A Thesis for the Degree of Ph.D. in Science

Existence of Densities and their Regularity for Solutions of Stochastic Differential Equations by Malliavin Calculus

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Preface

Malliavin suggested a new stochastic analysis at an International Symposium at Kyoto University in 1976. It was not enough accurate mathematically at the time. However, Ikeda and Watanabe noticed the importance of his idea. After that, by the contributions of many mathematicians including many Japanese, the analysis was progressed dramatically and established fully as a new method of stochastic analysis, which is called Malliavin calculus now. Malliavin calculus is another version of the theory of Sobolev spaces. In Malliavin calculus, one discusses integrals and differentials on infinite dimensional Gaussian spaces instead of Euclidian spaces with the Lebesgue measures. Because of fine properties of Gaussian measures, one can make analogies of the theory of Sobolev spaces, though Gaussian spaces might be infinite dimensional. It is also called "Analysis on Wiener spaces."

Malliavin calculus was established in order to know regularity properties of distributions of solutions of stochastic differential equations. Thereby we can see that the density has regularity according to the smoothness of the coefficients of the stochastic differential equation. The first result on this fact was proved by Kusuoka and Stroock [13, 14, 15]. Their theory was simplified and made arrangement later by many mathematicians, which can be found in [24]. On the other hand, there are variety of Malliavin calculus, because many mathematicians had tried to formulate original Malliavin's idea. One of them was given by Bismut [1]. After that, many applications and extensions of Malliavin calculus were established, one of which is Malliavin calculus to stochastic differential equations on manifolds (c.f. [26], [2]). Another important extension is Malliavin calculus for Lévy processes (c.f. [6]). This is useful for mathematical finance, and is a hot topic right now. A result of the present author [17] is related to this theory. There is an application to numerical analysis (c.f. [12]), which is also for mathematical finance and plays an important role in practical business.

The aim of this thesis is to apply Malliavin calculus to stochastic differential equations which have never been attacked. Two results are shown. The first result [16] is an application of Malliavin calculus to stochastic differential equations whose coefficients are not necessarily Lipschitz continuous. In Malliavin calculus, Lipschitz continuity of the coefficients is always assumed. If one tries to apply Malliavin calculus to equations with non-Lipschitz continuous coefficients, many difficulties will occur. The second result [17] is an application of Malliavin calculus to stochastic differential equations driven by rotation-invariant stable processes, where we mean stable processes in the sense of [23]. The theory by Di Nunno et al [6] can be applied to the equations driven by square-integrable Lévy processes. However, the present author makes another formulation of Malliavin calculus for such stable processes, because their theory is not applicable to the problem concerned. The formulation given by the author is also applicable for subordinated Brownian motions. Here, we mean subordinated Brownian motions in the sense as in Section 4.4. In this thesis, after a short review of Malliavin calculus for Brownian motions, the problems mentioned above will be discussed.

Contents

Chapter 1

Introduction

Malliavin calculus is well known as a method to prove regularity properties of distributions of solutions of stochastic differential equations, and one of the most important results is that regularity of the density of a stochastic differential equation is inherited by the smoothness of their coefficients under some suitable conditions of ellipticity. The present author applies Malliavin calculus to stochastic differential equations which it have never been applied to, and obtains two results.

The first result is to apply Malliavin calculus to equations whose coefficients are not necessarily Lipschitz continuous. Let $T > 0$, d and r positive integers, $(B(t))$ an r-dimensional Brownian motion, and

$$\sigma = (\sigma^i_j)_{i=1,\dots,d,j=1,\dots,r} \in C_b([0,T] \times \mathbf{R}; \mathbf{R}^d \otimes \mathbf{R}^r),$$
$$b = (b^i)_{i=1,\dots,d} \in C_b([0,T] \times \mathbf{R}^d; \mathbf{R}^d).$$

We consider the d-dimensional stochastic differential equation:

$$\begin{cases} dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt, \\ X(0) = x_0 \in \mathbf{R}^d. \end{cases}$$

We assume that this equation has some conditions about ellipticity, for example uniformly elliptic, Hörmander condition, and so on. As we see in Section 2.3, if $n \in \mathbf{N}$,

$$\sigma \in C_b^{0,n+2}([0,T] \times \mathbf{R}^d; \mathbf{R}^d \otimes \mathbf{R}^r), \quad b \in C_b^{0,n+2}([0,T] \times \mathbf{R}^d; \mathbf{R}^d),$$

then the distribution of the solution $P \circ X(t)^{-1}$ has its density, which belongs to $C_b^n(\mathbf{R}^d)$.

Concerning the existence of the density, as we see in Section 2.4, there is a result of Bouleau and Hirsch [4]. This result implies that under some conditions about ellipticity and Lipschitz continuity with a constant K

$$|\sigma(t,x) - \sigma(t,y)| + |b(t,x) - b(t,y)| \le K|x-y|, \quad x, y \in \mathbf{R}^d, t \in [0,T],$$

then the distribution of solution $P \circ X(t)^{-1}$ has its density function.

Roughly speaking, under some conditions about ellipticity, it seems that the solution has its density, even if Lipschitz continuity is not satisfied on the coefficients. The author considers if the solution has its density or not when the coefficients are not Lipschitz continuous. However, when stochastic differential equations whose coefficients are not Lipschitz continuous, the solutions would not belong to any Sobolev space in general. Hence, we prepare the class V_h which is larger than Sobolev spaces, and considered the relation between absolute continuity of random variables and the class V_h. This relation is associated with a theorem of Bouleau and Hirsch. Moreover, we obtain a sufficient condition for solutions of stochastic differential equations to belong to the class V_h, and show that solutions have their densities in a special case by using the class V_h.

The second result is a formulation of Malliavin calculus for stochastic differential equations driven by subordinated Brownian motions. As we see in Chapter 2, Malliavin calculus is well known as a method to know regularity properties of distributions of solutions of stochastic differential equations driven by Brownian motions, and we can see that the densities of the solution has the regularity according to the smoothness of the coefficients of equations. There is a natural interest in applying to the equation driven by stable processes. Consider the following N-dimensional stochastic differential equation:

$$\begin{cases} dX(t) = \displaystyle\sum_{k=1}^{r} \sigma_k(t, X(t-))dZ_k(t) + b(t, X(t))dt \\ X(0) = x_0, \end{cases}$$

where $\{Z_k\}$ are independent rotation-invariant stable processes, and the coefficients are Lipschitz continuous. The indices of the stable processes may be different. The definition of the stochastic integral can be found in [9], and the detail of the definition is given in Section 4.4. If the equation satisfy some conditions about ellipticity, it seems that the distribution of the solution has its density function at each time.

On the other hand, there is Malliavin calculus for Lévy processes (c.f. [6]). The method works in mathematical finance very well. However their theory is not applicable to the problem concerned. Another idea is needed for the problem concerned here.

By using subordination, the classical formulation of Malliavin calculus is applicable to functionals of rotation-invariant stable processes. This method enables us to prove that the ellipticity of a stochastic differential equation driven by subordinated Brownian motions implies existence of the density of the solution. In this thesis, we mean subordinated Brownian motions in the sense as in Section 4.4. We can find Malliavin calculus for equations driven by subordinated Brownian motions in [18]. In [18] the case that the number of subordinators is one and the subordinator is an increasing Lévy process with some condition is considered.

We consider the case including that the number of subordinators is more than one and the subordinators are not necessary increasing Lévy processes. We prove our theorems in a similar way to [24], and show regularity properties of distributions of solutions of equations driven by subordinated Brownian motions. The proof consists of two parts. One is Malliavin calculus for stochastic differential equations driven by Brownian motions with deterministic time change, and the other is the inheritance of the regularity of densities from those of conditional probabilities. That is because the discussion is simplified by considering the equation under the conditional probability given by the σ-field generated by the subordinators. Hence we make two steps to prove it. In the last section, we consider the case of stochastic differential equations driven by stable processes. We show that the ellipticity of equations driven by stable processes implies existence of the density of the solution. Moreover, in the case $r = 1$, we can also prove the regularity of the density according to the regularity of the coefficients.

In this thesis, we discuss the results. Before stating the results, we give a short review of Malliavin calculus for Brownian motions in Chapter 2. We discuss the first result and the second result mentioned above in Chapter 3 and in Chapter 4, respectively.

Chapter 2

Review of Malliavin calculus

In this chapter, we review the standard formulation of Malliavin calculus given in [24]. [19] and [6] are also elementary textbooks of Malliavin calculus, but the notation is different from [24]. In this section, we use the notation of [24].

2.1 Preliminaries

Let B be a Banach space, B^* be the dual space, and $\langle\ ,\ \rangle$ be the pairing of the components of them. For a Hilbert space H, we denote the inner product by $(\ ,\ )$ and the norm by $|\cdot|_H$. We often identify a Hilbert space and the dual space.

First we give the notation of functional spaces. Let

$$C_0([0,\infty);\mathbf{R}^d) := \{w\,;\,w \text{ is } \mathbf{R}^d\text{-valued continuous function}, w(0)=0\}$$

The probability measure on $C_0([0,\infty);\mathbf{R}^d)$ which is the law of a d-dimensional Brownian motion is called the Wiener measure. The pair $(C_0([0,\infty);\mathbf{R}^d),\mu)$ is called the Wiener space.

Let $T>0$. We restrict the Wiener space on $[0,T]$ and consider $(C_0([0,T];\mathbf{R}^d),\mu)$. We define a Hilbert space H_T by

$$\begin{aligned} H_T &:= \{h \in C_0([0,T];\mathbf{R}^d)\,;\, h \text{ is absolutely continuous} \\ &\qquad\qquad \text{with respect to the Lebesgue measure and } \dot{h} \in L^2([0,T];\mathbf{R}^d)\}. \end{aligned}$$

Here $\dot{h}$ means the derivative of h. Then H_T is embedded in $C_0([0,T];\mathbf{R}^d)$ continuously and densely. The triplet $(C_0([0,T];\mathbf{R}^d),H_T,\mu)$ is also called the Wiener space.

As an extension of the Wiener space, we have abstract Wiener spaces as follows.

Definition 2.1.1 *Let B be a separable Banach space and H a Hilbert space embedded in B continuously and densely. Let μ be a Gaussian measure on B satisfying that*

$$\int_B \exp\{\sqrt{-1}\langle w, \varphi\rangle\}\mu(dw) = \exp\{-\frac{1}{2}|\varphi|_{H^*}{}^2\}, \quad \varphi \in B^* \subseteq H^*.$$

The triplet (B, H, μ) is called an abstract Wiener space.

Then we have

$$\int_B \langle w, \varphi\rangle^2 \mu(dw) = |\iota^*\varphi|^2_{H^*}.$$

B^* is a subset of H^* by the embedding $\iota^* : B^* \to H^*$. Thus the mapping from H^* to a subspace of $L^2(\mu)$ is isometry. Therefore we can define $\langle w, h\rangle$ for $h \in H$. If $h, k \in H$ are orthogonal, then

$$\int_B \exp\{\sqrt{-1}\langle w, h+k\rangle\}\mu(dw) = \exp\{-\frac{1}{2}|h|_{H^*}{}^2\}\exp\{-\frac{1}{2}|k|_{H^*}{}^2\}.$$

Therefore $\langle w, h\rangle$ and $\langle w, k\rangle$ are independent under μ.

Second we define H-derivative which plays the most important role in Malliavin calculus. Let K be a separable Hilbert space, and $\mathcal{L}^n_{(2)}(H; \mathbf{R})$ be the n-linear Hilbert-Schmidt class operators on $H \times \ldots \times H$. $\mathcal{L}^n_{(2)}(H; \mathbf{R})$ is a Hilbert space with the inner product

$$(S, T)_{\mathcal{L}^n_{(2)}(H;\mathbf{R})} := \sum_{i_1,\ldots,i_n=1}^{\infty} S(e_{i_1}, e_{i_2}, \ldots, e_{i_n})T(e_{i_1}, e_{i_2}, \ldots, e_{i_n}), \quad S, T \in \mathcal{L}^n_{(2)}(H; \mathbf{R}),$$

where $\{e_i\}$ is H a complete orthonormal system.

Definition 2.1.2 *K-valued function F on B is H-differentiable at $x \in B$ if there exists an $h^* \in H^*$ such that*

$$\frac{d}{dt}F(x+th)\bigg|_{t=0} = \langle h, h^*\rangle, \quad h \in H.$$

We call h^* an H-differential of F at x, and denote it by $DF(x)$.

We define the class $\mathcal{S}$ by the total set of F satisfying that

$$F(x) = f(\langle x, \varphi_1\rangle, \ldots, \langle x, \varphi_n\rangle) \tag{2.1.1}$$

where $n \in \mathbf{N}, \varphi_1, \ldots, \varphi_n \in B^*, f \in C^\infty(\mathbf{R}^n)$, and all the derivatives of f are growing in polynomial order. We can define the class $\mathcal{S}(K)$ of K-valued functions similarly.

We define L^p-norm for K-valued functions F on B by

$$||f||_p := \left(\int_B |f(x)|^p_K \mu(dx)\right)^{\frac{1}{p}}.$$

Let $L^p(B,\mu;K)$ be the completion of $\mathcal{S}(K)$ by L^p-norm. Then H-derivative D can be regarded as a closed operator from $L^p(B,\mu;K)$ to $L^p(B,\mu;\mathcal{L}_{(2)}(H;K))$.

Now we define Sobolev spaces with respect to H-derivative.

Definition 2.1.3 *For $k \in \mathbf{N}$, we denote the completion of $\mathcal{S}$ with respect to the norm $\sum_{l=0}^{k} ||D^l f||_p$ by $W^{k,p}$, and call them Sobolev spaces.*

Next we consider H-derivative of stochastic integrals. Let $(B_k(t);[0,T])$ be independent d-dimensional Brownian motions associated with (B,H,μ) and $(\mathscr{F}_t)$ the σ-field generated by $(B_k(s); 0 \le s \le t, k = 1,2,\dots,r)$.

We define some classes of stochastic processes. Let $(C_0([0,T];\mathbf{R}^d), H_T, \mu)$ be the Wiener space, K a separable Hilbert space, and $p > 1$. In this section, we use $E[\cdot]$ as the expectation with respect to the Wiener measure.

We define $\mathcal{L}^p(w;K)$ by a class of $\mathbf{R}^d \otimes K$-valued $(\mathscr{F}_t)$-adapted processes $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_d)$ satisfying that

$$||\Phi||_{\mathcal{L}^p(w;K)} := E\left[\left\{\int_0^T |\Phi(t)|^2_{\mathbf{R}^d\otimes K} dt\right\}^{p/2}\right]^{1/p} < \infty,$$

where $|\Phi(t)|_{\mathbf{R}^d\otimes K}$ means the norm of $\Phi(t)$ on $\mathbf{R}^d \otimes K$ defined by

$$|\Phi(t)|^2_{\mathbf{R}^d\otimes K} := \sum_{i=1}^{d} |\Phi_i(t)|^2_K.$$

Then, $\mathcal{L}^p(w;K)$ is a Banach space with the norm $||\cdot||_{\mathcal{L}^p(w;K)}$, and we can define stochastic integral for elements of $\mathcal{L}^p(w;K)$.

We define $\mathcal{L}^p(dt;K)$ by a class of K-valued $(\mathscr{F}_t)$-adapted processes Ψ satisfying

$$||\Psi||_{\mathcal{L}^p(dt;K)} := \int_0^T E[|\Psi(t)|^p_K]^{1/p} dt < \infty.$$

We define $\mathcal{L}^{n,p}(w;K)$ by a class of $\mathbf{R}^d \otimes K$-valued $(\mathscr{F}_t)$-adapted processes $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_d)$ such that $\Phi(t) \in W^{n,p}(\mathbf{R}^d \otimes K)$ for all t, $D^k\Phi \in \mathcal{L}^p(w;\mathcal{L}^k_{(2)}(H;K))$ for $k = 1,2,\dots,n$, and

$$||\Phi||_{\mathcal{L}^{n,p}(w;K)} := E\left[\sum_{k=0}^{n}\left\{\int_0^T |D^k\Phi(t)|^2_{\mathbf{R}^d\otimes\mathcal{L}^k_{(2)}(H;K)} dt\right\}^{p/2}\right]^{1/p} < \infty.$$

We define $\mathcal{L}^{n,p}(dt;K)$ be a class of K-valued $(\mathscr{F}_t)$-adapted processes Ψ such that $\Psi(t) \in W^{n,p}(K)$ for all t, $D^k\Psi \in \mathcal{L}^p(dt;\mathcal{L}^k_{(2)}(H;K))$ for $k = 1,2,\dots,n$, and

$$||\Psi||_{\mathcal{L}^{n,p}(dt;K)} := \sum_{k=0}^{n}\int_0^T E[|D^k\Psi(t)|^p_{\mathcal{L}^k_{(2)}(H;K)}]^{1/p} dt < \infty.$$

Using the classes mentioned above, we can give the proposition about the H-derivatives of stochastic integrals as follows.

