde F_{13}|_{x_3=\Psi}}\frac{\partial^2 h}{\partial y_i\partial y_j} + H_{ij} \quad (i,j=1,2),\end{aligned} \tag{2.1.16}$$

$$\frac{\partial\Psi}{\partial t} = \frac{1}{\tilde F_{13}|_{x_3=\Psi}}\left\{-\frac{\partial h}{\partial t} + \int_\Psi^b \frac{\partial \tilde F_{13}}{\partial t}\,\mathrm{d}x_3\right\} =: -\frac{1}{\tilde F_{13}|_{x_3=\Psi}}\frac{\partial h}{\partial t} + E_1. \tag{2.1.17}$$

Now, rewriting the problem (2.1.1)–(2.1.7) in y-coordinates and denoting $(\mathbf{v}^{(h)*}, \bar w^{(h)*}, \theta^{(h)*}, S^{(h)*}, h)$ by $(\mathbf{u}, u_3, \tilde\theta, \tilde S, h)$ for brevity, we have

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} = L_{1,h}\mathbf{u} + \tilde{\mathbf{G}}_{1,h}(\mathbf{u}, u_3),\\ \nabla_{h,3}u_3 - (\nabla_{h,3}\tilde F_{13}^{(h)*})\dfrac{u_3}{\tilde F_{13}^{(h)*}} = \tilde G_{3,h}(\mathbf{u}),\\ \dfrac{\partial\tilde\theta}{\partial t} = L_{2,h}\tilde\theta + \tilde G_{4,h}(\mathbf{u}, u_3, \tilde\theta),\\ \dfrac{\partial \tilde S}{\partial t} = L_{3,h}\tilde S + \tilde G_{5,h}(\mathbf{u}, u_3, \tilde S) \quad \text{in } \tilde\Omega_T,\\ \dfrac{\partial h}{\partial t} = L_{4,h}h + \tilde G_{6,h}(\mathbf{u}, \tilde\theta) \quad \text{in } \mathbf{R}^2_T, \end{cases} \tag{2.1.18}$$

$$\begin{cases} B_h\mathbf{u} = \tilde{\mathbf{G}}_2(\mathbf{u}),\\ (\tilde\theta, \tilde S) = (\theta_e, S_e)|_{x_3=\Psi(y',t)} \quad \text{on } \tilde\Gamma_{sT},\\ (\mathbf{u}, u_3, \tilde\theta, \tilde S) = (\mathbf{0}, 0, \theta_b, S_b)|_{x_3=b(y')} \quad \text{on } \tilde\Gamma_{bT},\\ (\mathbf{u}, \tilde\theta, \tilde S)(y, 0) = (\mathbf{v}_0^{(h_0)*}, \theta_0^{(h_0)*}, S_0^{(h_0)*})(y) \quad \text{on } \tilde\Omega,\\ h(y', 0) = h_0(y') \quad \text{on } \mathbf{R}^2, \end{cases} \tag{2.1.19}$$

where

$$L_{1,h}\mathbf{u} := \mu_1 L_{11,h}\mathbf{u} + \mu_2 L_{12,h}\mathbf{u},$$

$$L_{11,h}\mathbf{u} := \left[l_{11,h} + 2l_{12,h} + |\mathbf{F}_5^{(h)*}|^2 \left(a^{33}\right)^2 \frac{\partial^2}{\partial y_3^2} \right] \mathbf{u},$$

$$L_{12,h}\mathbf{u} := \tilde{F}_{13}^{(h)*2} \left(a^{33}\right)^2 \frac{\partial^2 \mathbf{u}}{\partial y_3^2},$$

$$l_{11,h} := \nabla^2 + 2\mathbf{a}^3 \cdot \nabla \frac{\partial}{\partial y_3} + |\mathbf{a}^3|^2 \frac{\partial^2}{\partial y_3^2}, \qquad l_{12,h} := a^{33}\mathbf{F}_5^{(h)*} \cdot \nabla_h \frac{\partial}{\partial y_3},$$

$$\begin{aligned}
\tilde{\mathbf{G}}_{1,h}(\mathbf{u}, u_3) := \mu_1 \Bigg[& (\nabla_h^2 - l_{11,h}) + \nabla_h \cdot \left(a^{33}\mathbf{F}_5^{(h)*}\right) \frac{\partial}{\partial y_3} \\
& + a^{33}\mathbf{F}_5^{(h)*} \cdot \left(\frac{\partial \mathbf{a}^3}{\partial y_3} + \frac{\partial}{\partial y_3} \left(a^{33}\mathbf{F}_5^{(h)*}\right) \right) \frac{\partial}{\partial y_3} \Bigg] \mathbf{u} \\
& + \mu_2 \left[\left(a^{33}\tilde{F}_{13}^{(h)*} \frac{\partial}{\partial y_3} \right)^2 - (a^{33}\tilde{F}_{13}^{(h)*})^2 \frac{\partial^2}{\partial y_3^2} \right] \mathbf{u} \\
& - \left[(\mathbf{u} \cdot \nabla_h) + \left((\mathbf{F}_5^{(h)*} \cdot \mathbf{u}) a^{33} + u_3 a^{33} + F_6^{(h)*} a^{33} + A_1(y,t) \frac{\partial h}{\partial t} \right) \frac{\partial}{\partial y_3} \right] \mathbf{u} \\
& - f\mathbf{A}\mathbf{u} - \frac{1}{\varrho}\mathbf{F}_5^{(h)*} + \mathbf{F}_1'^{(h)*} \\
=: & \, \mu_1 \tilde{G}_{11,h}\mathbf{u} + \mu_2 \tilde{G}_{12,h}\mathbf{u} - \tilde{G}_{13,h}(\mathbf{u}, u_3)\mathbf{u} - f\mathbf{A}\mathbf{u} - \frac{1}{\varrho}\mathbf{F}_5^{(h)*} + \mathbf{F}_1'^{(h)*},
\end{aligned}$$

$$\tilde{G}_{3,h}(\mathbf{u}) := -\nabla_h \cdot \mathbf{u} - a^{33}\mathbf{F}_5^{(h)*} \cdot \frac{\partial \mathbf{u}}{\partial y_3},$$

$$L_{2,h}\tilde{\theta} := \mu_3 L_{11,h}\tilde{\theta} + \mu_4 L_{12,h}\tilde{\theta},$$

$$\tilde{G}_{4,h}(\mathbf{u}, u_3, \tilde{\theta}) := \mu_3 \tilde{G}_{11,h}\tilde{\theta} + \mu_4 \tilde{G}_{12,h}\tilde{\theta} - \tilde{G}_{13,h}(\mathbf{u}, u_3)\tilde{\theta} + F_2^{(h)*},$$

$$L_{3,h}\tilde{S} := \mu_5 L_{11,h}\tilde{S} + \mu_6 L_{12,h}\tilde{S},$$

$$\tilde{G}_{5,h}(\mathbf{u}, u_3, \tilde{S}) := \mu_5 \tilde{G}_{11,h}\tilde{S} + \mu_6 \tilde{G}_{12,h}\tilde{S} - \tilde{G}_{13,h}(\mathbf{u}, u_3)\tilde{S} + F_3^{(h)*},$$

$$B_h\mathbf{u} := \left\{ \mu_1 \left[(\mathbf{n}' \cdot \nabla_h)\mathbf{u} + (\mathbf{F}_5^{(h)*} \cdot \mathbf{n}') a^{33} \frac{\partial \mathbf{u}}{\partial y_3} \right] + \mu_2 \tilde{F}_{13}^{(h)*} a^{33} \frac{\partial \mathbf{u}}{\partial y_3} n_3 \right\}$$

$$- \left\{ \mu_1 \left[(\mathbf{n}' \cdot \nabla_h)\mathbf{u} \cdot \mathbf{n}' + (\mathbf{F}_5^{(h)*} \cdot \mathbf{n}') a^{33} \frac{\partial \mathbf{u}}{\partial y_3} \cdot \mathbf{n}' \right] \right.$$

$$\left. + \mu_2 \tilde{F}_{13}^{(h)*} \left(a^{33} \frac{\partial \mathbf{u}}{\partial y_3} \cdot \mathbf{n}' \right) n_3 \right\} \mathbf{n}',$$

$$\tilde{\mathbf{G}}_2(\mathbf{u}) := |\mathbf{u}|^\alpha \mathbf{u},$$

$$L_{4,h}h := \frac{\sigma L}{la(\theta_e)\,(1 + |\nabla\Psi(y', t)|^2)} \sum_{i,j=1}^{2} c_{ij} \frac{\partial^2 h}{\partial y_i \partial y_j},$$

$$c_{11} := 1 + \left(D_2 - \frac{1}{\tilde{F}_{13}|_{x_3=\Psi}} \frac{\partial h}{\partial y_2} \right)^2, \quad c_{22} := 1 + \left(D_1 - \frac{1}{\tilde{F}_{13}|_{x_3=\Psi}} \frac{\partial h}{\partial y_1} \right)^2,$$

$$c_{12} = c_{21} := -\left(\frac{1}{\tilde{F}_{13}|_{x_3=\Psi}} \frac{\partial h}{\partial y_1} - D_1 \right) \left(\frac{1}{\tilde{F}_{13}|_{x_3=\Psi}} \frac{\partial h}{\partial y_2} - D_2 \right),$$

$$\tilde{G}_{6,h}(\mathbf{u}, \tilde{\theta}) := \tilde{F}_{13}|_{x_3=\Psi(y',t)} \left[E_1 - \frac{\sigma L}{la(\theta_e)\,(1 + |\nabla\Psi(y', t)|^2)} \sum_{i,j=1}^{2} c_{ij} H_{ij} \right.$$

$$- \frac{1}{la(\theta_e)} \left\{ -\mu_3 \left(\nabla_h + a^{33}(h) \mathbf{F}_5^{(h)*}|_{y_3=\Psi(y',t)} \frac{\partial}{\partial y_3} \right) \tilde{\theta} \cdot \nabla\Psi(y', t) \right.$$

$$\left.\left. + \mu_4 \tilde{F}_{13}^{(h)*}|_{x_3=\Psi(y',t)} a^{33}(h) \frac{\partial \tilde{\theta}}{\partial y_3} + g_1^{(h)*} \left(1 + |\nabla\Psi(y', t)|^2 \right)^{\frac{1}{2}} |\mathbf{u}|^\alpha \tilde{\theta} \right\} \right],$$

$$\mathbf{n} = (n_1, n_2, n_3)^{\mathrm{T}} = \frac{\mathbf{a}}{|\mathbf{a}|}, \quad \mathbf{n}' = (n_1, n_2)^{\mathrm{T}}$$

$$\mathbf{a} = -\left(\; F_{51}^{(h)*}, \; F_{52}^{(h)*}, \; \tilde{F}_{13}^{(h)*} \; \right)^{\mathrm{T}},$$

where $la(\theta_e) = la(\theta_e(y', \Psi(y', t), t))$. It is to be noted that we can extend $(\mathbf{u}\big|_{t=0}, \tilde{\theta}\big|_{t=0}, \tilde{S}\big|_{t=0})(y)$ and d_0 into the half space $t > 0$ preserving the regularity, which is denoted by $(\bar{\mathbf{u}}_0, \bar{\tilde{\theta}}_0, \bar{\tilde{S}}_0, \bar{d}_0)$. We also define the extension of h_0 by

$$\bar{h}_0 = \bar{h}_0(y', t) = p_0 + \int_{\bar{d}_0}^{b} \tilde{F}_{13} \, \mathrm{d}x_3.$$

For the detail, see Section 2.3.

Then the problem (2.1.18), (2.1.19) can be rewritten as the following one for $(\mathbf{u}', u_3', \tilde{\theta}', \tilde{S}', h') := (\mathbf{u} - \bar{\mathbf{u}}_0, u_3, \tilde{\theta} - \bar{\tilde{\theta}}_0, \tilde{S} - \bar{\tilde{S}}_0, h - \bar{h}_0)$:

$$\begin{cases} \dfrac{\partial \mathbf{u}'}{\partial t} = L_{1,h}\mathbf{u}' + L_{1,h}\bar{\mathbf{u}}_0 - \dfrac{\partial \bar{\mathbf{u}}_0}{\partial t} + \tilde{\mathbf{G}}_{1,h}(\mathbf{u}, u_3), \\ \nabla_{h,3} u_3' - (\nabla_{h,3}\tilde{F}_{13}^{(h)*}) \dfrac{u_3'}{\tilde{F}_{13}^{(h)*}} = \tilde{G}_{3,h}(\mathbf{u}), \\ \dfrac{\partial \tilde{\theta}'}{\partial t} = L_{2,h}\tilde{\theta}' + L_{2,h}\bar{\tilde{\theta}}_0 - \dfrac{\partial \bar{\tilde{\theta}}_0}{\partial t} + \tilde{G}_{4,h}(\mathbf{u}, u_3, \tilde{\theta}), \\ \dfrac{\partial \tilde{S}'}{\partial t} = L_{3,h}\tilde{S}' + L_{3,h}\bar{\tilde{S}}_0 - \dfrac{\partial \bar{\tilde{S}}_0}{\partial t} + \tilde{G}_{5,h}(\mathbf{u}, u_3, \tilde{S}) \quad \text{in } \tilde{\Omega}_T, \\ \dfrac{\partial h'}{\partial t} = L_{4,h}h' + L_{4,h}\bar{h}_0 - \dfrac{\partial \bar{h}_0}{\partial t} + \tilde{G}_{6,h}(\mathbf{u}, \tilde{\theta}) \quad \text{in } \mathbf{R}_T^2, \end{cases} \tag{2.1.20}$$

$$\begin{cases} B_h\mathbf{u}' = -B_h\bar{\mathbf{u}}_0 + \tilde{\mathbf{G}}_2(\mathbf{u}), \\ (\tilde{\theta}', \tilde{S}') = (\theta_e|_{x_3=\Psi(y',t)} - \bar{\tilde{\theta}}_0, S_e|_{x_3=\Psi(y',t)} - \bar{\tilde{S}}_0) \quad \text{on } \tilde{\Gamma}_{sT}, \\ (\mathbf{u}', u_3', \tilde{\theta}', \tilde{S}') = (-\bar{\mathbf{u}}_0, 0, \theta_b|_{x_3=b(y')} - \bar{\tilde{\theta}}_0, S_b|_{x_3=b(y')} - \bar{\tilde{S}}_0) \quad \text{on } \tilde{\Gamma}_{bT}, \\ (\mathbf{u}', \tilde{\theta}', \tilde{S}')|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega}, \\ h'|_{t=0} = 0 \quad \text{on } \mathbf{R}^2, \end{cases} \tag{2.1.21}$$

where $(\mathbf{u}, \tilde{\theta}, \tilde{S}, h)$ is replaced by $(\mathbf{u}' + \bar{\mathbf{u}}_0, \tilde{\theta}' + \bar{\tilde{\theta}}_0, \tilde{S}' + \bar{\tilde{S}}_0, h' + \bar{h}_0)$ in the right-hand sides.

2.2 Main Theorem

In this section, we state the main theorem of this chapter. For the notation of function spaces, refer to the Appendix A (see also, for instance, [37], [38]).

Theorem 2.2.1. *Let $l \in (1/2, 1)$, and T be an arbitrary positive number. Assume that*

(i) $\alpha = 2$ *or* $\alpha > 2l + 1$;

(ii) $la(\cdot) : \mathbf{R} \to \mathbf{R}^+ \equiv \{k \in \mathbf{R} \big| k > 0\}$ *satisfies* $la(x) > 0$, *and is bounded and Lipschitz continuous together with its derivatives such that* $la \in C^{2+L}(\mathbf{R})$ *(i.e. continuously differentiable up to the second order, with the Lipschitz continuous second order derivatives) with the norm*

$$|||la||| := \sum_{i=0}^{2} \left[\sup_{x \in \mathbf{R}} \left| \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^i la(x) \right| + \left| \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^i la \right|^{(L)} \right] < \infty,$$

where $|la|^{(L)}$ *is Lipschitz coefficient of* la;

(iii) $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, θ_0, $S_0 \in \overline{W}_2^{1+l}(\Omega)$, $d_0 \in W_2^{\frac{3}{2}+l}(\mathbf{R}^2)$, $0 < \underline{\theta}_0 \le \theta_0(x)$ *and* $0 < \underline{S}_0 \le S_0(x)$ *with positive constants* $\underline{\theta}_0$ *and* $\underline{S}_0$, *respectively;*

(iv) θ_e, θ_b, S_e, $S_b \in \overline{W}_2^{2+l, \frac{2+l}{2}}(\mathbf{R}_T^3)$, $\frac{\partial \theta_e}{\partial x_3}$, $\frac{\partial \theta_b}{\partial x_3}$, $\frac{\partial S_e}{\partial x_3}$ $\frac{\partial S_b}{\partial x_3} \in W_2^{2+l, \frac{2+l}{2}}(\mathbf{R}_T^3)$, $\theta_e - \theta_0$, $\theta_b - \theta_0$, $S_e - S_0$, $S_b - S_0 \in \widetilde{W}_2^{2+l, \frac{2+l}{2}}(\mathbf{R}_T^3)$, $0 < \underline{\theta}_0 \le \theta_e(x)$, $\theta_b(x)$ *and* $0 < \underline{S}_0 \le S_e(x)$, $S_b(x)$;

(v) $b \in \overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)$ *for any* $d_0(x') - b(x') > c_0$ *for any* $x' \in \mathbf{R}^2$ *with a positive constant* c_0;

(vi) $\mathbf{F}_1$, F_2 *and* $F_3 \in \widetilde{W}_2^{l, \frac{l}{2}}(\mathbf{R}_T^3)$, *and their derivatives with respect to* x_3 *satisfy the Hölder condition with exponent* $\beta > l/2$ *with respect to* x_3 *(we call this property as condition* (A)*). For the function with this property, we introduce the notation*

$$|||f|||_T^2 := \|f\|^2_{\widetilde{W}_2^{l, \frac{l}{2}}(\mathbf{R}_T^3)} + \left(\left| \frac{\partial f}{\partial x_3} \right|^{(\beta)}_{x_3} \right)^2,$$

where $|f|^{(\beta)}_{x_3}$ *stands for the Hölder coefficient of* f *in* x_3 *with exponent* β *uniformly in* x' *and* t;

(vii) $g_1 \in \widetilde{W}_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_T^3)$;

(viii) $F_{13} \in \widetilde{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}_T^3)$, *and* $|F_{13}| < \varrho g$ *in* $\mathbf{R}_T^3$.

Moreover, the following compatibility conditions are satisfied:

$$\mathbf{v}_0'(x,0) = \mathbf{v}_0, \quad x \in \Omega,$$

$$\mathbf{T}(\mathbf{v}_0)\mathbf{n}\big|_{t=0} - \left(\mathbf{T}(\mathbf{v}_0)\mathbf{n}\big|_{t=0} \cdot \mathbf{n}'\big|_{t=0}\right)\mathbf{n}'\big|_{t=0} = |\mathbf{v}_0|^\alpha\, \mathbf{v}_0, \quad x \in \Gamma_s(0),$$

$$\theta_e(x', d_0, 0) = \theta_0(x), \quad S_e(x', d_0, 0) = S_0(x), \quad x \in \Gamma_s(0),$$

$$\theta_b(x', b(x'), 0) = \theta_0(x), \quad S_b(x', b(x'), 0) = S_0(x), \quad x \in \Gamma_b.$$

Then, there exists $T^* \in (0,T]$ *such that the problem* (2.1.20)–(2.1.21) *has a unique solution* $(\mathbf{u}', u_3, \tilde{\theta}', \tilde{S}', h') \in Z(T^*) := W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times \widetilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_{T^*}^2)$ *satisfying* $0 < \tilde{\theta} = \tilde{\theta}' + \bar{\tilde{\theta}}$ *and* $0 < \tilde{S} = \tilde{S}' + \bar{\tilde{S}}$ *on* $\tilde{\Omega}_{T^*}$.

2.3 Auxiliary Lemmas

In this section, we prepare some lemmas used in the proof of the main theorem in Section 2.5. It is to be noted that $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, θ_0, $S_0 \in \overline{W}_2^{1+l}(\Omega)$, $d_0 \in W_2^{\frac{3}{2}+l}(\mathbf{R}^2)$ $(1/2 < l < 1)$ imply $\mathbf{v}_0^{(h_0)*} \in W_2^{1+l}(\tilde{\Omega})$, $\theta_0^{(h_0)*}$, $S_0^{(h_0)*} \in \overline{W}_2^{1+l}(\tilde{\Omega})$. By the trace theorem, they are extensible into the half space $t > 0$ so that the extended functions $(\bar{\mathbf{u}}_0, \bar{\tilde{\theta}}_0, \bar{\tilde{S}}_0, \bar{d}_0)$ satisfy for some constant C (see, for instance, [46])

$$\begin{cases} \|\bar{\mathbf{u}}_0\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le C\|\mathbf{v}_0^{(h_0)*}\|_{W_2^{1+l}(\tilde{\Omega})}, \\ \|\bar{\tilde{\theta}}_0\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le C\|\theta_0^{(h_0)*}\|_{\overline{W}_2^{1+l}(\tilde{\Omega})}, \\ \|\bar{\tilde{S}}_0\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le C\|S_0^{(h_0)*}\|_{\overline{W}_2^{1+l}(\tilde{\Omega})}, \\ \|\bar{d}_0\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \le C\|d_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)}. \end{cases} \tag{2.3.1}$$

Now, making use of the well known inequalities

$$\begin{cases} ||uv||_{W_2^{1+l}(\tilde{\Omega})} \le c_1||u||_{W_2^{1+l}(\tilde{\Omega})}||v||_{W_2^{1+l}(\tilde{\Omega})} \quad \text{for } \forall u,\ v \in W_2^{1+l}(\tilde{\Omega}), \\ ||uv||_{W_2^{l}(\tilde{\Omega})} \le c_1||u||_{W_2^{1+l}(\tilde{\Omega})}||v||_{W_2^{l}(\tilde{\Omega})} \quad \text{for } \forall u \in W_2^{1+l}(\tilde{\Omega}),\ \forall v \in W_2^{l}(\tilde{\Omega}) \end{cases} \tag{2.3.2}$$

with a positive constant c_1 (see, for instance, [40]–[43]), we prepare some lemmas concerning the estimates used later. In the followings in this chapter, C's stand for constants depending on $\|b\|_{\overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)}$, $\|F_{13}\|_{\widetilde{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}_T^3)}$, $\|d_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)}$, $\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}$, $\|\theta_0\|_{\overline{W}_2^{1+l}(\mathbf{R}^3)}$, $\|S_0\|_{\overline{W}_2^{1+l}(\mathbf{R}^3)}$, and P's polynomials of their arguments with coefficients having the same dependency as C's. Proofs of Lemmas are given in Appendix B.

Lemma 2.3.1. *Let h', h'_1, $h'_2 \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$ and $h = h' + \bar{h}_0$, $h_i = h'_i + \bar{h}_0$ $(i = 1, 2)$. Then the following estimates hold:*

$$\|\Psi(\cdot;h)\|^2_{W_2^{i-\frac{1}{2}+l,\frac{i+l}{2}-\frac{1}{4}}(\mathbf{R}_T^2)} \le P\left(\|h'\|_{W_2^{i-\frac{1}{2}+l,\frac{i+l}{2}-\frac{1}{4}}(\mathbf{R}_T^2)}\right) \ (i = 0, 1, 2, 3), \quad (2.3.3)$$

$$\begin{aligned}&\|\Psi(\cdot;h_1) - \Psi(\cdot;h_2)\|^2_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}_T^2)} \\ &\quad \le P\left(\sum_{j=1}^{2}\|h'_j\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)\|h'_1 - h'_2\|^2_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}_T^2)} \quad (i = 0, 1, 2).\end{aligned} \quad (2.3.4)$$

Lemma 2.3.2. *Let $h' \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$ and $h = h' + \bar{h}_0$. Then the following inequality holds:*

$$\begin{aligned}&|\tilde{X}_3(y^{1\prime}, y_3^1, t; h) - \tilde{X}_3(y^{2\prime}, y_3^2, t; h)|^2 \\ &\quad \le P\left(\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)\left(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2\right).\end{aligned}$$

Now we turn to the estimates of the functions appearing in the conditions of Theorem 2.2.1.

Lemma 2.3.3. *Let h', h'_1, $h'_2 \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$, $h = h' + \bar{h}_0$, $h_i = h'_i + \bar{h}_0$ $(i = 1, 2)$ and $1/2 < l' < l$.*
(1) *For a function f satisfying condition* (A)*, the following estimates hold:*

$$\|f^{(h)*}\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} \le P\left(\|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\right)\|\!|f|\!\|_T^2, \quad (2.3.5)$$

$$\begin{aligned}&\|f^{(h_1)*} - f^{(h_2)*}\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} \\ &\quad \le P\left(\sum_{j=1}^{2}\|h'_j\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)\|h'_1 - h'_2\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\|\!|f|\!\|_T^2.\end{aligned} \quad (2.3.6)$$

(2) *For a function* $f \in \widetilde{W}_2^{1+l,\frac{1+l}{2}}(\mathbf{R}_T^3)$, *the following estimates hold:*

$$\|f^{(h)*}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\Big)\|f\|^2_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\mathbf{R}_T^3)}, \tag{2.3.7}$$

$$\|f^{(h_1)*} - f^{(h_2)*}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\sum_{j=1}^2 \|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big)\|h_1' - h_2'\|^2_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\|f\|^2_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\mathbf{R}_T^3)}. \tag{2.3.8}$$

(3) *For a function* $f \in \widetilde{W}_2^{2+l,1+\frac{l}{2}}(\mathbf{R}_T^3)$, *the following estimates hold:*

$$\|f^{(h)*}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\Big)\|f\|^2_{\widetilde{W}_2^{2+l,1+\frac{l}{2}}(\mathbf{R}_T^3)}, \tag{2.3.9}$$

$$\|f^{(h_1)*} - f^{(h_2)*}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\sum_{j=1}^2 \|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big)\|h_1' - h_2'\|^2_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\|f\|^2_{\widetilde{W}_2^{2+l,1+\frac{l}{2}}(\mathbf{R}_T^3)}. \tag{2.3.10}$$

Lemma 2.3.4. *Let* h', h_1', $h_2' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$ *and* $h = h' + \bar{h}_0$, $h_i = h_i' + \bar{h}_0$ $(i = 1, 2)$. *Then the following estimates hold:*

$$\|\mathbf{F}_5^{(h)*}\|^2_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big),$$

$$\|\mathbf{F}_5^{(h_1)*} - \mathbf{F}_5^{(h_2)*}\|^2_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\sum_{j=1}^2 \|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big) \times \|h_1' - h_2'\|^2_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}_T^2)} \quad (i = 0, 1, 2).$$

Similarly as Lemma 2.3.4, we can obtain the following lemma.