Proposition 2.1.4 *Let $A = (A_1, A_2, \ldots, A_d) \in \mathcal{L}^{n,p}(w; K)$, $B \in \mathcal{L}^{n,p}(dt; K)$, and $\Gamma = (\Gamma(t); 0 \le t \le T)$ K-valued $(\mathscr{F}_t)$-adapted processes such that $\Gamma(t) \in W^{n,p}(K)$ for all t, $D^k\Gamma$ is $\mathcal{L}^k_{(2)}(H; K)$-valued $(\mathscr{F}_t)$-adapted for $k = 1, 2, \ldots, n$, and*

$$\sum_{k=0}^{n} E\left[\sup_{0\le t\le T} |D^k\Gamma(t)|^p_{\mathcal{L}^k_{(2)}(H;K)}\right] < \infty.$$

Let

$$\Psi(t) := \sum_{i=1}^{d} \int_0^t A_i(s) dw^i_s + \int_0^t B(s)ds + \Gamma(t).$$

Then, $\Psi(t) \in W^{n,p}(K)$ for all t, $D^k\Psi$ is $\mathcal{L}^k_{(2)}(H; K)$-valued $(\mathscr{F}_t)$-adapted for $k = 1, 2, \ldots, n$, and

$$D\Psi(t)[h] = \sum_{i=1}^{d} \int_0^t DA^i(s)[h]dw^i_s + \sum_{i=1}^{d} \int_0^t h^i(s)A^i(s)ds + \int_0^t DB(s)ds + D\Gamma(t), \quad h \in H_T,$$

where $D\Psi(t)[h]$ means the value $D\Psi(t)$ at h.

2.2 Smoothness of distributions on Wiener spaces

In this section, we consider regularity property of Wiener functionals. The results of this section are of the H-differential version of the theory of Sobolev spaces with the Lebesgue measure. Let (B, H, μ) be an abstract Weiner space.

Theorem 2.2.1 *Let $p > N$. For an Wiener functional $F : B \longrightarrow \mathbf{R}^N$, we assume that there exists a vector $K = (K_1, K_2, \ldots, K_N)$ satisfying that*

$$E[\partial_j\phi \circ F] = E[(\phi \circ F)K_j], \quad \phi \in C^\infty_K, \quad j = 1, 2, \ldots, N.$$

Here C^∞_K is the total set of functions in C^∞ with compact support.

(i) *If $K \in L^p(\mu)$, then the image of the measure $\rho = \mu \circ F^{-1}$ has a bounded continuous density function $f \in C_b(\mathbf{R}^N)$ and there exists a constant $C = C(N, p)$ such that*

$$||f||_{C_b(\mathbf{R}^N)} \le C||K||^N_{L^p(\mu)}.$$

(ii) *Assume further that for* $H_0 \in L^p(\mu)$, *there exists an* $\mathbf{R}^d$*-valued function* $H = (H_1, H_2, \ldots, H_N) \in L^p(\mu)$ *satisfying that*

$$E[(\partial_j \phi \circ F)H_0] = E[(\phi \circ F)H_j], \quad \phi \in C_K^\infty, \quad j = 1, 2, \ldots, N.$$

Then $\nu = (H_0\mu) \circ F^{-1}$ *has a bounded continuous density function* $k \in C_b(\mathbf{R}^N)$ *and there exists a constant* $C = C(N, p)$ *such that*

$$||k||_{C_b(\mathbf{R}^N)} \leq C||H||_{L^p(\mu)}^{\frac{N}{p}} ||H_0||_{L^p(\mu)}^{1-\frac{N}{p}} ||f||_{C_b(\mathbf{R}^N)}^{1-\frac{1}{p}}.$$

(iii) *Let* α *be a multi-index. If, in addition, for* $0 < |\alpha| \leq n$ *there exists* $H_\alpha \in L^p(\mu)$ *satisfying*

$$E[(\partial_\alpha \phi \circ F)H_0] = E[(\phi \circ F)H_\alpha], \quad \phi \in C_K^\infty,$$

then $k \in C_b^{n-1}(\mathbf{R}^N)$ *and there exists a constant* $C = C(N, p)$ *such that*

$$||k||_{C_b^{n-1}(\mathbf{R}^N)} \leq C \left(||H_0||_{L^p(\mu)} + \sum_{0<|\alpha|\leq n} ||H_\alpha||_{L^p(\mu)} \right) ||f||_{C_b(\mathbf{R}^N)}^{1-\frac{1}{p}}.$$

Next we consider non-degeneracy of Wiener functionals.
Fix $F = (F^1, F^2, \ldots, F^N) : B \to \mathbf{R}^N$.

Definition 2.2.2 *Let* $p \geq 1$ *and* $F \in W^{1,p}(\mathbf{R}^N)$. *We define* $\Delta : B \to \mathbf{R}^N \otimes \mathbf{R}^N$ *by*

$$\Delta_{ij} := (DF^i, DF^j)_{H^*}.$$

Δ *is called Malliavin's covariance matrix.*

Then the non-degeneracy can be expressed as the integrability of $(\det \Delta)^{-1}$. We can know the diffusivity from it.

Now we give a formula of the integration by parts formula. Set

$$\Phi_i G := D^* \left(\sum_{j=1}^N (\det(\Delta^{-1}))_{ij} DF^j G \right),$$

where D^* is the dual operator of D in L^2. For a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ let

$$\Phi_\alpha G := \Phi_1^{\alpha_1} \circ \Phi_2^{\alpha_2} \circ \ldots \circ \Phi_N^{\alpha_N} G.$$

These definitions depend on F. We have an estimate of Φ_α as follows.

Proposition 2.2.3 *Let $k = 0, 1, 2, \ldots$, $p > 1$, α multi-index. We define*

$$M(\alpha, k) := \frac{1}{2}|\alpha|^2 + \frac{3}{2}|\alpha| + k|\alpha|.$$

Let $r > 1$. Assume that $F \in W^{k+|\alpha|+1,4Np}$, $\Delta^{-1} \in L^{2p}(\mu)$, and $G \in W^{k+|\alpha|,r}$.

Then $\Phi_\alpha G \in W^{k,s}$ where s is defined by

$$\frac{1}{s} = \frac{M(\alpha, k)}{p} + \frac{1}{r},$$

and there exists a constant $C = C(N, k, p)$ satisfying

$$||\Phi_\alpha G||_{k,s} \leq C||DF||_{k+|\alpha|,4Np}^{2NM(\alpha,k)}||\Delta^{-1}||_{2p}^{M(\alpha,k)}||G||_{k+|\alpha|,r}.$$

In particular, if $G = 1$, it holds for $r = \infty$, $||G||_{n,r} = 1$.

We can describe the integration by parts formula.

Proposition 2.2.4 *Let $p, r > 1$ satisfying $1 > \frac{2}{p} + \frac{1}{r}$. If $F \in W^{2,2Np}(\mathbf{R}^N)$, $(\det \Delta)^{-1} \in L^{2p}$, and $G \in W^{1,r}$, then*

$$E[(\partial_j \phi \circ F)G] = E[(\phi \circ F)\Phi_j G], \quad \phi \in C_K^\infty(\mathbf{R}^N), \quad j = 1, 2, \ldots, N.$$

Furthermore, for multi-index α, we choose p and r such that $1 > \frac{M(\alpha,0)}{p} + \frac{1}{r}$. If $F \in W^{|\alpha|+1,4Np}$, $\Delta^{-1} \in L^{2p}(\mu)$, and $G \in W^{|\alpha|,r}$, then

$$E[(\partial_\alpha \phi \circ F)G] = E[(\phi \circ F)\Phi_\alpha G], \quad \phi \in C_K^\infty(\mathbf{R}^N).$$

The integration by parts formula is a remarkable idea of Malliavin. It is also called Malliavin's trick. From the results above, we can conclude the following theorem about existence of densities and their regularities of Wiener functionals.

Theorem 2.2.5 *Let $p > N$. If $F \in W^{2,8Np}(\mathbf{R}^N)$ and $\Delta^{-1} \in L^{4p}$, then $\mu \circ F^{-1}$ is absolutely continuous with respect to the Lebesgue measure, and its density function f belongs to $C_b(\mathbf{R}^N)$ and there exists a constant $C = C(p, N)$ such that*

$$||f||_{C_b(\mathbf{R}^N)} \leq C||DF||_{1,8Np}^{4N^2}||\Delta^{-1}||_{4p}^{2N}.$$

Let $n \in \mathbf{N}$, $M = n^2/2 + 3n/2$. Assume that $F \in W^{n+1,4NMp}$ and $\Delta^{-1} \in L^{2Mp}(\mu)$. Then, it follows that $f \in C_b^{n-1}(\mathbf{R}^N)$ and there exists a constant $C' = C'(p, N, n)$ such that

$$||f||_{C_b^{n-1}(\mathbf{R}^N)} \leq C'\left(1 + ||DF||_{n,4NMp}^{2NM}||\Delta^{-1}||_{2Mp}^{M}\right)||f||_{C_b(\mathbf{R}^N)}^{1-\frac{1}{p}}.$$

2.3 Malliavin calculus for stochastic differential equations

In this section we give applications of Malliavin calculus for stochastic differential equations.

Let $(B_k(t); [0,T])$ be independent d-dimensional Brownian motions and $(\mathscr{F}_t)$ the σ-field generated by $(B_k(s); 0 \le s \le t, k = 1, 2, \ldots, r)$. We consider the following N-dimensional stochastic differential equation:

$$\begin{cases} dX(t) = \sum_{k=1}^{r} \sigma_k(t, X(t-))dB_k(t) + b(t, X(t))dt \\ X(0) = x_0, \end{cases} \tag{2.3.1}$$

where $\{\sigma_k\}$ are $\mathbf{R}^d \otimes \mathbf{R}^N$-valued measurable functions on $[0,T] \times \mathbf{R}^N$, b is also an $\mathbf{R}^N$-valued continuous function on $[0,T] \times \mathbf{R}^N$, and $x_0 \in \mathbf{R}^N$. Moreover they satisfy with a positive constant K

$$\max_k |\sigma_k(t,x) - \sigma_k(t,y)| + |b(t,x) - b(t,y)| \le K|x-y|, \quad x, y \in \mathbf{R}^N, \ t \in [0,T].$$

Then we have the following theorem.

Theorem 2.3.1 *The equation* (2.3.1) *has the unique* $(\mathscr{F}_t)$*-adapted solution* $X = (X(t))$ *satisfying that*

$$E\left[\sup_{0 \le t \le T} |X(t)|^p\right]^{\frac{1}{p}} \le x_0 e^{Mt},$$

for all $p > 1$ *where* M *is a constant depending on* r, p *and* K.

Now we apply Malliavin calculus to the solution $X = (X(t))$ of the equation (2.3.1).

Theorem 2.3.2 *Let* $n \in \mathbf{N}$. *We assume that* $\sigma_k \in C_b^{0,n}([0,T] \times \mathbf{R}^N; \mathbf{R}^d \otimes \mathbf{R}^N)$ *for* $k = 1, 2, \ldots, r$, $b \in C_b^{0,n}([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$. *Then we have* $X(t) \in W^{n,p}(\mathbf{R}^N)$ *for* $t \in [0,T]$, *and there exists a constant* M *depending on* r, p, n *and the bounds of the spatial derivatives of* σ_k *and* b *up to order* n *such that*

$$||X(t)||_{n,p} \le x_0 e^{Mt}, \quad t \in [0,T].$$

Next we consider the relation between the ellipticity of equations and the non-degeneracy of Malliavin covariance matrices.

Theorem 2.3.3 *We assume that* $\sigma_k \in C_b^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^d \otimes \mathbf{R}^N)$ *for* $k = 1, 2, \ldots, r$, $b \in C_b^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$, *and that there exists a positive constant* ε *such that*

$$\sum_{k=1}^{r} \sigma_k(0, x_0)\ {}^t\sigma_k(0, x_0) \geq \varepsilon.$$

Then, Malliavin covariance matrix $\Delta(t) = ((DX^i(t), DX^j(t))_{H^*})_{ij}$ *is invertible, and there exists a constant* C *satisfying that for all* $p > 1$

$$E[\det(\Delta(t))^{-p}] \leq Ct^{-Np}e^{Ct}, \quad t \in [0,T].$$

Thus, applying Sobolev's inequality with respect to H-derivative, we have the following theorem.

Theorem 2.3.4 *In* (2.3.1) *we assume that* $\sigma_k \in C_b^{0,n+2}([0,T] \times \mathbf{R}^N; \mathbf{R}^d \otimes \mathbf{R}^N)$ *for* $k = 1, 2, \ldots, r$, $b \in C_b^{0,n+2}([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$, *and there exists a positive constant* ε *such that*

$$\sum_{k=1}^{r} \sigma_k(0, x_0)\ {}^t\sigma_k(0, x_0) \geq \varepsilon.$$

Then the law $P(t, x_0, dy)$ *of* $X(t, x_0)$ *is absolutely continuous with respect to the Lebesgue measure and whose density function* $p(t, x, y)$ *has the estimate*

$$\max_{0 \leq l \leq m} \sup_{y \in \mathbf{R}^d} \left| \nabla_y^l p(t, x_0, y) \right| \leq c_1 \min\{\phi_i(t); i = 1, 2, \ldots, r\}^{-c_3} \exp\left\{ c_2 \left(t + \sum_{k=1}^{r} \phi_k(t) \right) \right\} \tag{2.3.2}$$

with positive constants c_1, c_2, c_3. *Moreover, if there exist positive constants* ε *and* t_0 *such that*

$$\sum_{k=1}^{r} \sigma_k(t, x)\ {}^t\sigma_k(t, x) \geq \varepsilon, \quad t \in [0, t_0],\ x \in \mathbf{R}^N,$$

then we can choose constants c_1, c_2, c_3 *in* (2.3.2) *dependently only on* t_0.

There are further known results. In the last theorem, we assume the non-degeneracy of the diffusion coefficient at starting point. However, this condition can be relaxed to Hörmander's condition. The result is also seen in [24]. On the other hand, there is a similar theorem for Itô's equations in [13]. Itô's equations mean stochastic differential equations whose coefficients depend on the past.

2.4 The method for absolute continuity by Bouleau and Hirsch

Bouleau and Hirsch studied existence of densities of solutions of stochastic differential equations when the coefficients have less regularity. They also used Malliavin calculus, and they applied coarea formula instead of the theory in Section 2.2. In this section, we give the outline of their theory.

The following theorem is the coarea formula. The precise arguments of the coarea formula can be found in [7].

Theorem 2.4.1 *Let m and n be positive integers satisfying that $m > n$, and $\mathcal{H}^{m-n}$ be the $(m-n)$-dimensional Hausdorff measure on $\mathbf{R}^n$. We assume $f : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ is Lipschitz continuous, and let $J_n f$ be the Jacobian of f. Then we have*

$$\int_{\mathbf{R}^m} g(x) J_m f(x) dx = \int_{\mathbf{R}^n} \left(\int_{f^{-1}(\{y\})} g(x) \mathcal{H}^{m-n}(dx) \right) dy$$

for all Borel measurable positive function g on $\mathbf{R}^m$.

The idea of Bouleau and Hirsch is to apply the theorem to Gaussian measures. Any non-degenerate finite-dimensional Gaussian measure is absolutely continuous with respect to the Lebesgue measure. Furthermore, the symmetrization of the Jacobian is associated with the Malliavin covariance. Thus, the absolute continuity follows from the non-degeneracy of Malliavin covariances. By the method, the following theorem is obtained. It is a simple version of a theorem given by Bouleau and Hirsch.