Lemma 2.3.5. *Let h', h_1' and $h_2' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$ and $h = h' + \bar{h}_0$, $h_i = h_i' + \bar{h}_0$ $(i = 1,2)$. Then the following estimates hold:*

$$
\begin{aligned}
&\|F_6^{(h)*}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big),\\
&\|F_6^{(h_1)*} - F_6^{(h_2)*}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\sum_{j=1}^{2}\|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big)\\
&\qquad \times \|h_1' - h_2'\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}.
\end{aligned}
$$

2.4 Linear Problems

Let us introduce the linear operators $L_{i,\bar{h}_0}$ $(i = 1,2,3,4)$, which are obtained from $L_{i,h}$ $(i = 1,2,3,4)$ with (h,Ψ) replaced by $(\bar{h}_0, \bar{d}_0)$. From the assumptions of Theorem 2.2.1, it is easily seen that the coefficients of $L_{i,\bar{h}_0}$ $(i = 1,2,3)$ belong to $W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$, and those of $L_{4,\bar{h}_0}$ to $\overline{W}_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$. In this section we consider the following linear problems.

$$
\begin{cases}
\dfrac{\partial \mathbf{u}'}{\partial t} - L_{1,\bar{h}_0}\mathbf{u}' = \mathbf{l}_1,\\[2mm]
\dfrac{\partial \tilde{\theta}'}{\partial t} - L_{2,\bar{h}_0}\tilde{\theta}' = l_4,\\[2mm]
\dfrac{\partial \tilde{S}'}{\partial t} - L_{3,\bar{h}_0}\tilde{S}' = l_5 \quad \text{in } \tilde{\Omega}_T,\\[2mm]
B_{\bar{h}_0}\mathbf{u}' = \mathbf{l}_2,\ (\tilde{\theta}', \tilde{S}') = (\bar{\theta}_e, \bar{S}_e) \quad \text{on } \tilde{\Gamma}_{sT},\\[2mm]
(\mathbf{u}', \tilde{\theta}, \tilde{S}) = (-\bar{\mathbf{u}}_0, \bar{\theta}_b, \bar{S}_b) \quad \text{on } \tilde{\Gamma}_{bT},\\[2mm]
(\mathbf{u}', \tilde{\theta}', \tilde{S}')\big|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega};
\end{cases}
\tag{2.4.1}
$$

$$
\begin{cases}
\dfrac{\partial h'}{\partial t} - L_{4,\bar{h}_0}h' = l_6 \quad \text{in } \mathbf{R}_T^2,\\[2mm]
h'\big|_{t=0} = 0 \quad \text{on } \mathbf{R}^2;
\end{cases}
\tag{2.4.2}
$$

$$
\begin{cases} \nabla_{\bar{h}_0,3} u_3' - (\nabla_{\bar{h}_0,3} \tilde{F}_{13}^{(\bar{h}_0)*}) \dfrac{u_3'}{\tilde{F}_{13}^{(\bar{h}_0)*}} = l_3 \quad \text{in } \tilde{\Omega}_T, \\ u_3' = 0 \quad \text{on } \tilde{\Gamma}_{bT}. \end{cases} \tag{2.4.3}
$$

For problems (2.4.1)–(2.4.2), we have

Lemma 2.4.1. (i) *Let* $\mathbf{l}_1 \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)$, $\mathbf{l}_2 \in W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, $l_4,\ l_5 \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)$, $\bar{\theta}_e,\ \bar{S}_e \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, $\bar{\mathbf{u}}_0 \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{bT})$ $\bar{\theta}_b,\ \bar{S}_b \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{bT})$, *and satisfy the compatibility conditions*

$$
\begin{cases} \bar{\mathbf{u}}_0 = \mathbf{0}, \quad \bar{\theta}_b(y,0) = 0, \quad \bar{S}_b(y,0) = 0, \quad y \in \tilde{\Gamma}_b, \\ \mathbf{l}_2\big|_{t=0} = \mathbf{0}, \quad \bar{\theta}_e(y,0) = \theta_0(y), \quad \bar{S}_e(y,0) = S_0(y), \quad y \in \tilde{\Gamma}_s. \end{cases}
$$

Then problem (2.4.1) *has a unique solution* $(\mathbf{u}', \tilde{\theta}', \tilde{S}') \in Z'(T) := W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$ $\times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$ *satisfying*

$$
\begin{aligned} \|(\mathbf{u}', \tilde{\theta}', \tilde{S}')\|_{Z'(T)} \leq C_1' \Big[& \|\mathbf{l}_1\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} + \|\mathbf{l}_2\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} + \sum_{i=4}^{5} \|l_i\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} \\ & + \|\bar{\theta}_e\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} + \|\bar{S}_e\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} \\ & + \|\bar{\mathbf{u}}_0\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{bT})} + \|\bar{\theta}_b\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{bT})} + \|\bar{S}_b\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{bT})} \Big]. \end{aligned} \tag{2.4.4}
$$

(ii) *For* $l_6 \in W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$, *problem* (2.4.2) *has a unique solution* $h' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$ *satisfying*

$$
\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \leq C_2' \|l_6\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}. \tag{2.4.5}
$$

Proof. Note that the operator $L_{i,\bar{h}_0}$ $(i = 1,2,3,4)$ is uniformly elliptic. Indeed, for $\xi' \equiv (\xi_1, \xi_2)^{\mathrm{T}}$, $(\xi'^{\mathrm{T}},\ \xi_3) \in \mathbf{R}^3 \backslash \{0\}$ and $\mathbf{a}_0^3 \equiv (a_0^{31},\ a_0^{32})^{\mathrm{T}} = \big(a^{13}(\bar{h}_0),\ a^{23}(\bar{h}_0)\big)^{\mathrm{T}}$, $a_0^{33} \equiv a^{33}(\bar{h}_0)$, the charasteristic polynomial of $L_{1,\bar{h}_0}$ is

$$
\begin{aligned} \mu_1 \left(|\xi'|^2 + 2\mathbf{a}_0^3 \cdot \xi' \xi_3 + |\mathbf{a}_0^3|^2 \xi_3^2\right) + 2\mu_1 a_0^{33} \left(\mathbf{F}_5^{(\bar{h}_0)*} \cdot \xi' \xi_3 + \mathbf{a}_0^3 \cdot \mathbf{F}_5^{(\bar{h}_0)*} \xi_3^2 \right) & \\ + \mu_1 |\mathbf{F}_5^{(\bar{h}_0)*}|^2 (a_0^{33})^2 \xi_3^2 + \mu_2 \tilde{F}_{13}^{(\bar{h}_0)*2} (a_0^{33})^2 \xi_3^2 & \\ = \mu_1 |\xi' + \mathbf{a}^3 \xi_3 + a^{33} \mathbf{F}_5^{(\bar{h}_0)*} \xi_3|^2 + \mu_2 \tilde{F}_{13}^{(\bar{h}_0)*2} (a_0^{33})^2 \xi_3^2 > 0. & \end{aligned}
$$

This means that $L_{1,\bar{h}_0}$ is uniformly elliptic. In exactly the same method, other operators $L_{i,\bar{h}_0}$ $(i = 2, 3, 4)$ are also uniformly elliptic. Then the general theory for linear partial differential equations of parabolic type [21] leads to the desired result. □

Lemma 2.4.2. *For* $l_3 \in W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)$*, the problem* (2.4.3) *has a unique solution* $u_3' \in \widetilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)$ *such that*

$$\|u_3'\|_{\widetilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)} \leq C_3' \|l_3\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)}.$$

Proof. Problem It is easy to see that by an integration with respect to x_3, (2.4.3) has an exact solution given by

$$u_3'(t, x', x_3) = \tilde{F}_{13}^{(\bar{h}_0)*}(t, x', x_3) \left[\int_{h_0}^{x_3} \frac{l_3}{a_0^{33} \tilde{F}_{13}^{(\bar{h}_0)*}} \, \mathrm{d}x_3 \right].$$

This directly implies $u_3' \in \widetilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)$. □

2.5 Nonlinear Problem (Proof of Theorem 2.2.1)

2.5.1 Successive Approximations

In this section, we prove Theorem 2.2.1 by an iteration method. Let

$$(\mathbf{u}'_{(0)}, u'_{3(0)}, \tilde{\theta}'_{(0)}, \tilde{S}'_{(0)}, h'_{(0)}) = (\mathbf{0}, 0, 0, 0, 0)$$

and $(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}, h'_{(m+1)})$ $(m = 0, 1, 2, \ldots)$ be a solution of the following problem for a given $(\mathbf{u}'_{(m)}, u'_{3(m)}, \tilde{\theta}'_{(m)}, \tilde{S}'_{(m)}, h'_{(m)}) \in Z(T)$.

$$\begin{cases} \dfrac{\partial \mathbf{u}'_{(m+1)}}{\partial t} - L_{1,\bar{h}_0}\mathbf{u}'_{(m+1)} = \left[L_{1,h_{(m)}} - L_{1,\bar{h}_0}\right]\mathbf{u}'_{(m+1)} + L_{1,h_{(m)}}\bar{\mathbf{u}}_0 \\ \qquad\qquad -\dfrac{\partial \bar{\mathbf{u}}_0}{\partial t} + \tilde{G}_{1,h_{(m)}}(\mathbf{u}_{(m)}, u_{3(m)}) =: \mathbf{l}_1^{(m+1)}, \\ \nabla_{\bar{h}_0,3} u'_{3(m+1)} - (\nabla_{\bar{h}_0,3}\tilde{F}_{13}^{(\bar{h}_0)*})\dfrac{u'_{3(m+1)}}{\tilde{F}_{13}^{(\bar{h}_0)*}} = -(\nabla_{\bar{h}_0,3} - \nabla_{h_{(m)},3})u'_{3(m+1)} \\ \qquad\qquad +\tilde{G}_{3,h_{(m)}}(\mathbf{u}_{(m+1)}, \mathbf{u}_{(m)}) =: l_3^{(m+1)}, \\ \dfrac{\partial \tilde{\theta}'_{(m+1)}}{\partial t} - L_{2,\bar{h}_0}\tilde{\theta}'_{(m+1)} = \left[L_{2,h_{(m)}} - L_{2,\bar{h}_0}\right]\tilde{\theta}'_{(m+1)} + L_{2,h_{(m)}}\bar{\tilde{\theta}}_0 \\ \qquad\qquad -\dfrac{\partial \bar{\tilde{\theta}}_0}{\partial t} + \tilde{G}_{4,h_{(m)}}(\mathbf{u}_{(m)}, u_{3(m)}, \tilde{\theta}_{(m)}) =: l_4^{(m+1)}, \\ \dfrac{\partial \tilde{S}'_{(m+1)}}{\partial t} - L_{3,\bar{h}_0}\tilde{S}'_{(m+1)} = \left[L_{3,h_{(m)}} - L_{3,\bar{h}_0})\right]\tilde{S}'_{(m+1)} + L_{3,h_{(m)}}\bar{\tilde{S}}_0 \\ \qquad\qquad -\dfrac{\partial \bar{\tilde{S}}_0}{\partial t} + \tilde{G}_{5,h_{(m)}}(\mathbf{u}_{(m)}, u_{3(m)}, \tilde{\theta}_{(m)}) =: l_5^{(m+1)} \quad \text{in } \tilde{\Omega}_T, \end{cases} \tag{2.5.1}$$

$$
\begin{cases}
B_{\bar{h}_0}\mathbf{u}'_{(m+1)} = -B_{h_{(m)}}\bar{\mathbf{u}}_0 + (B_{\bar{h}_0}\mathbf{u}'_{(m+1)} - B_{h_{(m)}}\mathbf{u}'_{(m+1)}) + \tilde{\mathbf{G}}_2(\mathbf{u}_{(m)}) \\
\qquad\qquad =: \mathbf{l}_2^{(m+1)}, \\
(\tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}) = (\theta_e - \bar{\tilde{\theta}}_0, S_e - \bar{\tilde{S}}_0) \quad \text{on } \tilde{\Gamma}_{sT}, \\
(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}) = (-\bar{\mathbf{u}}_0, 0, \theta_b - \bar{\tilde{\theta}}_0, S_b - \bar{\tilde{S}}_0) \quad \text{on } \tilde{\Gamma}_{bT}, \\
(\mathbf{u}'_{(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)})|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega},
\end{cases}
\tag{2.5.2}
$$

$$
\begin{cases}
\dfrac{\partial h'_{(m+1)}}{\partial t} - L_{4,\bar{h}_0} h'_{(m+1)} = (L_{4,h_{(m)}} - L_{4,\bar{h}_0}) h'_{(m+1)} + L_{4,h_{(m)}} \bar{h}_0 \\
\qquad\qquad - \dfrac{\partial \bar{h}_0}{\partial t} + \tilde{G}_{6,h_{(m)}}(\mathbf{u}_{(m)}, \tilde{\theta}_{(m+1)}, \tilde{\theta}_{(m)}) =: l_6^{(m+1)} \quad \text{in } \mathbf{R}^2_T, \\
h'_{(m+1)}|_{t=0} = 0 \quad \text{on } \mathbf{R}^2.
\end{cases}
\tag{2.5.3}
$$

Here $\mathbf{u}_{(m)} = \mathbf{u}'_{(m)} + \bar{\mathbf{u}}_0$, $\tilde{\theta}_{(m)} = \tilde{\theta}'_{(m)} + \bar{\tilde{\theta}}_0$, $\tilde{S}_{(m)} = \tilde{S}'_{(m)} + \bar{\tilde{S}}_0$, and

$$
\tilde{G}_{3,h_{(m)}}(\mathbf{u}_{(m+1)}, \mathbf{u}_{(m)}) = -\nabla_{h_{(m)}} \cdot \mathbf{u}_{(m+1)} - a^{33}(h_{(m)}) \mathbf{F}_5^{(h_{(m)})*} \cdot \frac{\partial \mathbf{u}_{(m)}}{\partial y_3},
$$

$$
\begin{aligned}
&\tilde{G}_{6,h_{(m)}}(\mathbf{u}_{(m)}, \tilde{\theta}_{(m+1)}, \tilde{\theta}_{(m)}) \\
&\quad = \tilde{F}_{13}(y', \Psi(y',t;h_{(m)}),t) \Bigg[E_1 - \frac{\sigma L}{la(\theta_e)\left(1 + |\nabla \Psi(y',t;h_{(m)})|^2\right)} \sum_{i,j=1}^{2} c_{ij} H_{ij} \\
&\qquad - \frac{1}{la(\theta_e)} \Bigg\{ -\mu_3 \left(\nabla_{h_{(m)}} + a^{33}(h_{(m)}) \mathbf{F}_5^{(h_{(m)})*} \frac{\partial}{\partial y_3} \right) \tilde{\theta}_{(m)} \cdot \nabla \Psi(y',t;h_{(m)}) \\
&\qquad + g_1^{(h_{(m)})*} \left(1 + \left| \nabla \Psi(y',t;h_{(m)}) \right|^2 \right)^{\frac{1}{2}} |\mathbf{u}_{(m)}|^{\alpha} \tilde{\theta}_{(m)} \Bigg\} \Bigg] \\
&\qquad - \frac{\mu_4}{la(\theta_e)} \Bigg[\left(F_{13}(y', \Psi(y',t;h_{(m)}),t)^2 - (\varrho g)^2 \right) a^{33}(h_{(m)}) \frac{\partial \tilde{\theta}_m}{\partial y_3} \\
&\qquad + (\varrho g)^2 a^{33}(h_{(m)}) \frac{\partial \tilde{\theta}_{m+1}}{\partial y_3} \Bigg]
\end{aligned}
$$

with $la(\theta_e) = la(\theta_e(y', \Psi(y', t; h_{(m)}), t))$.

The unique existence of $(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}, h'_{(m+1)})$ is guaranteed by Lemmas 5.1 and 5.2 and the fixed point arguments [21].

Now applying Lemmas 2.4.1 and 2.4.2 again, we estimate $(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}, h'_{(m+1)})$.

By the interpolation and Young's inequalities, it is easy to confirm that

$$\begin{aligned} \|u\|^2_{W_2^{k,0}(\tilde{\Omega}_T)} &\le \epsilon \|u\|^2_{W_2^{m,0}(\tilde{\Omega}_T)} + C_\epsilon \int_0^T \|u\|^2_{L_2(\tilde{\Omega})}\,\mathrm{d}t \\ &\le \epsilon \|u\|^2_{W_2^{m,0}(\tilde{\Omega}_T)} + C_\epsilon T^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L_2(\tilde{\Omega}_T)} \end{aligned} \tag{2.5.4}$$

for any $\epsilon > 0$ if $m > k$ and $u|_{t=0} = 0$. Using (2.3.2), (2.5.4) and Lemmas 2.3.1–2.3.5, we shall estimate the right-hand side of (2.5.1)–(2.5.3). In the followings, $P(\cdot)$ stands for the polynomial of its arguments, ϵ an arbitrary positive number, and $1/2 < l' < l$.

First, we state some lemmas concerning the estimates of the right-hand side of (2.5.1)–(2.5.3).

Lemma 2.5.1.

$$\begin{aligned} &\|\mathbf{l}_1^{(m+1)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} + \|l_4^{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Omega}_t)} + \|l_5^{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Omega}_t)} \\ &\quad \le (\epsilon + C_\epsilon t)\, P\left(\|u_{3(m)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \\ &\qquad \times \left(\left\|\mathbf{u}_{(m)}\right\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \left\|\mathbf{u}_{(m)}\right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \left\|\mathbf{u}_{(m+1)}\right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \right) \\ &\qquad\quad + P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}. \end{aligned}$$

Proof. First, making use of (2.3.2), (2.5.4), multiplicative inequalities and the fact that

$$\|a^{33}(h_{(m)})f\|_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_t)} \le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \|f\|_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_t)} \tag{2.5.5}$$

$(i=1,2)$ hold for any $f \in W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)$, we can show the inequalities

$$
\begin{aligned}
&\left\| \tilde{G}_{11,h_{(m)}} \mathbf{u}_{(m)} \right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\quad \leq \left\{ \left\| \sum_{i=1}^{2} \left(\frac{\partial a^{i3}(h_{(m)})}{\partial y_i} + a^{i3}(h_{(m)}) \frac{\partial a^{i3}(h_{(m)})}{\partial y_3} \right) \right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} \right. \\
&\qquad\qquad \left. + \left\| \nabla_{h_{(m)}} \cdot \left(a^{33}(h_{(m)}) \mathbf{F}_5^{(h_{(m)})*} \right) \right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} \right\} \left\| \frac{\partial \mathbf{u}_{(m)}}{\partial y_3} \right\|_{W_2^{1+l',\frac{1+l'}{2}}(\tilde{\Omega}_t)} \\
&\qquad + \| a^{33}(h_{(m)}) \mathbf{F}_5^{(h_{(m)})*} \|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \left(\left\| \frac{\partial \mathbf{a}^3(h_{(m)})}{\partial y_3} \right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} \right. \\
&\qquad\qquad \left. + \left\| \frac{\partial}{\partial y_3} \left(a^{33}(h_{(m)}) \mathbf{F}_5^{(h_{(m)})*} \right) \right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} \right) \left\| \frac{\partial \mathbf{u}_{(m)}}{\partial y_3} \right\|_{W_2^{1+l',\frac{1+l'}{2}}(\tilde{\Omega}_t)} \\
&\quad \leq (\epsilon + C_\epsilon t)\, P(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}) \left\| \mathbf{u}_{(m)} \right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} .
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\left\| \tilde{G}_{12,h_{(m)}} \mathbf{u}_{(m)} \right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\leq (\epsilon + C_\epsilon t)\, P(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}) \left\| F_{13}^{(h_{(m)})*} \right\|^2_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} \left\| \mathbf{u}_{(m)} \right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} .
\end{aligned}
$$

It is easy to estimate the lower order terms in $\mathbf{l}_1^{(m+1)}$, for example,

$$
\begin{aligned}
&\left\| \left(\mathbf{u}_{(m)} \cdot \nabla_{h_{(m)}} \right) \mathbf{u}_{(m)} \right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\qquad \leq \|\mathbf{u}_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \left\| \left(\nabla + \mathbf{a}^3(h_{(m)}) \frac{\partial}{\partial y_3} \right) \mathbf{u}_{(m)} \right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)},
\end{aligned}
$$

$$
\begin{aligned}
&\left\| F_6^{(h_{(m)})*} a^{33}(h_{(m)}) \frac{\partial \mathbf{u}_{(m)}}{\partial y_3} \right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\quad \leq \left\| F_6^{(h_{(m)})*} \right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} \left\| a^{33}(h_{(m)}) \frac{\partial \mathbf{u}_{(m)}}{\partial y_3} \right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} \\
&\quad \leq P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \left\| F_6^{(h_{(m)})*} \right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} \left\| \mathbf{u}_{(m)} \right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}
\end{aligned}
$$

by virtue of (2.3.2) and (2.5.5). Since the extended functions of the initial data have explicit representations

$$\frac{\partial}{\partial y_i}\mathbf{v}_0^{(h_0)*}(y',y_3) = \frac{\partial \mathbf{v}_0}{\partial x_i}(y',\tilde{X}_3(y',y_3,0;h_0))(1-\delta_{i3}) + \frac{\partial \mathbf{v}_0}{\partial x_3}(y',\tilde{X}_3(y',y_3,0;h_0))\frac{\partial \tilde{X}_3}{\partial y_i}(y',y_3,0;h_0) \quad (i=1,2,3),$$

we get

$$\begin{aligned}\left\| L_{1,h_{(m)}}\bar{\mathbf{u}}_0 - \frac{\partial \bar{\mathbf{u}}_0}{\partial t}\right\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} &\le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\bar{\mathbf{u}}_0\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\\ &\le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{u}_0^{(h_0)*}\|^2_{W_2^{1+l}(\tilde{\Omega})}\\ &\le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{v}_0\|^2_{W_2^{1+l}(\Omega)}.\end{aligned}$$

Moreover, the first term in the right-hand side of $\mathbf{l}_1^{(m+1)}$ is estimated by noting the following inequalities:

$$\begin{aligned}&\|(l_{11,h_{(m)}} - l_{11,\bar{h}_0})\mathbf{u}'_{(m+1)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}\\ &\qquad \le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{u}'_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)},\end{aligned}$$

and in general

$$\begin{aligned}&\|(f^{(h_{(m)})*} - f^{(\bar{h}_0)*})\nabla^2\mathbf{u}'_{(m+1)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}\\ &\quad \le (\epsilon + C_\epsilon t)P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|f\|_{W_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_t^3)}\|\mathbf{u}'_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\end{aligned}$$

for $f \in W_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_t^3)$. Consequently, with the aid of Lemmas 2.3.3–2.3.5, we have the desired estimate. The terms $l_4^{(m+1)}$, $l_5^{(m+1)}$ are estimated in exactly the same way as $\mathbf{l}_1^{(m+1)}$. □

Lemma 2.5.2. *For $\mathbf{l}_2^{(m+1)}$, we have*

$$\begin{aligned}\|\mathbf{l}_2^{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{st})} &\le (\epsilon + C_\epsilon t)\left(\|\mathbf{u}_{(m)}\|^{\alpha+1}_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|^3_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)\\ &\quad + (\epsilon + C_\epsilon t)\,P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{u}_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\\ &\quad + P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}.\end{aligned}$$

Proof. Note that

$$\| |\mathbf{u}_{(m)}|^\alpha |_{y_3=p_0} \|_{W_2^{\frac{1}{2}+l, \frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \leq C \|\mathbf{u}_{(m)}\|^2_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_t)} \left(1 + \|\mathbf{u}_{(m)}\|^{\alpha-2}_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_t)} \right) \tag{2.5.6}$$

holds for $\alpha \geq 2$. Indeed, we first show

$$\| |\mathbf{u}_{(m)}|^\alpha |_{y_3=p_0} \|_{W_2^{\frac{1}{2}+l,0}(\mathbf{R}_t^2)} \leq C \|\mathbf{u}_{(m)}\|^2_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_t)} \left(1 + \|\mathbf{u}_{(m)}\|^{\alpha-2}_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_t)} \right). \tag{2.5.7}$$

Since in the case $\alpha = 2$, (2.5.7) is obvious by (2.5.4), we consider the case $\alpha > 2$, *i.e.*, $\alpha = 2 + \delta$ with $\delta > 0$. Then we have

$$\begin{aligned}
&\left| \frac{\partial}{\partial y_i} \left(|\mathbf{u}_{(m)}|^\alpha \right) (y^{1\prime}, p_0, t) - \frac{\partial}{\partial y_i} \left(|\mathbf{u}_{(m)}|^\alpha \right) (y^{2\prime}, p_0, t) \right|^2 \\
&= \left| \alpha \left[|\mathbf{u}_{(m)}|^\delta \mathbf{u}_{(m)} \cdot \frac{\partial \mathbf{u}_{(m)}}{\partial y_i} (y^{1\prime}, p_0, t) - |\mathbf{u}_{(m)}|^\delta \mathbf{u}_{(m)} \cdot \frac{\partial \mathbf{u}_{(m)}}{\partial y_i} (y^{2\prime}, p_0, t) \right] \right|^2 \\
&\leq C \Bigg[\left| \left|\mathbf{u}_{(m)}\right|^\delta \mathbf{u}_{(m)}(y^{1\prime}, p_0, t) - \left|\mathbf{u}_{(m)}\right|^\delta \mathbf{u}_{(m)}(y^{2\prime}, p_0, t) \right|^2 \left| \frac{\partial \mathbf{u}_{(m)}}{\partial y_i} (y^{1\prime}, p_0, t) \right|^2 \\
&\qquad + |\mathbf{u}_{(m)}(y^{2\prime}, p_0, t)|^{2\delta+2} \left| \frac{\partial \mathbf{u}_{(m)}}{\partial y_i} (y^{1\prime}, p_0, t) - \frac{\partial \mathbf{u}_{(m)}}{\partial y_i} (y^{2\prime}, p_0, t) \right|^2 \Bigg].
\end{aligned}$$

The first term is estimated by using the mean value theorem, so that

$$\begin{aligned}
&\left| \left|\mathbf{u}_{(m)}\right|^\delta \mathbf{u}_{(m)}(y^{1\prime}, p_0, t) - \left|\mathbf{u}_{(m)}\right|^\delta \mathbf{u}_{(m)}(y^{2\prime}, p_0, t) \right|^2 \\
&\leq C \left[|\mathbf{u}_{(m)}(y^{1\prime}, p_0, t)| + |\mathbf{u}_{(m)}(y^{2\prime}, p_0, t)| \right]^{2\delta} \\
&\qquad \times \left| \mathbf{u}_{(m)}(y^{1\prime}, p_0, t) - \mathbf{u}_{(m)}(y^{2\prime}, p_0, t) \right|^2
\end{aligned}$$

This makes it possible to get the desired estimate for the first term. The second term and the lower order norm $\left\| \left|\mathbf{u}_{(m)}\right|^\alpha |_{y_3=p_0} \right\|_{L_2(0,t;W_2^1(\mathbf{R}^2))}$ are estimated easily, and finally we have (2.5.7). For $\left\| \left|\mathbf{u}_{(m)}\right|^\alpha |_{y_3=p_0} \right\|_{W_2^{0,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$, we have

$$\begin{aligned}
\left\| \left|\mathbf{u}_{(m)}\right|^\alpha |_{y_3=p_0} \right\|_{L_2(\mathbf{R}^2; W_2^{\frac{1}{4}+\frac{l}{2}}(0,t))} &\leq C \left\| \left|\mathbf{u}_{(m)}\right|^\alpha |_{y_3=p_0} \right\|_{L_2(\mathbf{R}^2; W_2^1(0,t))} \\
&\leq C \|\mathbf{u}_{(m)}\|^\alpha_{W_2^{2+l, 1+\frac{l}{2}}(\tilde{\Omega}_t)}.
\end{aligned}$$

Thus we have (2.5.6), and consequently

$$\||\mathbf{u}_m|^\alpha \mathbf{u}_m\big|_{y_3=p_0}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \le C\|\mathbf{u}_{(m)}\|^3_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\left(1+\|\mathbf{u}_{(m)}\|^{\alpha-2}_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right).$$

Other terms in $\mathbf{l}_2^{(m+1)}$ are estimated in the same way as the corresponding terms in $\mathbf{l}_1^{(m+1)}$. Thus we have the assertion of the lemma. □

Lemma 2.5.3. *For $l_3^{(m+1)}$, we have*

$$\begin{aligned}\|l_3^{(m+1)}\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Gamma}_{st})} &\le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{u}_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \\ &\quad + (\epsilon + C_\epsilon t)\,P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right) \\ &\quad \times\left[\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right].\end{aligned}$$

Proof. Since

$$\begin{aligned}&\|\tilde{G}_{3,h_{(m)}}(\mathbf{u}_{(m+1)},\mathbf{u}_{(m)})\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_t)} \\ &\le \left\|\big(\nabla+\mathbf{a}^3(h_{(m)})\big)\cdot\frac{\partial \mathbf{u}_{(m+1)}}{\partial y_3} + a^{33}(h_{(m)})\mathbf{F}_5^{(h_{(m)})*}\cdot\frac{\partial \mathbf{u}_{(m)}}{\partial y_3}\right\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_t)} \\ &\le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{u}_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \\ &+ (\epsilon+C_\epsilon t)\,P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{F}_5^{(h_{(m)})*}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\|\mathbf{u}_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\end{aligned}$$

by virtue of (2.5.4) and (2.5.5), we have the assertion of the lemma. □

Lemma 2.5.4. *For $l_6^{(m+1)}$, we have*

$$\begin{aligned}\|l_6^{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} &\le \|(L_{4,h_{(m)}} - L_{4,\bar{h}_0})h'_{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \\ &\quad + \|L_{4,h_{(m)}}\bar{h}_0\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} + \left\|\frac{\partial \bar{h}_0}{\partial t}\right\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} + \|\tilde{G}_{6,h_{(m)}}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\end{aligned}$$

$$\leq (\epsilon + C_\epsilon t)\, P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\left(1 + \|h'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\tilde{\mathbf{R}}_t^2)}\right)$$

$$\times\left[1 + \left(1 + \|\tilde{\theta}'_{(m)}\|_{W_2^{2+l',\frac{2+l'}{2}}(\tilde{\Omega}_t)}\right)\left(\|\mathbf{u}_{(m)}\|^\alpha_{W_2^{2+l',\frac{2+l'}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|^2_{W_2^{2+l',\frac{2+l'}{2}}(\tilde{\Omega}_t)}\right)\right]$$

$$+ C\left(1 + \|\theta_e\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)}\right)\left(1 + \|\tilde{\theta}'_{(m+1)}\|_{W_2^{2+l',\frac{2+l'}{2}}(\tilde{\Omega}_t)}\right)$$

$$+ P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|d_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)}.$$

Proof. We begin with estimating the term $\tilde{G}_{6,h_{(m)}}$. First, it is easy to get

$$\|la(\theta_e)\|^2_{\overline{W}_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}$$

$$\leq C|\!|\!|la|\!|\!|^2\left(1 + \|\Psi(\cdot; h_{(m)})\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)^2\left(1 + \|\theta_e\|^2_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)}\right)^2.$$

Second, the term containing $|\mathbf{u}_{(m)}|^\alpha$ is estimated in exactly the same way as $\tilde{\mathbf{G}}_2(\mathbf{u}_{(m)})$. Indeed, we have

$$\left\||\mathbf{u}_m|^\alpha\tilde{\theta}_m\big|_{y_3=p_0}\right\|_{W_2^{\frac{1}{2}+l,0}(\mathbf{R}_t^2)}$$

$$\leq C\left(1 + \|\tilde{\theta}'_m\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)\left(\|\mathbf{u}_m\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_m\|^\alpha_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)$$

by making use of (2.5.7), and we also have

$$\left\||\mathbf{u}_m|^\alpha\tilde{\theta}_m\big|_{y_3=p_0}\right\|_{W_2^{0,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \leq C\left(1 + \|\tilde{\theta}'_m\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)\|\mathbf{u}_m\|^\alpha_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}.$$

Then we easily obtain

$$\|\tilde{G}_{6,h_{(m)}}(\mathbf{u}_{(m)}, \tilde{\theta}_{(m+1)}, \tilde{\theta}_{(m)})\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \leq (\epsilon + C_\epsilon t)P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)$$

$$\times\left[1 + \left(1 + \|\tilde{\theta}'_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)\left(\|\mathbf{u}_{(m)}\|^\alpha_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)\right.$$

$$\left. + \|\tilde{\theta}'_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right],$$

and finally we have the desired estimate. □

2.5.2 Proof of Theorem 2.2.1

Now we proceed to the proof of Theorem 2.2.1. Introduce the notation

$$E_m(t) := \|(\mathbf{u}'_{(m)}, u'_{3(m)}, \tilde{\theta}'_{(m)}, \tilde{S}'_{(m)}, h'_{(m)})\|_{Z(t)},\ E'_m(t) := \|(\mathbf{u}'_{(m)}, \tilde{\theta}'_{(m)}, \tilde{S}'_{(m)})\|_{Z'(t)},$$

where $Z'(T) = W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$ as defined in the statement of Lemma 2.4.1. Applying Lemmas 2.4.1 and 2.4.2 to problems (2.5.1)–(2.5.3), and making use of Lemmas 2.5.1–2.5.4, we arrive at the inequalities

$$E'_{m+1}(t) \le C_1\Big[1 + (\epsilon + C_\epsilon t)\Big\{\phi_1(E_m(t)) + \phi_2(E_m(t))E'_{m+1}(t)\Big\}\Big] \tag{2.5.8}$$

and

$$\begin{aligned}
&\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|h'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \\
&\quad \le \tilde{C}_2\Big[1 + \phi_3(E_m(t))\Big(\|\mathbf{u}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{\theta}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}\Big) \\
&\qquad + (\epsilon + C_\epsilon t)\Big\{\phi_1(E_m(t)) + \phi_2(E_m(t)) \\
&\qquad\quad \times \Big(\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|h'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big)\Big\}\Big]
\end{aligned} \tag{2.5.9}$$

for any $t \in (0, T]$, where $\tilde{C}_2 = C_2 + C_3$ and ϕ_i $(i = 1, 2, 3)$ are monotonically increasing. Adding (2.5.8) and (2.5.9) multiplied by $1/(2\tilde{C}_2\phi_3(E_m(t))$, we get the inequality

$$E_{m+1}(t) \le C_4(t)\Big[1 + (\epsilon + C_\epsilon t)\Big\{\phi_1(E_m(t)) + \phi_2(E_m(t))E_{m+1}(t)\Big\}\Big]$$

with some constant $C_4(t)$ depending on t monotonically increasingly.