Theorem 2.4.2 *We consider the stochastic differential equation* (2.3.1), *and assume that $\{\sigma_k; k = 1, 2, \ldots, r\}$ and b are Lipschitz continuous in the spacial component. And there exists a positive constant ε such that*

$$\sum_{k=1}^{r} \sigma_k(0, x_0)\ {}^t\sigma_k(0, x_0) \geq \varepsilon.$$

Then, the law of $X(t)$ is absolutely continuous with respect to the Lebesgue measure.

Bouleau and Hirsch also used the result of the previous section for the non-degeneracy of Malliavin covariances.

Bouleau and Hirsch obtained further results. In the above theorem, we assume the non-degeneracy of the diffusion coefficient at starting point. However, this condition can be weaken to Hörmander condition. We remark that Hirsch showed a similar theorem for Itô's equation in [8].

Chapter 3

Existence of densities of solutions of stochastic differential equations by Malliavin calculus

In this chapter, we discuss application of Malliavin calculus to equations whose coefficients are not necessarily Lipschitz continuous. We will introduce a class V_h of random variables and discuss it in Section 3.1, and the relation between the solution of stochastic differential equation and the class V_h in Section 3.2.

3.1 Analysis in class V_h

First we introduce a class of random variables. When we consider stochastic differential equations whose coefficients are not necessarily Lipschitz continuous, the solutions would not belong to any Sobolev space in general. So we need a larger class than Sobolev space.

Let $(\Omega, \mathscr{F}, P)$ be a probability space which is an orthogonal product measure space of an abstract Wiener space (B, H, μ) and another probability space $(\Omega', \mathscr{F}', \nu)$. Of course, our argument includes the case that Ω' is trivial, for example, $\Omega' = \{0\}$. Throughout this chapter we identify $\omega \in \Omega$ as $(x, \omega') \in B \times \Omega'$.

Let F be a random variable on $(\Omega, \mathscr{F}, P)$. When the limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(x + \varepsilon h, \omega') - F(x, \omega'))$$

exists for $h \in H$, then we denote this limit by $D_h F(x, \omega')$. D_h is regarded as the derivative in direction h.

We prepare some notation. Let $h \in H$ be fixed and $\{h_k\}$ a complete orthonormal

normal system of H^* such that $h = h_1$. Since $B^* \subset H^*$ is a continuous embedding,

$$B \ni x \longmapsto (\langle x, h_1 \rangle, \langle x, h_2 \rangle, \ldots) \in \mathbf{R}^\infty$$

is injective. Here we denoted $\langle x, h \rangle$ in the sense of Wiener integral of 1-order. Hence let

$$y = \langle x, h_1 \rangle \in \mathbf{R}, \quad \tilde{x} = (\langle x, h_2 \rangle, \langle x, h_3 \rangle, \ldots) \in \mathbf{R}^\infty,$$

then we can identify x as $(y, \tilde{x})$.

Next we describe the measures of y and $\tilde{x}$. By the orthogonality of $\{h_k\}$ in H^*, if $k \neq j$, $\langle \tilde{x}, h_k \rangle$ and $\langle \tilde{x}, h_j \rangle$ are independent under μ. Since $\{\langle x, h_k \rangle\}$ is a Gaussian system under μ, $\{\langle x, h_k \rangle\}$ are independent. In particular, $y = \langle x, h_1 \rangle$ and $\tilde{x} = (\langle x, h_2 \rangle, \langle x, h_3 \rangle, \ldots)$ are independent under μ. We regard the measure space (B, μ) as an orthogonal measure space for y and $\tilde{x}$. Moreover we have the following decomposition:

$$B \cong \mathbf{R} \times \tilde{B}, \quad \mu \cong \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \otimes \tilde{\mu}.$$

Here we used the fact that $y = \langle x, h_1 \rangle$ has a normal distribution with mean 0 and variance 1 under μ. We denote partial derivative with respect to y by ∂_y.

Definition 3.1.1 *We define $V_h(B \times \Omega')$ by the total set of random variables F on $(\Omega, \mathscr{F}, P)$ such that there exists a random variable $\widehat{F}$ on $(\Omega, \mathscr{F}, P)$ satisfying that $F = \widehat{F}$ a.s. and $\widehat{F}(x + th, \omega')$ is a function of bounded variation on any finite interval with respect to t for all x and ω'. If Ω' is trivial, for example, $\Omega' = \{0\}$, then we denote it by $V_h(B)$ simply.*

Now we give a criterion for a random variable to belong to the class V_h.

Theorem 3.1.2 *Let $(\Omega, \mathscr{F}, P)$ be a probability space which is an orthogonal product measure space of an abstract Wiener space (B, H, μ) and another probability space $(\Omega', \mathscr{F}', \nu)$. Let $p > 1$, $h \in H$, and $F \in L^p(\Omega, \mathscr{F}, P)$. We assume that there exists a sequence $\{F_n : n \in \mathbf{N}\}$ in $L^p(\Omega, \mathscr{F}, P)$ so that*

(i) *F_n converges to F almost surely,*

(ii) *$\{F_n\}$ are uniformly bounded in $L^p(\Omega, \mathscr{F}, P)$,*

(iii) *$F_n(x + th, \omega')$ is absolutely continuous in t with respect to the one-dimensional Lebesgue measure for all x and n,*

(iv) *$\{D_h F_n\}$ are uniformly bounded in $L^1(\Omega, \mathscr{F}, P)$.*

Then $F \in V_h(B \times \Omega')$.

Proof. To simplify the notation, we assume that $|h|_H = 1$. Since $\{F_n\}$ are uniformly bounded in $L^p(\Omega, \mathscr{F}, P)$, $\{F_n\}$ are uniformly integrable. Since F_n converges to F almost surely, F_n also converges to F in $L^{1*}(\Omega, \mathscr{F}, P)$. For given a positive number M let us define a function $\phi \in C^\infty(\mathbf{R})$ such that

$$0 \le \phi \le 1, \quad 0 \le \phi' \le 2, \quad \phi(y) = \begin{cases} 1, & \text{if} \quad |y| \le M, \\ 0, & \text{if} \quad |y| \ge M+1. \end{cases}$$

Then for $t, s \in [-M, M]$ we have

$$\begin{aligned}
&\int_{\Omega'} \int_{\tilde{B}} \int_{\mathbf{R}} |F(y+t, \tilde{x}, \omega')\phi(y+t) - F(y+s, \tilde{x}, \omega')\phi(y+s)| \\
&\qquad\qquad\qquad\qquad \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy\tilde{\mu}(d\tilde{x})\nu(d\omega') \\
&= \lim_{n\to\infty} \int_{\Omega'} \int_{\tilde{B}} \int_{\mathbf{R}} |F_n(y+t, \tilde{x}, \omega')\phi(y+t) - F_n(y+s, \tilde{x}, \omega')\phi(y+s)| \\
&\qquad\qquad\qquad\qquad \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy\tilde{\mu}(d\tilde{x})\nu(d\omega') \\
&\le \liminf_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\Omega'} \int_{\tilde{B}} \int_{\mathbf{R}} |F_n(y+t, \tilde{x}, \omega')\phi(y+t) - F_n(y+s, \tilde{x}, \omega')\phi(y+s)| \\
&\qquad\qquad\qquad\qquad \times dy\tilde{\mu}(d\tilde{x})\nu(d\omega') \\
&= \liminf_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\Omega'} \int_{\tilde{B}} \int_{\mathbf{R}} \left| \int_s^t \partial_y[F_n(y+v, \tilde{x}, \omega')\phi(y+v)]dv \right| dy\tilde{\mu}(d\tilde{x})\nu(d\omega') \\
&\le \liminf_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_s^t \int_{\Omega'} \int_{\tilde{B}} \int_{\mathbf{R}} |(\partial_y F_n(y+v, \tilde{x}, \omega'))\phi(y+v) + F_n(y+v, \tilde{x}, \omega')\phi'(y+v)| \\
&\qquad\qquad\qquad\qquad \times dy\tilde{\mu}(d\tilde{x})\nu(d\omega')dv \\
&= \liminf_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_s^t \int_{\Omega'} \int_{\tilde{B}} \int_{\mathbf{R}} |(\partial_y F_n(y, \tilde{x}, \omega'))\phi(y) + F_n(y, \tilde{x}, \omega')\phi'(y)| \\
&\qquad\qquad\qquad\qquad \times dy\tilde{\mu}(d\tilde{x})\nu(d\omega')dv \\
&\le \frac{1}{\sqrt{2\pi}} |t-s| \sup_n \int_{\Omega'} \int_{\tilde{B}} \int_{-(M+2)}^{(M+2)} [\ |(\partial_y F_n(y, \tilde{x}, \omega')))\phi(y)| + |F_n(y, \tilde{x}, \omega')\phi'(y)|\ ] \\
&\qquad\qquad\qquad\qquad \times dy\tilde{\mu}(d\tilde{x})\nu(d\omega') \\
&\le 2|t-s| e^{\frac{(M+2)^2}{2}} \sup_n \int_{\Omega'} \int_{\tilde{B}} \int_{-(M+2)}^{(M+2)} [\ |(\partial_y F_n(y, \tilde{x}, \omega')))| + |F_n(y, \tilde{x}, \omega')|\ ] \\
&\qquad\qquad\qquad\qquad \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy\tilde{\mu}(d\tilde{x})\nu(d\omega') \\
&\le 2e^{\frac{(M+2)^2}{2}} |t-s| \sup_n (\ ||F_n||_{L^1(\Omega)} + ||D_h F_n||_{L^1(\Omega)}\ ).
\end{aligned}$$

Therefore for $s, t \in [-M, M]$

$$\int_{\Omega'} \int_{\tilde{B}} \int_{\mathbf{R}} |F(y+t, \tilde{x}, \omega')\phi(y+t) - F(y+s, \tilde{x}, \omega')\phi(y+s)|$$

$$\times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy\tilde{\mu}(d\tilde{x})\nu(d\omega')$$
$$\leq C_M|t-s|, \tag{3.1.1}$$

where C_M is a constant depending only on M and $\sup_n(||F_n||_{L^1(\Omega)} + ||D_hF_n||_{L^1(\Omega)})$. We define a functions $\{F_\phi^m\}$ on $B \times \Omega$ by

$$F_\phi^m(y, \tilde{x}, \omega') := 2^m \int_0^{\frac{1}{2^m}} F(y+v, \tilde{x}, \omega')\phi(y+v)dv, \qquad m = 1, 2, 3, \dots.$$

Then

$$\begin{aligned}
&\int_{\mathbf{R}} |\partial_y F_\phi^m(y, \tilde{x}, \omega')|dy \\
&\leq 2^m \int_{\mathbf{R}} |\partial_y \int_0^{\frac{1}{2^m}} F(y+u, \tilde{x}, \omega')\phi(y+u)du|dy \\
&= 2^m \int_{\mathbf{R}} |\partial_y \int_y^{y+\frac{1}{2^m}} F(u, \tilde{x}, \omega')\phi(u)du|dy \\
&= 2^m \int_{\mathbf{R}} \left| F\left(y + \frac{1}{2^m}, \tilde{x}, \omega'\right) \phi\left(y + \frac{1}{2^m}\right) - F(y, \tilde{x}, \omega')\phi(y)\right| dy \\
&\leq 2^m \int_{\mathbf{R}} \left\{ \left| F\left(y + \frac{1}{2^m}, \tilde{x}, \omega'\right) \phi\left(y + \frac{1}{2^m}\right) - F\left(y + \frac{1}{2^{m+1}}, \tilde{x}, \omega'\right) \phi\left(y + \frac{1}{2^{m+1}}\right)\right| \right. \\
&\qquad\qquad \left. + \left| F\left(y + \frac{1}{2^{m+1}}, \tilde{x}, \omega'\right) \phi\left(y + \frac{1}{2^{m+1}}\right) - F(y, \tilde{x}, \omega')\phi(y)\right| \right\} dy \\
&= 2^{m+1} \int_{\mathbf{R}} \left| F\left(y + \frac{1}{2^{m+1}}, \tilde{x}, \omega'\right) \phi\left(y + \frac{1}{2^{m+1}}\right) - F(y, \tilde{x}, \omega')\phi(y)\right| dy \\
&= 2^{m+1} \int_{\mathbf{R}} |\partial_y \int_0^{\frac{1}{2^{m+1}}} F(y+u, \tilde{x}, \omega')\phi(y+u)du|dy \\
&= \int_{\mathbf{R}} |\partial_y F_\phi^{m+1}(y, \tilde{x}, \omega')|dy.
\end{aligned}$$

This means that $\{\int_{\mathbf{R}} |\partial_y F_\phi^m(y, \tilde{x}, \omega')|dy\}$ are increasing in m. Thus, from (3.1.1), it follows the inequality

$$\begin{aligned}
&\int_{\Omega'} \int_{\tilde{B}} \left(\sup_m \int_{\mathbf{R}} |\partial_y F_\phi^m(y, \tilde{x}, \omega')|dy \right) \tilde{\mu}(d\tilde{x})\nu(d\omega') \\
&= \sup_m \int_{\Omega'} \int_{\tilde{B}} \left(\int_{\mathbf{R}} 2^m \left| F\left(y + \frac{1}{2^m}, \tilde{x}, \omega'\right) \phi\left(y + \frac{1}{2^m}\right) - F(y, \tilde{x}, \omega')\phi(y)\right| dy \right) \\
&\qquad\qquad \times \tilde{\mu}(d\tilde{x})\nu(d\omega') \\
&\leq \sqrt{2\pi} \exp\left(\frac{(M+1)^2}{2}\right) \sup_m \left\{ 2^m \int_{\Omega'} \int_{\tilde{B}} \int_{\mathbf{R}} \left| F\left(y + \frac{1}{2^m}, \tilde{x}, \omega'\right) \phi\left(y + \frac{1}{2^m}\right) \right.\right. \\
&\qquad\qquad \left.\left. - F(y, \tilde{x}, \omega')\phi(y)\right| \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy\tilde{\mu}(d\tilde{x})\nu(d\omega') \right\}
\end{aligned}$$

$$\leq \sqrt{2\pi} \exp\left(\frac{(M+1)^2}{2}\right) C_M.$$

This implies that for $\tilde{\mu} \times \nu$-almost all $(\tilde{x}, \omega')$

$$\sup_m \int_{\mathbf{R}} |\partial_y F_\phi^m(y, \tilde{x}, \omega')| dy < \infty.$$

On the other hand, by the definition of F_ϕ^m, there exists a function $\overline{F}(\cdot, \tilde{x}, \omega')$ for all $(\tilde{x}, \omega')$ so that

$$\lim_{m\to\infty} F_\phi^m(y, \tilde{x}, \omega') = \overline{F}(y, \tilde{x}, \omega')\phi(y),$$
$$\overline{F}(\cdot, \tilde{x}, \omega') = F(\cdot, \tilde{x}, \omega'), \quad dy\text{-a.e.}$$

Hence, by Corollary 5.3.4 of [30], $\overline{F}(\cdot, \tilde{x}, \omega')\phi$ is a function of bounded variation on $\mathbf{R}$ for $\tilde{\mu} \times \nu$-almost all $(\tilde{x}, \omega')$. For $\tilde{\mu} \times \nu$-almost all $(\tilde{x}, \omega)$ and for all $M > 0$, $\overline{F}(\cdot, \tilde{x}, \omega')$ is a function of bounded variation on $[-M, M]$, from which we have that $F \in V_h(B \times \Omega')$. □

Example 3.1.3 Let $(\Omega, \mathscr{F}, P)$ be a probability space, L a Lévy process on $(\Omega, \mathscr{F}, P)$, $\mathscr{F}^L$ a σ-field generated by L, and F an $\mathscr{F}^L$-measurable random variable on $(\Omega, \mathscr{F}, P)$. Then, from the Lévy-Itô decomposition and A3.2 in [29], we can regard F as a random variable on a product space generated by a space $(W, \mathscr{B}(W), \mu)$ of the part of Brownian motions and a space $(\Omega', \mathscr{F}, \nu)$ of the jump part. We denote the Cameron-Martin space associated with $(W, \mathscr{B}(W), \mu)$ by H. Now let $h \in H$ and $p > 1$. Assume that there exists a sequence of random variables $\{F_n : n \in \mathbf{N}\}$ in $L^p(W \times \Omega', \mu \otimes \nu)$ so that $\{F_n\}$ converges to F almost surely and $\{D_h F_n\}$ are uniformly bounded in $L^1(W \times \Omega', \mu \otimes \nu)$. Then $F \in V_h(W \times \Omega')$.