Let a positive constant M such that $C_4(T) < M$. Take ϵ first small enough so that

$$\epsilon C_4(T)\phi_2(M) < 1, \quad \epsilon C_4(T)\Big[\phi_1(M) + \phi_2(M)M\Big] < M - C_4(T)$$

hold, and then $T_1 \in (0, T]$ so that

$$C_4(T)C_\epsilon\phi_2(M)T_1 < 1 - C_4(T)\epsilon\phi_2(M),$$

$$C_4(T)C_\epsilon T_1\left\{\phi_1(M)+\phi_2(M)M\right\} < M - C_4(T) - \epsilon C_4(T)\left\{\phi_1(M)+\phi_2(M)M\right\}$$

hold. Consequently we obtain

$$E_{m+1}(T_1) < \frac{C_4(T)\Big\{(\epsilon + C_\epsilon T_1)\phi_1(M)+1\Big\}}{1 - C_4(T)(\epsilon + C_\epsilon T_1)\phi_2(M)} < M$$

from the assumption $E_m(T_1) < M$. By induction the sequence $\{(\mathbf{u}'_{(m)}, u'_{3(m)}, \tilde{\theta}'_{(m)}, \tilde{S}'_{(m)}, h'_{(m)})\}_{m=0}^{\infty}$ is well defined in $Z(T_1)$ and $E_m(T_1) < M$ for $m = 0, 1, 2, \ldots$.

Now we prove its convergence. Subtract (2.5.1)–(2.5.3) with m replaced by $(m-1)$ from (2.5.1)–(2.5.3). Then

$$\begin{aligned}
&(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{u}'_{3(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)}, \tilde{h}'_{(m+1)}) \\
&:= (\mathbf{u}'_{(m+1)} - \mathbf{u}'_{(m)}, u'_{3(m+1)} - u'_{3(m)}, \tilde{\theta}'_{(m+1)} - \tilde{\theta}'_{(m)}, \tilde{S}'_{(m+1)} - \tilde{S}'_{(m)}, h'_{(m+1)} - h'_{(m)})
\end{aligned}$$

satisfies the equations

$$\begin{cases}
\dfrac{\partial \tilde{\mathbf{u}}'_{(m+1)}}{\partial t} - L_{1,\bar{h}_0}\tilde{\mathbf{u}}'_{(m+1)} = \mathbf{l}_1^{(m+1)} - \mathbf{l}_1^{(m)}, \\[2ex]
\nabla_{h_0,3}\tilde{u}'_{3(m+1)} - (\nabla_{h_0,3}\tilde{F}_{13}^{(h_0)*})\dfrac{\tilde{u}'_{3(m+1)}}{\tilde{F}_{13}^{(h_0)*}} = l_3^{(m+1)} - l_3^{(m)}, \\[2ex]
\dfrac{\partial \tilde{\tilde{\theta}}'_{(m+1)}}{\partial t} - L_{2,\bar{h}_0}\tilde{\tilde{\theta}}'_{(m+1)} = l_4^{(m+1)} - l_4^{(m)}, \\[2ex]
\dfrac{\partial \tilde{\tilde{S}}'_{(m+1)}}{\partial t} - L_{3,\bar{h}_0}\tilde{\tilde{S}}'_{(m+1)} = l_5^{(m+1)} - l_5^{(m)} \quad \text{in } \tilde{\Omega}_{T_1},
\end{cases}$$

$$\begin{cases}
B_{\bar{h}_0}\tilde{\mathbf{u}}'_{(m+1)} = \mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}, \\[2ex]
(\tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)}) = (\theta_e|_{x_3=\Psi(\cdot;h_{(m)})} - \theta_e|_{x_3=\Psi(\cdot;h_{(m-1)})}, \\[2ex]
\qquad\qquad S_e|_{x_3=\Psi(\cdot;h_{(m)})} - S_e|_{x_3=\Psi(\cdot;h_{(m-1)})}) \quad \text{on } \tilde{\Gamma}_{sT_1}, \\[2ex]
(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{u}'_{3(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)}) = (\mathbf{0}, 0, 0, 0) \quad \text{on } \tilde{\Gamma}_{bT_1}, \\[2ex]
(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)})|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega},
\end{cases}$$

$$\begin{cases} \dfrac{\partial \tilde{h}'_{(m+1)}}{\partial t} - L_{4,\bar{h}_0}\tilde{h}'_{(m+1)} = l_6^{(m+1)} - l_6^{(m)} \quad \text{in } \mathbf{R}^2_{T_1}, \\ \tilde{h}'_{(m+1)}\big|_{t=0} = 0 \quad \text{on } \mathbf{R}^2. \end{cases}$$

Then Lemmas 2.4.1 and 2.4.2 yield for any $t \le T_1$ the estimates

$$\begin{aligned} &\|(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)})\|_{Z'(t)} \\ &\quad \le C_1 \Big[\|\mathbf{l}_1^{(m+1)} - \mathbf{l}_1^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}\|_{W_2^{\frac12+l,\frac14+\frac{l}{2}}(\tilde{\Gamma}_{st})} \\ &\qquad + \|l_4^{(m+1)} - l_4^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} + \|l_5^{(m+1)} - l_5^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\ &\qquad + \|\theta_e(y', \Psi(y',t;h_{(m)}),t) - \theta_e(y', \Psi(y',t;h_{(m-1)}),t)\|_{W_2^{\frac32+l,\frac34+\frac{l}{2}}(\mathbf{R}^2_t)} \\ &\qquad + \|S_e(y', \Psi(y',t;h_{(m)}),t) - S_e(y', \Psi(y',t;h_{(m-1)}),t)\|_{W_2^{\frac32+l,\frac34+\frac{l}{2}}(\mathbf{R}^2_t)} \Big], \end{aligned}$$

$$\|\tilde{h}'_{(m+1)}\|_{W_2^{\frac52+l,\frac54+\frac{l}{2}}(\mathbf{R}^2_t)} \le C_2 \|l_6^{(m+1)} - l_6^{(m)}\|_{W_2^{\frac12+l,\frac14+\frac{l}{2}}(\mathbf{R}^2_t)},$$

$$\|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} \le C_3 \|l_3^{(m+1)} - l_3^{(m)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}.$$

Each term in the right-hand side of the above inequalities except for $\|\mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}\|_{W_2^{\frac12+l,\frac14+\frac{l}{2}}(\tilde{\Gamma}_{st})}$ and $\|l_6^{(m+1)} - l_6^{(m)}\|_{W_2^{\frac12+l,\frac14+\frac{l}{2}}(\mathbf{R}^2_t)}$ can be estimated just as we have done for $\|\mathbf{l}_1^{(m+1)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}$, $\|l_i^{(m+1)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}$ $(i = 4, 5)$ and $\|l_3^{(m+1)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}$. Then we have

$$\begin{aligned} &\|\mathbf{l}_1^{(m+1)} - \mathbf{l}_1^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} + \sum_{i=4}^{5} \|l_i^{(m+1)} - l_i^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\ &\quad \le (\epsilon + C_\epsilon t) \Big[P\Big(\big\|\big(\mathbf{u}'_{(m+1)}, \mathbf{u}'_{(m)}, \mathbf{u}'_{(m-1)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}, \|u'_{3(m-1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}, \\ &\qquad\qquad \big\|\big(\tilde{\theta}'_{(m+1)}, \tilde{\theta}'_{(m)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}, \big\|\big(\tilde{S}'_{(m+1)}, \tilde{S}'_{(m)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}, \\ &\qquad\qquad \big\|\big(h'_{(m)}, h'_{(m-1)}\big)\big\|_{W_2^{\frac52+l,\frac54+\frac{l}{2}}(\mathbf{R}^2_t)} \Big) \Big\|\big(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\tilde{\theta}}'_{(m)}, \tilde{\tilde{S}}'_{(m)}, \tilde{h}'_{(m)}\big)\Big\|_{Z(t)} \end{aligned}$$

$$+ P\left(\|\tilde{h}'_{(m-1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right) \left\|(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)})\right\|_{Z'(t)}\Bigg],$$

$$\|l_3^{(m+1)} - l_3^{(m)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}$$

$$\leq (\epsilon + C_\epsilon t)\Bigg[P\Bigg(\left\|(\mathbf{u}'_{(m+1)}, \mathbf{u}'_{(m)})\right\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}, \|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)},$$

$$\left\|(h'_{(m)}, h'_{(m-1)})\right\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Bigg)$$

$$\times\left(\|\tilde{\mathbf{u}}'_{(m)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|\tilde{h}'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)$$

$$+ P\left(\|\tilde{h}'_{(m-1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\tilde{\mathbf{u}}'_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\Bigg].$$

In estimating $\left\|l_6^{(m+1)} - l_6^{(m)}\right\|_{W_2^{\frac{1}{2}+l,0}(\mathbf{R}_t^2)}$, the terms except the one containing $\left|\mathbf{u}_{(m)}\right|^\alpha \theta_{(m)} - \left|\mathbf{u}_{(m-1)}\right|^\alpha \theta_{(m-1)}$ are rather easy to do. Hence we show only its estimates. Let $f(\mathbf{u},\theta) := |\mathbf{u}|^\alpha\theta|_{y3=p_0}$. Then the mean value theorem implies

$$\begin{aligned}
&f(\mathbf{u}_{(m)}, \tilde{\theta}_{(m)}) - f(\mathbf{u}_{(m-1)}, \tilde{\theta}_{(m-1)}) \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} f\left(s(\mathbf{u}_{(m)}, \tilde{\theta}_{(m)}) + (1-s)(\mathbf{u}_{(m-1)}, \tilde{\theta}_{(m-1)})\right)\mathrm{d}s \\
&= \int_0^1 \left[\sum_{i=1}^2 \frac{\partial f}{\partial u_i}(\mathbf{u}_s, \theta_s)(u_{(m)i} - u_{(m-1)i}) + \frac{\partial f}{\partial \theta}(\mathbf{u}_s, \theta_s)(\tilde{\theta}_{(m)} - \tilde{\theta}_{(m-1)})\right]\mathrm{d}s \\
&= \int_0^1 \left[\alpha\,|\mathbf{u}_s|^{\alpha-2}\,\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s + |\mathbf{u}_s|^\alpha\,\tilde{\tilde{\theta}}'_{(m)}\right]\mathrm{d}s,
\end{aligned}$$

where $\mathbf{u}_s = s\mathbf{u}_{(m)} + (1-s)\mathbf{u}_{(m-1)}$, $\theta_s = s\tilde{\theta}_{(m)} + (1-s)\tilde{\theta}_{(m-1)}$. For the estimate $\|f(\mathbf{u}_{(m)}, \theta_{(m)}) - f(\mathbf{u}_{(m-1)}, \theta_{(m-1)})\|_{L_2(0,t;\dot{W}_2^{\frac{1}{2}+l}(\mathbf{R}^2))}$, it is sufficient to estimate the term

$$\int_0^1 \left[\frac{\partial}{\partial y_i}\left(|\mathbf{u}_s|^{\alpha-2}\,\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s\right)(y^{1\prime}, p_0, t) - \frac{\partial}{\partial y_i}\left(|\mathbf{u}_s|^{\alpha-2}\,\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s\right)(y^{2\prime}, p_0, t)\right]\mathrm{d}s.$$

Let

$$G_i(t,y') := \frac{\partial}{\partial y_i}\int_0^1 |\mathbf{u}_s|^{\alpha-2}\,\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s \mathrm{d}s$$

$$= \int_0^1 \Bigg[(\alpha-2)\,|\mathbf{u}_s|^{\alpha-4}\left(\mathbf{u}_s\cdot\frac{\partial \mathbf{u}_s}{\partial y_i}\right)(\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s$$

$$+\,|\mathbf{u}_s|^{\alpha-2}\,\frac{\partial}{\partial y_i}\left((\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s\right)\Bigg]\mathrm{d}s,$$

and estimate

$$\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\frac{|G_i(y^{1\prime},t)-G_i(y^{2\prime},t)|^2}{|y^{1\prime}-y^{2\prime}|^{1+2l}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}$$

$$=\int\int_{|y^{1\prime}-y^{2\prime}|>1}\frac{|G_i(y^{1\prime},t)-G_i(y^{2\prime},t)|^2}{|y^{1\prime}-y^{2\prime}|^{1+2l}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}$$

$$+\int\int_{|y^{1\prime}-y^{2\prime}|\le 1}\frac{|G_i(y^{1\prime},t)-G_i(y^{2\prime},t)|^2}{|y^{1\prime}-y^{2\prime}|^{1+2l}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}. \tag{2.5.10}$$

Estimating the second term in (2.5.10) is more difficult than the first one, and we first estimate it. Denoting $K_i(t,y') := |\mathbf{u}_s|^{\alpha-4}\left(\mathbf{u}_s\cdot\dfrac{\partial \mathbf{u}_s}{\partial y_i}\right)(t,y',p_0)$, we have

$$|G_i(y^{1\prime},t)-G_i(y^{2\prime},t)|^2$$

$$\le C\Bigg[\int_0^1 \left|K_i(y^{1\prime},t)-K_i(y^{2\prime},t)\right|^2 |\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}(y^{1\prime},p_0,t)|^2|\theta_s(y^{1\prime},p_0,t)|^2\,\mathrm{d}s$$

$$+\int_0^1 |K_i(y^{2\prime},t)|^2|\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}(y^{1\prime},p_0,t)-\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}(y^{2\prime},p_0,t)|^2|\theta_s(y^{1\prime},p_0,t)|^2\,\mathrm{d}s$$

$$+\int_0^1 |K_i(y^{2\prime},t)|^2|\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}(y^{2\prime},p_0,t)|^2|\theta_s(y^{1\prime},p_0,t)-\theta_s(y^{2\prime},p_0,t)|^2\,\mathrm{d}s$$

$$+\int_0^1 \left||\mathbf{u}_s|^{\alpha-2}(y^{1\prime},p_0,t)-|\mathbf{u}_s|^{\alpha-2}(y^{2\prime},p_0,t)\right|^2\left|\frac{\partial}{\partial y_i}\left((\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s\right)(y^{1\prime},p_0,t)\right|^2\mathrm{d}s$$

$$+\int_0^1 |\mathbf{u}_s(y^{2\prime},p_0,t)|^{2(\alpha-2)}\Bigg|\frac{\partial}{\partial y_i}\left((\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s\right)(y^{1\prime},p_0,t)$$

$$-\frac{\partial}{\partial y_i}\left((\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s\right)(y^{2\prime},p_0,t)\Bigg|^2\mathrm{d}s\Bigg]. \tag{2.5.11}$$

Among the terms in the right-hand side of (2.5.11), the first term is the most difficult due to its singularity, so that we show its estimate. Take $0<\sigma<1$, which will be determined later. Then, by applying the mean value theorem, there exists $0 < \hat{s} < 1$ such that

$$\begin{aligned}&\left|K_i(y^{1\prime},t) - K_i(y^{2\prime},t)\right|^2 \\ &\qquad = |\nabla K_i(\,\hat{s}y^{1\prime} + (1-\hat{s})y^{2\prime},t)|^{2\sigma}|y^{1\prime} - y^{2\prime}|^{2\sigma}\left|K_i(y^{1\prime},t) - K_i(y^{2\prime},t)\right|^{2-2\sigma}.\end{aligned}$$

Taking into account

$$\frac{\partial}{\partial y_j}K_i = (\alpha-4)\,|\mathbf{u}_s|^{\alpha-6}\left(\mathbf{u}_s\cdot\frac{\partial \mathbf{u}_s}{\partial y_j}\right)\left(\mathbf{u}_s\cdot\frac{\partial \mathbf{u}_s}{\partial y_i}\right) + |\mathbf{u}_s|^{\alpha-4}\frac{\partial}{\partial y_j}\left(\mathbf{u}_s\cdot\frac{\partial \mathbf{u}_s}{\partial y_i}\right),$$

and putting $z' := y^{1\prime} - y^{2\prime}$ in (2.5.11), we have for the first term,

$$\begin{aligned}&\int_0^t \mathrm{d}t\int_{\mathbf{R}^2}\int_{|z'|\le 1}\frac{|K_i(y^{1\prime},t) - K_i(y^{1\prime}+z',t)|^2|\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s(y^{1\prime},p_0,t)|^2}{|z'|^{1+2l}}\,\mathrm{d}y^{1\prime}\mathrm{d}z' \\
&\le C\sum_{j=1}^{3}\int_0^t \mathrm{d}t\int_{\mathbf{R}^2}\int_{|z'|\le 1}\frac{1}{|z'|^{1+2l}} \\
&\quad\times\Bigg[|\mathbf{u}_s(y^{1\prime}+z',p_0,t)|^{2\sigma(\alpha-4)}\left|\frac{\partial\mathbf{u}_s}{\partial y_j}(y^{1\prime}+z',p_0,t)\right|^{2\sigma}\left|\frac{\partial\mathbf{u}_s}{\partial y_i}(y^{1\prime}+z',p_0,t)\right|^{2\sigma} \\
&\qquad\qquad + |\mathbf{u}_s(y^{1\prime}+z',p_0,t)|^{2\sigma(\alpha-3)}\left|\frac{\partial^2\mathbf{u}_s}{\partial y_i\partial y_j}(y^{1\prime}+z',p_0,t)\right|^{2\sigma}\Bigg] \\
&\quad\times 2\left(\sup_{\mathbf{R}^2_t}|\mathbf{u}_s(\cdot,p_0,\cdot)|^{\alpha-3}\sup_{\mathbf{R}^2}\left|\frac{\partial\mathbf{u}_s}{\partial y_i}(\cdot,p_0,t)\right|\right)^{2-2\sigma} \\
&\quad\times|z'|^{2\sigma}|\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s(y^{1\prime},p_0,t)|^2\,\mathrm{d}z'\mathrm{d}y^{1\prime} \\
&\le C\sum_{j=1}^{3}\sup_{\mathbf{R}^2_t}|\theta_s(\cdot,p_0,\cdot)|^2\int_{|z'|\le 1}\frac{|z'|^{2\sigma}}{|z'|^{1+2l}}\,\mathrm{d}z' \\
&\quad\times\Bigg[\sup_{\mathbf{R}^2_t}|\mathbf{u}_s(\cdot,p_0,\cdot)|^{2(\alpha-\sigma-2)}\sup_t\int_{\mathbf{R}^2}|\tilde{\mathbf{u}}'_{(m)}(y^{1\prime},p_0,t)|^2\,\mathrm{d}y^{1\prime} \\
&\quad\times\left(\int_0^t\sup_{\mathbf{R}^2}\left|\frac{\partial\mathbf{u}_s}{\partial y_i}(\cdot,p_0,t)\right|^{2p}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^t\sup_{\mathbf{R}^2}\left|\frac{\partial\mathbf{u}_s}{\partial y_j}(\cdot,p_0,t)\right|^{2\sigma p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} \\
&\quad+\sup_{\mathbf{R}^2_t}|\mathbf{u}_s(\cdot,p_0,\cdot)|^{2(\alpha-2)}\sup_{\mathbf{R}^2_t}|\tilde{\mathbf{u}}'_{(m)}(\cdot,p_0,\cdot)|^2\end{aligned}$$

$$\times \left(\left(\int_0^t \sup_{\mathbf{R}^2} \left| \frac{\partial \mathbf{u}_s}{\partial y_i}(\cdot, p_0, t) \right|^{(2-2\sigma)q} \mathrm{d}t \right)^{\frac{1}{q}} \left(\int_0^t \left\| \frac{\partial^2 \mathbf{u}_s}{\partial y_i \partial y_j}(\cdot, p_0, t) \right\|_{L_2(\mathbf{R}^2)}^{2\sigma q'} \mathrm{d}t \right)^{\frac{1}{q'}} \right]. \tag{2.5.12}$$

Here we applied the Hölder inequality, and $1/p + 1/p' = 1/q + 1/q' = 1$. It is to be noted that the integral

$$\int_{|z'| \le 1} \frac{|z'|^{2\sigma}}{|z'|^{1+2l}} \, \mathrm{d}z' = \int_0^1 \frac{r^{2\sigma+1}}{r^{1+2l}} \, \mathrm{d}r$$

is determined as a finite value for $\sigma > l - 1/2$ and $W_2^{\frac{l}{2}-\frac{1}{4}}(0,t) \subset L_{2p}(0,t) \bigcap L_{2\sigma p'}(0,t) \bigcap L_{(2-2\sigma)q}(0,t) \bigcap L_{2\sigma q'}(0,t)$ with $\sigma \le \frac{2\eta}{1-2\eta}$, $\eta = l/2 - 1/4$, $1 \le p \le \frac{1}{1-2\eta}$, $\frac{1}{1-2\sigma\eta'} \le q < \frac{1}{(2-2\sigma)\eta'}$, $\eta' = 3/4 - l/2$. Now if we take σ such that $l - 1/2 < \sigma \le \min\{\alpha - 2, \frac{2\eta}{1-2\eta}\}$, then the right most hand side of (2.5.12) is determined as a finite value. The first term in (2.5.10) can be estimated in the same manner by taking $\sigma = 0$. These give an estimate of $\left\| f(\mathbf{u}_{(m)}, \tilde{\theta}_{(m)}) - f(\mathbf{u}_{(m-1)}, \tilde{\theta}_{(m-1)}) \right\|_{L_2(0,t;\dot{W}_2^{\frac{1}{2}+l,0}(\mathbf{R}_t^2))}$. Adding this to the lower order norms, which are easy to get, yields the estimate of $\| f(\mathbf{u}_{(m)}, \tilde{\theta}_{(m)}) - f(\mathbf{u}_{(m-1)}, \tilde{\theta}_{(m-1)}) \|_{W_2^{\frac{1}{2}+l,0}(\mathbf{R}_t^2)}$. Similarly one can obtain the estimate of $\| f(\mathbf{u}_{(m)}, \tilde{\theta}_{(m)}) - f(\mathbf{u}_{(m-1)}, \tilde{\theta}_{(m-1)}) \|_{W_2^{0,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$. Consequently, we have

$$\begin{aligned} \| l_6^{(m+1)} - l_6^{(m)} \|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \le (\epsilon + C_\epsilon t) \, P\Big(& \| (\mathbf{u}'_{(m)}, \mathbf{u}'_{(m-1)}) \|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}, \\ & \| (\tilde{\theta}'_{(m)}, \tilde{\theta}'_{(m-1)}) \|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}, \| h'_{(m)} \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \Big) \\ & \times \left(\| \tilde{\mathbf{u}}'_{(m)} \|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \| \tilde{\theta}'_{(m)} \|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \| \tilde{h}'_{(m)} \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \\ & + P\left(\| h'_{(m-1)} \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \| \tilde{\tilde{\theta}}'_{(m+1)} \|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}. \end{aligned}$$

$\left\| \mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)} \right\|_{W_2^{\frac{1}{2}+l,\frac{l}{2}+\frac{1}{4}}(\tilde{\Gamma}_{st})}$ is estimated in exactly the same way as above,

$$\begin{aligned} \left\| \mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)} \right\|_{W_2^{\frac{1}{2}+l,\frac{l}{2}+\frac{1}{4}}(\tilde{\Gamma}_{st})} \le (\epsilon + C_\epsilon t) \, P\Big(& \| (\mathbf{u}'_{(m)}, \mathbf{u}'_{(m-1)}) \|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}, \\ & \| h'_{(m)} \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \Big) \left(\| \tilde{\mathbf{u}}'_{(m)} \|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \| \tilde{h}'_{(m)} \|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right). \end{aligned}$$

In this case, we need only $\alpha > l+1/2$. Using the notation $Z'(T) = W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$ again and denoting

$$\tilde{E}_m(t) := \|(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\tilde{\theta}}'_{(m)}, \tilde{\tilde{S}}'_{(m)}, \tilde{h}'_{(m)})\|_{Z(t)}, \;\; \tilde{E}'_m(t) := \|(\tilde{\mathbf{u}}'_{(m)}, \tilde{\tilde{\theta}}'_{(m)}, \tilde{\tilde{S}}'_{(m)})\|_{Z'(t)},$$

we get for any $t \in (0, T_1]$,

$$\begin{aligned} \tilde{E}'_{m+1}(t) \le C_1(\epsilon + C_\epsilon t)\Big[&\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)\tilde{E}'_m(t) \\ &+ \phi_5(E_{m-1}(T_1))\tilde{E}'_{m+1}(t) + \phi_6(E_{m+1}(T_1) + E_m(T_1))\|\tilde{h}'_{(m)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big], \end{aligned} \tag{2.5.13}$$

$$\begin{aligned} &\|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|\tilde{h}'_{(m+1)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \\ &\le \tilde{C}_2\Big[\phi_7(E_m(T_1))\Big\{\|\tilde{\mathbf{u}}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{\tilde{\theta}}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}\Big\} \\ &+ (\epsilon + C_\epsilon t)\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)\tilde{E}_m(t) \\ &+ (\epsilon + C_\epsilon t)\phi_5(E_{m-1}(T_1))\Big\{\|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|h'_{(m+1)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big\}\Big], \end{aligned} \tag{2.5.14}$$

where $\tilde{C}_2 = C_2 + C_3$ and ϕ_i $(i = 4, 5, 6, 7)$ are monotonically increasing in their arguments.

Adding (2.5.13) and (2.5.14) multiplied by $1/(2\tilde{C}_2\phi_7(E_m(T_1))$, we get the estimate

$$\begin{aligned} \tilde{E}_{m+1}(t) \le C_5(T_1)(\epsilon + C_\epsilon t)\big\{&\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)\tilde{E}_m(t) \\ &+\phi_5(E_{m-1}(T_1))\tilde{E}_{m+1}(t) + \phi_6(E_{m+1}(T_1) + E_m(T_1))\tilde{E}_m(t)\big\} \end{aligned} \tag{2.5.15}$$

for any $t \in (0, T_1]$ with $C_5(t)$ having the same property as $C_4(t)$. Taking ϵ small enough again so that

$$\epsilon C_5(T_1)\Big[\phi_4(3M) + \phi_5(M) + \phi_6(2M)\Big] < 1$$

holds, and then $T_2 \in (0, T_1]$ so that

$$C_5(T_1)C_\epsilon\phi_5(M)T_2 < 1 - C_5(T_1)\epsilon\phi_5(M),$$

$$C_5(T_1)C_\epsilon\left[\phi_4(3M)+\phi_5(M)+\phi_6(2M)\right]T_2$$
$$< 1-\epsilon C_5(T_1)\left[\phi_4(3M)+\phi_5(M)+\phi_6(2M)\right]$$

hold. For these ϵ and T_2, we obtain

$$\tilde{E}_{m+1}(T_2) \le r\tilde{E}_m(T_2), \quad r = \frac{C_5(T_1)(\epsilon + C_\epsilon T_2)\Big[\phi_4(3M)+\phi_6(2M)\Big]}{1 - C_5(T_1)(\epsilon + C_\epsilon T_2)\phi_5(M)} \in (0,1).$$

Then we can verify that $\{(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\bar{\theta}}'_{(m)}, \tilde{\bar{S}}'_{(m)}, \tilde{h}'_{(m)})\}_{m=0}^{\infty}$ is a Cauchy sequence in $Z(T_2)$. Therefore the limit function

$$(\tilde{\mathbf{u}}', \tilde{u}'_3, \tilde{\bar{\theta}}', \tilde{\bar{S}}', \tilde{h}') = \lim_{m\to\infty}(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\bar{\theta}}'_{(m)}, \tilde{\bar{S}}'_{(m)}, \tilde{h}'_{(m)})$$

exists in $Z(T_2)$, which is our desired solution.

Now we shall show that $0 < \underline{\theta}_0/2 \le \tilde{\theta}(y,t)$ and $0 < \underline{S}_0/2 \le \tilde{S}(y,t)$ hold by taking the time interval small enough again. Since $\tilde{\theta}' = \tilde{\theta} - \tilde{\bar{\theta}}_0 \in W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)$, we have

$$\tilde{\theta}(y,t) \ge \tilde{\bar{\theta}}_0\big|_{t=0}(y) - \Big(|\tilde{\theta}'(y,t)| + |\tilde{\bar{\theta}}_0(y,t) - \tilde{\bar{\theta}}_0(y,0)|\Big)$$
$$\ge \underline{\theta}_0 - t^\gamma\left(\sup_{y\in\tilde{\Omega}}|\tilde{\theta}'(y,t)|_t^{(\gamma)} + \sup_{y\in\tilde{\Omega}}|\tilde{\bar{\theta}}_0(y,t)|_t^{(\gamma)}\right),$$

where $|f|_t^{(\gamma)}$ stands for the Hölder coefficient of f with respect to t with exponent $\gamma \in \left\{0<\gamma<\frac{l}{2}-\frac{1}{4}\right\}$. Note that the Sobolev embedding inequality leads to

$$\sup_{y\in\tilde{\Omega}}|\tilde{\theta}'(y,t)|_t^{(\gamma)} \le \|\tilde{\theta}'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T^*})}, \quad \sup_{y\in\tilde{\Omega}}|\tilde{\bar{\theta}}_0(y,t)|_t^{(\gamma)} \le \|\tilde{\bar{\theta}}'_0\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T^*})}.$$

If we take

$$T_3 = \left(\frac{\underline{\theta}_0}{2\left(\|\tilde{\theta}'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T^*})} + \|\tilde{\bar{\theta}}'_0\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T^*})}\right)}\right)^{\frac{1}{\gamma}},$$

then we have $\theta(t,x) > \underline{\theta}_0/2$ on $[0,T_3]$. A similar argument holds for $\tilde{S}$. Denote again the time interval by $[0,T_3]$ on which both $\underline{\theta}_0/2 < \tilde{\theta}(y,t) < \infty$ and $\underline{S}_0/2 < \tilde{S}(y,t) < \infty$ hold. $T^* = \min\{T_2, T_3\}$ provides the desired result.

Uniqueness of the solution can be proved by virtue of an analogous inequality to (2.5.15).

This completes the proof of Theorem 2.2.1.

Chapter 3

Primitive Equations for the Atmosphere

In Chapter 3, we study an initial-boundary value problem of the primitive equations for the atmosphere in the three-dimensional strip. Since the unknown functions θ and h appear in the hydrostatic equation, the estimates of functions after the coordinate transform are more complex than those in the case of the ocean. Due to the same reason, we should note that more regularity for θ is necessary in the present case.

3.1 Formulation of the Problem

In this section, we formulate the free boundary problem of primitive equations for the atmosphere. As in the case of the ocean, our problem can be formulated in the strip-like region by adopting f-plane approximation. By $x = (x_1, x_2, x_3)$, we denote orthogonal Cartesian coordinate system with x_3 being the vertical direction. Let the ocean surface (unknown free boundary) and the upper boundary of the atmosphere be described by $x_3 = d(x', t)$ and $x_3 = H$ $(x' = (x_1, x_2))$, respectively, where H is a positive constant satisfying $H > d_0(x') \equiv d(x', 0)$, and $d(x', t)$ is assumed to be a function satisfying $d(x', t) < H$ for any $x' \in \mathbf{R}^2$ and $t \geq 0$.