Now we give an application of V_h. The following theorem tells the relation between the class V_h and the absolute continuity. It is associated with that of Bouleau and Hirsch.

Theorem 3.1.4 *Let $(\Omega, \mathscr{F}, P)$ be a probability space which is an orthogonal product measure space of an abstract Wiener space (B, H, μ) and another probability space $(\Omega', \mathscr{F}', \nu)$. Let F be a random variable such that $F \in V_h(B \times \Omega')$. If $\widehat{F}$ is the modification of F appeared in the definition of $V_h(B \times \Omega')$, then the measure*

$$(|D_h \widehat{F}| P) \circ \widehat{F}^{-1}$$

is absolutely continuous with respect to the one-dimensional Lebesgue measure.

Proof. Since $\widehat{F}(\cdot, \tilde{x}, \omega')$ is a function of bounded variation on any finite interval, a limit function

$$\overline{F}(y, \tilde{x}, \omega') := \lim_{\varepsilon \downarrow 0} \widehat{F}(y + \varepsilon, \tilde{x}, \omega')$$

exists, which is a right-continuous version of $\widehat{F}(\cdot, \tilde{x}, \omega')$. Hence we have

$$\overline{F} = F, \quad P\text{-a.e.}$$

Fix a constant $M > 0$. To translate the domain, we define a function $\overline{F}_M(y, z)$ on $[0, 2M] \times \tilde{B} \times \Omega'$ by

$$\overline{F}_M(y, \tilde{x}, \omega') := \overline{F}(y - M, \tilde{x}, \omega'), \qquad y \in [0, 2M],\ \tilde{x} \in \tilde{B},\ \omega' \in \Omega'.$$

Then for every nonnegative continuous function f on $\mathbf{R}$ we have

$$\begin{aligned} &\int_{\Omega'} \int_{\tilde{B}} \int_{-M}^{M} f(\widehat{F}(y, \tilde{x}, \omega')) |\partial_y \widehat{F}(y, \tilde{x}, \omega')| dy \tilde{\mu}(d\tilde{x}) \nu(d\omega') \\ &= \int_{\Omega'} \int_{\tilde{B}} \int_{0}^{2M} f(\overline{F}_M(y, \tilde{x}, \omega')) |\partial_y \overline{F}_M(y, \tilde{x}, \omega')| dy \tilde{\mu}(d\tilde{x}) \nu(d\omega'). \end{aligned} \tag{3.1.2}$$

Now we interpolate the discontinuous points of $\overline{F}_M(y, \tilde{x}, \omega')$ linearly with respect to y, and make a continuous function. First we fix $\tilde{x}$ and ω'. Let $C \subset [0, 2M]$ be the set of continuous points of $\overline{F}_M(y, \tilde{x}, \omega')$ with respect to y, and $\{\xi_k\} \subset [0, 2M]$ discontinuous points of $\overline{F}_M(y, \tilde{x}, \omega')$ with respect to y. Set

$$\begin{aligned} j_{\tilde{x}, \omega'} &: [0, 2M] \longrightarrow \mathbf{R}, \\ j_{\tilde{x}, \omega'}(y) &:= \overline{F}_M(y, \tilde{x}, \omega') - \overline{F}_M(y-, \tilde{x}, \omega'), \end{aligned}$$

where

$$\overline{F}_M(y-, \tilde{x}, \omega') := \lim_{\varepsilon \downarrow 0} \overline{F}_M(y - \varepsilon, \tilde{x}, \omega'),$$

and

$$\begin{aligned} J_{\tilde{x}, \omega'} &: [0, 2M] \longrightarrow \mathbf{R}, \\ J_{\tilde{x}, \omega'}(y) &:= \sum_{0 < \xi_k \le y} |j_{\tilde{x}, \omega'}(\xi_k)|. \end{aligned}$$

Define τ by

$$\tau : [0, 2M + J_{\tilde{x}, \omega'}(2M)] \longrightarrow [0, 2M],$$

$$\tau(\tilde{y}) := \begin{cases} \inf\{u \in [0, 2M]; u + J_{\tilde{x}, \omega'}(u) > \tilde{y}\}, & \text{if} \quad \tilde{y} \in [0, 2M + J_{\tilde{x}, \omega'}(2M)), \\ 2M, & \text{if} \quad \tilde{y} = 2M + J_{\tilde{x}, \omega'}(2M). \end{cases}$$

Since τ is an inverse function of the increasing function $\cdot + J_{\tilde{x},\omega'}(\cdot)$, it is continuous and increasing. Then the function $\tilde{F}_M(\tilde{y}, \tilde{x}, \omega')$ defined on $[0, 2M + J_{\tilde{x},\omega'}(2M)] \times \tilde{B}$

$$\tilde{F}_M(\tilde{y}, \tilde{x}, \omega') := \overline{F}_M(\tau(\tilde{y}), \tilde{x}, \omega') + \mathrm{sgn}(j_{\tilde{x},\omega'}(\tau(\tilde{y})))\{\tilde{y} - (J_{\tilde{x},\omega'}(\tau(\tilde{y})) + \tau(\tilde{y}))\},$$

$$\mathrm{sgn}(u) := \begin{cases} 1, & \text{if} \quad u > 0, \\ 0, & \text{if} \quad u = 0, \\ -1, & \text{if} \quad u < 0, \end{cases}$$

is continuous with respect to $\tilde{y}$. When $\tau(\tilde{y}) \in C$,

$$\tilde{F}_M(\tilde{y}, \tilde{x}, \omega') = \overline{F}_M(\tau(\tilde{y}), \tilde{x}, \omega').$$

From the fact that the total set of discontinuous points of a function of bounded variation is a null set with the Lebesgue measure, for every nonnegative continuous function f it holds

$$\begin{aligned}
&\int_0^{2M} f(\overline{F}_M(y, \tilde{x}, \omega'))|\partial_y \overline{F}_M(y, \tilde{x}, \omega')|dy \\
&= \int_0^{2M} f(\overline{F}_M(y, \tilde{x}, \omega'))|\partial_y \overline{F}_M(y, \tilde{x}, \omega')|\mathbf{1}_C(y)dy \\
&= \int_0^{2M+J_{\tilde{x},\omega'}(2M)} f(\overline{F}_M(\tau(\tilde{y}), \tilde{x}, \omega'))|(\partial_y \overline{F}_M)(\tau(\tilde{y}), \tilde{x}, \omega')|\mathbf{1}_C(\tau(\tilde{y}))d\tau(\tilde{y}) \\
&= \int_0^{2M+J_{\tilde{x},\omega'}(2M)} f(\tilde{F}_M(\tilde{y}, \tilde{x}, \omega'))\mathbf{1}_C(\tau(\tilde{y}))|(\partial_y \overline{F}_M)(\tau(\tilde{y}), \tilde{x}, \omega')|d\tau(\tilde{y}) \\
&\le \int_0^{2M+J_{\tilde{x},\omega'}(2M)} f(\tilde{F}_M(\tilde{y}, \tilde{x}, \omega'))\mathbf{1}_C(\tau(\tilde{y}))|d_{\tilde{y}} \overline{F}_M(\tau(\tilde{y}), \tilde{x}, \omega')| \\
&\le \int_0^{2M+J_{\tilde{x},\omega'}(2M)} f(\tilde{F}_M(\tilde{y}, \tilde{x}, \omega'))|d_{\tilde{y}} \tilde{F}_M(\tilde{y}, \tilde{x}, \omega')|. \qquad (3.1.3)
\end{aligned}$$

By Theorem 6.4 of Chapter IX in [22], we have the following lemma.

Lemma 3.1.5 *Let ψ be a function of bounded variation on $[a, b]$, and $N^{\psi}_{[a,b]}(c)$ the number of crossing points on $[a, b]$ between the graphs $y = \psi(x)$ and $y = c$ for $c \in \mathbf{R}$. Then*

$$\int_a^b f(\psi(x))|d\psi(x)| = \int_{-\infty}^{\infty} f(y) N^{\psi}_{[a,b]}(y)dy, \quad f \in C(\mathbf{R}).$$

From this lemma we have

$$\int_0^{2M+J_{\tilde{x},\omega'}(2M)} f(\tilde{F}_M(\tilde{y}, \tilde{x}, \omega'))|d_{\tilde{y}} \tilde{F}_M(\tilde{y}, \tilde{x}, \omega')| = \int_{-\infty}^{\infty} f(u) N^{\tilde{F}_M(\cdot, \tilde{x}, \omega')}_{[0, 2M+J_{\tilde{x},\omega'}(2M)]}(u)du,$$

This and (3.1.3) yield

$$\int_0^{2M} f(\overline{F}_M(y,\tilde{x},\omega'))|\partial_y \overline{F}_M(y,\tilde{x},\omega')|dy \le \int_{-\infty}^{\infty} f(u) N^{\tilde{F}_M(\cdot,\tilde{x},\omega')}_{[0,2M+J_{\tilde{x},\omega'}(2M)]}(u)du$$

for $\tilde{\mu}\times\nu$-almost every $(\tilde{x},\omega')$. From this and (3.1.2) one can derive

$$\begin{aligned}\int_{\Omega'}\int_{\tilde{B}}\int_{-M}^{M} & f(\widehat{F}(y,\tilde{x},\omega'))|\partial_y \widehat{F}(y,\tilde{x},\omega')|dy\tilde{\mu}(d\tilde{x})\nu(d\omega') \\ & \le \int_{\Omega'}\int_{\tilde{B}}\int_{-\infty}^{\infty} f(u) N^{\tilde{F}_M(\cdot,\tilde{x},\omega')}_{[0,2M+J_{\tilde{x},\omega'}(2M)]}(u)du\tilde{\mu}(d\tilde{x})\nu(d\omega') \end{aligned} \tag{3.1.4}$$

for every nonnegative continuous function f, and hence for every nonnegative Lebesgue measurable function f. It is easily seen that

$$\int_{\Omega'}\int_{\tilde{B}}\int_{-M}^{M} \mathbf{1}_A(\widehat{F}(y,\tilde{x},\omega'))|\partial_y \widehat{F}(y,\tilde{x},\omega')|dy\tilde{\mu}(d\tilde{x})\nu(d\omega') = 0$$

holds for any null set A in $\mathbf{R}$. Since this equation holds for all $M>0$,

$$\int_{\Omega'}\int_{\tilde{B}}\int_{\mathbf{R}} \mathbf{1}_A(\widehat{F}(y,\tilde{x},\omega'))|\partial_y \widehat{F}(y,\tilde{x},\omega')|dy\tilde{\mu}(d\tilde{x})\nu(d\omega') = 0.$$

This means

$$E[\mathbf{1}_A(\widehat{F})|D_h\widehat{F}|] = 0.$$

□

3.2 Applications to stochastic differential equations

In this section we consider if solutions of stochastic differential equations whose coefficients are not Lipschitz continuous have their densities or not. We begin with the following lemma which plays the most important role in this chapter.

Lemma 3.2.1 *Let r be a positive integer, $(\Omega,\mathscr{F},P)$ a probability space, $(B(t))$ an r-dimensional Brownian motion on $(\Omega,\mathscr{F},P)$, $(\mathscr{F}_t)$ a reference family, $\sigma=(\sigma_j)_{j=1,2,\dots,r}$ an $\mathbf{R}^r$-valued adapted function on $[0,T]\times\Omega$, and b an adapted function on $[0,T]\times\Omega$. Let a one-dimensional $(\mathscr{F}_t)$-adapted continuous process $X=(X(t))$ on $(\Omega,\mathscr{F},P)$ satisfy the equation*

$$X(t) = x_0 + \sum_{j=1}^{r}\int_0^t \sigma_j(s,\omega)X(s)dB^j(s) + \int_0^t b(s,\omega)ds,$$

where x_0 is a constant. Assume that

$$\max_j \sup_{t,\omega} |\sigma_j(t,\omega)| < \infty,$$

and there exist constants M, K and a finite measure η on $[0,T]$ satisfying

$$|b(t,\omega)| \le M + K\left(\int_0^t |X(s)|d\eta(s) + |X(t)|\right), \quad (t,\omega) \in [0,T]\times\Omega,$$

Then there exists a constant C which depends on only T, x_0, M, K, and $\eta([0,T])$ such that

$$E[|X(t)|] \le C, \qquad t \in [0,T].$$

Proof. We choose $1 > a_1 > a_2 > \ldots > 0$ such that

$$\int_{a_1}^1 \frac{1}{u}du = 1\,, \int_{a_2}^{a_1} \frac{1}{u}du = 2\,, \ldots\,, \int_{a_m}^{a_{m-1}} \frac{1}{u}du = m\,, \ldots.$$

It is obvious that $a_m \to 0$ as $m \to \infty$. For $\{a_m\}$, we can define continuous functions ψ_m on $[0,\infty)$ such that

$$\operatorname{supp}\psi_m \subset (a_m, a_{m-1})\,, \quad 0 \le \psi_m(u) \le \frac{2}{mu}\,, \quad \int_{a_m}^{a_{m-1}} \psi_m(u)du = 1.$$

It is easily seen that the functions

$$\varphi_m(y) = \int_0^{|y|} du \int_0^z \psi_m(z)dz$$

defined on $\mathbf{R}$ have the properties

$$\varphi_m \in C^2(\mathbf{R})\,, \quad |\varphi_m'(y)| \le 1\,, \quad \varphi_m(y) \nearrow |y| \quad (m \longrightarrow \infty).$$

Itô's formula leads to

$$\begin{aligned}
&E[\varphi_m(X(t))] \\
&= \varphi_m(x_0) + \frac{1}{2}\sum_{j=1}^r \int_0^t E[\varphi_m''(X(s))(\sigma_j(s,\omega))^2(X(s))^2]ds + \int_0^t E[\varphi_m'(X(s))b(s,\omega)]ds.
\end{aligned}$$

The second term of the right hand side is estimated as follows;

$$\begin{aligned}
&\left|\frac{1}{2}\sum_{j=1}^r \int_0^t E[\varphi_m''(X(s))(\sigma_j(s,\omega))^2(X(s))^2]ds\right| \\
&\quad \le \frac{1}{2}\sum_{j=1}^r \int_0^t E[|\varphi_m''(X(s))|(\sigma_j(s,\omega))^2|X(s)|^2]ds \\
&\quad \le \frac{1}{m}\sum_{j=1}^r \int_0^t E[(\sigma_j(s,\omega))^2|X(s)|]ds \\
&\quad \longrightarrow 0 \qquad (m \longrightarrow \infty).
\end{aligned}$$

For the third term

$$
\begin{aligned}
\left|\int_0^t E[\varphi_m'(X(s))b(s,\omega)]ds\right| &\leq \int_0^t E[|b(s,\omega)|]ds \\
&\leq Mt + K\int_0^t (\int_0^s E[|X(u)|]d\eta(u) + E[|X(s)|])ds \\
&\leq Mt + K(\eta([0,T])+1)\int_0^t \sup_{0\leq u\leq s} E[|X(u)|]ds.
\end{aligned}
$$

Therefore, letting $m \longrightarrow \infty$, (3.2.1) yields

$$
\sup_{0\leq s\leq t} E[|X(s)|] \leq |x_0| + Mt + K(\eta([0,T])+1)\int_0^t \sup_{0\leq u\leq s} E[|X(u)|]ds.
$$

Applying Gronwall's lemma to this inequality, we have

$$
\sup_{0\leq s\leq t} E[|X(s)|] \leq K(|x_0| + MT)e^{K(\eta([0,T])+1)T}.
$$

□

Now we give some notation. We denote Sobolev space with respect to H-derivative with indices k and p by $W^{k,p}$, and the total set of smooth functions on $C([0,T];\mathbf{R}^d)$ by $C^\infty(C([0,T];\mathbf{R}^d))$, where the smoothness means that in the sense of Gâteau derivative. The precise definition of $C^\infty(C([0,T];\mathbf{R}^d))$ can be seen in [13]. We define $C_b^\infty(C([0,T];\mathbf{R}^d))$ by the total set of the elements of $C^\infty(C([0,T];\mathbf{R}^d))$ whose derivatives are bounded. We denote partial derivative with respect to spatial component by ∂_x. For real numbers a and b, we denote $\max\{a,b\}$ and $\min\{a,b\}$ by $a\vee b$ and $a\wedge b$, respectively. Let r be a positive integer and T be a positive number. For fixed r and T, let

$$
\begin{aligned}
W &:= \{w\in C([0,T];\mathbf{R}^r); w(0)=0\}, \\
H &:= \{h\in W;\quad h \text{ is absolute continuous and } \int_0^T \dot{h}^j(t)^2dt < \infty,\quad j=1,2,\ldots,r\},
\end{aligned}
$$

and let μ be the Wiener measure on W. Clearly (W,H,μ) is an abstract Wiener space.