Then the domain $\Omega(t)$ of the atmosphere at time t is represented as $\{(x', x_3)| \; x' \in \mathbf{R}^2, d(x', t) < x_3 < H\}$. The equations that we consider in this chapter are as follows:

$$
\begin{cases}
\dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + w\dfrac{\partial \mathbf{v}}{\partial x_3} - \dfrac{1}{\varrho}\left[\mu_1\triangle\mathbf{v} + \mu_2\dfrac{\partial^2\mathbf{v}}{\partial x_3^2}\right] + f\mathbf{A}\mathbf{v} = -\dfrac{1}{\varrho}\nabla p + \mathbf{F}_1,\\
\dfrac{\partial p}{\partial x_3} = -\varrho g,\\
\dfrac{\partial \varrho}{\partial t} + \nabla\cdot(\varrho\mathbf{v}) + \dfrac{\partial}{\partial x_3}(\varrho w) = 0,\\
\dfrac{\partial \theta}{\partial t} + (\mathbf{v}\cdot\nabla)\theta + w\dfrac{\partial \theta}{\partial x_3} - \left[\mu_3\Delta\theta + \mu_4\dfrac{\partial^2\theta}{\partial x_3^2}\right] = F_2,\\
\dfrac{\partial q}{\partial t} + (\mathbf{v}\cdot\nabla)q + w\dfrac{\partial q}{\partial x_3} - \left[\mu_5\Delta q + \mu_6\dfrac{\partial^2 q}{\partial x_3^2}\right] = F_3, \quad x\in\Omega(t),\ t>0.
\end{cases} \tag{3.1.1}
$$

The first equations stand for the equations of motion of the atmosphere in the horizontal directions, while the second one is derived by applying the hydrostatic approximation to the vertical component of the equations of motion. The third one is the continuity equation, and the fourth and the fifth ones are diffusion equations of the heat and the moisture, respectively. We describe only the notations different from those in Chapter 2; In (3.1.1) $\mathbf{F}_1$ is the horizontal components of the external forces given in $\mathbf{R}^3\times[0,\infty)$. ϱ is the density, q is the moisture; F_2 and F_3 are the sources of heat and moisture, respectively; (μ_5,μ_6) are given by scaling sums of turbulent and molecular diffusivities of moisture. In addition to (3.1.1), we use the equation of state for the ideal gas, $p=\varrho R\theta$. The conditions on the free surface $\Gamma_s(t)=\{x\in\mathbf{R}^3|x_3=d(x',t),t>0\}$ are imposed as follows:

$$
\begin{cases}
\mathbf{T}(\mathbf{v})\mathbf{n} - (\mathbf{T}(\mathbf{v})\mathbf{n}\cdot\mathbf{n}')\mathbf{n}' = |\mathbf{v}|^\alpha\mathbf{v},\\
-\left(\mu_3\nabla\theta\cdot\mathbf{n}' + \mu_4\dfrac{\partial\theta}{\partial x_3}n_3\right) = -la(\theta_e)\mathcal{V} + g_1|\mathbf{v}|^\alpha\theta + \sigma LK,\\
(\theta,q,p) = (\theta_e,q_e,p_0),
\end{cases} \tag{3.1.2}
$$

where

$$
\mathbf{T}(\mathbf{v}) = \begin{pmatrix} \mu_1\dfrac{\partial v_1}{\partial x_1} & \mu_1\dfrac{\partial v_1}{\partial x_2} & \mu_2\dfrac{\partial v_1}{\partial x_3}\\ \mu_1\dfrac{\partial v_2}{\partial x_1} & \mu_1\dfrac{\partial v_2}{\partial x_2} & \mu_2\dfrac{\partial v_2}{\partial x_3} \end{pmatrix} \tag{3.1.3}
$$

is a part of the stress tensor, $\mathbf{n} = (n_1, n_2, n_3)^{\mathrm{T}} = \left(\mathbf{n}'^{\mathrm{T}},\ n_3\right)^{\mathrm{T}}$ is the unit inward normal vector to $\Gamma_s(t)$ at time t, and p_0 is a positive constant which means the atmospheric pressure at the ocean surface. The first condition in (3.1.2) means a balance of the wind shear using bulk formulae, while the second one a balance of the heat flux at the ocean surface, including the effect of the evaporation and condensation. Since $\mathcal{V} = \frac{\partial d}{\partial t}/\sqrt{1+|\nabla d|^2}$, $(3.1.2)_2$ can be written as an equation for d

$$\frac{\partial d}{\partial t} = L_{4,d}d + G_{6,d}(\mathbf{v}, \theta), \quad x' \in \mathbf{R}^2,\ t > 0, \tag{3.1.4}$$

where

$$L_{4,d}\tilde{d} := \frac{\sigma L}{la(\theta_e)(1+|\nabla d|^2)}\left\{\left(1+\left(\frac{\partial d}{\partial x_2}\right)^2\right)\frac{\partial^2 \tilde{d}}{\partial x_1^2} - 2\frac{\partial d}{\partial x_1}\frac{\partial d}{\partial x_2}\frac{\partial^2 \tilde{d}}{\partial x_1 \partial x_2} + \left(1+\left(\frac{\partial d}{\partial x_1}\right)^2\right)\frac{\partial^2 \tilde{d}}{\partial x_2^2}\right\},$$

$$G_{6,d}(\mathbf{v}, \theta) := \frac{1}{la(\theta_e)}\left[\left(-\mu_3 \nabla\theta\big|_{\Gamma_s(t)} \cdot \nabla d + \mu_4 \frac{\partial \theta}{\partial x_3}\big|_{\Gamma_s(t)}\right) + g_1\sqrt{1+|\nabla d|^2}|\mathbf{v}|^{\alpha}\theta\big|_{\Gamma_s(t)}\right].$$

The conditions at the upper surface of the atmosphere are

$$(\mathbf{v}, w, \theta, q)(x,t) = (\mathbf{0}, 0, \theta_H, q_H)(x,t), \quad x \in \Gamma_H := \{(x', H) | x' \in \mathbf{R}^2\},\ t > 0. \tag{3.1.5}$$

Initial conditions are

$$(\mathbf{v}, \theta, q)(x, 0) = (\mathbf{v}_0, \theta_0, q_0)(x), \quad x \in \Omega := \Omega(0), \tag{3.1.6}$$

$$d(x', 0) = d_0(x'), \quad x' \in \mathbf{R}^2. \tag{3.1.7}$$

Let us introduce the p-coordinate system. From $(3.1.1)_2$ with the equation of state and $(3.1.2)_3$, it is easily seen that p can be represented as

$$p = p(x', x_3, t) = p_0 \exp\left(-\int_{d(x',t)}^{x_3} \frac{g}{R\theta(x', x_3, t)} \mathrm{d}x_3\right). \tag{3.1.8}$$

We denote the pressure at the upper boundary of the atmosphere by

$$h = h(x', t) = p(x', H, t) = p_0 \exp\left(-\int_d^H \frac{g}{R\theta(x', x_3, t)} \mathrm{d}x_3\right), \tag{3.1.9}$$

and $h_0 = h_0(x') = h(x', 0)$. It is natural to assume $d_0(x') < H$ for any $x' \in \mathbf{R}^2$.

Since (3.1.8) leads to $\partial p/\partial x_3 = -pg/R\theta < 0$, we can define a map

$$y_3 \longmapsto p_0 \exp\left(-\int_{y_3}^{H} \frac{g}{R\theta}\,\mathrm{d}x_3\right) =: \Phi_1(y_3;\theta),$$

for which there exists an inverse function

$$d = \Psi(x', t; \theta, h) := \Phi_1^{-1}(h;\theta)$$

first, and a map

$$y_3 \longmapsto p_0 \exp\left(-\int_{\Psi(y';\theta,h)}^{y_3} \frac{g}{R\theta}\,\mathrm{d}x_3\right) =: \Phi_2(y_3;\theta,h),$$

for which there exists an inverse function

$$x_3 = X_3(x', p, t; \theta, h) := \Phi_2^{-1}(p;\theta,h).$$

We can take p as an independent variable in place of x_3. From (3.1.8), it is easily seen that

$$\nabla p = p\left(\frac{g\nabla d}{R\theta_e(x', d(x',t),t)} + \int_d^{x_3} \frac{g\nabla\theta}{R\theta^2}\,\mathrm{d}x_3\right) =: \mathbf{F}_5(x', x_3, t), \quad (3.1.10)$$

$$\frac{\partial p}{\partial t} = p\left(\frac{g}{R\theta_e(x', d(x',t),t)}\frac{\partial d}{\partial t} + \int_d^{x_3} \frac{g}{R\theta^2}\frac{\partial\theta}{\partial t}\,\mathrm{d}x_3\right) =: F_6(x', x_3, t). \quad (3.1.11)$$

Notice that after introducing p-coordinates, the ocean surface becomes flat and is represented by the equation $p = p_0$, while the upper surface given by $p = h(x', t)$ is unknown. For a function $F(x', x_3, t)$, in order to indicate explicitly the dependence on $\theta^* = \theta \circ X_3$ and h after this transform, we use the notation like

$$F^{(\theta^*,h)}(x', p, t) := F(x', X_3(x', p, t; \theta, h), t),$$
$$\theta^*(x', p, t) = \theta(x', X_3(x', p, t; \theta, h), t).$$

By introducing

$$\bar{w} := dp/dt = \partial p/\partial t + (\mathbf{v}\cdot\nabla)p + w\partial p/\partial x_3 = F_6 + \mathbf{v}\cdot\mathbf{F}_5 + w\partial p/\partial x_3,$$

$$\partial p/\partial x_3 = -gp/R\theta,$$

w is represented as

$$w = -\frac{R\theta(\bar{w} - F_6 - \mathbf{v}\cdot\mathbf{F}_5)}{gp}.$$

From $(3.1.1)_2$ and $(3.1.1)_3$, we immediately arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial p}{\partial x_3}\right) + \frac{\partial p}{\partial x_3}\nabla\cdot\mathbf{v} + \frac{\partial p}{\partial x_3}\frac{\partial w}{\partial x_3} = 0. \tag{3.1.12}$$

Since the first term is represented as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial p}{\partial x_3}\right) = \frac{\partial}{\partial x_3}\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right) - \frac{\partial \mathbf{v}}{\partial x_3}\cdot\nabla p - \frac{\partial w}{\partial x_3}\frac{\partial p}{\partial x_3},$$

(3.1.12) becomes

$$\frac{\partial \bar{w}}{\partial x_3} - \frac{\partial \mathbf{v}}{\partial x_3}\cdot\nabla p + \frac{\partial p}{\partial x_3}\nabla\cdot\mathbf{v} = 0.$$

After rewriting this in p-coordinates, we have

$$\frac{\partial p}{\partial x_3}\frac{\partial \bar{w}^*}{\partial p} - \frac{\partial p}{\partial x_3}\frac{\partial \mathbf{v}^*}{\partial p}\cdot\mathbf{F}_5^{(\theta^*,h)} + \frac{\partial p}{\partial x_3}\left(\nabla\cdot\mathbf{v}^* + \mathbf{F}_5^{(\theta^*,h)}\frac{\partial \mathbf{v}^*}{\partial p}\right) = 0$$

for $(\mathbf{v}^*, \bar{w}^*)$ defined by $(\mathbf{v}^*, \bar{w}^*)(x', p, t) = (\mathbf{v}, \bar{w})(x', x_3, t)$. Hence, we obtain

$$\nabla\cdot\mathbf{v}^* + \partial\bar{w}^*/\partial p = 0$$

(cf. [23]). Moreover, we introduce another mapping $(x', p) \longmapsto (y', y_3)$ defined in (2.1.11). By composing these transformations, the regions

$$\bigcup_{0\le t\le T}(\Omega(t)\times\{t\}),\ \bigcup_{0\le t\le T}(\Gamma_H\times\{t\}),\ \bigcup_{0\le t\le T}(\Gamma_s(t)\times\{t\})$$

are transformed onto the regions

$$\tilde{\Omega}_T := \tilde{\Omega}\times[0,T],\ \tilde{\Gamma}_{HT} := \tilde{\Gamma}_H\times[0,T],\ \tilde{\Gamma}_{sT} := \tilde{\Gamma}_s\times[0,T],$$

respectively, where

$$\tilde{\Omega} = \{(y', y_3) | y' \in \mathbf{R}^2,\ p_0 < y_3 < h_0(y')\},$$
$$\tilde{\Gamma}_H = \{(y', y_3) | y' \in \mathbf{R}^2,\ y_3 = h_0(y')\},$$
$$\tilde{\Gamma}_s = \{(y', y_3) | y' \in \mathbf{R}^2,\ y_3 = p_0\}.$$

We denote the inverse of transposed matrix of the Jacobian matrix of this transform by

$$\big(J[(x',p)/(y',y_3)]^{\mathrm{T}}\big)^{-1} = (a^{ij}) = (a^{ij}(h)) \quad (i,j=1,2,3).$$

In the followings, the operator $(\nabla_h,\ \nabla_{h,3})$ is the same as that in Chapter 1, and

$$\tilde{X}_3(y',y_3,t;\theta^{**},h) := X_3(x',p,t;\theta^*,h)\big|_{x'=y',p=p(y,t)},$$

$$\theta^{**}(y',y_3,t) := \theta^*(x',p,t)\big|_{x'=y',p=p(y,t)},$$

$$f^{(\theta^{**},h)}(y',y_3,t) := f^{(\theta^*,h)}(x',p,t)\big|_{x'=y',p=p(y,t)}.$$

Now let us derive the explicit representation of $\mathbf{F}_5^{(\theta^{**},h)}$ and $F_6^{(\theta^{**},h)}$. Representing the integral term in (3.1.9) by p-coordinate system, we have

$$\mathbf{F}_5^{(\theta^*,h)}(x',p,t) = p\bigg(\frac{g\nabla\Psi}{R\theta_e(x',\Psi(x',t),t)} - \int_{p_0}^{p}\frac{1}{p\theta^*}\left(\nabla\theta^* + \mathbf{F}_5^{(\theta^*,h)}\frac{\partial\theta^*}{\partial p}\right)\,\mathrm{d}p\bigg).$$

We have the following boundary condition from this integral equation

$$\mathbf{F}_5^{(\theta^*,h)}(x',p,t)|_{p=p_0} = \frac{p_0 g\nabla\Psi(x',t)}{R\theta_e(x',\Psi(x',t),t)}.$$

We derive the explicit representation

$$\mathbf{F}_5^{(\theta^*,h)}(x',p,t) = \frac{p}{\theta^*}\bigg(\frac{g\nabla\Psi(x',t)}{R} - \int_{p_0}^{p}\frac{\nabla\theta^*}{p}\,\mathrm{d}p\bigg),$$

and hence

$$\begin{aligned}
&\mathbf{F}_5^{(\theta^{**},h)}(y',y_3,t)\\
&\quad= \frac{p(y,t)}{\theta^{**}}\bigg\{\frac{g\nabla\Psi(y',t)}{R} - \frac{p_0-h}{p_0-h_0}\int_{p_0}^{y_3}\frac{1}{p(y,t)}\left(\nabla\theta^{**} + \mathbf{a}^3\frac{\partial\theta^{**}}{\partial y_3}\right)\,\mathrm{d}y_3\bigg\}\\
&\quad=: \frac{p(y,t)}{\theta^{**}}\frac{g}{R}\nabla\Psi(y',t) + \mathbf{C}_1(y,t). \qquad (3.1.13)
\end{aligned}$$

Here, and henceforth, ∇ stands for the gradient operator with respect to y'. Similarly, we obtain

$$F_6^{(\theta^*,h)}(x',p,t) = \frac{p}{\theta^*}\bigg(\frac{g}{R}\frac{\partial\Psi}{\partial t}(x',t) - \int_{p_0}^{p}\frac{1}{p}\frac{\partial\theta^*}{\partial t}\,\mathrm{d}p\bigg),$$

$$F_6^{(\theta^{**},h)}(y', y_3, t)$$
$$= \frac{p(y,t)}{\theta^{**}} \left\{ \frac{g}{R} \frac{\partial \Psi}{\partial t}(y', t) - \frac{p_0 - h}{p_0 - h_0} \int_{p_0}^{y_3} \frac{1}{p(y,t)} \left(\frac{\partial \theta^{**}}{\partial t} + \frac{\partial y_3}{\partial t} \frac{\partial \theta^{**}}{\partial y_3} \right) \mathrm{d}y_3 \right\}.$$
$$=: \frac{p(y,t)}{\theta^{**}} \frac{g}{R} \frac{\partial \Psi}{\partial t}(y', t) + \tilde{C}_1(y,t), \quad \left(\frac{\partial y_3}{\partial t} \right)^* = A_1(y,t) \frac{\partial h}{\partial t}. \tag{3.1.14}$$

Differentiate the relation (3.1.9) with respect to x' and t, and rewrite them in the y-coordinate system. Then we have

$$\frac{\nabla h}{h} = \nabla \Psi \frac{g}{R\theta_e} - \int_{p_0}^{h_0} \frac{1}{\theta^{**} p(y,t)}$$
$$\times \left[\nabla \theta^{**} + (A_1 \nabla h + \mathbf{B}_1) \frac{\partial \theta^{**}}{\partial y_3} + \mathbf{F}_5 \left(\frac{\partial y_3}{\partial p} \right)^* \frac{\partial \theta^{**}}{\partial y_3} \right] \left(\frac{p_0 - h}{p_0 - h_0} \right) \mathrm{d}y_3,$$
$$\frac{1}{h} \frac{\partial h}{\partial t} = \frac{\partial \Psi}{\partial t} \frac{g}{R\theta_e} - \int_{p_0}^{h_0} \frac{1}{\theta^{**} p(y,t)}$$
$$\times \left[\frac{\partial \theta^{**}}{\partial t} + A_1 \frac{\partial h}{\partial t} \frac{\partial \theta^{**}}{\partial y_3} + F_6 \left(\frac{\partial y_3}{\partial p} \right)^* \frac{\partial \theta^{**}}{\partial y_3} \right] \left(\frac{p_0 - h}{p_0 - h_0} \right) \mathrm{d}y_3,$$

where A_1 amd B_1 are the same as defined in Section 2.1, right before (2.1.12), and

$$\left(\frac{\partial y_3}{\partial p} \right)^* = \frac{p_0 - h_0(y')}{p_0 - h(y', t)}.$$

Inserting (3.1.13) and (3.1.14) into the above equalities, and noting that

$$\frac{g}{R\theta_e} - \int_{p_0}^{h_0} \frac{g}{R(\theta^{**})^2} \frac{\partial \theta^{**}}{\partial y_3} = \frac{g}{R\theta_H(y', H, t)}$$

holds, we have the following equalities:

$$\nabla \Psi = \frac{1}{F(y',t)} \left(E(y',t) \nabla h + \mathbf{K}_1(y',t) \right), \tag{3.1.15}$$

$$\frac{\partial^2 \Psi}{\partial y_i \partial y_j} = \frac{E}{F} \frac{\partial^2 h}{\partial y_i \partial y_j} + H_{ij} \quad (i,j = 1,2), \tag{3.1.16}$$

$$\frac{\partial \Psi}{\partial t} = \frac{1}{F(y',t)} \left(E(y',t) \frac{\partial h}{\partial t} + K_2(y',t) \right), \tag{3.1.17}$$

where

$$F(y',t) = \frac{g}{R\theta_H(y', H, t)},$$

$$E(y',t) = \frac{1}{h(y',t)} + \frac{p_0 - h}{p_0 - h_0} \int_{p_0}^{h_0} \frac{(A_1 + D_1)}{p(y,t)\theta^{**}} \frac{\partial \theta^{**}}{\partial y_3} \, \mathrm{d}y_3,$$

$$D_1(y,t) = -\frac{p(y,t)}{\theta^{**}} \int_{p_0}^{y_3} \frac{A_1}{p(y,t)} \frac{\partial \theta^{**}}{\partial y_3} \, \mathrm{d}y_3,$$

$$\begin{aligned} \mathbf{K}_1(y',t) &= \frac{p_0 - h}{p_0 - h_0} \int_{p_0}^{h_0} \left[\frac{1}{p(y,t)\theta^{**}} \left(\nabla \theta^{**} + \mathbf{B}_1 \frac{\partial \theta^{**}}{\partial y_3} \right) + \mathbf{L}_1 \right] \, \mathrm{d}y_3 \\ &=: (K_{11}, \ K_{12})^{\mathrm{T}}, \end{aligned}$$

$$\mathbf{L}_1(y,t) = -\frac{1}{\theta^{**2}} \left(\int_{p_0}^{y_3} \frac{1}{p(y,t)} \left(\nabla \theta^{**} + \mathbf{B}_1 \frac{\partial \theta^{**}}{\partial y_3} \right) \, \mathrm{d}y_3 \right) \frac{\partial \theta^{**}}{\partial y_3},$$

$$H_{ij} = \frac{\partial}{\partial y_i} \left(\frac{E}{F} \right) \frac{\partial h}{\partial y_j} + \frac{\partial}{\partial y_i} \left(\frac{K_{1j}}{F} \right) \quad (i,j = 1,2),$$

$$\begin{aligned} K_2(y',t) &= \frac{p_0 - h}{p_0 - h_0} \int_{p_0}^{h_0} \left(\frac{1}{p(y,t)\theta^{**}} \frac{\partial \theta^{**}}{\partial t} + L_2 \right) \, \mathrm{d}y_3, \\ L_2(y,t) &= -\frac{1}{\theta^{**2}} \left(\int_{p_0}^{y_3} \frac{1}{p(y,t)} \frac{\partial \theta^{**}}{\partial t} \, \mathrm{d}y_3 \right) \frac{\partial \theta^{**}}{\partial y_3}. \end{aligned}$$

Now, let the problem (3.1.1)–(3.1.7) be rewritten in y-coordinates for $(\mathbf{u}, u_3, \tilde{\theta}, \tilde{q}, h)$, $(\mathbf{u}, u_3)(y', y_3, t) = (\mathbf{v}^*, \bar{w}^*)(x', p, t)$, $\tilde{\theta} = \theta^{**}$, $\tilde{q}(y', y_3, t) = q^*(x', p, t) = q(x', x_3, t)$, $h(y', t) = h(x', t)$. Then we have

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} = L_{1,\tilde{\theta},h} \mathbf{u} + \tilde{\mathbf{G}}_{1,\tilde{\theta},h}(\mathbf{u}, u_3, \tilde{\theta}), \\ \nabla_{h,3} u_3 = \tilde{G}_{3,h}(\mathbf{u}), \\ \dfrac{\partial \tilde{\theta}}{\partial t} = L_{2,\tilde{\theta},h} \tilde{\theta} + \tilde{G}_{4,\tilde{\theta},h}(\mathbf{u}, u_3, \tilde{\theta}), \\ \dfrac{\partial \tilde{q}}{\partial t} = L_{3,\tilde{\theta},h} \tilde{q} + \tilde{G}_{5,\tilde{\theta},h}(\mathbf{u}, u_3, \tilde{q}) \quad \text{in } \tilde{\Omega}_T, \\ \dfrac{\partial h}{\partial t} = L_{4,\tilde{\theta},h} h + \tilde{G}_{6,\tilde{\theta},h}(\mathbf{u}, \tilde{\theta}) \quad \text{in } \mathbf{R}^2_T. \end{cases} \tag{3.1.18}$$

$$\begin{cases} B_{\tilde{\theta},h}\mathbf{u} = \tilde{\mathbf{G}}_2(\mathbf{u}), \\ (\tilde{\theta},\tilde{q}) = (\theta_e, q_e)|_{x_3=\Psi} \quad \text{on } \tilde{\Gamma}_{sT}, \\ (\mathbf{u}, u_3, \tilde{\theta}, \tilde{q}) = (\mathbf{0}, 0, \theta_H, q_H)|_{x_3=H} \quad \text{on } \tilde{\Gamma}_{HT}, \\ (\mathbf{u}, \tilde{\theta}, \tilde{q})(y,0) = (\mathbf{v}_0^{(\tilde{\theta}_0,h_0)}, \theta_0^{(\tilde{\theta}_0,h_0)}, q_0^{(\tilde{\theta}_0,h_0)})(y) \quad \text{on } \tilde{\Omega}, \\ h(y',0) = h_0(y') \quad \text{on } \mathbf{R}^2, \end{cases} \tag{3.1.19}$$

where

$$L_{1,\tilde{\theta},h}\mathbf{u} := \mu_1 L_{11,\tilde{\theta},h}\mathbf{u} + \mu_2 L_{12,\tilde{\theta},h}\mathbf{u},$$

$$L_{11,\tilde{\theta},h}\mathbf{u} = \frac{R\tilde{\theta}}{p(y,t)}\left[l_{11,h}\mathbf{u} + 2l_{12,\tilde{\theta},h}\mathbf{u} + |\mathbf{F}_5^{(\tilde{\theta},h)}|^2(a^{33})^2\frac{\partial^2\mathbf{u}}{\partial y_3^2}\right],$$

$$L_{12,\tilde{\theta},h}\mathbf{u} = \frac{g^2 p(y,t)}{R\tilde{\theta}}(a^{33})^2\frac{\partial^2\mathbf{u}}{\partial y_3^2},$$

$$l_{11,h} = \nabla^2 + 2\mathbf{a}^3\cdot\nabla\frac{\partial}{\partial y_3} + |\mathbf{a}^3|^2\frac{\partial^2}{\partial y_3^2}, \qquad l_{12,\tilde{\theta},h} = a^{33}\mathbf{F}_5^{(\tilde{\theta},h)}\cdot\nabla_h\frac{\partial}{\partial y_3},$$

$$\begin{aligned}
\tilde{\mathbf{G}}_{1,\tilde{\theta},h}(\mathbf{u},u_3,\tilde{\theta}) := {}& \frac{\mu_1 R\tilde{\theta}}{p(y,t)}\Bigg[(\nabla_h^2 - l_{11,h}) + \nabla_h\cdot\left(a^{33}\mathbf{F}_5^{(\tilde{\theta},h)}\right)\frac{\partial}{\partial y_3} \\
&+ a^{33}\mathbf{F}_5^{(\tilde{\theta},h)}\cdot\left(\frac{\partial\mathbf{a}^3}{\partial y_3} + \frac{\partial}{\partial y_3}\left(a^{33}\mathbf{F}_5^{(\tilde{\theta},h)}\right)\right)\frac{\partial}{\partial y_3}\Bigg]\mathbf{u} \\
&+ \frac{\mu_2 R\tilde{\theta}}{p(y,t)}\left[\left(a^{33}\frac{p(y,t)g}{R\tilde{\theta}}\frac{\partial}{\partial y_3}\right)^2 - \left(a^{33}\frac{p(y,t)g}{R\tilde{\theta}}\right)^2\frac{\partial^2}{\partial y_3^2}\right]\mathbf{u} \\
&- \Bigg[(\mathbf{u}\cdot\nabla_h) + (\mathbf{F}_5^{(\tilde{\theta},h)}\cdot\mathbf{u})a^{33}\frac{\partial}{\partial y_3} + a^{33}\left(u_3 - F_6^{(\tilde{\theta},h)} - \mathbf{u}\cdot\mathbf{F}_5^{(\tilde{\theta},h)}\right)\frac{\partial}{\partial y_3} \\
&\quad + F_6^{(\tilde{\theta},h)}a^{33}\frac{\partial}{\partial y_3} + A_1\frac{\partial h}{\partial t}\frac{\partial}{\partial y_3}\Bigg]\mathbf{u} - f\mathbf{A}\mathbf{u} - \frac{R\tilde{\theta}}{p(y,t)}\mathbf{F}_5^{(\tilde{\theta},h)} + \mathbf{F}_1^{(\tilde{\theta},h)} \\
=: {}& \mu_1\tilde{G}_{11,\tilde{\theta},h}\mathbf{u} + \mu_2\tilde{G}_{12,\tilde{\theta},h}\mathbf{u} - \tilde{G}_{13,\tilde{\theta},h}(\mathbf{u},u_3)\mathbf{u} - f\mathbf{A}\mathbf{u} - \frac{R\tilde{\theta}}{p(y,t)}\mathbf{F}_5^{(\tilde{\theta},h)} + \mathbf{F}_1^{(\tilde{\theta},h)},
\end{aligned}$$

$$\tilde{G}_{3,h}(\mathbf{u}) := -\nabla_h \cdot \mathbf{u} - \mathbf{a}^3 \cdot \frac{\partial \mathbf{u}}{\partial y_3},$$

$$L_{2,\tilde{\theta},h}\tilde{\theta} := \mu_3 L_{11,\tilde{\theta},h}\tilde{\theta} + \mu_4 L_{12,\tilde{\theta},h}\tilde{\theta},$$

$$\tilde{G}_{4,\tilde{\theta},h}(\mathbf{u}, u_3, \tilde{\theta}) := \mu_3 \tilde{G}_{11,\tilde{\theta},h}\tilde{\theta} + \mu_4 \tilde{G}_{12,\tilde{\theta},h}\tilde{\theta} - \tilde{G}_{13,\tilde{\theta},h}(\mathbf{u}, u_3)\tilde{\theta} + F_2^{(\tilde{\theta},h)},$$

$$L_{3,\tilde{\theta},h}\tilde{q} := \mu_5 L_{11,\tilde{\theta},h}\tilde{q} + \mu_6 L_{12,\tilde{\theta},h}\tilde{q},$$

$$\tilde{G}_{5,\tilde{\theta},h}(\mathbf{u}, u_3, \tilde{q}) := \mu_5 \tilde{G}_{11,\tilde{\theta},h}\tilde{q} + \mu_6 \tilde{G}_{12,\tilde{\theta},h}\tilde{q} - \tilde{G}_{13,\tilde{\theta},h}(\mathbf{u}, u_3)\tilde{q} + F_3^{(\tilde{\theta},h)},$$

$$\begin{aligned} B_{\tilde{\theta},h}\mathbf{u} := & \left\{ \mu_1 \left[(\mathbf{n}' \cdot \nabla_h)\mathbf{u} + (\mathbf{F}_5^{(\tilde{\theta},h)} \cdot \mathbf{n}') a^{33} \frac{\partial \mathbf{u}}{\partial y_3} \right] - \frac{\mu_2 p_0 g}{R\theta_e\big|_{x_3=\Psi(y',t)}} a^{33} \frac{\partial \mathbf{u}}{\partial y_3} n_3 \right\} \\ & - \left\{ \mu_1 \left[(\mathbf{n}' \cdot \nabla_h)\mathbf{u} \cdot \mathbf{n}' + (\mathbf{F}_5^{(\tilde{\theta},h)} \cdot \mathbf{n}') a^{33} \frac{\partial \mathbf{u}}{\partial y_3} \cdot \mathbf{n}' \right] \right. \\ & \left. - \frac{\mu_2 p_0 g}{R\theta_e\big|_{x_3=\Psi(y',t)}} \left(a^{33} \frac{\partial \mathbf{u}}{\partial y_3} \cdot \mathbf{n}' \right) n_3 \right\} \mathbf{n}', \end{aligned}$$

$$\tilde{\mathbf{G}}_2(\mathbf{u}) := |\mathbf{u}|^\alpha \mathbf{u},$$

$$L_{4,\tilde{\theta},h} h := \frac{F\sigma L}{Ela(\theta_e)\left(1 + |\nabla\Psi(y',t)|^2\right)} \sum_{i,j=1}^{2} c_{ij} \frac{\partial^2 h}{\partial y_i \partial y_j},$$

$$c_{11} := 1 + \left(\frac{E}{F} \frac{\partial h}{\partial y_2} + K_{12} \right)^2, \quad c_{22} := 1 + \left(\frac{E}{F} \frac{\partial h}{\partial y_1} + K_{11} \right)^2,$$

$$c_{12} = c_{21} := -\left(\frac{E}{F} \frac{\partial h}{\partial y_1} + K_{11} \right) \left(\frac{E}{F} \frac{\partial h}{\partial y_2} + K_{12} \right),$$

$$\tilde{G}_{6,\tilde{\theta},h}(\mathbf{u},\tilde{\theta}) := \frac{1}{E}\Bigg[-K_2 + \frac{F\sigma L}{la(\theta_e)\,(1+|\nabla\Psi(y',t)|^2)}\sum_{i,j=1}^{2} c_{ij}H_{ij}$$

$$- \frac{F}{la(\theta_e)}\Bigg\{-\mu_3\left(\nabla_h + a^{33}(h)\mathbf{F}_5^{(\tilde{\theta},h)}\frac{\partial}{\partial y_3}\right)\tilde{\theta}\big|_{y_3=p_0}\cdot\nabla\Psi(y',t)$$

$$+ \frac{\mu_4 p_0 g}{R\theta_e\big|_{x_3=\Psi(y',t)}}a^{33}(h)\frac{\partial\tilde{\theta}}{\partial y_3}\big|_{y_3=p_0}$$

$$+ g_1^{(\tilde{\theta},h)}\left(1+|\nabla\Psi(y',t)|^2\right)^{\frac{1}{2}}|\mathbf{u}|^{\alpha}\tilde{\theta}\big|_{y_3=p_0}\Bigg\}\Bigg],$$

$$\mathbf{n} = (n_1,n_2,n_3)^{\mathrm{T}} = \frac{\mathbf{a}}{|\mathbf{a}|},\quad \mathbf{n}' = (n_1,n_2)^{\mathrm{T}},\quad \mathbf{a} = \left(\ -F_{51}^{(h)*},\ -F_{52}^{(h)*},\ \tilde{F}_{13}^{(h)*}\ \right)^{\mathrm{T}}$$

with $la(\theta_e) = la(\theta_e(y',\Psi(y',t),t))$. It is to be noted that the functions $(\mathbf{u}\big|_{t=0}$, $\tilde{\theta}\big|_{t=0}, \tilde{S}\big|_{t=0})(y)$ and $d_0(y') = d_0(x')$ are extensible into the half space $t>0$ preserving the regularity, which are denoted by $(\bar{\mathbf{u}}_0, \bar{\tilde{\theta}}_0, \bar{\tilde{q}}_0)$, and $\bar{d}_0$, respectively. We also denote the extension of $\theta_0(x)$ and $h_0(y')$ by

$$\bar{\theta}_0(x,t),\quad \bar{h}_0 = \bar{h}_0(y',t) = p_0\exp\left(-\int_{\bar{d}_0}^{H}\frac{g}{R\bar{\theta}_0}\,\mathrm{d}x_3\right),$$

respectively.