The following lemma is a version of Lemma 3.2.1 about H-derivative of a stochastic differential equation.

Lemma 3.2.2 *Let $T>0$ be fixed. Let d and r be positive integers, (W,H,P) the r-dimensional Wiener space, $(B(t))$ the r-dimensional Brownian motion associated with (W,H,P), $\mathscr{B}(W)$ a Borel σ-field of W, $(\mathscr{F}_t)$ a reference family,*

$$
\begin{gathered}
\sigma = (\sigma_j^i)_{i=1,\ldots,d,j=1,\ldots,r} \in C_b([0,T]\times\mathbf{R};\mathbf{R}^d\otimes\mathbf{R}^r),\quad \sigma_j^i(t,\cdot)\in C^\infty(\mathbf{R}),\quad t\in[0,T], \\
b = (b^i)_{i=1,\ldots,d} \in C_b([0,T]\times C([0,T];\mathbf{R}^d);\mathbf{R}^d), \\
b^i(t,\cdot)\in C^\infty(C([0,T];\mathbf{R}^d)),\quad t\in[0,T].
\end{gathered}
$$

We assume that a d-dimensional $(\mathscr{F}_t)$-adapted continuous process $X = (X(t))$ on $(W, \mathscr{B}(W), P)$ satisfies the stochastic differential equation:

$$\begin{cases} dX^i(t) = \displaystyle\sum_{j=1}^{r} \sigma_j^i(t, X^i(t))dB^j(t) + b^i(t, X)dt, \quad i = 1, 2, \dots d, \\ X(0) = x_0 \in \mathbf{R}^d. \end{cases}$$

Moreover, we assume that there exist constants M, K and a finite measure η on $[0, T]$ satisfying that

$$\max_{i,j} |\sigma_j^i(t, x)| \leq M, \quad (t, x) \in [0, T] \times \mathbf{R},$$

$$\max_i |b^i(t, w) - b^i(t, w')| \leq K \left(\int_0^t |w(s) - w'(s)| d\eta(s) + |w(t) - w'(t)| \right),$$

$$t \in [0, T], \ w, w' \in C([0, T]; \mathbf{R}^d).$$

Then, for all t in $[0, T]$, $k = 1, 2, \dots$, and $p \geq 1$, $X(t)$ belongs to $W^{k,p}$, and there exists a constant C which depends on only M, K, and $\eta([0, T])$ such that

$$E[|D_h X^i(t)|] \leq C|h|_H, \qquad h \in H, \quad i = 1, 2, \dots, d.$$

Proof. By the Lipschitz continuity of the coefficients and theorem (2.19) of [13], X can be expressed as a functional on (W, H, μ) which is the Wiener space generated by the Brownian motion $(B(t))$, and we have $X(t) \in W^{k,p}$ for any positive integer k and $p \geq 1$. Hence it's sufficient to prove the existence of a constant C. Let $h \in H$ be fixed. Consider the H-derivative of the stochastic differential equation for X, then we have

$$\begin{aligned} D_h X^i(t) = \sum_{j=1}^{r} \int_0^t \partial_x \sigma_j^i(s, X^i(s)) D_h X^i(s) dB^j(s) \\ + \sum_{j=1}^{r} \int_0^t \dot{h}^j(s) \sigma_j^i(s, X(s)) ds + \int_0^t D_h b^i(s, X) ds, \\ i = 1, 2, \dots, d. \end{aligned}$$

From the condition of b, it follows that for almost all w

$$\begin{aligned} &|D_h b^i(t, X(\cdot, w))| \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |b^i(t, X(\cdot, w + \varepsilon h)) - b^i(t, X(w))| \\ &\leq \lim_{\varepsilon \to 0} \frac{K}{\varepsilon} (\int_0^t |X(s, w + \varepsilon h) - X(s, w)| d\eta(s) + |X(t, w + \varepsilon h) - X(t, w)|) \\ &= K(\int_0^t |D_h X(s, w)| d\eta(s) + |D_h X(t, w)|). \end{aligned}$$

Hence we can show the estimate by similar discussion to the proof of Lemma 3.2.1. $\square$

Now we will give a sufficient condition for solutions of stochastic differential equations to belong to the class V_h. The advantage of the following theorem is that we assume only bounded on the diffusion coefficient σ.

Theorem 3.2.3 *Let d and r be positive integers, $(B(t))$ an r-dimensional Brownian motion, and*

$$\sigma = (\sigma^i_j)_{i=1,\dots,d,j=1,\dots,r} \in C_b([0,T] \times \mathbf{R}; \mathbf{R}^d \otimes \mathbf{R}^r),$$
$$b = (b^i)_{i=1,\dots,d} \in C_b([0,T] \times C([0,T]; \mathbf{R}^d); \mathbf{R}^d).$$

Assume that there exist constants M, K and a Radon measure η on $[0,T]$ satisfying

$$\max_{i,j} |\sigma^i_j(t,x)| \le M, \quad (t,x) \in [0,T] \times \mathbf{R},$$
$$\max_i |b^i(t,w) - b^i(t,w')| \le K \left(\int_0^t |w(s) - w'(s)| d\eta(s) + |w(t) - w'(t)| \right),$$
$$t \in [0,T], \ w, w' \in C([0,T]; \mathbf{R}^d).$$

In addition, we assume the d-dimensional stochastic differential equation:

$$\begin{cases} dX^i(t) = \displaystyle\sum_{j=1}^r \sigma^i_j(t, X^i(t)) dB^j(t) + b^i(t,X)dt, \quad i = 1,2,\dots d, \\ X(0) = x_0 \in \mathbf{R}^d \end{cases}$$

has pathwise uniqueness.

Then the solution $(X(t))$ can be defined on the Wiener space (W,H,μ) and $X^i(t)$ is in $V_h(W)$ for all t in $[0,T]$, $i = 1,2,\dots,d$, and $h \in H$. Moreover, if we denote the version of $X^i(t)$ appeared in Definition 3.1.1 by $\widehat{X^i(t)}$, then

$$\left(\left| D_h \widehat{X^i(t)} \right| \mu \right) \circ X^i(t)^{-1}$$

is absolutely continuous with respect to the one-dimensional Lebesgue measure.

Proof. Pathwise uniqueness of the equation implies that the solution X can be expressed as a functional on the Wiener space (W,H,μ) generated by the Brownian motion $(B(t))$. By Lemma 5.2 of [8], there exist the sequences $\{\sigma_n\}$ and $\{b_n\}$

$$\{\sigma_n\} \subset C_b([0,T] \times \mathbf{R}; \mathbf{R}^d \otimes \mathbf{R}^r), \quad \{b_n\} \subset C_b([0,T] \times C([0,T]; \mathbf{R}); \mathbf{R}^d)$$

which satisfy that

$$\{\sigma_n(t,\cdot)\} \subset C_b^\infty(\mathbf{R}), \quad \lim_{n\to\infty} ||\sigma_n(t,\cdot) - \sigma(t,\cdot)||_{C_b(\mathbf{R};\mathbf{R}^d\times\mathbf{R}^r)} = 0$$
$$\max_{i,j} |(\sigma_n)^i_j(t,x)| \le M, \quad (t,x) \in [0,T]\times\mathbf{R},$$
$$\{b_n(t\cdot,\cdot)\} \subset C_b^\infty(C([0,T];\mathbf{R}^d)),$$
$$\lim_{n\to\infty} |b_n(t,w) - b(t,w)| = 0, \quad t\in[0,T], \quad w \in C([0,T];\mathbf{R}^d),$$
$$|b_n(t,w) - b_n(t,w')| \le K\left(\int_0^t |w(s) - w'(s)|d\eta(s) + |w(t) - w'(t)|\right),$$
$$t\in[0,T], \quad w,w' \in C([0,T];\mathbf{R}^d).$$

Let $\{X_n\}$ be the strong solutions of the stochastic differential equations mentioned above with coefficients σ and b replaced by σ_n and b_n, respectively. By [10], we have for all $t\in[0,T]$

$$X_n(t) \longrightarrow X(t) \quad \text{a.s..}$$

On the other hand, by a standard method of stochastic differential equations, we have for all $t\in[0,T]$

$$\sup_n E[|X_n(t)|^2] < \infty.$$

Therefore we can use Theorem 3.1.2 so that we have $X^i(t) \in V_h(W)$ for all $t\in[0,T]$ and $i = 1,2,\ldots,d$. The last assertion follows from Theorem 3.1.4. □

In the arguments above, we did not assume the ellipticity. In the case that the coefficients are Lipschitz continuous, it is known that some conditions about ellipticity of a multi-dimensional stochastic differential equation implies the positivity of $|\det(DX^i(t), DX^j(t))_H|$. In general when the coefficients are not Lipschitz continuous, these facts may not hold. However in a special case we can show the positivity of $|D_hX^i(t)|$ for a special h as follows.

Theorem 3.2.4 *Let r be a positive integer, and $(B(t))$ an r-dimensional Brownian motion. Assume that the one-dimensional stochastic differential equation:*

$$\begin{cases} dX(t) = \sum_{j=1}^r \sigma_j(t,X(t))dB^j(t) + b(t,X(t))dt \\ X(0) = x_0 \in \mathbf{R} \end{cases}$$

has pathwise uniqueness, when the coefficients

$$\sigma = (\sigma_j)_{j=1,\ldots,r} \in C_b([0,T]\times\mathbf{R};\mathbf{R}^r), \quad b\in C_b([0,T]\times\mathbf{R};\mathbf{R}),$$

are assumed to satisfy

$$\max_j |\sigma_j(t,x)| \le M, \quad (t,x) \in [0,T] \times \mathbf{R},$$
$$|b(t,x) - b(t,y)| \le K|x-y|, \quad x,y \in \mathbf{R}, \quad t \in [0,T]$$

with some constants M and K. In addition we assume that there is a closed subset S of $[0,T] \times \mathbf{R}$ satisfying that σ_j is in $C^{0,2}(([0,T] \times \mathbf{R}) \setminus S)$ for all $j = 1,2,\dots,r$, and $\sum_{j=1}^r \sigma_j$ is positive on $([0,T] \times \mathbf{R}) \setminus S$. We set $S^t := \{x; (t,x) \in S\}$.

Then

$$\mu \circ X(t)^{-1}\big|_{\mathbf{R} \setminus S^t}$$

is absolutely continuous to the one-dimensional Lebesgue measure restricted on $\mathbf{R} \setminus S^t$ for all t in $[0,T]$.

Proof. It is sufficient to prove the case $t = T$. Theorem 3.2.3 implies that the solution $(X(t))$ can be defined on the Wiener space (W,H,μ) and $X(T) \in V_h(W)$. For simplicity we also use $X(T)$ in the place of $\widehat{X}(t)$ in the definition of $V_h(W)$. Now we can choose $\sigma^{(n)} = (\sigma_j^{(n)})_{j=1,2,\dots,r} \in C_b([0,T] \times \mathbf{R}; \mathbf{R}^r)$ and $b^{(n)} \in C_b([0,T] \times \mathbf{R}; \mathbf{R})$ for $n = 1,2,3,\dots$ so that

$$\sigma_j^{(n)}(t,\cdot) \in C_b^\infty(\mathbf{R}; \mathbf{R}^r), \quad j = 1,2,\dots,r, \quad t \in [0,T],$$
$$\lim_{n\to\infty} \sup_{t\in[0,T],x\in\mathbf{R}} |\sigma_j^{(n)}(t,x) - \sigma_j(t,x)| = 0, \quad j = 1,2,\dots r,$$
$$b^{(n)}(t,\cdot) \in C_b^\infty(\mathbf{R}; \mathbf{R}),$$
$$\lim_{n\to\infty} \sup_{t\in[0,T],x\in\mathbf{R}} |b^{(n)}(t,x) - b(t,x)| = 0,$$
$$|b^{(n)}(t,x) - b^{(n)}(t,y)| \le K|x-y|, \quad x,y \in \mathbf{R}, \quad t \in [0,T],$$

and for any closed subinterval I of $\mathbf{R} \setminus S$

$$\lim_{n\to\infty} \sup_{t\in[0,T],x\in I} \left|\partial_x \sigma_j^{(n)}(t,x) - \partial_x \sigma_j(t,x)\right| = 0, \quad j = 1,2,\dots r,$$
$$\lim_{n\to\infty} \sup_{t\in[0,T],x\in I} \left|\partial_x^2 \sigma_j^{(n)}(t,x) - \partial_x^2 \sigma_j(t,x)\right| = 0, \quad j = 1,2,\dots r,$$
$$\lim_{n\to\infty} \sup_{t\in[0,T],x\in I} \left|\partial_x b^{(n)}(t,x) - \partial_x b(t,x)\right| = 0.$$

Let X_n be the solution of the following stochastic differential equation:

$$\begin{cases} dX_n(t) = \displaystyle\sum_{j=1}^r \sigma_j^{(n)}(t, X_n(t)) dB^j(t) + b^{(n)}(t, X_n(t)) dt \\ X_n(0) = x_0. \end{cases}$$

Then, from the result in [10] we have

$$\lim_{n\to\infty} E\left[\sup_{t\in[0,T]} |X_n(t)-X(t)|^2\right]=0.$$

Hence, we can choose a subsequence of $\{X_n\}$ which converges to X in the topology of $C([0,T])$ almost surely. For simplicity we denote the subsequence by $\{X_n\}$ again. Thus we have

$$\lim_{n\to\infty}\sup_{t\in[0,T]} |X_n(t)-X(t)|=0, \quad \text{a.s.} \tag{3.2.1}$$

Calculating the H-derivative of $X_n(T)$, we have

$$\begin{aligned} D_hX_n(T) &= \sum_{j=1}^r \int_0^T \partial_x\sigma_j^{(n)}(s,X_n(s))D_hX_n(t)dB^j(s) \\ &\quad +\sum_{j=1}^r \int_0^T \dot{h}^j(s)\sigma_j^{(n)}(s,X_n(s))ds+\int_0^T \partial_x b^{(n)}(s,X_n(s))D_hX_n(s)ds. \end{aligned}$$

For given X_n, this equation can be regarded as a stochastic differential equation of $DX_n(\cdot)[h]$ with linear coefficients. Thus by Problem 6.15 of Chapter 5 in [11] we have

$$\begin{aligned} D_hX_n(T) &= \int_0^T \left(\sum_{j=1}^r \dot{h}^j(s)\sigma_j^{(n)}(s,X_n(s))\right)\exp\left(\sum_{j=1}^r\int_s^T \partial_x\sigma_j^{(n)}(u,X_n(u))dB^j(u)\right. \\ &\quad \left. -\frac{1}{2}\int_s^T\sum_{j=1}^r[\partial_x\sigma_j^{(n)}(u,X_n(u))]^2du+\int_s^T \partial_x b^{(n)}(u,X_n(u))du\right)ds. \end{aligned}$$

Take

$$h^j(t):=t, \quad t\in[0,T], \quad j=1,2,\dots,r.$$

Then $h\in H$,

$$\begin{aligned} D_hX_n(T) &= \int_0^T \left(\sum_{j=1}^r \sigma_j^{(n)}(s,X_n(s))\right)\exp\left(\sum_{j=1}^r\int_s^T \partial_x\sigma_j^{(n)}(u,X_n(u))dB^j(u)\right. \\ &\quad \left. -\frac{1}{2}\int_s^T\sum_{j=1}^r[\partial_x\sigma_j^{(n)}(u,X_n(u))]^2du+\int_s^T \partial_x b^{(n)}(u,X_n(u))du\right)ds, \end{aligned} \tag{3.2.2}$$

and

$$D_hX_n(T)\geq 0, \quad n=1,2,3,\dots.$$

To obtain some information about the exponential part, we consider the time-reversal process of (X_n,B) by following the argument given in Section 4 of Chapter VII in [21].

Let Δ be the point of one point compactification of $\mathbf{R}^{r+2}$ and Z_n an $r+2$-dimensional Markov process defined by

$$Z_n(t) := \begin{pmatrix} X_n(t) \\ t \\ B^1(t) \\ \vdots \\ B^j(t) \end{pmatrix}, \quad \text{if} \quad 0 < t < T, \quad \text{and} \quad Z_n(t) := \Delta, \quad \text{if} \quad t \geq T.$$

We denote the starting point of Z_n by $\tilde{x}_0$. Clearly $Z_n(0) = \tilde{x}_0 = (x_0, 0, \ldots, 0)$. Let $\overline{W^{r+2}}$ be $C([0,\infty); \mathbf{R}^{r+2} \cup \Delta)$, $\mathscr{B}(\overline{W^{r+2}})$ a Borel σ-field of $\overline{W^{r+2}}$, and P_{Z_n} a probability measure on $(\overline{W^{r+2}}, \mathscr{B}(\overline{W^{r+2}}))$ the law of Z_n. We define ζ by

$$\zeta(w) := \inf\{t > 0; w^2(t) > T\}, \quad w \in \overline{W^{r+2}},$$

where w^2 means the second component of w. It is clear that ζ is the lifetime and $\zeta = T$ *a.s.* under P_{Z_n}. Moreover, ζ becomes co-optional time, because of the definition of ζ. Since Z_n is a Markov process, we can define a semi-group associated to P_{Z_n}. Let $\{T_t\}$ be the Feller semi-group on $C_\infty(\mathbf{R}^{r+2} \cup \Delta)$, where

$$C_\infty(\mathbf{R}^{r+2} \cup \Delta) := \{f \in C(\mathbf{R}^{r+2} \cup \Delta) \ ; \ \lim_{|x| \to \infty} f(x) = 0\}.$$

We define a measure ν on $\mathbf{R}^{r+2} \cup \Delta$ by

$$\int_{\mathbf{R}^{r+2}} f(x) \nu(dx) = \int_0^\infty T_s f(\tilde{x}_0) ds, \quad f \in C_\infty(\mathbf{R}^{r+2} \cup \Delta).$$

By the definition of ζ, this integration is well-defined. Then, it is easy to see that $\{T_t\}$ is a strong continuous and contractive semi-group on $L^2(\nu)$. Next we define $\widehat{T}_t$ by the dual operator of T_t on $L^2(\nu)$. Then we have the following lemma.

Lemma 3.2.5 *$\{\widehat{T}_t\}$ is a strong continuous and contractive semi-group on $L^2(\nu)$.*

Proof of Lemma 3.2.5. It is clear that $\{\widehat{T}_t\}$ is a semi-group on $L^2(\nu)$. Contractivity of $\{\widehat{T}_t\}$ on $L^2(\nu)$ follows from that of $\{T_t\}$. Thus it is sufficient to show the strong continuity on $L^2(\nu)$. Let f and $g \in C_\infty(\mathbf{R}^{r+2} \cup \Delta)$. Then

$$\int_{\mathbf{R}^{r+2} \cup \Delta} f(\widehat{T}_t g) d\nu = \int_{\mathbf{R}^{r+2} \cup \Delta} (T_t f) g d\nu \longrightarrow \int_{\mathbf{R}^{r+2} \cup \Delta} f g d\nu, \quad \text{as} \quad t \to 0. \tag{3.2.3}$$

For all $f \in C_\infty(\mathbf{R}^{r+2} \cup \Delta)$

$$\int_{\mathbf{R}^{r+2} \cup \Delta} (\widehat{T}_t f)^2 d\nu$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^{r+2}\cup\Delta} (T_t\widehat{T}_t f) f d\nu \\
&= E[\int_0^\infty (T_t\widehat{T}_t f)(Z_n(s)) f(Z_n(s)) ds] \\
&= E[\int_0^\infty E[\widehat{T}_t f(Z_n(t+s))|\mathscr{F}_s] f(Z_n(s)) ds] \\
&= E[\int_0^\infty (\widehat{T}_t f)(Z_n(t+s)) f(Z_n(s)) ds] \\
&= E[\int_0^\infty (\widehat{T}_t f)(Z_n(t+s)) f(Z_n(t+s)) ds] \\
&\qquad -E[\int_0^\infty (\widehat{T}_t f)(Z_n(t+s))(f(Z_n(t+s)) - f(Z_n(s))) ds]. \quad (3.2.4)
\end{aligned}
$$

By the contractivity of $\{\widehat{T}_t\}$ on $L^2(\nu)$ and (3.2.3), we have

$$
\begin{aligned}
&\left| E[\int_0^\infty (\widehat{T}_t f)(Z_n(t+s)) f(Z_n(t+s)) ds] - E[\int_0^\infty f(Z_n(t+s))^2 ds] \right| \\
&= \left| E[\int_0^\infty \{((\widehat{T}_t - I)f)(Z_n(t+s))\} f(Z_n(t+s)) ds] \right| \\
&\leq \left| E[\int_0^\infty \{((\widehat{T}_t - I)f)(Z_n(s))\} f(Z_n(s)) ds] \right| + \left| E[\int_0^t \{((\widehat{T}_t - I)f)(Z_n(s))\} f(Z_n(s)) ds] \right| \\
&\leq \left| \int_{\mathbf{R}^{r+2}\cup\Delta} ((\widehat{T}_t - I)f) f d\nu \right| + E[\int_0^t \{((\widehat{T}_t - I)f)(Z_n(s))\}^2 ds]^{\frac{1}{2}} E[\int_0^t f(Z_n(s))^2 ds]^{\frac{1}{2}} \\
&\leq \left| \int_{\mathbf{R}^{r+2}\cup\Delta} f((T_t - I)f) d\nu \right| + ||(\widehat{T}_t - I)f||_{L^2(\nu)} ||f||_\infty \sqrt{t} \\
&\longrightarrow 0, \quad \text{as} \quad t \to 0. \qquad (3.2.5)
\end{aligned}
$$

On the other hand, by the contractivity of $\{\widehat{T}_t\}$ on $L^2(\nu)$, we have

$$
\begin{aligned}
&\left| E[\int_0^\infty (\widehat{T}_t f)(Z_n(t+s))(f(Z_n(t+s)) - f(Z_n(s))) ds] \right| \\
&\leq E[\int_0^\infty \{(\widehat{T}_t f)(Z_n(t+s))\}^2 ds]^{\frac{1}{2}} E[\int_0^\infty \{f(Z_n(t+s)) - f(Z_n(s))\}^2 ds]^{\frac{1}{2}} \\
&\leq E[\int_0^\infty \{(\widehat{T}_t f)(Z_n(s))\}^2 ds]^{\frac{1}{2}} E[\int_0^T \{f(Z_n(t+s)) - f(Z_n(s))\}^2 ds]^{\frac{1}{2}} \\
&= ||\widehat{T}_t f||_{L^2(\nu)} E[\int_0^T \{f(Z_n(t+s)) - f(Z_n(s))\}^2 ds]^{\frac{1}{2}} \\
&\longrightarrow 0 \quad \text{as} \quad t \to 0. \qquad (3.2.6)
\end{aligned}
$$

Therefore, (3.2.4), (3.2.5), and (3.2.6) yield

$$
\lim_{t\to 0} \left| \int_{\mathbf{R}^{r+2}\cup\Delta} (\widehat{T}_t f)^2 d\nu - E[\int_0^\infty f(Z_n(t+s))^2 ds] \right| = 0.
$$

From

$$\left|E[\int_0^\infty f(Z_n(t+s))^2 ds] - E[\int_0^\infty f(Z_n(s))^2 ds]\right|$$
$$= \left|E[\int_0^t f(Z_n(s))^2 ds]\right| \le ||f||_\infty^2 t \longrightarrow 0, \quad \text{as} \quad t \to 0,$$

it follows that

$$\lim_{t\to 0}\left|\int_{\mathbf{R}^{r+2}\cup\Delta} (\widehat{T}_t f)^2 d\nu - \int_{\mathbf{R}^{r+2}\cup\Delta} f^2 d\nu\right| = 0.$$

Therefore, by this equation and (3.2.3)

$$\begin{aligned}
&\int_{\mathbf{R}^{r+2}\cup\Delta} |\widehat{T}_t f - f|^2 d\nu \\
&= \int_{\mathbf{R}^{r+2}\cup\Delta} (\widehat{T}_t f)^2 d\nu - 2\int_{\mathbf{R}^{r+2}\cup\Delta} (\widehat{T}_t f) f d\nu + \int_{\mathbf{R}^{r+2}\cup\Delta} f^2 d\nu \\
&= \int_{\mathbf{R}^{r+2}\cup\Delta} (\widehat{T}_t f)^2 d\nu - \int_{\mathbf{R}^{r+2}\cup\Delta} f^2 d\nu - 2\left(\int_{\mathbf{R}^{r+2}\cup\Delta} (\widehat{T}_t f) f d\nu - \int_{\mathbf{R}^{r+2}\cup\Delta} f^2 d\nu\right) \\
&\longrightarrow 0, \quad \text{as} \quad t \to 0.
\end{aligned}$$

□

Now we continue to prove Theorem 3.2.4. Lemma 3.2.5 implies that a Markov process associated with $\{\widehat{T}_t\}$. Let

$$\widehat{Z}_n(t) := \begin{cases} Z_n(\zeta-), & \text{if} \quad t = 0, \\ Z_n(\zeta - t), & \text{if} \quad 0 < t < \zeta, \\ \Delta, & \text{if} \quad t \ge \zeta. \end{cases}$$

Let $\widehat{\mathscr{F}_t^n}$ be $\sigma(\widehat{Z}_n(s); s \le t)$. Then, from Theorem (4.5) of Chapter VII in [21], it follows that the process $\widehat{Z}_n$ is a Markov process with respect to $(\widehat{\mathscr{F}_t^n})$ associated with transition semi-group $\{\widehat{T}_t\}$. On the other hand, $\zeta = T$. The processes $(\widehat{X}_n(t); t \in [0,T])$ and $(\widehat{B}(t); t \in [0,T])$ defined by

$$\begin{aligned}
\widehat{X}_n(t) &:= X_n(T-t), \quad t \in [0,T], \\
\widehat{B}(t) &:= B(T-t), \quad t \in [0,T],
\end{aligned}$$

are $(\widehat{\mathscr{F}_t^n})$-adapted processes. Moreover, we define $(\tilde{B}(t); t \in [0,T])$ by

$$\tilde{B}(t) := \widehat{B}(t) - \widehat{B}(0).$$

Since $(\widehat{B}(t); t \in [0,T])$ is a Gaussian process, so is $(\tilde{B}(t); t \in [0,T])$. By checking its mean and its covariance, it is easy to see that $(\tilde{B}(t); t \in [0,T])$ is an $(\widehat{\mathscr{F}}_t^n)$-Brownian motion. By Exercise(2.18) of Chapter IV in [21], we have

$$\begin{aligned}
&\sum_{j=1}^{r} \int_{T-t}^{T} \partial_x \sigma_j^{(n)}(s, X_n(s)) dB^j(s) \\
&= \sum_{j=1}^{r} \int_0^t \partial_x \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) d\widehat{B}^j(s) \\
&\qquad\qquad + \sum_{j=1}^{r} \int_0^t \partial_x^2 \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) ds \quad \text{a.s.} \\
&= \sum_{j=1}^{r} \int_0^t \partial_x \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) d\tilde{B}^j(s) \\
&\qquad\qquad + \sum_{j=1}^{r} \int_0^t \partial_x^2 \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) ds \quad \text{a.s.}
\end{aligned} \tag{3.2.7}$$

Note that all of stochastic integrals here are in the sense of Itô integral. Let m be any positive integer. Let

$$\begin{aligned}
\tau_n^m \;:=\; \inf \Bigg\{ t > 0; \max_{1 \le j \le r} \left\{ \left| \partial_x \sigma_j^{(n)}(T-t, \widehat{X}_n(t)) \right| \vee \left| \partial_x^2 \sigma_j^{(n)}(T-t, \widehat{X}_n(t)) \right| \right\} > m, \\
\sum_{j=1}^{r} \sigma_j^{(n)}(T-t, \widehat{X}_n(t)) < \frac{1}{m}, \text{ or } (T-t, \widehat{X}_n(t)) \in S \Bigg\} \wedge T
\end{aligned}$$

for each $m = 1, 2, 3, \ldots$. Then τ_n^m is an $(\widehat{\mathscr{F}}_t^n)$-stopping time for every $m = 1, 2, 3, \ldots$. Hence,

$$\begin{aligned}
E[\sup_{t \in [0,T]} | \int_0^{\tau_n^m \wedge t} \partial_x \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) d\tilde{B}^j(s)|^2] \le m^2 T, \qquad & j = 1, 2, \ldots, r, \\
\sup_{t \in [0,T]} | \int_0^{\tau_n^m \wedge t} \partial_x^2 \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) ds| \le MmT, \qquad & j = 1, 2, \ldots, r.
\end{aligned}$$

From these one can derive

$$\begin{aligned}
E[\liminf_{n \to \infty} \sup_{t \in [0,T]} | \int_0^{\tau_n^m \wedge t} \partial_x \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) d\tilde{B}^j(s)|^2] \le m^2 T, \qquad & j = 1, 2, \ldots, r, \\
\liminf_{n \to \infty} \sup_{t \in [0,T]} | \int_0^{\tau_n^m \wedge t} \partial_x^2 \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) \sigma_j^{(n)}(T-s, \widehat{X}_n(s)) ds| \le MmT, \quad & j = 1, 2, \ldots, r.
\end{aligned}$$

Therefore we have

$$\liminf_{n\to\infty}\sup_{t\in[0,T]}\Big|\int_0^{\tau_n^m\wedge t}\partial_x\sigma_j^{(n)}(T-s,\widehat{X}_n(s))d\tilde{B}^j(s)\Big|<\infty\quad\text{a.s.},\qquad j=1,2,\ldots,r,$$

$$\liminf_{n\to\infty}\sup_{t\in[0,T]}\Big|\int_0^{\tau_n^m\wedge t}\partial_x^2\sigma_j^{(n)}(T-s,\widehat{X}_n(s))\sigma_j^{(n)}(T-s,\widehat{X}_n(s))ds\Big|<\infty,\qquad j=1,2,\ldots,r.$$

Let

$$\begin{aligned}\tau^m \ :=\ \inf\Big\{t>0;\ &\max_{1\le j\le r}\left\{|\partial_x\sigma_j(T-t,X(T-t))|\vee\left|\partial_x^2\sigma_j(T-t,X(T-t))\right|\right\}>m,\\ &\sum_{j=1}^r\sigma_j(T-t,X(T-t))<\frac{1}{m},\ \text{or}\ (T-t,X(T-t))\in S\Big\}\wedge T\end{aligned}$$

for $m=1,2,3,\ldots$. Since $\sum_{j=1}^r\sigma_j(T-t,X(T-t))>\frac{1}{2m}$ for all t in a neighborhood of τ^m, oscillation occurs in the neighborhood of τ^m. Because of this fact, (3.2.1), and the definition of τ_n^m, it follows that

$$\lim_{n\to\infty}\tau_n^m=\tau^m\quad\text{a.s.}$$

Hence, by (3.2.1) again, there exists a subsequence $\{n(k)\}$ of $\mathbf{N}$ such that

$$\begin{aligned}&\lim_{k\to\infty}\sup_{t\in[0,T]}\Big|\int_0^{\tau_{n(k)}^m\wedge t}\partial_x\sigma_j^{(n(k))}(T-s,\widehat{X}_{n(k)}(s))d\tilde{B}^j(s)\Big|<\infty\quad\text{a.s.},\\ &\qquad\qquad j=1,2,\ldots,r,\\ &\lim_{k\to\infty}\sup_{t\in[0,T]}\Big|\int_0^{\tau_{n(k)}^m\wedge t}\partial_x^2\sigma_j^{(n(k))}(T-s,\widehat{X}_{n(k)}(s))\sigma_j^{(n(k))}(T-s,\widehat{X}_{n(k)}(s))ds\Big|<\infty\quad\text{a.s.},\\ &\qquad\qquad j=1,2,\ldots,r.\end{aligned}$$