Then the problem (3.1.18)-(3.1.19) becomes the following one for $(\mathbf{u}', u_3', \tilde{\theta}', \tilde{q}', h') := (\mathbf{u}-\bar{\mathbf{u}}_0, u_3, \tilde{\theta}-\bar{\tilde{\theta}}_0, \tilde{q}-\bar{\tilde{q}}_0, h-\bar{h}_0)$:

$$\begin{cases} \dfrac{\partial\mathbf{u}'}{\partial t} = L_{1,\tilde{\theta},h}\mathbf{u}' + L_{1,\tilde{\theta},h}\bar{\mathbf{u}}_0 - \dfrac{\partial\bar{\mathbf{u}}_0}{\partial t} + \tilde{\mathbf{G}}_{1,\tilde{\theta},h}(\mathbf{u},u_3,\tilde{\theta}), \\ \nabla_{h,3}u_3' = \tilde{G}_{3,h}(\mathbf{u}), \\ \dfrac{\partial\tilde{\theta}'}{\partial t} = L_{2,\tilde{\theta},h}\tilde{\theta}' + L_{2,\tilde{\theta},h}\bar{\tilde{\theta}}_0 - \dfrac{\partial\bar{\tilde{\theta}}_0}{\partial t} + \tilde{G}_{4,\tilde{\theta},h}(\mathbf{u},u_3,\tilde{\theta}), \\ \dfrac{\partial\tilde{q}'}{\partial t} = L_{3,\tilde{\theta},h}\tilde{q}' + L_{3,\tilde{\theta},h}\bar{\tilde{q}}_0 - \dfrac{\partial\bar{\tilde{q}}_0}{\partial t} + \tilde{G}_{5,\tilde{\theta},h}(\mathbf{u},u_3,\tilde{q}) \quad \text{in } \tilde{\Omega}_T, \\ \dfrac{\partial h'}{\partial t} = L_{4,\tilde{\theta},h}h' + L_{4,\tilde{\theta},h}\bar{h}_0 - \dfrac{\partial\bar{h}_0}{\partial t} + \tilde{G}_{6,\tilde{\theta},h}(\mathbf{u},\tilde{\theta}) \quad \text{in } \mathbf{R}_T^2, \end{cases} \tag{3.1.20}$$

$$
\begin{cases}
B_{\tilde{\theta},h}\mathbf{u}' = -B_{\tilde{\theta},h}\bar{\mathbf{u}}_0 + \tilde{\mathbf{G}}_2(\mathbf{u}), \\
(\tilde{\theta}', \tilde{q}') = (\theta_e|_{x_3=\Psi} - \bar{\tilde{\theta}}_0, q_e|_{x_3=\Psi} - \bar{\tilde{q}}_0) \quad \text{on } \tilde{\Gamma}_{sT}, \\
(\mathbf{u}', u_3', \tilde{\theta}', \tilde{q}') = (-\bar{\mathbf{u}}_0, 0, \theta_H|_{x_3=H} - \bar{\tilde{\theta}}_0, q_H|_{x_3=H} - \bar{\tilde{q}}_0) \quad \text{on } \tilde{\Gamma}_{uT}, \\
(\mathbf{u}', \tilde{\theta}', \tilde{q}')|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega}, \\
h'|_{t=0} = 0 \quad \text{on } \mathbf{R}^2,
\end{cases} \tag{3.1.21}
$$

where $(\mathbf{u}, \tilde{\theta}, \tilde{q}, h)$ in the right-hand sides is replaced by $(\mathbf{u}' + \bar{\mathbf{u}}_0, \tilde{\theta}' + \bar{\tilde{\theta}}_0, \tilde{q}' + \bar{\tilde{q}}_0, h + \bar{h}_0)$.

3.2 Main Theorem

Now we state the main theorem in this chapter.

Theorem 3.2.1. *Let $l \in (1/2, 1)$, and T be an arbitrary positive number. Assume that*

(i) $\alpha = 2$ *or* $\alpha > 2l + 1$;

(ii) $la(\cdot) : \mathbf{R} \to \mathbf{R}^+ \equiv \{k \in \mathbf{R} \big| k > 0\}$ *satisfies* $la(x) > 0$, *and* $la \in C^{2+L}(\mathbf{R})$ *(i.e. continuously differentiable up to the second order, with the Lipschitz continuous second order derivatives) with the norm*

$$
\|\!|la|\!\| := \sum_{i=0}^{2} \left[\sup_{x \in \mathbf{R}} \left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i la(x) \right| + \left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i la \right|^{(L)} \right] < \infty,
$$

where $|la|^{(L)}$ is Lipschitz coefficient of la;

(iii) $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, $\theta_0 \in \overline{W}_2^{2+l}(\Omega)$, $q_0 \in \overline{W}_2^{1+l}(\Omega)$, $d_0 \in W_2^{\frac{3}{2}+l}(\mathbf{R}^2)$, $0 < \underline{\theta}_0 \leq \theta_0(x)$ *and* $0 < \underline{q}_0 \leq q_0(x)$ *with some positive constants* $\underline{\theta}_0$ *and* $\underline{q}_0$;

(iv) θ_e, $\theta_H \in \overline{W}_2^{3+l, \frac{3+l}{2}}(\mathbf{R}_T^3)$, q_e, $q_H \in \overline{W}_2^{2+l, \frac{2+l}{2}}(\mathbf{R}_T^3)$, $\frac{\partial q_e}{\partial x_3}$, $\frac{\partial q_H}{\partial x_3} \in \widetilde{W}_2^{2+l, \frac{2+l}{2}}(\mathbf{R}_T^3)$, $\theta_e - \theta_0$, $\theta_H - \theta_0 \in W_2^{3+l, \frac{3+l}{2}}(\mathbf{R}_T^3)$, $q_e - q_0$, $q_H - q_0 \in \widetilde{W}_2^{2+l, 1+\frac{l}{2}}(\mathbf{R}_T^3)$,

$0 < \underline{\theta}_0 \leq \theta_e(x)$, $\theta_H(x)$, $0 < \underline{q}_0 \leq q_e(x)$, $q_H(x)$ *with the same constants as in* (iii)*;*

(v) $H - d_0(x') > c_0 > 0$ *on* $\mathbf{R}^2$*;*

(vi) $F_2 \in \widetilde{W}_2^{1+l,\frac{1+l}{2}}(\mathbf{R}_T^3)$, $\mathbf{F}_1$, $F_3 \in \widetilde{W}_2^{l,\frac{l}{2}}(\mathbf{R}_T^3)$, *and their derivatives with respect to* x_3 *satisfy the Hölder condition with exponent* $\beta > l/2$ *with respect to* x_3 *(we call this property as condition* (A)*). For the function* f *with this property, we introduce the notation:*

$$\|\!| f \|\!|_{0,T}^2 := \|f\|^2_{\widetilde{W}_2^{l,\frac{l}{2}}(\mathbf{R}_T^3)} + \left(\left| \frac{\partial f}{\partial x_3} \right|_{x_3}^{(\beta)} \right)^2,$$

where $|f|_{x_3}^{(\beta)}$ *stands for the Hölder coefficient of* f *in* x_3 *with exponent* β *uniformly in* x' *and* t*;*

(vii) $g_1 \in \widetilde{W}_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_T^3)$.

Moreover, the following compatibility conditions are satisfied:

$$\frac{\partial \varrho}{\partial t}\Big|_{t=0,x_3=d_0} + \nabla \cdot (\varrho|_{t=0,x_3=d_0} \mathbf{v}_0)|_{x_3=d_0} + \frac{\partial}{\partial x_3}(\varrho w)\Big|_{t=0,x_3=d_0} = 0,$$

$$\mathbf{v}_0'(x,0) = \mathbf{v}_0, \quad x \in \Omega,$$

$$\mathbf{T}(\mathbf{v}_0)\mathbf{n}\big|_{t=0} - \left(\mathbf{T}(\mathbf{v}_0)\mathbf{n}\big|_{t=0} \cdot \mathbf{n}'\big|_{t=0}\right) \mathbf{n}'\big|_{t=0} = |\mathbf{v}_0|^\alpha \, \mathbf{v}_0, \quad x \in \Gamma_s(0),$$

$$\theta_e(x', d_0, 0) = \theta_0(x), \quad q_e(x', d_0, 0) = q_0(x), \quad x \in \Gamma_s(0),$$

$$\theta_H(x', H, 0) = \theta_0(x), \quad q_H(x', H, 0) = q_0(x), \quad x \in \Gamma_b,$$

$$\frac{\partial \theta_H}{\partial t}\Big|_{t=0,x_3=H} + (\mathbf{v}_0|_{x_3=H} \cdot \nabla)\theta_0|_{x_3=H} + w|_{t=0,x_3=H} \frac{\partial \theta_0}{\partial x_3}$$
$$- \left[\mu_3 \Delta\theta_0|_{x_3=H} + \mu_4 \frac{\partial^2 \theta_0}{\partial x_3^2}\Big|_{x_3=H} \right] = F_2|_{t=0,x_3=H},$$

$$\frac{\partial \theta_e}{\partial t}\Big|_{t=0,x_3=d_0} + (\mathbf{v}_0|_{x_3=d_0} \cdot \nabla)\theta_0|_{x_3=d_0} + w|_{t=0,x_3=d_0} \frac{\partial \theta_0}{\partial x_3}\Big|_{x_3=d_0}$$
$$- \left[\mu_3 \Delta\theta_0|_{x_3=d_0} + \mu_4 \frac{\partial^2 \theta_0}{\partial x_3^2}\Big|_{x_3=d_0} \right] = F_2|_{t=0,x_3=d_0},$$

Then, there exists $T^* \in (0,T]$ *such that the problem* (3.1.20)–(3.1.21) *has a unique solution* $(\mathbf{u}', u_3, \tilde{\theta}', \tilde{q}', h') \in Z(T^*) := W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times \widetilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times$

$W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_{T^*}) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_{T^*})$, *satisfying* $0 < \tilde{\theta} = \tilde{\theta}' + \bar{\tilde{\theta}}$ *and* $0 < \tilde{q} = \tilde{q}' + \bar{\tilde{q}}$ *on* $\tilde{\Omega}_{T^*}$.

3.3 Auxiliary Lemmas

In this section, we state some lemmas used in the proof of the main theorem in Section 3.5 without proofs. Since $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, $\theta_0 \in \overline{W}_2^{2+l}(\Omega)$, $q_0 \in \overline{W}_2^{1+l}(\Omega)$ and $d_0 \in W_2^{\frac{3}{2}+l}(\mathbf{R}^2)$ $(1/2 < l < 1)$ imply $\mathbf{v}_0^{(\tilde{\theta}_0,h_0)} \in W_2^{1+l}(\tilde{\Omega})$, $\theta_0^{(\tilde{\theta}_0,h_0)} \in \overline{W}_2^{2+l}(\tilde{\Omega})$, $q_0^{(\tilde{\theta}_0,h_0)} \in \overline{W}_2^{1+l}(\tilde{\Omega})$, respectively, the extended functions $(\bar{\mathbf{u}}_0, \bar{\tilde{\theta}}_0, \bar{\tilde{q}}_0, \bar{d}_0)$ introduced in the end of Section 3.2 satisfy

$$\begin{cases} \|\bar{\mathbf{u}}_0\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \leq C\|\mathbf{u}_0^{(\tilde{\theta}_0,h_0)}\|_{W_2^{1+l}(\tilde{\Omega})}, \\ \|\bar{\tilde{\theta}}_0\|_{\overline{W}_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)} \leq C\|\theta_0^{(\tilde{\theta}_0,h_0)}\|_{\overline{W}_2^{2+l}(\tilde{\Omega})}, \\ \|\bar{\tilde{q}}_0\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \leq C\|q_0^{(\tilde{\theta}_0,h_0)}\|_{\overline{W}_2^{1+l}(\tilde{\Omega})}. \\ \|\bar{d}_0\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)} \leq C\|d_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)} \end{cases} \tag{3.3.1}$$

for some constant C (see, for instance, [46]). In the followings, C's stand for constants depending on $\|d_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)}$, $\|\mathbf{v}_0\|_{W_2^{2+l}(\Omega)}$, $\|\theta_0\|_{\overline{W}_2^{2+l}(\tilde{\Omega})}$, $\|q_0\|_{\overline{W}_2^{1+l}(\tilde{\Omega})}$, and $C(\cdot)$'s monotone increasing functions of their arguments.

Lemma 3.3.1. *Let* $h' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)$, $\tilde{\theta}' \in W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)$, $h = h' + \bar{h}_0$, $\tilde{\theta} = \tilde{\theta}' + \bar{\tilde{\theta}}_0$ *and* $1/2 < l' < l$. *Then the following estimates hold:*

$$\|\Psi(\cdot;\tilde{\theta},h)\|_{W_2^{l-\frac{1}{2},\frac{l}{2}-\frac{1}{4}}(\mathbf{R}^2_T)} \leq C\left(\|h'\|_{W_2^{l-\frac{1}{2},\frac{l}{2}-\frac{1}{4}}(\mathbf{R}^2_T)}, \|\tilde{\theta}'\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)}\right), \tag{3.3.2}$$

$$\|\Psi(\cdot;\tilde{\theta},h)\|_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}^2_T)} \leq C\left(\|h'\|_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}^2_T)}, \|\tilde{\theta}'\|_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_T)}\right) \quad (i = 0,1,2). \tag{3.3.3}$$

Lemma 3.3.2. *Let* $h' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)$, $\tilde{\theta}' \in W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)$, $h = h' + \bar{h}_0$,

$\tilde{\theta} = \tilde{\theta}' + \bar{\tilde{\theta}}_0$. *Then the following estimates hold:*

$$\|\mathbf{F}_5^{(\tilde{\theta},h)}\|_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_T)} \le C\Big(\|\tilde{\theta}'\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)}, \|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big)$$
$$(i = 0, 1, 2),$$

$$\|F_6^{(\tilde{\theta},h)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \le C\Big(\|\tilde{\theta}'\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)}, \|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big).$$

Lemma 3.3.3. *Let* $h_1', \ h_2' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$, $\tilde{\theta}_1', \ \tilde{\theta}_2' \in W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)$, $h_i = h_i' + \bar{h}_0$, $\tilde{\theta}_i = \tilde{\theta}_i' + \bar{\tilde{\theta}}_0$ $(i = 1, 2)$, $\tilde{\tilde{\theta}}' = \tilde{\theta}_1' - \tilde{\theta}_2'$, *and* $\tilde{h}' = h_1' - h_2'$, *and* $1/2 < l' < l$. *Then the following estimates hold:*

$$\begin{aligned}
&\big|\Psi(y^{1\prime}, y_3^1; \tilde{\theta}_1, h_1) - \Psi(y^{2\prime}, y_3^2; \tilde{\theta}_2, h_2)\big| \\
&\quad\le C\Big(\sum_{j=1}^{2} \|\tilde{\theta}_j'\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)}, \sum_{j=1}^{2} \|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big) \\
&\qquad\times \Big(\|\tilde{\tilde{\theta}}'\|_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}_T^2)}\Big)|y^{1\prime} - y^{2\prime}|^2. \qquad (3.3.4)
\end{aligned}$$

Lemma 3.3.4. *Let* $h', \ h_1', \ h_2' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$, $\tilde{\theta}', \ \tilde{\theta}_1', \ \tilde{\theta}_2' \in W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)$, $h = h' + \bar{h}_0$, $\tilde{\theta} = \tilde{\theta}' + \bar{\tilde{\theta}}_0$, $h_i = h_i' + \bar{h}_0$, $\tilde{\theta}_i = \tilde{\theta}_i' + \bar{\tilde{\theta}}_0$ $(i = 1, 2)$, $\tilde{\tilde{\theta}}' = \tilde{\theta}_1' - \tilde{\theta}_2'$, *and* $\tilde{h}' = h_1' - h_2'$, *and* $1/2 < l' < l$. *Then the following estimates hold:*

$$\begin{aligned}
\|\mathbf{F}_5^{(\tilde{\theta}_1,h_1)} - \mathbf{F}_5^{(\tilde{\theta}_2,h_2)}\|_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_T)} &\le C\Big(\sum_{j=1}^{2} \|\theta_j'\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)}, \sum_{j=1}^{2} \|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big) \\
&\quad\times \Big(\|\tilde{\tilde{\theta}}'\|_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}_T^2)}\Big), \\
\|F_6^{(\tilde{\theta}_1,h_1)} - F_6^{(\tilde{\theta}_2,h_2)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} &\le C\Big(\sum_{j=1}^{2} \|\theta_j'\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)}, \sum_{j=1}^{2} \|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big) \\
&\quad\times \Big(\|\tilde{\tilde{\theta}}\|_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}_T^2)}\Big).
\end{aligned}$$

Lemma 3.3.5. *Let* $h' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$, $\tilde{\theta}' \in W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)$, $h = h' + \bar{h}_0$,

$\tilde{\theta} = \tilde{\theta}' + \bar{\theta}_0$. *The following inequalities hold:*

$$|\tilde{X}_3(y^{1\prime}, y_3^1, t; \tilde{\theta}, h) - \tilde{X}_3(y^{2\prime}, y_3^2, t; \tilde{\theta}, h)|^2$$
$$\leq C\left(\|\tilde{\theta}'\|_{W_2^{3+l', \frac{3+l'}{2}}(\tilde{\Omega}_T)}, \|h'\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\tilde{\Omega}_T)} \right) \left(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2 \right),$$

$$|\tilde{X}_3(y', y_3, t; \tilde{\theta}, h) - \tilde{X}_3(y', y_3, t-\tau; \tilde{\theta}, h)|^2$$
$$\leq C\left(\|\tilde{\theta}'\|_{W_2^{3+l', \frac{3+l'}{2}}(\tilde{\Omega}_T)}, \|h'\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\tilde{\Omega}_T)} \right) |\tau|^2.$$

Lemma 3.3.6. *Let* $h' \in W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$, $\tilde{\theta}' \in W_2^{3+l, \frac{3+l}{2}}(\tilde{\Omega}_T)$, $h = h' + \bar{h}_0$, $\tilde{\theta} = \tilde{\theta}' + \bar{\bar{\theta}}_0$, *and* $1/2 < l' < l$. *Then the following inequalities hold:*

$$\left\| \nabla X_3|_{p=p(y,t)} \right\|_{W_2^{2+l, 1+\frac{l}{2}}(\tilde{\Omega}_T)} \leq C\left(\|\tilde{\theta}'\|_{W_2^{3+l', \frac{3+l'}{2}}(\mathbf{R}_T^2)}, \|h'\|_{W_2^{\frac{5}{2}+l', \frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)} \right),$$
$$\left\| \frac{\partial X_3}{\partial t}\bigg|_{p=p(y,t)} \right\|_{W_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_T)} \leq C\left(\|\tilde{\theta}'\|_{W_2^{3+l', \frac{3+l'}{2}}(\mathbf{R}_T^2)}, \|h'\|_{W_2^{\frac{5}{2}+l', \frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)} \right).$$

Now let $\bar{X}_3(y', y_3, t) = \tilde{X}_3(y', y_3, t; \tilde{\theta}_1, h_1) - \tilde{X}_3(y', y_3, t; \tilde{\theta}_2, h_2)$. For this, we have

Lemma 3.3.7. *Let* h', h_1', $h_2' \in W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$, $\tilde{\theta}'$, $\tilde{\theta}_1'$, $\tilde{\theta}_2' \in W_2^{3+l, \frac{3+l}{2}}(\tilde{\Omega}_T)$, $h = h' + \bar{h}_0$, $\tilde{\theta} = \tilde{\theta}' + \bar{\bar{\theta}}_0$, $h_i = h_i' + \bar{h}_0$, $\tilde{\theta}_i = \tilde{\theta}_i' + \bar{\bar{\theta}}_0$ $(i = 1, 2)$, $\tilde{\bar{\theta}}' = \tilde{\theta}_1' - \tilde{\theta}_2'$, *and* $\tilde{h}' = h_1' - h_2'$. *Then the following inequalities hold:*

$$|\bar{X}_3(y', y_3, t)|^2 \leq C\left(\sum_{i=1}^2 \|\tilde{\theta}_i'\|_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_T)}, \sum_{i=1}^2 \|h_i'\|_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \right)$$
$$\times \left[\|\tilde{\bar{\theta}}'\|^2_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|^2_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \right],$$

$$|\bar{X}_3(y^{1\prime}, y_3^1, t) - \bar{X}_3(y^{2\prime}, y_3^2, t)|^2$$
$$\leq C\left(\sum_{i=1}^2 \|\tilde{\theta}_i'\|_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_T)}, \sum_{i=1}^2 \|h_i'\|_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \right)$$
$$\times \left[\|\tilde{\bar{\theta}}'\|^2_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|^2_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \right] \left(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2 \right),$$

$$
\begin{aligned}
&|\bar{X}_3(y', y_3, t) - \bar{X}_3(y', y_3, t-\tau)|^2 \\
&\qquad \le C\left(\sum_{i=1}^{2} \|\tilde{\theta}'_i\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)}, \sum_{i=1}^{2} \|h'_i\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right) \\
&\qquad\qquad \times \left[\|\tilde{\tilde{\theta}}'\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right]|\tau|^2.
\end{aligned}
$$

The following lemma will be used in the proof of Lemma 3.3.9, whose proof is found in that of Lemma 2.3.4, given in Appendix B.1.4.

Lemma 3.3.8. *There exists a constant* δ*,* $0 < \delta < \min\left\{2 - 2l, \frac{2l(2l-1)}{3-2l}\right\}$*, such that the following inequality holds:*

$$
\|fg\|_{W_2^{l,0}(\tilde{\Omega}_T)} \le C(1 + \sup_{\mathbf{R}^2} |h_0|)^{\delta} \|f\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} \|g\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}. \tag{3.3.5}
$$

Now we turn to the estimates of the functions appearing in the assumptions of Theorem 3.2.1.

Lemma 3.3.9. *Let* $h', h'_1, h'_2 \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$*,* $\tilde{\theta}', \tilde{\theta}'_1, \tilde{\theta}'_2 \in W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)$*,* $h = h' + \bar{h}_0$*,* $\tilde{\theta} = \tilde{\theta}' + \bar{\bar{\theta}}_0$*,* $h_i = h'_i + \bar{h}_0$*,* $\tilde{\theta}_i = \tilde{\theta}'_i + \bar{\bar{\theta}}_0$ $(i = 1, 2)$*,* $\tilde{\tilde{\theta}}' = \tilde{\theta}'_1 - \tilde{\theta}'_2$*,* $\tilde{h}' = h'_1 - h'_2$ *and* $1/2 < l' < l$*.*
(1) *For a function* f *satisfying condition* (A)*, the following estimates hold:*

$$
\|f^{(\tilde{\theta},h)}\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} \le C\left(\|\tilde{\theta}'\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)}, \|h'\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right) |\!|\!| f |\!|\!|_T^2, \tag{3.3.6}
$$

$$
\begin{aligned}
\|f^{(\tilde{\theta}_1,h_1)} - f^{(\tilde{\theta}_2,h_2)}\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} &\le C\left(\sum_{j=1}^{2} \|\tilde{\theta}'_j\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)}, \sum_{j=1}^{2} \|h'_j\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right) \\
&\quad \times \left(\|\tilde{\tilde{\theta}}'\|^2_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right) |\!|\!| f |\!|\!|_T^2.
\end{aligned} \tag{3.3.7}
$$

(2) *For a function* $f \in \widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)$*, the following estimates hold:*

$$
\|f^{(\tilde{\theta},h)}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \le C\left(\|\tilde{\theta}'\|_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_T)}, \|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\right) \|f\|^2_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\mathbf{R}_T^3)}, \tag{3.3.8}
$$

$$
\|f^{(\tilde{\theta}_1,h_1)} - f^{(\tilde{\theta}_2,h_2)}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \le C\Big(\sum_{j=1}^2 \|\tilde{\theta}'_j\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)}, \sum_{j=1}^2 \|h'_j\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big)
$$
$$
\times \Big(\|\tilde{\bar{\theta}}'\|^2_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|^2_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\Big)\|f\|^2_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\mathbf{R}_T^3)}. \tag{3.3.9}
$$

(3) *For a function* $f \in \widetilde{W}_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$*, the following estimates hold:*

$$
\|f^{(\tilde{\theta},h)}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le C\Big(\|\tilde{\theta}'\|_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_T)}, \|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\Big)\|f\|^2_{\widetilde{W}_2^{2+l,1+\frac{l}{2}}(\mathbf{R}_T^3)}, \tag{3.3.10}
$$

$$
\|f^{(\tilde{\theta}_1,h_1)} - f^{(\tilde{\theta}_2,h_2)}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le C\Big(\sum_{j=1}^2 \|\tilde{\theta}'_j\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)}, \|h'_j\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big)
$$
$$
\times \Big(\|\tilde{\bar{\theta}}'\|^2_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|^2_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\Big)\|f\|^2_{\widetilde{W}_2^{2+l,1+\frac{l}{2}}(\mathbf{R}_T^3)}. \tag{3.3.11}
$$

3.4 Linear Problems

Let us introduce the linear operators $L_{i,\bar{\bar{\theta}}_0,\bar{h}_0}$ $(i = 1, 2, 3, 4)$, which are obtained from $L_{i,\tilde{\theta},h}$ $(i = 1, 2, 3, 4)$ with $(\tilde{\theta}, h, \Psi)$ replaced by $(\bar{\bar{\theta}}_0, \bar{h}_0, \bar{d}_0)$. From the assumptions of Theorem 3.2.1, it is easily seen that the coefficients of $L_{i,\bar{\bar{\theta}}_0,\bar{h}_0}$ $(i = 1, 2, 3)$ belong to $W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$, and those of $L_{4,\bar{\bar{\theta}}_0,\bar{h}_0}$ to $\overline{W}_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$. In this section we consider the following linear problems.

$$
\begin{cases}
\dfrac{\partial \mathbf{u}'}{\partial t} - L_{1,\bar{\bar{\theta}}_0,\bar{h}_0}\mathbf{u}' = \mathbf{l}_1, \\
\dfrac{\partial \tilde{\theta}'}{\partial t} - L_{2,\bar{\bar{\theta}}_0,\bar{h}_0}\tilde{\theta}' = l_4, \\
\dfrac{\partial \tilde{q}'}{\partial t} - L_{3,\bar{\bar{\theta}}_0,\bar{h}_0}\tilde{q}' = l_5 \quad \text{in } \tilde{\Omega}_T, \\
B_{\bar{\bar{\theta}}_0,\bar{h}_0}\mathbf{u}' = \mathbf{l}_2,\ (\tilde{\theta}', \tilde{q}') = (\bar{\theta}_e, \bar{q}_e) \quad \text{on } \tilde{\Gamma}_{sT}, \\
(\mathbf{u}', \tilde{\theta}', \tilde{q}') = (-\bar{\mathbf{u}}_0, \theta_H|_{x_3=H} - \bar{\bar{\theta}}_0, q_H|_{x_3=H} - \bar{\bar{q}}_0) \quad \text{on } \tilde{\Gamma}_{uT}, \\
(\mathbf{u}', \tilde{\theta}', \tilde{q}')|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega};
\end{cases} \tag{3.4.1}
$$

$$\begin{cases} \dfrac{\partial h'}{\partial t} - L_{4,\bar{\bar{\theta}}_0,\bar{h}_0} h' = l_6 \quad \text{in } \mathbf{R}^2_T, \\ h'\big|_{t=0} = 0 \quad \text{on } \mathbf{R}^2; \end{cases} \tag{3.4.2}$$

$$\begin{cases} \nabla_{\bar{h}_0,3} u_3' = l_3 \quad \text{in } \tilde{\Omega}_T, \\ u_3' = 0 \quad \text{on } \tilde{\Gamma}_{uT}. \end{cases} \tag{3.4.3}$$

For problems (3.4.1), (3.4.2), we have

Lemma 3.4.1. (i) *Let* $\mathbf{l}_1 \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)$, $\mathbf{l}_2 \in W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, $l_4 \in W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)$ $l_5 \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)$, $\bar{\theta}_e$, $\bar{q}_e \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, $\bar{\theta}_H$, $\bar{q}_H \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{HT})$, *and satisfy the compatibility conditions*

$$\begin{cases} \bar{\mathbf{u}}_0 = \mathbf{0}, \quad \bar{\theta}_H(y,0) = 0, \quad \bar{q}_H(y,0) = 0, \\ \dfrac{\partial \theta_H}{\partial t}\Big|_{t=0,\ y_3=h_0} = l_4\big|_{t=0, y_3=h_0}, \quad x \in \tilde{\Gamma}_H, \\ \mathbf{l}_2|_{t=0} = \mathbf{0}, \quad \bar{\theta}_e(y,0) = 0, \quad \bar{q}_e(y,0) = 0, \\ \dfrac{\partial \theta_e}{\partial t}\Big|_{t=0,\ y_3=p_0} = l_4\big|_{t=0, y_3=p_0}, \quad x \in \tilde{\Gamma}_s. \end{cases}$$

Then the problem (3.4.1) *has a unique solution* $(\mathbf{u}', \tilde{\theta}', \tilde{q}') \in Z'(T)$, $Z'(T) \equiv W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T) \times W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$ *satisfying*

$$\begin{aligned} \|(\mathbf{u}, \tilde{\theta}', \tilde{q}')\|_{Z'(T)} \le C_1' \Big[& \|\mathbf{l}_1\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} + \|\mathbf{l}_2\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} + \|l_4\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \\ & + \|l_5\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} + \|\bar{\theta}_e\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} + \|\bar{q}_e\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} \\ & + \|\bar{\mathbf{u}}_0\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{uT})} + \|\bar{\theta}_H\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\tilde{\Gamma}_{uT})} + \|\bar{q}_H\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{uT})} \Big]. \end{aligned} \tag{3.4.4}$$

(ii) *For* $l_6 \in W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}^2_T)$ *and* $h_0 \in W_2^{\frac{3}{2}+l}(\mathbf{R}^2)$, *problem* (3.4.2) *has a unique solution* $h' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)$ *satisfying*

$$\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)} \le C_2' \|l_6\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}. \tag{3.4.5}$$

The proof of Lemma 3.4.1 is similar to Lemma 2.4.1, and we omit it here.