If we set

$$Y_{n(k)}(t,T):=\sum_{j=1}^r\int_{T-t}^T\partial_x\sigma_j^{(n(k))}(s,X_{n(k)}(s))dB^j(s),$$

then by (3.2.7) there is a random variable C such that for almost all w

$$\sup_{t\in[\tau_{n(k)}^m(w),T]}|Y_{n(k)}(t,T)(w)|<C(w),\quad k=1,2,3,\ldots.$$

On the other hand, by the definition of $\tau_n^m(w)$, for almost all w

$$\sup_{t\in[\tau_{n(k)}^m(w),T]}\int_t^T\sum_{j=1}^r[\partial_x\sigma_j^{(n(k))}(u,X_{n(k)}(u,w))]^2du\le rm^2T,\quad k=1,2,3,\ldots.$$

This and (3.2.2) yield that for almost all w

$$\begin{aligned}
&D_h X_{n(k)}(T,w) \\
&\geq \int_{\tau^m_{n(k)}(w)}^T \sum_{j=1}^r \sigma_j^{(n(k))}(s, X_{n(k)}(s,w)) \exp\left(Y_{n(k)}(s,T) \right. \\
&\left. -\frac{1}{2}\int_s^T \sum_{j=1}^r [\partial_x \sigma_j^{(n(k))}(u, X_{n(k)}(u,w))]^2 du + \int_s^T \partial_x b^{(n(k))}(u, X_{n(k)}(u,w))du \right) ds \\
&\geq \exp\left(-C(w) - \frac{1}{2} r m^2 T - KT\right) \sum_{j=1}^r \int_{\tau^m_n(w)}^T \sigma_j^{(n)}(s, X_n(s,w))ds \\
&\geq \frac{1}{2m}(T - \tau^m_n(w)) \exp\left(-C(w) - \frac{1}{2} r m^2 T - KT\right).
\end{aligned}$$

Hence for almost all w

$$\liminf_{k\to\infty} D_h X_{n(k)}(T,w) \geq \frac{1}{2m} \exp\left(-C(w) - \frac{1}{2} r m^2 T - KT\right) \liminf_{k\to\infty}(T - \tau^m_n(w)).$$

If w satisfies $X(T,w) \in S$ and $\liminf_{k\to\infty} \tau^m_{n(k)}(w) < T$, then

$$\liminf_{k\to\infty} D_h X_{n(k)}(T,w) > 0.$$

On the other hand, for almost all w with respect to μ

$$\begin{aligned}
D_h X(T,w) &= \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}(X(T, w+\varepsilon h) - X(T,w)) \\
&= \liminf_{\varepsilon\to 0} \liminf_{k\to\infty} \frac{1}{\varepsilon}(X_{n(k)}(T, w+\varepsilon h) - X_{n(k)}(T,w)) \\
&= \liminf_{\varepsilon\to 0} \liminf_{k\to\infty} \frac{1}{\varepsilon} \int_0^\varepsilon D_h X_{n(k)}(T, w+uh)du \\
&\geq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \liminf_{k\to\infty} D_h X_{n(k)}(T, w+uh)du \\
&= \overline{\liminf_{k\to\infty} D_h X_{n(k)}(T,w)}^h,
\end{aligned}$$

where $\overline{\liminf_{k\to\infty} D_h X_{n(k)}(T,w)}^h$ means a right-continuous version of $\liminf_{k\to\infty} D_h X_{n(k)}(T,w)$ for direction h. Since any non-degenerate Gaussian measure is absolute continuous to the Lebesgue measure, we have

$$D_h X(T) \geq \liminf_{k\to\infty} D_h X_{n(k)}(T) \quad \text{a.s.}$$

From this it follows that there exists a null set $N_1(m)$ such that if w satisfies $X(T,w) \in S$, $\liminf_{k\to\infty} \tau^m_{n(k)}(w) < T$ and $w \notin N_1(m)$, and hence

$$D_h X(T,w) > 0.$$

Now we define S_m^T by

$$S_m^T := \left\{ x \in \mathbf{R};\, \max_{1\le j\le r} \left[|\partial_x \sigma_j(T,x)| \vee \left|\partial_x^2 \sigma_j(T,x)\right| \right] > m \text{ and } \sum_{j=1}^r \sigma_j(T,x) > \frac{1}{m} \right\}.$$

Then, by the definition of $\tau_{n(k)}^m$ and (3.2.1), there exists a null set $N_2(m)$ such that if $X(T,w) \in S_m^T$ and $w \notin N_2(m)$, $\liminf_{k\to\infty} \tau_{n(k)}^m(w) < T$. Thus, if w satisfies that $X(T,w) \in S_m^T$ and $w \notin N_2(m)$, then

$$D_h X(T,w) > 0.$$

This implies that

$$D_h X(T,w) > 0, \quad w \in (X(T))^{-1}(S^T) \cap \left(\bigcup_{m\in\mathbf{N}} N_2(m) \right)^c.$$

By Theorem 3.2.3 we have the conclusion of Theorem 3.2.4. $\square$

Example 3.2.6 Consider a one-dimensional stochastic differential equation:

$$\begin{cases} dX(t) = \sqrt{X(t)}dB(t) + b(t,X(t))dt \\ X(0) = x_0 \in [0,\infty), \end{cases}$$

where $(B(t))$ be a one-dimensional Brownian motion,

$$b \in C_b([0,T]\times[0,\infty);\mathbf{R}), \quad b(t,0) > 0, \quad t \in [0,T],$$

and there exists constants K satisfying that

$$|b(t,x) - b(t,y)| \le K|x-y|, \quad x,y \in [0,\infty) \text{ and } t \in [0,T].$$

Let $(X(t))$ be the solution of the stochastic differential equation. Then, the distribution of $X(t)$ has its density function for all t in $[0,T]$.

In fact, the condition of the coefficients implies that there exists a solution $(X(t))$ with state space $[0,\infty)$. Moreover according to the result in [28] tells that the stochastic differential equation has pathwise uniqueness. Theorem 3.2.4 is applicable to it with $S = [0,T]\times\{0\}$, so that

$$\mu \circ X(t)^{-1}\big|_{(0,\infty)}$$

is absolutely continuous to the one-dimensional Lebesgue measure restricted on $(0,\infty)$ for all t in $[0,T]$. Because of the condition of coefficients, it can be seen that $\mu\circ X(t)^{-1}(\{0\}) = 0$. Therefore the distribution of $X(t)$ has its density function for all t in $[0,T]$.

Chapter 4

Malliavin calculus for stochastic differential equations driven by subordinated Brownian motions

In this chapter we discuss Malliavin calculus for stochastic differential equations driven by subordinated Brownian motions. We prepare the techniques for calculating the integrals with deterministic time change in Section 4.1. These techniques enable us to apply to the standard stochastic calculus to our case. In Section 4.2, we discuss Malliavin calculus for stochastic differential equations with deterministic time change. In Section 4.3, we discuss the inheritance of regularity of densities from those of conditional probabilities. That is the reason why we consider stochastic differential equations with deterministic time change. In Section 4.4, we derive the general results from Section 4.2 and Section 4.3. In Section 4.5 we discuss the most interesting example: stochastic differential equations driven by rotation-invariant stable processes.

Throughout this chapter, we use $\{C_j; j = 0, 1, 2, \ldots\}$ as positive constants and the dependent parameters are written such as $C_0(p)$.

4.1 Malliavin calculus for functionals of Brownian motions with deterministic time change

For a fixed positive number T, let ϕ be a right-continuous and increasing function on $[0, T]$ with $\phi(0) = 0$, where "increasing" means $\phi(t_1) < \phi(t_2)$ for $t_1 < t_2$ through this chapter.

We define the inverse function ϕ^{-1} by

$$\phi^{-1}(s) := \begin{cases} \inf\{t; \phi(t) > s\}, & \text{if} \quad s \in [0, \phi(T)), \\ T, & \text{if} \quad s = \phi(T). \end{cases}$$

Set

$$\begin{aligned} W &:= \{w\,;\, w \text{ is } \mathbf{R}^d\text{-valued continuous function on } [0, \phi(T)], w(0) = 0\}, \\ H &:= \{h \in C([0, \phi(T)]; \mathbf{R}^d);\ h \text{ is absolutely continuous and } \dot{h} \in L^2([0, \phi(T)]; \mathbf{R}^d)\}, \end{aligned}$$

and let μ be the Wiener measure on W. The triplet (W, H, μ) is an abstract Wiener space. Hence we can apply Malliavin calculus to the functionals on (W, H, μ). Let $(B(t))$ be the canonical d-dimensional Brownian motion associated with (W, H, μ), $\mathscr{F}_t$ the σ-field generated by $(B(s); 0 \le s \le \phi(t))$, D the H-derivative operator, and D_h the differential in direction h for each $h \in H$. Then for all $h \in H$ we have

$$\begin{aligned} D_h B(\phi(t)) &= h(\phi(t)), \quad t \in [0, T], \\ D_h \int_0^T f(t) dB(\phi(t)) &= \int_0^T f(t) dh(\phi(t)), \quad f \in C([0, T]). \end{aligned}$$

Here the integral of the left-hand side is in the sense of stochastic integrals by $(\mathscr{F}_t)$-martingales, and that of the right-hand side is in the sense of Stieltjes integrals. More generally, we have an analogue of Proposition 6.1 in [24]. We need some lemmas and some notation before we state the analogue.

Lemma 4.1.1 *Let f be a right-continuous function with left limits. Then we have*

$$\int_0^T f(t-) d\phi(t) = \int_0^{\phi(T)} f(\phi^{-1}(s)-) ds.$$

Proof. Since ϕ is a function of bounded variation, the contribution of the small jumps for the integrals are sufficiently small. We assume that the number N of the jumps of ϕ is finite. Let $\{\xi_i; i = 1, 2, \ldots, N-1\}$ be the discontinuous points of ϕ, $\xi_0 := 0$, $\xi_N := T$, and $\{t_{i,j}; j = 0, 1, \ldots, N_i\}$ a partition of $[\xi_{i-1}, \xi_i]$ for $i = 1, 2, \ldots, N$. We denote $\max_{i,j}(t_{i,j} - t_{i,j-1})$ by Δ. Then,

$$\begin{aligned} &\int_0^T f(t-) d\phi(t) \\ &= \lim_{\Delta \to 0} \sum_{i=1}^{N} \Bigg[\sum_{j=1}^{N_i - 1} f(t_{i,j-1}-)\{\phi(t_{i,j}) - \phi(t_{i,j-1})\} + f(t_{i,N_i-1}-)\{\phi(t_{i,N_i}-) - \phi(t_{i,N_i-1})\} \\ &\qquad\qquad + f(t_{i,N_i}-)\{\phi(t_{i,N_i}) - \phi(t_{i,N_i}-)\} \Bigg]. \end{aligned}$$

If we set $s_{i,j} := \phi(t_{i,j})$, then $\phi^{-1}(s_{i,j}) = t_{i,j}$ for $j = 0, 1, \ldots, N_i - 1$. Therefore

$$\begin{aligned}
&\int_0^T f(t-)d\phi(t) \\
&= \lim_{\Delta\to 0}\sum_{i=1}^{N}\left[\sum_{j=1}^{N_i-1} f(\phi^{-1}(s_{i,j-1})-)\{s_{i,j}-s_{i,j-1}\} + f(\phi^{-1}(s_{i,N_i-1})-)\{\phi(t_{i,N_i}-)-s_{i,N_i-1}\}\right. \\
&\qquad\qquad\qquad\left. + f(\phi^{-1}(s_{i,N_i})-)\{\phi(t_{i,N_i})-\phi(t_{i,N_i}-)\}\right] \\
&= \sum_{i=1}^{N}\int_{\phi(\xi_{i-1})}^{\phi(\xi_i-)} f(\phi^{-1}(s)-)ds + \sum_{i=1}^{N} f(\phi^{-1}(s_{i,N_i})-)\{\phi(\xi_i)-\phi(\xi_i-)\}.
\end{aligned}$$

Since $\phi^{-1}(s)$ is a constant on $[\phi(\xi_i-), \phi(\xi_i)]$,

$$f(\phi^{-1}(s_{i,N_i})-)\{\phi(\xi_i)-\phi(\xi_i-)\} = \int_{\phi(\xi_i-)}^{\phi(\xi_i)} f(\phi^{-1}(s)-)ds, \quad i = 1, 2, \ldots, N.$$

Thus we have

$$\int_0^T f(t-)d\phi(t) = \int_0^{\phi(T)} f(\phi^{-1}(s)-)ds.$$

□

Similarly we have the following lemma.

Lemma 4.1.2 *Let Ψ be an $(\mathscr{F}_t)$-adapted right-continuous process with left limits satisfying that*

$$E\left[\int_0^T |\Psi(s-)|^2 d\phi(s)\right] < \infty.$$

Then we have

$$\int_0^T \Psi(t-)dB(\phi(t)) = \int_0^{\phi(T)} \Psi(\phi^{-1}(s)-)dB(s) \quad a.s.$$

Here the integral of the left-hand side is in the sense of stochastic integrals by $(\mathscr{F}_t)$-martingales, and that of the right-hand side is in the sense of stochastic integrals by $(\mathscr{F}_t^B)$-martingales, where $(\mathscr{F}_t^B)$ is the σ-field generated by $(B_s; 0 \le s \le t)$.

Let $A(t)$ be $[B(\phi(\cdot)), B(\phi(\cdot))](t)$ where the definition of $[\cdot,\cdot]$ is in Section 6 of Chapter II in [20]. We show the following lemma which is a version of Burkholder's inequality (c.f. Theorem 92 of Chapter VII of [5]).

Lemma 4.1.3 *Let p be a positive number, and Ψ an $(\mathscr{F}_t)$-adapted right-continuous process with left limits satisfying*

$$E\left[\int_0^T |\Psi(s-)|^2 d\phi(s)\right] < \infty.$$

Then, we have

$$\begin{aligned}\left[\left(\int_0^T \Psi(s-)^2 dA(s)\right)^{\frac{p}{2}}\right] &\leq C_0(p)E\left[\sup_{0\leq t\leq T}\left|\int_0^t \Psi(s-)dB(\phi(s))\right|^p\right] \\ &\leq C_1(p)E\left[\left(\int_0^T \Psi(s-)^2 d\phi(s)\right)^{\frac{p}{2}}\right].\end{aligned}$$

Proof. Theorem 92 of Chapter VII of [5] implies the first estimate. Hence we prove the second estimate. Let

$$M(t) := \int_0^t \Psi(\phi^{-1}(s)-)dB(s).$$

Then, M is a continuous martingale. From Burkholder's inequality, Lemma 4.1.1, and Lemma 4.1.2 it holds with a constant $C(p)$ that

$$\begin{aligned}E\left[\sup_{0\leq t\leq T}\left|\int_0^t \Psi(s-)dB(\phi(s))\right|^p\right] &= E\left[\sup_{0\leq t\leq T}|M(\phi(t))|^p\right] \\ &\leq E\left[\sup_{0\leq t\leq \phi(T)}|M(t)|^p\right] \\ &\leq C(p)E\left[\langle M(t)\rangle(\phi(T))^{\frac{p}{2}}\right] \\ &= C(p)E\left[\left(\int_0^{\phi(T)} \Psi(\phi^{-1}(s)-)^2 ds\right)^{\frac{p}{2}}\right] \\ &= C(p)E\left[\left(\int_0^T \Psi(s-)^2 d\phi(s)\right)^{\frac{p}{2}}\right].\end{aligned}$$

□

The following lemma is a version of Gronwall's inequality.