Lemma 3.4.2. *For* $l_3 \in W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)$*, the problem* (3.4.3) *has a unique solution* $u_3' \in \widetilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)$ *such that*

$$\|u_3'\|_{\widetilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)} \leq C_3'\|l_3\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)}.$$

Proof. It is easy to see that by the integration with respect to x_3, problem (3.4.3) has an exact solution given by

$$u_3'(y', x_3, t) = \frac{1}{a_0^{33}} \int_{h_0}^{x_3} l_3 \, \mathrm{d}y_3.$$

This directly leads to $u_3' \in \widetilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)$. □

3.5 Nonlinear Problem (Proof of Theorem 3.2.1)

3.5.1 Successive Approximations

In this section, we prove Theorem 3.2.1 by an iteration method. Let

$$(\mathbf{u}'_{(0)}, u'_{3(0)}, \tilde{\theta}'_{(0)},\ \tilde{q}'_{(0)}, h'_{(0)}) = (\mathbf{0}, 0, 0, 0, 0)$$

and $(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{q}'_{(m+1)}, h'_{(m+1)})$ $(m = 0, 1, 2, \ldots)$ be a solution of the following problem for a given $(\mathbf{u}'_{(m)}, u'_{3(m)}, \tilde{\theta}'_{(m)}, \tilde{q}'_{(m)}, h'_{(m)}) \in Z(T)$.

$$\left\{\begin{aligned}
&\frac{\partial \mathbf{u}'_{(m+1)}}{\partial t} - L_{1,\bar{\tilde{\theta}}_0,\bar{h}_0}\mathbf{u}'_{(m+1)} = \big[L_{1,\tilde{\theta}_{(m)},h_{(m)}} - L_{1,\bar{\tilde{\theta}}_0,\bar{h}_0}\big]\mathbf{u}'_{(m+1)} + L_{1,\tilde{\theta}_{(m)},h_{(m)}}\bar{\mathbf{u}}_0 \\
&\qquad\qquad -\frac{\partial \bar{\mathbf{u}}_0}{\partial t} + \tilde{\mathbf{G}}_{1,\tilde{\theta}_{(m)},h_{(m)}}(\mathbf{u}_{(m)}, u_{3(m)}, \tilde{\theta}_{(m)}) =: \mathbf{l}_1^{(m+1)}, \\
&\nabla_{\bar{h}_0,3}u'_{3(m+1)} = -(\nabla_{\bar{h}_0,3} - \nabla_{h_{(m)},3})u'_{3(m+1)} + \tilde{G}_{3,h_{(m)}}(\mathbf{u}_{(m+1)}, \mathbf{u}_{(m)}) =: l_3^{(m+1)}, \\
&\frac{\partial \tilde{\theta}'_{(m+1)}}{\partial t} - L_{2,\bar{\tilde{\theta}}_0,\bar{h}_0}\tilde{\theta}'_{(m+1)} = \big[L_{2,\tilde{\theta}_{(m)},h_{(m)}} - L_{2,\bar{\tilde{\theta}}_0,\bar{h}_0}\big]\tilde{\theta}'_{(m+1)} + L_{2,\tilde{\theta}_{(m)},h_{(m)}}\bar{\tilde{\theta}}_0 \\
&\qquad\qquad -\frac{\partial \bar{\tilde{\theta}}_0}{\partial t} + \tilde{G}_{4,\tilde{\theta}_{(m)},h_{(m)}}(\mathbf{u}_{(m)}, u_{3(m)}, \tilde{\theta}_{(m)}) =: l_4^{(m+1)}, \\
&\frac{\partial \tilde{q}'_{(m+1)}}{\partial t} - L_{3,\bar{\tilde{\theta}}_0,\bar{h}_0}\tilde{q}'_{(m+1)} = \big[L_{3,\tilde{\theta}_{(m)},h_{(m)}} - L_{3,\bar{\tilde{\theta}}_0,\bar{h}_0})\big]\tilde{q}'_{(m+1)} + L_{3,\tilde{\theta}_{(m)},h_{(m)}}\bar{\tilde{q}}_0 \\
&\qquad\qquad -\frac{\partial \bar{\tilde{q}}_0}{\partial t} + \tilde{G}_{5,\tilde{\theta}_{(m)},h_{(m)}}(\mathbf{u}_{(m)}, u_{3(m)}, \tilde{q}_{(m)}) =: l_5^{(m+1)} \quad \text{in } \tilde{\Omega}_T,
\end{aligned}\right. \tag{3.5.1}$$

$$\left\{\begin{aligned}
&B_{\bar{\tilde{\theta}}_0,\bar{h}_0}\mathbf{u}'_{(m+1)} = -B_{\tilde{\theta}_{(m)},h_{(m)}}\bar{\mathbf{u}}_0 + (B_{\bar{\tilde{\theta}}_0,\bar{h}_0}\mathbf{u}'_{(m+1)} - B_{\tilde{\theta}_{(m)},h_{(m)}}\mathbf{u}'_{(m+1)}) \\
&\qquad\qquad +\tilde{\mathbf{G}}_2(\mathbf{u}_{(m)}) =: \mathbf{l}_2^{(m+1)}, \\
&(\tilde{\theta}'_{(m+1)}, \tilde{q}'_{(m+1)}) = (\theta_e - \bar{\tilde{\theta}}_0, q_e - \bar{\tilde{q}}_0) \quad \text{on } \tilde{\Gamma}_{sT}, \\
&(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{q}'_{(m+1)}) = (-\bar{\mathbf{u}}_0, 0, \theta_H - \bar{\tilde{\theta}}_0, q_H - \bar{\tilde{q}}_0) \quad \text{on } \tilde{\Gamma}_{uT}, \\
&(\mathbf{u}'_{(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{q}'_{(m+1)})|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega},
\end{aligned}\right. \tag{3.5.2}$$

$$
\begin{cases}
\dfrac{\partial h'_{(m+1)}}{\partial t} - L_{4,\bar{\tilde{\theta}}_0,\bar{h}_0} h'_{(m+1)} = (L_{4,\tilde{\theta}_{(m)},h_{(m)}} - L_{4,\bar{\tilde{\theta}}_0,\bar{h}_0}) h'_{(m+1)} + L_{4,\tilde{\theta}_{(m)},h_{(m)}} \bar{h}_0 \\
\qquad\qquad - \dfrac{\partial \bar{h}_0}{\partial t} + \tilde{G}_{6,\tilde{\theta}_{(m)},h_{(m)}}(\mathbf{u}_{(m)}, \tilde{\theta}_{(m+1)}, \tilde{\theta}_{(m)}) =: l_6^{(m+1)} \quad \text{in } \mathbf{R}_T^2, \\
h'_{(m+1)}|_{t=0} = 0 \quad \text{on } \mathbf{R}^2.
\end{cases} \tag{3.5.3}
$$

Here $\mathbf{u}_{(m)} = \mathbf{u}'_{(m)} + \bar{\mathbf{u}}_0$, $\tilde{\theta}_{(m)} = \tilde{\theta}'_{(m)} + \bar{\tilde{\theta}}_0$, $\tilde{q}_{(m)} = \tilde{q}'_{(m)} + \bar{\tilde{q}}_0$, and

$$
\tilde{G}_{3,h_{(m)}}(\mathbf{u}_{(m+1)}, \mathbf{u}_{(m)}) = -\nabla_{h_{(m)}} \cdot \mathbf{u}_{(m+1)} - \mathbf{a}^3(h_{(m)}) \cdot \frac{\partial \mathbf{u}_{(m)}}{\partial y_3},
$$

$$
\begin{aligned}
&\tilde{G}_{6,\tilde{\theta}_{(m)},h_{(m)}}(\mathbf{u}_{(m)}, \tilde{\theta}_{(m+1)}, \tilde{\theta}_{(m)}) \\
&:= \frac{1}{E(\tilde{\theta}_{(m)}, h_{(m)})} \Bigg[-K_2(\tilde{\theta}_{(m)}, h_{(m)}) \\
&\quad + \frac{F\sigma L}{la(\theta_e)\left(1 + |\nabla\Psi(y', t; \tilde{\theta}_{(m)}, h_{(m)})|^2\right)} \sum_{i,j=1}^{2} c_{ij} H_{ij}(\tilde{\theta}_{(m)}, h_{(m)}) \\
&\quad - \frac{F}{la(\theta_e)} \Bigg\{ -\mu_3 \left(\nabla_{h_{(m)}} + a^{33}(h_{(m)}) \mathbf{F}_5^{(\tilde{\theta}_{(m)},h_{(m)})} \frac{\partial}{\partial y_3} \right) \\
&\qquad\qquad \tilde{\theta}_{(m)}\Big|_{y_3=p_0} \cdot \nabla\Psi(y', t; \tilde{\theta}_{(m)}, h_{(m)}) \\
&\qquad\qquad - \frac{\mu_4 p_0 g}{R\theta_e\big|_{x_3=\Psi}} a^{33}(h_{(m)}) \frac{\partial \tilde{\theta}_{(m+1)}}{\partial y_3}\Bigg|_{y_3=p_0} \\
&\qquad + g_1^{(\tilde{\theta}_{(m)},h_{(m)})} \left(1 + \left| \nabla\Psi(y', t; \tilde{\theta}_{(m)}, h_{(m)}) \right|^2 \right)^{\frac{1}{2}} |\mathbf{u}_{(m)}|^{\alpha} \tilde{\theta}_{(m)}\Big|_{y_3=p_0} \Bigg\} \Bigg]
\end{aligned}
$$

with $la(\theta_e) = la(\theta_e(y', \Psi(y', t; \tilde{\theta}_{(m)}, h_{(m)}), t))$.

By the same way as in Chapter 1, we estimate the right-hand side of (3.5.1)–(3.5.3). In the followings, ϵ is an arbitrary positive number, and $1/2 < l' < l$.

Similar estimates as those in Lemmas 2.5.1–2.5.3 hold for this problem. We state the statements without proof.

Lemma 3.5.1. *For* $\mathbf{l}_1^{(m+1)}$*, we have*

$$
\begin{aligned}
&\|\mathbf{l}_1^{(m+1)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} + \|l_4^{(m+1)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Omega}_t)} + \|l_5^{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\leq (\epsilon + C_\epsilon t)\, C\left(\|u_{3(m)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)}, \|\tilde{\theta}'_{(m)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right) \\
&\qquad \times \left(\left\|\mathbf{u}_{(m)}\right\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \left\|\mathbf{u}_{(m)}\right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \left\|\mathbf{u}_{(m+1)}\right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right) \\
&\qquad\quad + C\left(\|\tilde{\theta}'_{(m)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}.
\end{aligned}
$$

Lemma 3.5.2. *For* $\mathbf{l}_2^{(m+1)}$*, we have*

$$
\begin{aligned}
&\|\mathbf{l}_2^{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{st})} \leq (\epsilon + C_\epsilon t)\left(\|\mathbf{u}_{(m)}\|^{\alpha+1}_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|^3_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right) \\
&\quad + (\epsilon + C_\epsilon t)\, C\left(\|\tilde{\theta}'_{(m)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\left\|\mathbf{u}_{(m+1)}\right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \\
&\quad + C\|\mathbf{v}_0\|_{W_2^{1+l}(\tilde{\Omega})}.
\end{aligned}
$$

Lemma 3.5.3. *For* $l_3^{(m+1)}$*, we have*

$$
\begin{aligned}
&\|l_3^{(m+1)}\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Gamma}_{st})} \\
&\qquad \leq C\left(\|\tilde{\theta}'_{(m)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|\mathbf{u}_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \\
&\qquad\quad + (\epsilon + C_\epsilon t)\, C\left(\|\tilde{\theta}'_{(m)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right) \\
&\qquad\qquad \times \left[\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right].
\end{aligned}
$$

Lemma 3.5.4. *For $l_6^{(m+1)}$, we have*

$$\|l_6^{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$$
$$\leq (\epsilon + C_\epsilon t)\, C \left(\|\tilde{\theta}'_{(m)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right)$$
$$\times \left(1 + \|h'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\tilde{\mathbf{R}}_t^2)}\right) \left[1 + \left(1 + \|\tilde{\theta}'_{(m)}\|_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_t)}\right)\right.$$
$$\left.\times \left(\|\mathbf{u}_{(m)}\|^\alpha_{W_2^{2+l',\frac{2+l'}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|^2_{W_2^{2+l',\frac{2+l'}{2}}(\tilde{\Omega}_t)} \right) + \|\tilde{\theta}'_{(m+1)}\|_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_t)} \right]$$
$$+ C \left(1 + \|d_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)}\right) \left(1 + \|\theta_0\|_{\overline{W}_2^{2+l}(\mathbf{R}^3)}\right).$$

3.5.2 Proof of Theorem 3.2.1

Based on Lemmas 3.5.1–3.5.4, we can easily show the boundedness of $\|(\mathbf{u}'_{(m)}, \tilde{\theta}'_{(m)}, \tilde{q}'_{(m)})\|_{Z'(t)}$. Let us introduce the notation

$$E_m(t) := \|(\mathbf{u}'_{(m)}, u'_{3(m)}, \tilde{\theta}'_{(m)}, \tilde{q}'_{(m)}, h'_{(m)})\|_{Z(t)}, \quad E'_m(t) := \|(\mathbf{u}'_{(m)}, \tilde{\theta}'_{(m)}, \tilde{q}'_{(m)})\|_{Z'(t)}.$$

Applying Lemmas 3.4.1 and 3.4.2 to problems (3.5.1)–(3.5.3), and making use of Lemmas 3.5.1–3.5.4, we arrive at the inequalities

$$E'_{m+1}(t) \leq C_1 \left[1 + (\epsilon + C_\epsilon t) \Big\{ \phi_1(E_m(t)) + \phi_2(E_m(t)) E'_{m+1}(t) \Big\} \right] \tag{3.5.4}$$

and

$$\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|h'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$$
$$\leq \tilde{C}_2 \left[1 + \phi_3(E_m(t)) \left(\|\mathbf{u}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{\theta}'_{(m+1)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} \right) \right.$$
$$+ (\epsilon + C_\epsilon t) \Big\{ \phi_1(E_m(t)) + \phi_2(E_m(t))$$
$$\left. \times \left(\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|h'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \Big\} \right] \tag{3.5.5}$$

for any $t \in (0, T]$, $\tilde{C}_2 = C_2 + C_3$, and ϕ_i $(i = 1, 2, 3)$ are monotonically increasing. Adding (3.5.4) and (3.5.5) multiplied by $1/(2\tilde{C}_2 \phi_3(E_m(t))$, we get the inequality

$$E_{m+1}(T_1)\Big\{1 - C_4(t)(\epsilon + C_\epsilon t)\phi_2(E_m(t))\Big\} < C_4(t)\Big\{(\epsilon + C_\epsilon t)\phi_1(E_m(t)) + 1\Big\}$$

with a constant $C_4(t)$ depending on t monotonically increasingly.

Let a positive constant M such that $C_4(T) < M$. Take ϵ first small enough so that

$$\epsilon C_4(T)\phi_2(M) < 1, \quad \epsilon C_4(T)\Big[\phi_1(M) + \phi_2(M)M\Big] < M - C_4(T)$$

hold, and then $T_1 \in [0, T]$ so that

$$C_4(T)C_\epsilon\phi_2(M)T_1 < 1 - C_4(T)\epsilon\phi_2(M),$$

$$C_4(T)C_\epsilon T_1 \{\phi_1(M) + \phi_2(M)M\} < M - C_4(T) - \epsilon C_4(T)\{\phi_1(M) + \phi_2(M)M\}$$

hold. Consequently we obtain $E_{m+1}(T_1) < M$ from the assumption $E_m(T_1) < M$. By induction $\{(\mathbf{u}'_{(m)}, u'_{3(m)}, \tilde{\theta}'_{(m)}, \tilde{q}'_{(m)}, h'_{(m)})\}_{m=0}^{\infty}$ is well defined in $Z(T_1)$ and $E_m(T_1) < M$ for $m = 0, 1, 2, \ldots$.

Now we prove its convergence. Subtract (3.5.1)-(3.5.3) with m replaced by $(m-1)$ from (3.5.1)-(3.5.3). Then

$$\begin{aligned}&(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{u}'_{3(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{q}}'_{(m+1)}, \tilde{h}'_{(m+1)})\\ &:= (\mathbf{u}'_{(m+1)} - \mathbf{u}'_{(m)}, u'_{3(m+1)} - u'_{3(m)}, \tilde{\theta}'_{(m+1)} - \tilde{\theta}'_{(m)}, \tilde{q}'_{(m+1)} - \tilde{q}'_{(m)}, h'_{(m+1)} - h'_{(m)})\end{aligned}$$

satisfies the equations

$$\begin{cases} \dfrac{\partial \tilde{\mathbf{u}}'_{(m+1)}}{\partial t} - L_{1,\bar{\tilde{\theta}}_0,\bar{h}_0}\tilde{\mathbf{u}}'_{(m+1)} = \mathbf{l}_1^{(m+1)} - \mathbf{l}_1^{(m)}, \\ \nabla_{h_0,3}\tilde{u}'_{3(m+1)} = l_3^{(m+1)} - l_3^{(m)}, \\ \dfrac{\partial \tilde{\tilde{\theta}}'_{(m+1)}}{\partial t} - L_{2,\bar{\tilde{\theta}}_0,\bar{h}_0}\tilde{\tilde{\theta}}'_{(m+1)} = l_4^{(m+1)} - l_4^{(m)}, \\ \dfrac{\partial \tilde{\tilde{q}}'_{(m+1)}}{\partial t} - L_{3,\bar{\tilde{\theta}}_0,\bar{h}_0}\tilde{\tilde{q}}'_{(m+1)} = l_5^{(m+1)} - l_5^{(m)} \quad \text{in } \tilde{\Omega}_{T_1}, \end{cases}$$

$$
\begin{cases}
B_{\bar{\tilde{\theta}}_0,\bar{h}_0}\tilde{\mathbf{u}}'_{(m+1)} = \mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}, \\
(\tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{q}}'_{(m+1)}) = (\theta_e|_{x_3=\Psi(\cdot;\tilde{\tilde{\theta}}_{(m)},h_{(m)})} - \theta_e|_{x_3=\Psi(\cdot;\tilde{\tilde{\theta}}_{(m-1)},h_{(m-1)})}, \\
\qquad\qquad q_e|_{x_3=\Psi(\cdot;\tilde{\tilde{\theta}}_{(m)},h_{(m)})} - q_e|_{x_3=\Psi(\cdot;\tilde{\tilde{\theta}}_{(m-1)},h_{(m-1)})}) \quad \text{on } \tilde{\Gamma}_{sT_1}, \\
(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{u}'_{3(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{q}}'_{(m+1)}) = (\mathbf{0}, 0, 0, 0) \quad \text{on } \tilde{\Gamma}_{uT_1}, \\
(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{q}}'_{(m+1)})|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega},
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial \tilde{h}'_{(m+1)}}{\partial t} - L_{4,\bar{\tilde{\theta}}_0,\bar{h}_0}\tilde{h}'_{(m+1)} = l_6^{(m+1)} - l_6^{(m)} \quad \text{in } \mathbf{R}^2_{T_1}, \\
\tilde{h}'_{(m+1)}\big|_{t=0} = 0 \quad \text{on } \mathbf{R}^2.
\end{cases}
$$

Then Lemmas 3.4.1 and 3.4.2 yield for any $t \le T_1$ the estimates

$$
\begin{aligned}
&\|(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{q}}'_{(m+1)})\|_{Z'(t)} \\
&\quad \le C_1\Big[\|\mathbf{l}_1^{(m+1)} - \mathbf{l}_1^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{st})} \\
&\qquad + \|l_4^{(m+1)} - l_4^{(m)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|l_5^{(m+1)} - l_5^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\qquad + \|\theta_e(y', \Psi(y', t; \tilde{\tilde{\theta}}_{(m)}, h_{(m)}), t) \\
&\qquad\qquad - \theta_e(y', \Psi(y', t; \tilde{\tilde{\theta}}_{(m-1)}, h_{(m-1)}), t)\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \\
&\qquad + \|q_e(y', \Psi(y', t; \tilde{\tilde{\theta}}_{(m)}, h_{(m)}), t) \\
&\qquad\qquad - q_e(y', \Psi(y', t; \tilde{\tilde{\theta}}_{(m-1)}, h_{(m-1)}), t)\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big], \\
&\|\tilde{h}'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \le C_2\|l_6^{(m+1)} - l_6^{(m)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}, \\
&\|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} \le C_3\|l_3^{(m+1)} - l_3^{(m)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}.
\end{aligned}
$$

Each term in the right-hand side of the above inequalities except for $\|\mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{st})}$ and $\|l_6^{(m+1)} - l_6^{(m)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$ can be estimated just as we have done for $\|\mathbf{l}_1^{(m+1)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}$, $\|l_4^{(m+1)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}$, $\|l_5^{(m+1)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}$ and $\|l_3^{(m+1)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}$. Thus we have

$$\begin{aligned}
&\|\mathbf{l}_1^{(m+1)}-\mathbf{l}_1^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}+\|l_4^{(m+1)}-l_4^{(m)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}+\|l_5^{(m+1)}-l_5^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}\\
&\quad\le(\epsilon+C_\epsilon t)\,C\Big(\big\|\big(\mathbf{u}'_{(m+1)},\mathbf{u}'_{(m)},\mathbf{u}'_{(m-1)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)},\|u'_{3(m-1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)},\\
&\qquad\big\|\big(\tilde{\theta}'_{(m+1)},\tilde{\theta}'_{(m)}\big)\big\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)},\big\|\big(\tilde{q}'_{(m+1)},\tilde{q}'_{(m)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)},\\
&\qquad\big\|\big(h'_{(m)},h'_{(m-1)}\big)\big\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big)\,\Big\|\big(\tilde{\mathbf{u}}'_{(m)},\tilde{u}'_{3(m)},\tilde{\tilde{\theta}}'_{(m)},\tilde{\tilde{q}}'_{(m)},\tilde{h}'_{(m)}\big)\Big\|_{Z(t)},
\end{aligned}$$

$$\begin{aligned}
&\|l_3^{(m+1)}-l_3^{(m)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}\\
&\quad\le(\epsilon+C_\epsilon t)\,C\Big(\big\|\big(\mathbf{u}'_{(m+1)},\mathbf{u}'_{(m)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)},\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)},\\
&\qquad\big\|\big(h'_{(m)},h'_{(m-1)}\big)\big\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big)\\
&\qquad\times\Big(\|\tilde{\mathbf{u}}'_{(m)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}+\|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}+\|\tilde{h}'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big)\\
&\qquad+C\Big(\|\tilde{h}'_{(m-1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big)\,\|\tilde{\mathbf{u}}'_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}.
\end{aligned}$$

The following estimates have already been obtained in Chapter 2:

$$\begin{aligned}
&\|l_6^{(m+1)}-l_6^{(m)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\\
&\quad\le(\epsilon+C_\epsilon t)\,C\Big(\big\|\big(\mathbf{u}'_{(m)},\mathbf{u}'_{(m-1)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)},\\
&\qquad\qquad\big\|\big(\tilde{\theta}'_{(m)},\tilde{\theta}'_{(m-1)}\big)\big\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)},\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big)\\
&\qquad\times\Big(\|\tilde{\mathbf{u}}'_{(m)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}+\|\tilde{\tilde{\theta}}'_{(m)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}+\|\tilde{h}'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big)\\
&\qquad+C\Big(\|h'_{(m-1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big)\,\|\tilde{\tilde{\theta}}'_{(m+1)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)},
\end{aligned}$$

$$\left\|\mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}\right\|_{W_2^{\frac{1}{2}+l,\frac{l}{2}+\frac{1}{4}}(\tilde{\Gamma}_{st})}$$

$$\le (\epsilon + C_\epsilon t)\, C\Bigg(\|\tilde{\theta}'_{(m)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}, \|\tilde{\theta}'_{(m-1)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)},$$

$$\|h'_{(m-1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Bigg)$$

$$\times \Bigg[\Bigg(\|\tilde{\tilde{\theta}}'_{(m)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} + \|\tilde{h}'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Bigg)$$

$$\times \Bigg(\|\mathbf{u}'_{(m)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}'_{(m-1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}\Bigg) + \|\tilde{\mathbf{u}}'_{(m)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}\Bigg].$$

In this case, we need only $\alpha > l + 1/2$. Introducing the notations

$$\tilde{E}_m(t) := \|(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\tilde{\theta}}'_{(m)}, \tilde{\tilde{q}}'_{(m)}, \tilde{h}'_{(m)})\|_{Z(t)}, \quad \tilde{E}'_m(t) := \|(\tilde{\mathbf{u}}'_{(m)}, \tilde{\tilde{\theta}}'_{(m)}, \tilde{\tilde{q}}'_{(m)})\|_{Z'(t)},$$

we get for any $t \in (0, T_1]$,

$$\tilde{E}'_{m+1}(t) \le C_1(\epsilon + C_\epsilon t)\Big[\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)\tilde{E}'_m(t)$$
$$+ \phi_5(E_{m-1}(T_1))\tilde{E}'_{m+1}(t) + \phi_6(E_{m+1}(T_1) + E_m(T_1))\|\tilde{h}'_{(m)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big], \tag{3.5.6}$$

$$\|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|\tilde{h}'_{(m+1)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$$
$$\le \tilde{C}_2\Bigg[\phi_7(E_m(T_1))\Bigg(\|\tilde{\mathbf{u}}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{\tilde{\theta}}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}\Bigg)$$
$$+ (\epsilon + C_\epsilon t)\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)\tilde{E}_m(t)$$
$$+ (\epsilon + C_\epsilon t)\phi_5(E_{m-1}(T_1))\Bigg\{ \|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|h'_{(m+1)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Bigg\}\Bigg], \tag{3.5.7}$$

where ϕ_i $(i = 4, 5, 6, 7)$ are monotonically increasing in their arguments.

Adding (3.5.6) and (3.5.7) multiplied by $1/(2\tilde{C}_2\phi_7(E_m(T_1))$, we get the estimate

$$\tilde{E}_{m+1}(t) \le C_4(T_1)(\epsilon + C_\epsilon t)\big\{\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)\tilde{E}_m(t)$$
$$+\phi_5(E_{m-1}(T_1))\tilde{E}_{m+1}(t) + \phi_6(E_{m+1}(T_1) + E_m(T_1))\tilde{E}_m(t)\big\} \tag{3.5.8}$$

and hence

$$\left\{1 - C_4(T_1)(\epsilon + C_\epsilon t)\phi_5(E_{m-1}(T_1))\right\}\tilde{E}_{m+1}(t)$$

$$\le C_4(T_1)(\epsilon + C_\epsilon t)\Big\{\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)$$

$$+ \phi_6(E_{m+1}(T_1) + E_m(T_1))\Big\}\tilde{E}_m(t)$$

for any $t \in (0, T_1]$. Take ϵ small enough again to satisfy

$$\epsilon C_4(T_1)\Big[\phi_4(3M) + \phi_5(M) + \phi_6(2M)\Big] < 1,$$

and then $T_2 \in (0, T_1]$ so that

$$C_4(T_1)C_\epsilon\phi_5(M)T_2 < 1 - C_4(T_1)\epsilon\phi_5(M),$$

$$C_4(T_1)C_\epsilon\left[\phi_4(3M) + \phi_5(M) + \phi_6(2M)\right]T_2$$

$$< 1 - \epsilon C_4(T_1)\left[\phi_4(3M) + \phi_5(M) + \phi_6(2M)\right]$$

hold. For these ϵ and T_2, we obtain

$$\tilde{E}_{m+1}(T_2) \le r\tilde{E}_m(T_2), \quad r = \frac{C_4(T_1)(\epsilon + C_\epsilon T_2)\Big[\phi_4(3M) + \phi_6(2M)\Big]}{1 - C_4(T_1)(\epsilon + C_\epsilon T_2)\phi_5(M)} \in (0, 1).$$

This means that $\{(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\bar{\theta}}'_{(m)}, \tilde{\bar{q}}'_{(m)}, \tilde{h}'_{(m)})\}_{m=0}^{\infty}$ is a Cauchy sequence in $Z(T_2)$. Therefore the limit function

$$(\mathbf{u}', u_3', \tilde{\theta}', \tilde{q}', h') = \lim_{m\to\infty}(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\bar{\theta}}'_{(m)}, \tilde{\bar{q}}'_{(m)}, \tilde{h}'_{(m)})$$

exists in $Z(T_2)$, which is our desired solution.

Now we shall show that $0 < \underline{\theta}_0/2 \le \tilde{\theta}(y,t)$ and $0 < \underline{q}_0/2 \le \tilde{q}(y,t)$ hold by taking the time interval small enough again. Indeed, since $\tilde{\theta}' = \tilde{\theta} - \tilde{\bar{\theta}}_0 \in W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)$, we have

$$\tilde{\theta}(y,t) \ge \tilde{\bar{\theta}}_0\big|_{t=0}(y) - \Big(|\tilde{\theta}'(y,t)| + |\tilde{\bar{\theta}}_0(y,t) - \tilde{\bar{\theta}}_0(y,0)|\Big)$$

$$\ge \underline{\theta}_0 - t^\gamma\left(\sup_{y\in\tilde{\Omega}}|\tilde{\theta}'(y,t)|_t^{(\gamma)} + \sup_{y\in\tilde{\Omega}}|\tilde{\bar{\theta}}_0(y,t)|_t^{(\gamma)}\right),$$

where $|f|_t^{(\gamma)}$ stands for the Hölder coefficient of f with respect to t with exponent $0 < \gamma < \frac{l}{2} - \frac{1}{4}$. Sobolev embedding theorem implies $\sup_{y\in\tilde{\Omega}} |\tilde{\theta}'(y,t)|_t^{(\gamma)} \leq \|\tilde{\theta}'\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_{T_2})}$, $\sup_{y\in\tilde{\Omega}} |\bar{\tilde{\theta}}_0(y,t)|_t^{(\gamma)} \leq \|\bar{\tilde{\theta}}_0'\|_{\overline{W}_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_{T_2})}$. If we take

$$T_3 = \left(\frac{\underline{\theta}_0}{2\left(\|\tilde{\theta}'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T_2})} + \|\bar{\tilde{\theta}}_0'\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T_2})} \right)} \right)^{\frac{1}{\gamma}},$$

then we have $\theta(t,x) > \underline{\theta}_0/2$ on $[0,T_3]$. A similar argument holds for $\tilde{q}$. Denote again the time interval by $[0,T_3]$ on which both $\underline{\theta}_0/2 < \tilde{\theta}(y,t)$ and $\underline{q}_0/2 < \tilde{q}(y,t)$ hold. $T^* = \min\{T_2, T_3\}$ provides the desired result.

Uniqueness of the solution can be proved by virtue of an analogous inequality to (3.5.8).

This completes the proof of the main theorem.

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Appendix A

Anisotropic Sobolev–Slobodetskiĭ spaces

Let G be a domain in $\mathbf{R}^n$ and l a non-negative number. By $W_2^l(G)$ we mean a space of functions $u(x), x \in G$, equipped with the norm $\|u\|^2_{W_2^l(G)} = \sum\limits_{|\alpha|<l} \|D^\alpha u\|^2_{L_2(G)} + \|u\|^2_{\dot{W}_2^l(G)}$, where

$$\begin{cases} \|u\|^2_{\dot{W}_2^l(G)} = \sum\limits_{|\alpha|=l} \|D^\alpha u\|^2_{L_2(G)} = \sum\limits_{|\alpha|=l} \int_G |D^\alpha u(x)|^2 \mathrm{d}x \quad \text{if } l \text{ is an integer,} \\ \|u\|^2_{\dot{W}_2^l(G)} = \sum\limits_{|\alpha|=[l]} \int_G \int_G \dfrac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\{l\}}} \mathrm{d}x\mathrm{d}y \\ \qquad\qquad \text{if } l \text{ is a non-integer, } l = [l] + \{l\},\ 0 < \{l\} < 1. \end{cases}$$

We also define the following function spaces with $m > 1$:

$$\overline{W}_2^m(G) = \left\{ f(x) \middle| \|f\|^2_{\overline{W}_2^{m,\frac{m}{2}}(\tilde{\Omega}_T)} := \sup_{x\in G} |f|^2 + \|f\|^2_{\dot{W}_2^{m-[m]}(G)} + \|Df\|^2_{W_2^{m-1}(G)} < \infty \right\}.$$

Next we introduce anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{l,\frac{l}{2}}(G_T) := W_2^{l,0}(G_T) \cap W_2^{0,\frac{l}{2}}(G_T)$ ($G_T := G \times [0,T]$), whose norms are defined by

$$\begin{aligned} \|u\|^2_{W_2^{l,\frac{l}{2}}(G_T)} &= \int_0^T \|u(\cdot,t)\|^2_{W_2^l(G)} \mathrm{d}t + \int_G \|u(x,\cdot)\|^2_{W_2^{\frac{l}{2}}(0,T)} \mathrm{d}x \\ &=: \|u\|^2_{W_2^{l,0}(G_T)} + \|u\|^2_{W_2^{0,\frac{l}{2}}(G_T)}. \end{aligned}$$

We also define function spaces

$$\widetilde{W}_2^{l,\frac{l}{2}}(G_T) = \left\{ f \in W_2^{l,\frac{l}{2}}(G_T) \middle| \frac{\partial f}{\partial x_3} \in W_2^{l,\frac{l}{2}}(G_T) \right\}$$

with the norm

$$\|f\|^2_{\widetilde{W}_2^{l,\frac{l}{2}}(G_T)} = \|f\|^2_{W_2^{l,\frac{l}{2}}(G_T)} + \left\| \frac{\partial f}{\partial x_3} \right\|^2_{W_2^{l,\frac{l}{2}}(G_T)},$$

and for $m > 2$,

$$\overline{W}_2^{m,\frac{m}{2}}(G_T) = \left\{ f(x,t) \middle| \|f\|^2_{\overline{W}_2^{m,\frac{m}{2}}(\tilde{\Omega}_T)} := \sup_{G_T} |f|^2 + \sup_{x \in G} \|f\|^2_{\dot{W}_2^{\frac{m-[m]}{2}}(0,T)} \right.$$

$$\left. + \sup_{t \in (0,T)} \|f\|^2_{\dot{W}_2^{m-[m]}(G)} + \|D_x f\|^2_{W_2^{m-1,\frac{m-1}{2}}(G_T)} + \|D_t f\|^2_{W_2^{m-2,\frac{m-2}{2}}(G_T)} < \infty \right\},$$

where D_x and D_t represent the differential operators with respect to x and t, respectively. The norms of the vector spaces and the product spaces are defined by the standard vector norm and the sum of the norms of each space, respectively.

Appendix B

Proofs of Lemmas

B.1 Lemmas in Chapter 2

B.1.1 Proof of Lemma 2.3.1

Making use of the relation (2.1.17), we have

$$\left\|\frac{\partial \Psi}{\partial t}\right\|^2_{L_2(\mathbf{R}^2)} \le C\bigg(\left\|\frac{\partial h}{\partial t}\right\|^2_{L_2(\mathbf{R}^2)} + \|E_1\|^2_{L_2(\mathbf{R}^2)}\bigg). \tag{B.1.1}$$

This and

$$\|\Psi\|^2_{L_2(\mathbf{R}^2_T)} \le \|d_0\|^2_{L_2(\mathbf{R}^2)} + t\left\|\frac{\partial \Psi}{\partial t}\right\|^2_{L_2(\mathbf{R}^2_T)} \tag{B.1.2}$$

yield the estimate of $\|\Psi\|^2_{L_2(\mathbf{R}^2_T)}$.

The fractional norm $\|\Psi\|^2_{W_2^{l-\frac{1}{2},0}(\mathbf{R}^2_T)}$ is easily estimated from the inequality

$$\begin{aligned}|\Psi(y^{1\prime},t;h) - \Psi(y^{2\prime},t;h)|^2 \le C\left(\frac{1}{\inf_{x,t}|\tilde{F}_{13}|}\right)^2\bigg[&|h(y^{1\prime},t) - h(y^{2\prime},t)|^2 \\ &+ |b(y^{1\prime}) - b(y^{2\prime})|^2 \sup_{x_3}|\tilde{F}_{13}(y^{1\prime},x_3,t)|^2 \\ &+ \sup_{x_3}|\tilde{F}_{13}(y^{1\prime},x_3,t) - \tilde{F}_{13}(y^{2\prime},x_3,t)|^2\left(|b(y^{2\prime})|^2 + |\Psi(y^{1\prime},t;h)|^2\right)\bigg]\end{aligned} \tag{B.1.3}$$

with $h = p_0 + \int_d^b \tilde{F}_{13}\ \mathrm{d}x_3$. Indeed, applying the Hölder inequality and the Sobolev embedding theorem, one can easily confirm that

$$\begin{aligned}
&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \frac{|\Psi(y^{1\prime},t;h)|^2 |F_{13}(y^{1\prime},x_3,t) - F_{13}(y^{2\prime},x_3,t)|^2}{|y^{1\prime}-y^{2\prime}|^{1+2l}} \mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime} \\
&\leq \int_{\mathbf{R}^2} \frac{\|\Psi(y^{1\prime},t;h)\|^2_{L_p(\mathbf{R}^2)} \|F_{13}(y^{1\prime},x_3,t) - F_{13}(y^{1\prime}-z',x_3,t)\|^2_{L_q(\mathbf{R}^2)}}{|z'|^{1+2l}} \mathrm{d}z' \\
&\leq C\|\Psi(t)\|^2_{L_p(\mathbf{R}^2)} \left(\|\nabla F_{13}(\cdot,x_3,t)\|^2_{\dot{W}_2^{l-\frac{1}{2}}(\mathbf{R}^2)} + \|F_{13}(\cdot,x_3,t)\|^2_{\dot{W}_2^{l-\frac{1}{2}}(\mathbf{R}^2)} \right)
\end{aligned}$$

$(1/p + 1/q = 1/2)$ holds with a postive constant C. Other terms in (B.1.3) can be estimated in the same way.

For $\|\Psi\|_{W_2^{0,\frac{l}{2}-\frac{1}{4}}(\mathbf{R}^2_T)}$, using the relation $h = p_0 + \int_d^b \tilde{F}_{13}\ \mathrm{d}x_3$ again, we have

$$\begin{aligned}
\|\Psi\|^2_{W_2^{0,\frac{l}{2}-\frac{1}{4}}(\mathbf{R}^2_T)} \leq C\Bigg[&\|h\|^2_{W_2^{0,\frac{l}{2}-\frac{1}{4}}(\mathbf{R}^2_T)} \\
&+ \left(\|b\|^2_{\overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)} + \sup_{\mathbf{R}^2_T} |\Psi|^2 \right) \|F_{13}\|^2_{W_2^{l,\frac{l}{2}}(\mathbf{R}^3_T)} \Bigg].
\end{aligned}$$

Making use of these facts, we arrive at (2.3.3) with $i = 0$.

For $i = 1$, we make use of (2.1.15). Since

$$\begin{aligned}
&|\nabla\Psi(y^{1\prime},t;h) - \nabla\Psi(y^{2\prime},t;h)| \\
&\leq C\Bigg[\left| F_{13}(y^{1\prime},\Psi(y^{1\prime},t;h),t) - F_{13}(y^{2\prime},\Psi(y^{2\prime},t;h),t) \right| \\
&\qquad \times \left| -\nabla h + \tilde{F}_{13}(y',b(y'),t)\nabla b + \int_b^d \nabla\tilde{F}_{13}\ \mathrm{d}x_3 \right| \\
&\quad + |\nabla h(y^{1\prime},t) - \nabla h(y^{2\prime},t)| \\
&\quad + \left| \tilde{F}_{13}(y^{1\prime},b(y^{1\prime}),t)\nabla b(y^{1\prime}) - \tilde{F}_{13}(y^{2\prime},b(y^{2\prime}),t)\nabla b(y^{2\prime}) \right| \\
&\quad + \sup_{\mathbf{R}^3_T} |\nabla\tilde{F}_{13}| \Big(|\Psi(y^{1\prime},t;h) - \Psi(y^{2\prime},t;h)| + |b(y^{1\prime}) - b(y^{2\prime})| \Big) \\
&\quad + \big(\sup_{\mathbf{R}^2_T} |\Psi| + \sup_{\mathbf{R}^2} |b| \big) \sup_{x_3} |\nabla\tilde{F}_{13}(y^{1\prime},x_3,t) - \nabla\tilde{F}_{13}(y^{2\prime},x_3,t)| \Bigg],
\end{aligned}$$

the estimate of $\|\nabla\Psi(\cdot,t;h)\|_{W_2^{l-\frac{1}{2}}(\mathbf{R}^2)}$ follows from tracing the argument in the case of (2.3.3) with $i=0$. Similarly $\|\Psi\|_{W_2^{0,\frac{l}{2}+\frac{1}{4}}(\mathbf{R}^2_T)}$ is estimated as in the case $i=0$, and hence we have the estimate of $\|\Psi\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(\mathbf{R}^2_T)}$.

For the higher order norms ($i=2,3$) of Ψ, one can easily estimate them by (2.1.16)–(2.1.17).

Estimates in (2.3.4) are obtained in exactly the same way as (2.3.3).

B.1.2 Proof of Lemma 2.3.2

Since $p = p_0 + \int_{\Psi}^{X_3(x',p,t)} \tilde{F}_{13}\,\mathrm{d}x_3$ by the definition of X_3,

$$
\begin{aligned}
p-\tilde{p} &= \int_{X_3(x^{1\prime},p,t)}^{X_3(x^{2\prime},\tilde{p},t)} \tilde{F}_{13}(x^{1\prime},x_3,t)\,\mathrm{d}x_3 \\
&\quad + \int_{\Psi(x^{1\prime},t;h)}^{X_3(x^{2\prime},\tilde{p},t)} \left[\tilde{F}_{13}(x^{1\prime},x_3,t)-\tilde{F}_{13}(x^{2\prime},x_3,t)\right]\,\mathrm{d}x_3 \\
&\quad + \int_{\Psi(x^{2\prime},t;h)}^{\Psi(x^{1\prime},t;h)} \tilde{F}_{13}(x^{2\prime},x_3,t)\,\mathrm{d}x_3
\end{aligned}
$$

holds. Each term in the right-hand side is estimated as follows:

$$
\left|\int_{X_3(x^{1\prime},p,t)}^{X_3(x^{2\prime},\tilde{p},t)} \tilde{F}_{13}(x^{1\prime},x_3,t)\,\mathrm{d}x_3\right| \geq C|X_3(x^{1\prime},p,t)-X_3(x^{2\prime},\tilde{p},t)|,
$$

$$
\begin{aligned}
&\left|\int_{\Psi(x^{1\prime},t;h)}^{X_3(x^{2\prime},\tilde{p},t)} \left[\tilde{F}_{13}(x^{1\prime},x_3,t)-\tilde{F}_{13}(x^{2\prime},x_3,t)\right]\,\mathrm{d}x_3\right| \\
&\qquad \leq \left(\sup_{\mathbf{R}^2}|b|+\sup_{\mathbf{R}^2_T}|\Psi|\right)\sup_{x_3}|F_{13}(x^{1\prime},x_3,t)-F_{13}(x^{2\prime},x_3,t)| \\
&\qquad \leq \left(\sup_{\mathbf{R}^2}|b|+\sup_{\mathbf{R}^2_T}|\Psi|\right)\sup_{\mathbf{R}^3_T}|\nabla F_{13}||x^{1\prime}-x^{2\prime}|,
\end{aligned}
$$

$$
\left|\int_{\Psi(x^{2\prime},t;h)}^{\Psi(x^{1\prime},t;h)} \tilde{F}_{13}(x^{2\prime},x_3,t)\,\mathrm{d}x_3\right| \leq \left(\sup_{\mathbf{R}^3_T}|F_{13}|+\varrho g\right)\sup_{\mathbf{R}^2_T}|\nabla\Psi||x^{1\prime}-x^{2\prime}|.
$$

Thus we have

$$
\begin{aligned}
&|X_3(x^{1\prime},p,t;h)-X_3(x^{2\prime},\tilde{p},t;h)|^2 \\
&\le C\Bigg[\left(\|b\|^2_{\widetilde{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)}+\|\Psi\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_\mathbf{T})}\right)\|F_{13}\|^2_{\widetilde{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}^3_T)} \\
&\quad+\left(1+\|F_{13}\|^2_{\widetilde{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}^3_T)}\right)\|\Psi\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_\mathbf{T})}\Bigg]\left(|x^{1\prime}-x^{2\prime}|^2+|p-\tilde{p}|^2\right).
\end{aligned}
$$

From this and (2.1.11) the assertion is derived easily.

B.1.3 Proof of Lemma 2.3.3

To prove (2.3.5), (2.3.7), (2.3.9), we use the relations

$$f^{(h)*}(y',y_3,t)=f(y',\tilde{X}_3(y',y_3,t;h),t),$$

$$
\begin{aligned}
&\int_{\tilde{\Omega}}\int_{\tilde{\Omega}}\frac{|f^{(h)*}(y^{1\prime},y_3^1,t)-f^{(h)*}(y^{2\prime},y_3^2,t)|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\mathrm{d}y_3^1\mathrm{d}y_3^2 \\
&=\int_{\tilde{\Omega}}\int_{\tilde{\Omega}}\frac{|f(y^{1\prime},\tilde{X}_3(y^{1\prime},y_3^1,t;h),t)-f(y^{2\prime},\tilde{X}_3(y^{2\prime},y_3^2,t;h),t)|^2}{(|y^{1\prime}-y^{2\prime}|^2+|\tilde{X}_3(y^{1\prime},y_3^1,t;h)-\tilde{X}_3(y^{2\prime},y_3^2,t;h)|^2)^{\frac{3+2l}{2}}} \\
&\quad\times\frac{(|y^{1\prime}-y^{2\prime}|^2+|\tilde{X}_3(y^{1\prime},y_3^1,t;h)-\tilde{X}_3(y^{2\prime},y_3^2,t;h)|^2)^{\frac{3+2l}{2}}}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\mathrm{d}y_3^1\mathrm{d}y_3^2.
\end{aligned}
$$

Applying Lemma 2.3.2, changing the variables $\tilde{X}_3(y^{1\prime},y_3^1,t;h)$ and $\tilde{X}_3(y^{2\prime},y_3^2,t;h)$ in the integrand to $\tilde{y}_3^1$ and $\tilde{y}_3^2$, respectively , and noting

$$\frac{\partial(y',y_3)}{\partial(y',\tilde{y}_3^i)}=\tilde{F}_{13}(y^{i\prime},\tilde{y}_3^i,t)\frac{p_0-h_0}{p_0-h},$$

we can get the estimate

$$\|f^{(h)*}(t)\|^2_{\dot{W}_2^l(\tilde{\Omega})}\le P\left(\|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}^2_T)}\right)\|f(t)\|^2_{\dot{W}_2^l(\mathbf{R}^3)}. \tag{B.1.4}$$

The estimate of $\|f^{(h)*}(y)\|^2_{\dot{W}_2^{\frac{l}{2}}(0,T)}$ can be obtained in a similar way, and hence we arrive at the estimate of $\|f^{(h)*}\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)}$.

Higher order derivatives of $f^{(h)*}$ can be estimated analogously.

To prove (2.3.6), (2.3.8), (2.3.10), we use the following expression of $\tilde{f} := f^{(h_1)*}(y', y_3, t) - f^{(h_2)*}(y', y_3, t)$ derived from the mean value theorem (see, for instance, [36]):

$$\tilde{f}(y', y_3, t) = \bar{X}_3(y', y_3, t) \int_0^1 \frac{\partial f}{\partial x_3}(y', \tilde{X}_3(y', y_3, t; h_2) + s\bar{X}_3(y', y_3, t))\, \mathrm{d}s,$$

$$\bar{X}_3(y', y_3, t) = \tilde{X}_3(y', y_3, t; h_1) - \tilde{X}_3(y', y_3, t; h_2).$$

B.1.4 Proof of Lemma 2.3.4

First, we show

$$\|\tilde{f}\|_{W_2^{l,0}(\tilde{\Omega})} \leq P\left(\sum_{j=1}^{2} \|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right) \|h_1' - h_2'\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \|\!| f |\!\|_T.$$

Indeed, the above expression yields

$$\begin{aligned}
&|\tilde{f}(y^{1\prime}, y_3^1, t) - \tilde{f}(y^{2\prime}, y_3^2, t)|^2 \\
&\leq \left| \int_0^1 \left\{ \frac{\partial f}{\partial x_3}\Big(y^{1\prime}, s\tilde{X}_3(y^{1\prime}, y_3^1, t; h_1) + (1-s)\tilde{X}_3(y^{1\prime}, y_3^1, t; h_2), t\Big) \right.\right. \\
&\qquad \left.\left. - \frac{\partial f}{\partial x_3}\Big(y^{2\prime}, s\tilde{X}_3(y^{2\prime}, y_3^2, t; h_1) + (1-s)\tilde{X}_3(y^{2\prime}, y_3^2, t; h_2), t\Big) \right\} \mathrm{d}s \right|^2 \\
&\qquad \times \left|\bar{X}_3(y^{1\prime}, y_3^1, t)\right|^2 \\
&\quad + \left| \int_0^1 \frac{\partial f}{\partial x_3}\Big(y^{2\prime}, s\tilde{X}_3(y^{2\prime}, y_3^2, t; h_1) + (1-s)\tilde{X}_3(y^{2\prime}, y_3^2, t; h_2), t\Big)\, \mathrm{d}s \right|^2 \\
&\qquad \times \left|\bar{X}_3(y^{1\prime}, y_3^1, t) - \bar{X}_3(y^{2\prime}, y_3^2, t)\right|^2 =: I_1 + I_2. \qquad \text{(B.1.5)}
\end{aligned}$$

On the other hand, from $p(y, t; h) = \frac{(p_0 - h)(y_3 - h_0)}{p_0 - h_0} + h$ and (2.1.8), we derive the relation

$$p(y, t; h_1) - p(y, t; h_2) = \int_{X_3(y', p(y,t;h_2), t; h_2)}^{X_3(y', p(y,t;h_1), t; h_1)} \tilde{F}_{13}\, \mathrm{d}x_3 + \int_{\Psi(y',t;h_1)}^{\Psi(y',t;h_2)} \tilde{F}_{13}\, \mathrm{d}x_3,$$

which easily leads to

$$\left|\bar{X}_3(y, t)\right|^2 \leq C \left(|p(y, t; h_1) - p(y, t; h_2)|^2 + \left| \int_{\Psi(y',t;h_1)}^{\Psi(y',t;h_2)} \tilde{F}_{13}\, \mathrm{d}x_3 \right|^2 \right).$$

Noting that

$$
\begin{aligned}
|p(y,t;h_1) - p(y,t;h_2)| &= \left| \frac{(p_0 - y_3)(h_1' - h_2')}{p_0 - h_0} \right| \leq |h_1' - h_2'|, \\
|\Psi(y',t;h_1) - \Psi(y',t;h_2)| &\leq \left| \frac{h_1' - h_2'}{\inf_{x_3} \tilde{F}_{13}(y', x_3, t)} \right|,
\end{aligned}
$$

we obtain

$$
\left| \bar{X}_3(y,t) \right|^2 \leq C \left(1 + \|\tilde{F}_{13}\|^2_{W_2^{3+l, \frac{3+l}{2}}(\mathbf{R}_T^3)} \right) \left| (h_1' - h_2')(y',t) \right|^2 . \tag{B.1.6}
$$

Inserting (B.1.6) into I_1 in (B.1.5), and proceeding to evaluate in the same way as (B.1.4), we have

$$
\begin{aligned}
&\int_0^T \mathrm{d}t \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \frac{I_1}{(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2)^{\frac{3+2l}{2}}} \, \mathrm{d}y^{1\prime} \mathrm{d}y^{2\prime} dy_3^1 dy_3^2 \\
&\leq P \left(\sum_{i=1}^{2} \|h'\|^2_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \right) \left\| \frac{\partial f}{\partial x_3} \right\|^2_{W_2^{l, \frac{l}{2}}(\mathbf{R}_T^3)} \|\tilde{h}\|^2_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} .
\end{aligned} \tag{B.1.7}
$$

In the similar manner as above, using a notation $\tilde{h} := h_1' - h_2'$, we obtain

$$
\begin{aligned}
&\left[p(y^{1\prime}, y_3^1, t; h_1) - p(y^{1\prime}, y_3^1, t; h_2) \right] - \left[p(y^{2\prime}, y_3^2, t; h_1) - p(y^{2\prime}, y_3^2, t; h_2) \right] \\
&\quad = \left[\frac{1}{p_0 - h_0(y^{1\prime})} - \frac{1}{p_0 - h_0(y^{2\prime})} \right] (p_0 - y_3^1) \tilde{h}(y^{1\prime}, t) \\
&\qquad + \frac{(y_3^2 - y_3^1) \tilde{h}(y^{1\prime}, t)}{p_0 - h_0(y^{2\prime})} + \frac{p_0 - y_3^2}{p_0 - h_0(y^{2\prime})} [\tilde{h}(y^{1\prime}, t) - \tilde{h}(y^{2\prime}, t)], \\
&\left| \int_{\Psi(y^{1\prime},t;h_1)}^{\Psi(y^{1\prime},t;h_2)} \tilde{F}_{13}(y^{1\prime}, x_3, t) \, \mathrm{d}x_3 - \int_{\Psi(y^{2\prime},t;h_1)}^{\Psi(y^{2\prime},t;h_2)} \tilde{F}_{13}(y^{2\prime}, x_3, t) \, \mathrm{d}x_3 \right| \\
&\quad \leq \sup_{x_3} \left| \tilde{F}_{13}(y^{1\prime}, x_3, t) \right| \left| \Psi(y^{1\prime}, t; h_1) - \Psi(y^{1\prime}, t; h_2) \right| \\
&\qquad + \sup_{x_3} \left| \tilde{F}_{13}(y^{2\prime}, x_3, t) \right| \left| \Psi(y^{2\prime}, t; h_1) - \Psi(y^{2\prime}, t; h_2) \right| .
\end{aligned}
$$

These yield

$$
\begin{aligned}
&\left|\bar X_3(y^{1\prime}, y_3^1, t) - \bar X_3(y^{2\prime}, y_3^2, t)\right|^2 \\
&\le C\Bigg[\left(1 + \sup_{\mathbf{R}^2} |h_0 + \|F_{13}\|_{W_2^{3+l, \frac{3+l}{2}}(\mathbf{R}_T^3)}|\right)^2 \left|h_0(y^{1\prime}) - h_0(y^{2\prime})\right|^2 |\tilde h(y^{1\prime}, t)|^2 \\
&\quad + \left|y_3^1 - y_3^2|^2\right| \tilde h(y^{1\prime}, t)|^2 + \left(1 + \sup_{\mathbf{R}^2} |h_0|\right)^2 \left|\tilde h(y^{1\prime}, t) - \tilde h(y^{2\prime}, t)\right|^2 \\
&\quad + \left(1 + \|F_{13}\|_{W_2^{3+l, \frac{3+l}{2}}(\mathbf{R}_T^3)}\right)^2 \left(|\tilde h(y^{1\prime}, t)|^2 + |\tilde h(y^{2\prime}, t)|^2\right) \\
&\qquad \times \sup_{x_3} |F_{13}(y^{1\prime}, x_3, t) - F_{13}(y^{2\prime}, x_3, t)|^2. \qquad \text{(B.1.8)}
\end{aligned}
$$

As for I_2 in (B.1.5), we need to estimate the right-hand side of (B.1.8). For the terms except the second, we make use of the estimate

$$
\|fg\|_{W_2^{l,0}(\tilde\Omega_T)} \le C(1 + \sup_{\mathbf{R}^2} |h_0|)^\delta \|f\|^2_{W_2^{l, \frac{l}{2}}(\tilde\Omega_T)} \|g\|^2_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \qquad \text{(B.1.9)}
$$

for any $f \in W_2^{l,\frac{l}{2}}(\tilde\Omega_T)$ and $g \in W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$ in general with some positive constant δ. Indeed, let $\gamma = 2 - 2l - \delta$ with δ, $0 < \delta < \min\{2l(2l-1)/(3-2l),\ 2-2l\}$. Then, it is easy to see that

$$
\begin{aligned}
&\int_{\tilde\Omega} \frac{|g(y^{1\prime}, t) - g(y^{2\prime}, t)|^2}{(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2)^{\frac{3+2l}{2}}}\, \mathrm{d}y^{1\prime}\mathrm{d}y_3^1 \\
&\quad \le 4\pi \sup_{\mathbf{R}^2} |g(t)|^\gamma \sup_{\mathbf{R}^2} |\nabla g(t)|^{2-\gamma} \int_{\tilde\Omega} \frac{|y^{1\prime} - y^{2\prime}|^{2-\gamma}}{(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2)^{\frac{3+2l}{2}}}\, \mathrm{d}y^{1\prime}\mathrm{d}y_3^1 \\
&\quad \le 4\pi \sup_{\mathbf{R}^2} |g(t)|^\gamma \sup_{\mathbf{R}^2} |\nabla g(t)|^{2-\gamma} B\left(2 - \frac{\gamma}{2},\ l - \frac{1}{2} + \frac{\gamma}{2}\right) \\
&\qquad \times \int_{p_0}^{\sup |h_0|} |y_3^1 - y_3^2|^{1-\gamma-2l}\, \mathrm{d}y_3^1,
\end{aligned}
$$

where $B(x, y)$ is the beta function. This leads to the estimate

$$
\begin{aligned}
&\int_{\tilde\Omega}\int_{\tilde\Omega} \frac{|f(y^{2\prime}, y_3^2, t)|^2 |g(y^{1\prime}, t) - g(y^{2\prime}, t)|^2}{(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2)^{\frac{3+2l}{2}}}\, \mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\mathrm{d}y_3^1\mathrm{d}y_3^2 \\
&\qquad \le C(1 + \sup_{\mathbf{R}^2} |h_0|)^\delta \|g(t)\|^\gamma_{W_2^{\frac{1}{2}+l}(\mathbf{R}^2)} \|g(t)\|^{2-\gamma}_{W_2^2(\mathbf{R}^2)} \|f(t)\|^2_{L_2(\tilde\Omega)}.
\end{aligned}
$$

Applying the Hölder inequality yields the estimate

$$\int_0^T \|g(t)\|^\gamma_{W_2^{\frac{1}{2}+l}(\mathbf{R}^2)} \|g(t)\|^{2-\gamma}_{W_2^2(\mathbf{R}^2)} \|f(t)\|^2_{L_2(\tilde{\Omega})}\,\mathrm{d}t$$
$$\leq \sup_t \|g(t)\|^\gamma_{W_2^{\frac{1}{2}+l}(\mathbf{R}^2)} \|g\|^{2-\gamma}_{L_{\frac{2-\gamma}{l}}(0,T;W_2^2(\mathbf{R}^2))} \|f\|^2_{L_{\frac{2}{1-l}}(0,T;L_2(\tilde{\Omega}))}.$$

The assumption on δ and the Sobolev embedding theorem imply $W_2^{\frac{l}{2}}(0,T) \subset L_{\frac{2}{1-l}}(0,T)$ and $W_2^{\frac{1}{2}\left(l-\frac{1}{2}\right)}(0,T) \subset L_{\frac{2-\gamma}{l}}(0,T)$, so that (B.1.9) holds. For the second term in the right-hand side of (B.1.8), it is sufficient to consider

$$\int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{|f(y^{2\prime},y_3^2,t)|^2|y_3^1-y_3^2|^2|\tilde{h}(y^{1\prime},t)|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\mathrm{d}y_3^1\mathrm{d}y_3^2$$

in the region $|y_3^1-y_3^2| \neq 0$ with $f \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)$. Then we have

$$\int_0^T \mathrm{d}t \int_{p_0}^{\sup|h_0|} \frac{|y_3^1-y_3^2|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y_3^1$$
$$\times \int_{\tilde{\Omega}} |f(y^{2\prime},y_3^2)|^2\,\mathrm{d}y^{2\prime}\mathrm{d}y_3^2 \int_{\mathbf{R}^2} |\tilde{h}|^2\,\mathrm{d}y^{1\prime}$$
$$\leq \frac{\pi\|f\|^2_{L_2(\tilde{\Omega}_T)} \sup_t \|\tilde{h}(t)\|^2_{L_2(\mathbf{R}^2)}}{1+2l} \int_{p_0}^{\sup|h_0|} |y_3^1-y_3^2|^{1-2l}\,\mathrm{d}y_3^1$$
$$\leq \frac{\pi(p_0+\sup_{\mathbf{R}^2}|h_0|)^{3-2l}}{(1+2l)(1-l)} \|f\|^2_{L_2(\tilde{\Omega}_T)} \|\tilde{h}\|^2_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}.$$

Therefore, we can get the estimate of the second term of (B.1.8), and consequently $\|\tilde{f}\|_{W_2^{l,0}(\tilde{\Omega}_T)}$.

The estimate of $\|\tilde{f}\|_{L_2(\tilde{\Omega};\dot{W}_2^{\frac{l}{2}}(0,T))}$ is obtained in a similar manner under the assumption of the Hölder continuity of $\partial f/\partial x_3$. Actually, in this case, the right-hand side of (B.1.5) is replaced by

$$
\begin{aligned}
&\Bigg[\bigg|\int_0^1 \bigg\{ \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t; h_1) + (1-s)\tilde{X}_3(y', y_3, t; h_2), t\Big) \\
&\qquad - \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t; h_1) + (1-s)\tilde{X}_3(y', y_3, t; h_2), t-\tau\Big) \bigg\}\, \mathrm{d}s\bigg|^2 \\
&\quad + \bigg|\int_0^1 \bigg\{ \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t; h_1) + (1-s)\tilde{X}_3(y', y_3, t; h_2), t-\tau\Big) \\
&\qquad - \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t-\tau; h_1) + (1-s)\tilde{X}_3(y', y_3, t-\tau; h_2), t-\tau\Big) \bigg\}\, \mathrm{d}s\bigg|^2 \Bigg] \\
&\quad \times \big|\bar{X}_3(y', y_3, t)\big|^2 \\
&\quad + \bigg|\int_0^1 \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t-\tau; h_1) + (1-s)\tilde{X}_3(y', y_3, t-\tau; h_2), t-\tau\Big)\, \mathrm{d}s\bigg|^2 \\
&\qquad\quad \times \big|\bar{X}_3(y', y_3, t) - \bar{X}_3(y', y_3, t-\tau)\big|^2 \\
&\quad =: J_1 + J_2 + J_3. \qquad\qquad (\mathrm{B.1.10})
\end{aligned}
$$

It is easily seen that J_1 and J_3 in (B.1.10) can be estimated in exactly the same way as in (B.1.5). For J_2, we have

$$
\begin{aligned}
&\bigg|\int_0^1 \bigg\{ \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t; h_1) + (1-s)\tilde{X}_3(y', y_3, t; h_2), t-\tau\Big) \\
&\qquad - \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t-\tau; h_1) + (1-s)\tilde{X}_3(y', y_3, t-\tau; h_2), t-\tau\Big) \bigg\}\, \mathrm{d}s\bigg|^2 \\
&\qquad \le C\left(\left|\frac{\partial f}{\partial x_3}\right|_{x_3}^{(\beta)}\right)^2 \sum_{i=1}^{2} \Big|\tilde{X}_3(y', y_3, t; h_i) - \tilde{X}_3(y', y_3, t-\tau; h_i)\Big|^{2\beta}.
\end{aligned}
$$

In the same way as the proof of Lemma 2.3.2, we have

$$
\begin{aligned}
&|\tilde{X}_3(y', y_3, t; h) - \tilde{X}_3(y', y_3, t-\tau; h)| \\
&\quad \leq C\Bigg[|p(y,t) - p(y,t-\tau)| \\
&\qquad + \left(\sup_{\mathbf{R}^2}|b| + \sup_{\mathbf{R}^2_T}|\Psi|\right)\sup_{x_3}|F_{13}(y', x_3, t) - F_{13}(y', x_3, t-\tau)| \\
&\qquad + \left(1 + \sup_{\mathbf{R}^3_T}|F_{13}|\right)|\Psi(y', t; h) - \Psi(y', t-\tau; h)|\Bigg].
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\int_{\tilde{\Omega}} \mathrm{d}y \int_0^T \int_0^t \frac{J_2}{\tau^{1+l}}\, \mathrm{d}\tau\, \mathrm{d}t &\leq C\left(\left|\frac{\partial f}{\partial x_3}\right|^{(\beta)}_{x_3}\right)^2 T^{2\beta - l}\sum_{i=1}^{2}\Bigg[\|h_i'\|^{2\beta}_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)} \\
&+ \left(\|b\|^{2\beta}_{\overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)} + \|\Psi(\cdot; h_i)\|^{2\beta}_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\right)\|F_{13}\|^{2\beta}_{W_2^{3+l, \frac{3+l}{2}}(\mathbf{R}^3_T)} \\
&+ \left(1 + \|F_{13}\|^{2\beta}_{W_2^{3+l, \frac{3+l}{2}}(\mathbf{R}^3_T)}\right)\|\Psi(\cdot; h_i)\|^{2\beta}_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\Bigg]\|h_1' - h_2'\|^2_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}.
\end{aligned}
$$

These complete the estimate of $\|\tilde{f}\|_{W_2^{0, \frac{l}{2}}(\tilde{\Omega})}$, and hence $\|\tilde{f}\|_{W_2^{l, \frac{l}{2}}(\tilde{\Omega})}$.

Higher order norms can be estimated in a similar manner.

B.1.5 Proof of Lemma 2.3.5

According to the explicit form of $\mathbf{F}_5^{(h)*}$ given in (2.1.13), it is sufficient to estimate the second term $\mathbf{C}_1$. Since $\nabla_y F_{13}^{(h)*} \in W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_T)$ and

$$
\|\mathbf{a}^3(h_{(m)})\|_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_T)} \leq P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\right), \tag{B.1.11}
$$

we have

$$
\|\mathbf{C}_1\|_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_T)} \leq P\left(\|h'\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\right)\|F_{13}\|^2_{W_2^{2+l, \frac{2+l}{2}}(\mathbf{R}^3_T)}
$$

with the aid of multiplicative inequalities. The latter assertion is proved in the same way.

B.2 Lemmas in Chapter 3

B.2.1 Proof of Lemma 3.3.1

Making use of the relation (3.1.17), we have

$$\left\|\frac{\partial\Psi}{\partial t}\right\|_{L_2(\mathbf{R}^2_T)} \le \exp\left[C\left(\|h'\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\right)\right] \times C\left(\|h'\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}, \|\tilde{\theta}'\|_{W_2^{3+l',\frac{3+l'}{2}}(\tilde{\Omega}_T)}\right).$$

This and

$$\|\Psi\|^2_{L_2(\mathbf{R}^2_T)} \le \|d_0\|^2_{L_2(\mathbf{R}^2)} + t\left\|\frac{\partial\Psi}{\partial t}\right\|^2_{L_2(\mathbf{R}^2_T)} \tag{B.2.1}$$

yield the estimate of $\|\Psi\|^2_{L_2(\mathbf{R}^2_T)}$. The fractional norm $\|\Psi\|_{W_2^{l-\frac{1}{2},0}(\mathbf{R}^2_T)}$ is easily estimated from the inequality

$$|\Psi(y^{1\prime},t;\tilde{\theta},h) - \Psi(y^{2\prime},t;\tilde{\theta},h)|^2 \le C\exp\left(\sup_{\mathbf{R}^2_T}|h'|\right)\left[|h(y^{1\prime},t)-h(y^{2\prime},t)|^2 + \left(1+\sup_{\mathbf{R}^2_T}|h'|^2\right)\sup_{y_3}\left|\tilde{\theta}(y^{1\prime},y_3,t)-\tilde{\theta}(y^{2\prime},y_3,t)\right|^2\right]. \tag{B.2.2}$$

Indeed, it is sufficient to confine the following inequalities derived from the relation (3.1.9).

$$\left|\Psi(y^{1\prime},t;\tilde{\theta},h) - \Psi(y^{2\prime},t;\tilde{\theta},h)\right| \le C\left[\frac{|h(y^{1\prime},t)-h(y^{2\prime},t)|}{\inf_{\mathbf{R}^2_T}|h|} + \left(1+\sup_{\mathbf{R}^2_T}|h'|\right)\sup_{y_3}|\tilde{\theta}(y^{1\prime},y_3,t)-\tilde{\theta}(y^{2\prime},y_3,t)|\right],$$

and

$$\inf_{\mathbf{R}^2_T}|h| \ge p_0\exp\left(-\frac{g}{R\inf_{\tilde{\Omega}_T}\tilde{\theta}}\left(H+\sup_{\mathbf{R}^2_T}|\Psi|\right)\right).$$

Estimate of $\|\Psi\|_{W_2^{0,\frac{l}{2}-\frac{1}{4}}(\mathbf{R}^2_T)}$ can be achieved in a similar manner. Making use of these facts, we arrive at (3.3.3).

For $i = 0$ in (3.3.4), we make use of (3.1.15). Since

$$|\nabla\Psi(y^{1\prime}, t; \tilde{\theta}, h) - \nabla\Psi(y^{2\prime}, t; \tilde{\theta}, h)|^2 \leq \left[\left| \frac{E(y^{1\prime}, t) - E(y^{2\prime}, t)}{F(y^{1\prime}, t)} \right|^2 \left| \nabla h(y^{1\prime}, t) \right|^2 \right.$$

$$+ \left| E(y^{2\prime}, t) \right|^2 \left| \frac{1}{F(y^{1\prime}, t)} - \frac{1}{F(y^{2\prime}, t)} \right|^2 |\nabla h(y^{1\prime}, t)|^2$$

$$\left. + \left| \frac{E(y^{2\prime}, t)}{F(y^{2\prime}, t)} \right|^2 |\nabla h(y^{1\prime}, t) - \nabla h(y^{2\prime}, t)|^2 + \left| \frac{\mathbf{K}_1(y^{1\prime}, t)}{F(y^{1\prime}, t)} - \frac{\mathbf{K}_1(y^{2\prime}, t)}{F(y^{2\prime}, t)} \right|^2 \right],$$

the estimate of $\|\nabla\Psi(\cdot; \tilde{\theta}, h)\|_{W_2^{l-\frac{1}{2}}(\mathbf{R}^2)}$ follows from tracing the argument in the case of (3.3.3). Also, $\|\Psi\|_{W_2^{0,\frac{l}{2}+\frac{1}{4}}(\mathbf{R}_T^2)}$ can be estimated just in the same way as that for $\|\Psi\|_{W_2^{0,\frac{l}{2}-\frac{1}{4}}(\mathbf{R}_T^2)}$, and hence we have the estimate of $\|\Psi\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(\mathbf{R}_T^2)}$. The higher order norms ($i = 2$) of Ψ can be easily estimated from (3.1.16)–(3,1.17).

B.2.2 Proof of Lemma 3.3.2

According to the explicit form of $\mathbf{F}_5^{(\tilde{\theta},h)}$ given in (3.1.13), it is sufficient to estimate the second term $\mathbf{C}_1$. Making use of

$$\|\mathbf{a}^3(h)\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \leq C \left(\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \right), \tag{B.2.3}$$

we have

$$\|\mathbf{C}_1\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \leq C \left(\|\tilde{\theta}'\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_T)}, \|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \right)$$

with the aid of Lemma 3.3.1 and the multiplicative inequalities. The second inequality is proved in the same way with the aid of (3.1.14).

B.2.3 Proof of Lemma 3.3.3

It is easy to prove by making use of Lemmas 3.3.1–3.3.2, and rewriting θ in the integral terms by $\tilde{\theta}$.

B.2.4 Proof of Lemma 3.3.4

One can easily prove by applying Lemma 3.3.3 to (3.1.13), (3.1.14), and rewriting θ in the integral terms by $\tilde{\theta}$.

B.2.5 Proof of Lemma 3.3.5

From the definition of X_3,

$$\log\left(\frac{p}{p_0}\right) = -\int_{\Psi(x',t;\tilde{\theta},h)}^{X_3(x',p,t;\tilde{\theta},h)} \frac{g}{R\theta}\mathrm{d}x_3$$

holds, which implies

$$\begin{aligned}\log\left(\frac{p}{p_0}\right) - \log\left(\frac{\tilde{p}}{p_0}\right) &= \frac{g}{R}\int_{\Psi(x^{2\prime},t;\tilde{\theta},h)}^{X_3(x^{2\prime},\tilde{p},t;\tilde{\theta},h)} \left(\frac{1}{\theta(x^{2\prime},x_3,t)} - \frac{1}{\theta(x^{1\prime},x_3,t)}\right)\mathrm{d}x_3 \\ &+ \frac{g}{R}\int_{\Psi(x^{2\prime},t;\tilde{\theta},h)}^{\Psi(x^{1\prime},t;\tilde{\theta},h)} \frac{1}{\theta(x^{1\prime},x_3,t)}\mathrm{d}x_3 + \frac{g}{R}\int_{X_3(x^{2\prime},\tilde{p},t;\tilde{\theta},h)}^{X_3(x^{1\prime},p,t;\tilde{\theta},h)} \frac{1}{\theta(x^{1\prime},x_3,t)}\mathrm{d}x_3.\end{aligned}$$

Applying the mean value theorem to the left-hand side leads to

$$\begin{aligned}&|X_3(x^{1\prime},p,t;\tilde{\theta},h) - X_3(x^{2\prime},\tilde{p},t;\tilde{\theta},h)|^2 \\ &\quad\le C\sup_{\mathbf{R}^3_T}|\theta|^2\Bigg[|\Psi(x^{1\prime},t;\tilde{\theta},h) - \Psi(x^{2\prime},t;\tilde{\theta},h)|^2 \\ &\qquad + \left(1 + \|\Psi\|^2_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\right)\sup_{x_3}\left|\theta(x^{1\prime},x_3,t) - \theta(x^{2\prime},x_3,t)\right|^2 + |p-\tilde{p}|^2\Bigg].\end{aligned}$$

From this and (2.1.11) the first assertion is derived. In the same way the second one is proved.

B.2.6 Proof of Lemma 3.3.6

Making use of (3.1.8), we have

$$\nabla X_3|_{p=p(y,t)} = \frac{R\tilde{\theta}}{p(y,t)g}\mathbf{F}_5^{(\tilde{\theta},h)},$$

from which we have the first assertion. The second one is proved similarly.

B.2.7 Proof of Lemma 3.3.7

Denoting by $p(y,t) = p(y,t;h) = \frac{(p_0-h)(y_3-h_0)}{p_0-h_0} + h$, we derive from (3.1.8)

$$\log(p(y,t;\tilde\theta_1,h_1)) - \log(p(y,t;\tilde\theta_2,h_2))$$

$$= \left[\int_{\Psi(y',t;\tilde\theta_2,h_2)}^{\Psi(y',t;\tilde\theta_1,h_1)} + \int_{X_3(y',p(y,t;\tilde\theta_2,h_2);\tilde\theta_2,h_2)}^{X_3(y',p(y,t;\tilde\theta_1,h_1);\tilde\theta_1,h_1)}\right] \frac{g}{R\theta}\,\mathrm{d}x_3$$

$$+ \int_{\Psi(y',t;\tilde\theta_1,h_1)}^{X_3(y',p(y,t;\tilde\theta_1,h_1),t;\tilde\theta_1,h_1)} \left(\frac{1}{\theta_2(y',x_3,t)} - \frac{1}{\theta_1(y',x_3,t)}\right) \mathrm{d}x_3.$$

From this it is easy to get

$$\left|\bar X_3(y,t)\right|^2 \le C\left(1+\|\tilde\theta_2'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde\Omega_T)}\right)\left[\exp\left(1+\sum_{i=1}^{2}\|h_i'\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)\right.$$

$$\left.\times C\left(\sum_{i=1}^{2}\|\tilde\theta_i'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde\Omega_T)}, \sum_{i=1}^{2}\|h_i'\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)+1\right]$$

$$\times\left(|\tilde h'(y',t)|^2 + \sup_{x_3}|\tilde{\tilde\theta}'(y',x_3,t)|^2\right),$$

which leads to the first assertion. The second and the third inequalities can be proved in the similar way.

B.2.8 Proof of Lemma 3.3.9

To prove (3.3.6), (3.3.8), (3.3.10), we use the relations [18]

$$f^{(\tilde\theta,h)}(y',y_3,t) = f(y',\tilde X_3(y',y_3,t;\theta,h),t),$$

$$\int_{\tilde\Omega}\int_{\tilde\Omega} \frac{|f^{(\tilde\theta,h)}(y^{1\prime},y_3^1,t) - f^{(\tilde\theta,h)}(y^{2\prime},y_3^2,t)|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\mathrm{d}y_3^1\mathrm{d}y_3^2$$

$$= \int_{\tilde\Omega}\int_{\tilde\Omega} \frac{|f(y^{1\prime},\tilde X_3(y^{1\prime},y_3^1,t;\tilde\theta,h),t) - f(y^{2\prime},\tilde X_3(y^{2\prime},y_3^2,t;\tilde\theta,h),t)|^2}{(|y^{1\prime}-y^{2\prime}|^2+|\tilde X_3(y^{1\prime},y_3^1,t;\tilde\theta,h)-\tilde X_3(y^{2\prime},y_3^2,t;\tilde\theta,h)|^2)^{\frac{3+2l}{2}}}$$

$$\times \frac{(|y^{1\prime}-y^{2\prime}|^2+|\tilde X_3(y^{1\prime},y_3^1,t;\tilde\theta,h)-\tilde X_3(y^{2\prime},y_3^2,t;\tilde\theta,h)|^2)^{\frac{3+2l}{2}}}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\mathrm{d}y_3^1\mathrm{d}y_3^2.$$

Applying Lemma 3.3.7, changing variables $\tilde X_3(y^{1\prime},y_3^1,t;\tilde\theta,h)$ and $\tilde X_3(y^{2\prime},y_3^2,t;\tilde\theta,h)$ to $\tilde y_3^1$ and $\tilde y_3^2$, respectively in the integrand, and noting $\frac{\partial(y',y_3)}{\partial(y',\tilde y_3^i)} = \frac{pg}{R\tilde\theta(y',\tilde y_3^i,t)}\frac{p_0-h_0}{p_0-h}$ $(i =$

1, 2), we can get the estimate

$$\|f^{(\tilde{\theta},h)}(t)\|^2_{\dot{W}_2^l(\tilde{\Omega})} \le \tilde{C}\left(\|\tilde{\theta}'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)}, \|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}^2_T)}\right)\|f(t)\|^2_{\dot{W}_2^l(\mathbf{R}^3)}. \tag{B.2.4}$$

The estimate of $\|f^{(\tilde{\theta},h)}\|^2_{\dot{W}_2^{\frac{l}{2}}(0,T)}$ can be obtained in a similar way, and hence we arrive at the estimate of $\|f^{(\tilde{\theta},h)}\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)}$.

Higher order derivatives of $f^{(\tilde{\theta},h)}$ can be estimated analogously.

To prove (3.3.7), (3.3.8) and (3.3.9), we use the expression of

$$\tilde{f}(y', y_3, t) := f^{(\tilde{\theta}_1,h_1)}(y', y_3, t) - f^{(\tilde{\theta}_2,h_2)}(y', y_3, t)$$

derived from the mean value theorem (see, for instance, [17], [36]):

$$\begin{aligned}
&\tilde{f}(y', y_3, t) = \bar{X}_3(y', y_3, t)\\
&\quad\times\int_0^1 \frac{\partial f}{\partial x_3}\left(y', s\tilde{X}_3(y', y_3, t; \tilde{\theta}_1, h_1) + (1-s)\tilde{X}_3(y', y_3, t; \tilde{\theta}_2, h_2), t\right)\mathrm{d}s.
\end{aligned}$$

First, we show

$$\begin{aligned}
\|\tilde{f}\|_{W_2^{l,0}(\tilde{\Omega})} \le \tilde{C}&\left(\sum_{j=1}^2\|\tilde{\theta}_j'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)}, \sum_{j=1}^2\|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\right)\\
&\times\left(\|\tilde{\tilde{\theta}}'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} + \|\tilde{h}'\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\right)\|\!|f|\!\|_T.
\end{aligned}$$

Indeed, we have

$$\begin{aligned}
&|\tilde{f}(y^{1\prime}, y_3^1, t) - \tilde{f}(y^{2\prime}, y_3^2, t)|^2\\
&\le\Bigg|\int_0^1\Bigg\{\frac{\partial f}{\partial x_3}\left(y^{1\prime}, s\tilde{X}_3(y^{1\prime}, y_3^1, t; \tilde{\theta}_1, h_1) + (1-s)\tilde{X}_3(y^{1\prime}, y_3^1, t; \tilde{\theta}_2, h_2), t\right)\\
&\quad-\frac{\partial f}{\partial x_3}\left(y^{2\prime}, s\tilde{X}_3(y^{2\prime}, y_3^2, t; \tilde{\theta}_1, h_1) + (1-s)\tilde{X}_3(y^{2\prime}, y_3^2, t; \tilde{\theta}_2, h_2), t\right)\Bigg\}\mathrm{d}s\Bigg|^2\\
&\qquad\times\left|\bar{X}_3(y^{1\prime}, y_3^1, t)\right|^2\\
&\quad+\left|\int_0^1\frac{\partial f}{\partial x_3}\left(y^{2\prime}, s\tilde{X}_3(y^{2\prime}, y_3^2, t; \tilde{\theta}_1, h_1) + (1-s)\tilde{X}_3(y^{2\prime}, y_3^2, t; \tilde{\theta}_2, h_2), t\right)\mathrm{d}s\right|^2\\
&\qquad\times\left|\bar{X}_3(y^{1\prime}, y_3^1, t) - \bar{X}_3(y^{2\prime}, y_3^2, t)\right|^2 =: I_1 + I_2.
\end{aligned} \tag{B.2.5}$$

Applying the first inequality of Lemma 3.3.7 to I_1, we have

$$\begin{aligned}
&\int_0^T \mathrm{d}t \int_{\tilde\Omega}\int_{\tilde\Omega} \frac{I_1}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\, \mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}dy_3^1dy_3^2 \\
&\quad \le C\exp\left(\|h'\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right) \\
&\quad\quad \times \tilde C\left(\|\tilde\theta'\|^2_{2+l,\frac{2+l}{2}(\tilde\Omega_T)}, \|h'\|^2_{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}(\mathbf{R}_T^2)}\right)\left\|\frac{\partial f}{\partial x_3}\right\|^2_{W_2^{l,\frac{l}{2}}(\tilde\Omega_T)} \\
&\quad\quad\quad \times\left(\|\tilde\theta\|^2_{W_2^{2+l,1+\frac{l}{2}}(\tilde\Omega_T)}+\|\tilde h\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right). \qquad \text{(B.2.6)}
\end{aligned}$$

As for I_2 in (B.2.5), we use the second inequality of Lemma 3.3.7, in which it is sufficient to consider

$$\int_{\tilde\Omega}\int_{\tilde\Omega}\frac{|f(y^{2\prime},y_3^2,t)|^2|y_3^1-y_3^2|^2|\tilde h(y^{1\prime},t)|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\mathrm{d}y_3^1\mathrm{d}y_3^2$$

in the region $|y_3^1-y_3^2|\neq 0$ when $f\in W_2^{l,\frac{l}{2}}(\tilde\Omega_T)$. Then, with the aid of Lemma 3.3.8, we have

$$\begin{aligned}
&\int_0^T \mathrm{d}t\int_{p_0}^{\sup h_0}\frac{|y_3^1-y_3^2|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y_3^1 \\
&\quad\quad \times\int_{\tilde\Omega}|f(y^{2\prime},y_3^2)|^2\,\mathrm{d}y^{2\prime}\mathrm{d}y_3^2\int_{\mathbf{R}^2}|\tilde h|^2\,\mathrm{d}y^{1\prime} \\
&\quad\quad \le\frac{\pi\|f\|^2_{L_2(\tilde\Omega_T)}\sup_t\|\tilde h(t)\|^2_{L_2(\mathbf{R}^2)}}{1+2l}\int_{p_0}^{\sup h_0}|y_3^1-y_3^2|^{1-2l}\,\mathrm{d}y_3^1 \\
&\quad\quad \le\frac{\pi(p_0+\sup_{\mathbf{R}^2}h_0)^{3-2l}}{(1+2l)(1-l)}\|f\|^2_{L_2(\tilde\Omega_T)}\|\tilde h\|^2_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}.
\end{aligned}$$

From this, one can derive the estimate for I_2, and consequently $\|\tilde f\|_{W_2^{l,0}(\tilde\Omega_T)}$.

The estimate of $\|\tilde f\|_{L_2(\tilde\Omega;\dot W_2^{\frac{l}{2}}(0,T))}$ is obtained in a similar manner under the assumption of the Hölder continuity of $\partial f/\partial x_3$. Actually, in this case, the right-hand side of (B.2.5) is replaced by

$$
\begin{aligned}
&\Bigg[\bigg|\int_0^1 \bigg\{ \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t; \tilde{\theta}_1, h_1) + (1-s)\tilde{X}_3(y', y_3, t; \tilde{\theta}_2, h_2), t\Big) \\
&\qquad - \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t; \tilde{\theta}_1, h_1) + (1-s)\tilde{X}_3(y', y_3, t; \tilde{\theta}_2, h_2), t-\tau\Big)\bigg\}\,\mathrm{d}s\bigg|^2 \\
&\quad + \bigg|\int_0^1 \bigg\{ \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t; \tilde{\theta}_1, h_1) + (1-s)\tilde{X}_3(y', y_3, t; \tilde{\theta}_2, h_2), t-\tau\Big) \\
&\qquad - \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t-\tau; \tilde{\theta}_1, h_1) \\
&\qquad\quad + (1-s)\tilde{X}_3(y', y_3, t-\tau; \tilde{\theta}_2, h_2), t-\tau\Big)\bigg\}\,\mathrm{d}s\bigg|^2\Bigg] \left|\bar{X}_3(y', y_3, t)\right|^2 \\
&\quad + \bigg|\int_0^1 \bigg\{ \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t-\tau; \tilde{\theta}_1, h_1) \\
&\qquad\quad + (1-s)\tilde{X}_3(y', y_3, t-\tau; \tilde{\theta}_2, h_2), t-\tau\Big)\bigg\}\,\mathrm{d}s\bigg|^2 \\
&\qquad \times \left|\bar{X}_3(y', y_3, t) - \bar{X}_3(y', y_3, t-\tau)\right|^2 \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{B.2.7}
$$

It is easily seen that J_1 and J_3 in (B.2.7) can be estimated in exactly the same way as in (B.2.5) with the aid of Lemma 3.3.6–3.3.7. For J_2, we have

$$
\begin{aligned}
&\bigg|\int_0^1 \bigg\{ \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t; \tilde{\theta}_1, h_1) + (1-s)\tilde{X}_3(y', y_3, t; \tilde{\theta}_2, h_2), t-\tau\Big) \\
&- \frac{\partial f}{\partial x_3}\Big(y', s\tilde{X}_3(y', y_3, t-\tau; \tilde{\theta}_1, h_1) + (1-s)\tilde{X}_3(y', y_3, t-\tau; \tilde{\theta}_2, h_2), t-\tau\Big)\bigg\}\,\mathrm{d}s\bigg|^2 \\
&\qquad \le C \left|\frac{\partial f}{\partial x_3}\right|_{x_3}^{(\beta)2} \sum_{i=1}^{2} \left|\tilde{X}_3(y', y_3, t; h_i) - \tilde{X}_3(y', y_3, t-\tau; h_i)\right|^{2\beta}.
\end{aligned}
$$

Making use of the second inequality in Lemma 3.3.5, we have

$$\int_{\tilde{\Omega}} \mathrm{d}y \int_0^T \int_0^t \frac{J_2}{\tau^{1+l}} \, \mathrm{d}\tau \, \mathrm{d}t \le C \left| \frac{\partial f}{\partial x_3} \right|_{x_3}^{(\beta)2} T^{2\beta - l} \sum_{i=1}^{2} \Bigg[\|h_i'\|^{2\beta}_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}$$

$$+ \left(1 + \|\Psi(\cdot; \theta_i, h_i)\|^{2\beta}_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right) \|\theta_i\|^{2\beta}_{W_2^{3+l, \frac{3+l}{2}}(\tilde{\Omega}_T)}$$

$$+ \left(1 + \|\tilde{\theta}_i'\|^{2\beta}_{W_2^{3+l, \frac{3+l}{2}}(\tilde{\Omega}_T)}\right) \|\Psi(\cdot; \theta_i, h_i)\|^{2\beta}_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \Bigg] \|\tilde{h}'\|^2_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}.$$

These complete the estimate of $\|\tilde{f}\|_{W_2^{0, \frac{l}{2}}(\tilde{\Omega}_T)}$, and hence $\|\tilde{f}\|_{W_2^{l, \frac{l}{2}}(\tilde{\Omega}_T)}$.

Higher order norms can be estimated in a similar manner.