Lemma 4.1.4 *Let α, β be positive constants, f a right-continuous positive function on $[0,T]$ with left limits. If*

$$f(t) \leq \alpha + \beta\int_0^t f(u-)d\phi(u), \quad t\in[0,T],$$

then

$$f(t) \leq \alpha e^{\beta\phi(t)}, \quad t\in[0,T].$$

Proof. Since $\phi^{-1}(\phi(t)) = t$, it follows from Lemma 4.1.1 that

$$f(\phi^{-1}(\phi(t))) \leq \alpha + \beta \int_0^{\phi(t)} f(\phi^{-1}(u))du, \quad t \in [0,T],$$

which implies that

$$f(\phi^{-1}(\phi(t-))) \leq \alpha + \beta \int_0^{\phi(t-)} f(\phi^{-1}(u))du, \quad t \in [0,T].$$

Since $\phi^{-1}(s) = \phi^{-1}(\phi(t-))$ for $s \in [\phi(t-), \phi(t)]$, we have

$$f(\phi^{-1}(s)) \leq \alpha + \beta \int_0^{s} f(\phi^{-1}(u))du, \quad s \in [0,\phi(T)].$$

Applying Gronwall's inequality to $f \circ \phi^{-1}$, we have

$$f(\phi^{-1}(s)) \leq \alpha e^{\beta s}, \quad s \in [0,\phi(T)],$$

and hence have the assertion of Lemma 4.1.4 by letting $s = \phi(t)$ and the equality $\phi^{-1}(\phi(t)) = t$. $\square$

We prepare some notation. Let $p > 1$, n a positive integer, K a Hilbert space, $W^{n,p}(K)$ the Sobolev space of K-valued functions associated with H-derivative with indices n and p, and $\mathscr{L}_2^n(H;K)$ the total set of K-valued n-linear operators of Hilbert-Schmidt class on $\overbrace{H \times \ldots \times H}^{n}$. Now let us introduce two classes of stochastic processes. We define $\mathcal{L}^{n,p}(dB(\phi);K)$ by the total set of $(\mathscr{F}_t)$-predictable $\mathbf{R}^d \otimes K$-valued functions α satisfying that $\alpha(t) \in W^{n,p}(\mathbf{R}^d \otimes K)$ for all $t \in [0,T]$ and

$$||\alpha||_{\mathcal{L}^{n,p}(dB(\phi);K)} := E\left[\sum_{k=0}^{n} \left\{ \int_0^T |D^k \alpha(t-)|^2_{\mathscr{L}_2^k(H;\mathbf{R}^d \otimes K)} d\phi(t) \right\}^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty.$$

Next we define $\mathcal{L}^{n,p}(d\phi;K)$ be the total set of $(\mathscr{F}_t)$-predictable K-valued functions β satisfying that $\beta(t) \in W^{n,p}(K)$ for all $t \in [0,T]$ and

$$||\beta||_{\mathcal{L}^{n,p}(d\phi;K)} := \sum_{k=0}^{n} \int_0^T E\left[|D^k \beta(t-)|^p_{\mathscr{L}_2^k(H;K)} \right]^{\frac{1}{p}} d\phi(t) < \infty.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathcal{L}^{n,p}(dB(\phi);K)$, $\beta \in \mathcal{L}^{n,p}(d\phi;K)$, and $\gamma = (\gamma(t); 0 \leq t \leq T)$ an $(\mathscr{F}_t)$-adapted K-valued process. We assume that $\gamma(t) \in W^{n,p}(K)$ for all $t \in [0,T]$, and $D^k\gamma$ is an $(\mathscr{F}_t)$-adapted $\mathscr{L}_2^k(H;K)$-valued function such that

$$\sum_{k=0}^{n} E\left[\sup_{0 \leq t \leq T} |D^k \gamma(t)|^p_{\mathscr{L}_2^k(H;K)} \right] < \infty.$$

Define Φ by

$$\Phi(t) := \int_0^t \alpha(s-)dB(\phi(s)) + \int_0^t \beta(s-)d\phi(s) + \gamma(t).$$

Then, the following proposition holds.

Proposition 4.1.5 *$\Phi(t) \in W^{n,p}(K)$ for all $t \in [0,T]$, $D^k\Phi$ are $(\mathscr{F}_t)$-adapted $\mathscr{L}_2^k(H;K)$-valued processes for $k = 0, 1, \ldots n$, and there exists a constant C such that*

$$\begin{aligned} & E[\sup_{0\le t\le T} |D^k\Phi(t)|^p_{\mathcal{L}_2^k(H;K)}]^{\frac{1}{p}} \\ & \le C\left(||\alpha||_{\mathcal{L}^{n,p}(dB(\phi);K)} + ||\beta||_{\mathcal{L}^{n,p}(d\phi;K)} + \sum_{k=0}^n E\left[\sup_{0\le t\le T} |D^k\gamma(t)|^p_{\mathscr{L}_2^k(H;K)}\right]\right). \end{aligned}$$

Furthermore, $D\Phi(t)$ is given by

$$\begin{aligned} D\Phi(t)[h] = \int_0^t D\alpha(s-)[h]dB(\phi(s)) + \int_0^t \alpha(s-)dh(\phi(s)) + \int_0^t D\beta(s-)[h]d\phi(s) \\ +D\gamma(t)[h], \quad h \in H. \end{aligned}$$

Here the equality is in the sense of elements of $L^p(H \otimes K)$. Therefore, if we denote one of the complete orthonormal systems of H by $\{h^\lambda\}$, then

$$\begin{aligned} D\Phi(t) &= \int_0^t D\alpha(s-)dB(\phi(s)) \\ &\quad + \sum_\lambda h^\lambda \otimes \int_0^{\phi(t)} \alpha(\phi^{-1}(s)-)\dot{h}^\lambda(s)ds + \int_0^t D\beta(s-)d\phi(s) + D\gamma(t). \end{aligned}$$

Proof. To prove the first assertion, we use induction on n. For $n = 0$, by Lemma 4.1.3, we have

$$\begin{aligned} E\left[\sup_{0\le t\le T}\left|\int_0^t \alpha(s-)dB(\phi(s))\right|_K^p\right] &\le C_1(p)E\left[\left\{\int_0^T |\alpha(s-)|^2_{\mathbf{R}^d\otimes K}d\phi(s)\right\}^{\frac{p}{2}}\right] \\ &= ||\alpha||^p_{\mathcal{L}^{0,p}(dB(\phi);K)}. \end{aligned}$$

The other terms are estimated easily. Thus we have the first assertion for $n = 0$.

Assuming the result for $n - 1$, we will show the estimate for n. We check only the estimate of the stochastic integral, since that of the integral with respect to $d\phi$ follows similarly and clearly the part of γ follows. To simplify the notation, let $d = 1$. We will show it in the case that α is a step function such as

$$\alpha(t) = \alpha(t_j), \quad \text{for} \quad t \in [t_j, t_j + 1),$$

where $0 = t_0 < t_1 < \ldots < t_N = T$. For the general case it is obtained by taking the limit. Note that α is left-continuous.

In this case, the stochastic integral is expressed as

$$\int_0^T \alpha(t)dB(\phi(t)) = \sum_{j=0}^{N-1} \alpha(t_j)\left\{B(\phi(t_{j+1})) - B(\phi(t_j))\right\}.$$

It follows that for $h \in H$

$$\begin{aligned}
&D_h \int_0^T \alpha(t)dB(\phi(t)) \\
&= \sum_{j=0}^{N-1} D_h\alpha(t_j)\left\{B(\phi(t_{j+1})) - B(\phi(t_j))\right\} + \sum_{j=0}^{N-1} \alpha(t_j)\left\{h(\phi(t_{j+1})) - h(\phi(t_j))\right\} \\
&= \int_0^T D_h\alpha(t)dB(\phi(t)) + \int_0^T \alpha(t)dh(\phi(t)).
\end{aligned}$$

Let $I_\alpha[h] := \int_0^T \alpha(t)dh(\phi(t))$. Now we show that $I_\alpha \in W^{n,p}(H \otimes K)$. Similarly to the proof of Lemma 4.1.1 we have

$$I_\alpha[h] = \int_0^{\phi(T)} \alpha(\phi^{-1}(s))dh(s),$$

and hence

$$I_\alpha = \sum_\lambda h_\lambda \otimes \int_0^{\phi(T)} \alpha(\phi^{-1}(s))\dot{h}_\lambda(s)ds.$$

From this it follows that

$$D^k I_\alpha = \sum_\lambda h_\lambda \otimes \int_0^{\phi(T)} D^k\alpha(\phi^{-1}(s))\dot{h}_\lambda(s)ds,$$

and

$$\begin{aligned}
\left|D^k I_\alpha\right|^2_{\mathcal{L}_2^k(H;H\otimes K)} &= \sum_\lambda \left|\int_0^{\phi(T)} D^k\alpha(\phi^{-1}(s))\dot{h}_\lambda(s)ds\right|^2_{\mathscr{L}_2^k(H;K)} \\
&= \int_0^{\phi(T)} |D^k\alpha(\phi^{-1}(s))|^2_{\mathscr{L}_2^k(H;K)}ds \\
&= \int_0^T |D^k\alpha(t)|^2_{\mathscr{L}_2^k(H;K)}d\phi(t)
\end{aligned}$$

by virtue of Lemma 4.1.1. Therefore we have that

$$I_\alpha \in W^{n,p}(H \otimes K),$$
$$||D^k I_\alpha||_p \leq ||\alpha||_{\mathcal{L}^{k,p}(dB(\phi);K)},$$

and

$$D\left(\int_0^T \alpha(t)dB(\phi(t))\right)[h] = \int_0^T D\alpha(t)[h]dB(\phi(t)) + \int_0^T \alpha(t)dh(\phi(t)) \qquad (4.1.1)$$

in the sense of elements of $L^p(H\otimes K)$. It is easy to see that the equation (4.1.1) is also hold with T replaced by $t\in[0,T]$. Hence we have

$$D\Phi(t)[h] = \int_0^t D\alpha(s-)[h]dB(\phi(s)) + \int_0^t \alpha(s-)dh(\phi(s)) + \int_0^t D\beta(s-)[h]d\phi(s) + D\gamma(t)[h], \quad h\in H,$$

in the sense of elements of $L^p(H\otimes K)$, and the second assertion is obtained. Now we note that $D\Phi$ satisfies the assumption of $n-1$. Indeed, the third term satisfies the assumption of γ for $n-1$. Therefore, by the assumption of induction, for $k=1,2,\dots,n-1$, we have

$$\begin{aligned}
&E\left[\sup_{0\le t\le T}|D^k D\Phi(t)|_{\mathscr{L}_2^k(H;H\otimes K)}\right]^{\frac{1}{p}}\\
&\le C_2(p)\Big\{||D\alpha||_{\mathcal{L}^{n-1,p}(dB(\phi);H\otimes K)} + ||D\beta||_{\mathcal{L}^{n-1,p}(dB(\phi);H\otimes K)}\\
&\qquad\qquad + E\left[\sup_{0\le t\le T}\left|D^k I_\alpha(t)\right|^p_{\mathscr{L}_2^k(H;H\otimes K)}\right]^{\frac{1}{p}} + E\left[\sup_{0\le t\le T}\left|D^k D\gamma(t)\right|^p_{\mathscr{L}_2^k(H;H\otimes K)}\right]^{\frac{1}{p}}\Bigg\}\\
&\le 2C_2(p)\left\{||\alpha||_{\mathcal{L}^{n,p}(dB(\phi);K)} + ||\beta||_{\mathcal{L}^{n,p}(dB(\phi);K)} + E\left[\sup_{0\le t\le T}\left|D^{k+1}\gamma(t)\right|^p_{\mathscr{L}_2^{k+1}(H;K)}\right]^{\frac{1}{p}}\right\}.
\end{aligned}$$

Thus we have the conclusion for n. □

4.2 Malliavin calculus for stochastic differential equations with deterministic time change

We fix $T>0$. Let r be a positive integer, $d_1,d_2,\dots,d_r$ positive integers, $\phi_1,\phi_2,\dots,\phi_r$ right-continuous increasing functions on $[0,T]$ starting at 0. Set

$$\begin{aligned}
W_k &:= \{w\,;\, w \text{ is } \mathbf{R}^{d_k}\text{-valued continuous function on } [0,\phi_k(T)], w(0)=0\},\\
H_k &:= \{h\in C([0,\phi_k(T)];\mathbf{R}^{d_k});\ h \text{ is absolutely continuous and } \dot h\in L^2([0,\phi_k(T)];\mathbf{R}^{d_k})\},
\end{aligned}$$

and let μ_k be Wiener measure on W_k for $k=1,2,\dots,r$. We define the probability space (W,P) by

$$\begin{aligned}
W &:= W_1\times W_2\times\ldots\times W_r,\\
P &:= \mu_1\otimes\mu_2\otimes\ldots\otimes\mu_r.
\end{aligned}$$

If we set

$$H := H_1 \otimes H_2 \otimes \ldots \otimes H_r,$$

then (W, H, P) is an abstract Wiener space. Let $(B_k(t))$ be the canonical d_k-dimensional Brownian motion associated with (W_k, H_k, μ_k) for $k = 1, 2, \ldots, r$. Clearly, $B_1, B_2, \ldots, B_r$ are independent under P.

Next we consider stochastic differential equations with deterministic time change. Let $Z_k(t) := B_k(\phi_k(t))$ for $t \in [0, T]$ and $k = 1, 2, \ldots, r$ and $\mathscr{F}_t$ the σ-field generated by $(Z_k(s); 0 \leq s \leq t, k = 1, 2, \ldots, r)$. Then, Z_k is a square-integrable $(\mathscr{F}_t)$-martingale for every $k = 1, 2, \ldots, r$. We consider the following N-dimensional stochastic differential equation:

$$\begin{cases} dX(t) = \displaystyle\sum_{k=1}^{r} \sigma_k(t, X(t-))dZ_k(t) + b(t, X(t))dt, \\ X(0) = x_0, \end{cases} \tag{4.2.1}$$

where σ_k is an $\mathbf{R}^{d_k} \otimes \mathbf{R}^N$-valued continuous function on $[0, T] \times \mathbf{R}^N$ for $k = 1, 2, \ldots, r$, b is an $\mathbf{R}^N$-valued continuous function on $[0, T] \times \mathbf{R}^N$, and $x_0 \in \mathbf{R}^N$. We assume the estimate

$$\begin{aligned} \max_k |\sigma_k(t, x) - \sigma_k(t, y)| + |b(t, x) - b(t, y)| < K|x - y|, \qquad & x, y \in \mathbf{R}^N,\ t \in [0, T], \\ \max_k |\sigma_k(t, x)| + |b(t, x)| < K(1 + |x|), \qquad & x \in \mathbf{R}^N,\ t \in [0, T] \end{aligned}$$

with a positive constant K. Then we have the following theorem.

Theorem 4.2.1 *The equation* (4.2.1) *has a unique* $(\mathscr{F}_t)$*-adapted solution* $X = (X(t))$ *satisfying for any* $p > 1$

$$E\left[\sup_{0 \leq t \leq T} |X(t)|^p\right] \leq x_0 \exp\left\{M\left(T + \sum_{k=1}^{r} \phi_k(T)\right)\right\}, \tag{4.2.2}$$

where M *is a constant depending on* r, p *and* K*.*

Proof. It is sufficient to show (4.2.2) in the case of $p \geq 2$. We use Picard's successive approximation. Let $(\mathscr{F}_t)$-adapted right-continuous processes $\{X_n\}$ with left limits be defined by

$$\begin{aligned} X_0(t) &:= x_0, \\ X_{n+1}(t) &:= x_0 + \int_0^t \sum_{k=1}^{r} \sigma_k(s, X_n(s-))dZ_k(s) + \int_0^t b(s, X_n(s))ds. \end{aligned}$$

It is to be noted that the discontinuous points of X_n correspond with the discontinuous points of ϕ for any n almost surely. Now we show that there exists a constant M depending on p and K such that

$$E\left[\sup_{0\le s\le t}|X_{n+1}(s)-X_n(s)|^p\right]^{\frac{1}{p}}\le\frac{x_0}{2^n}\exp\left\{M\left(t+\sum_{k=1}^r\phi_k(t)\right)\right\}\tag{4.2.3}$$

by induction on n. When $n=1$, it is easy to see that the inequality (4.2.3) holds for sufficiently large M. We assume the inequality (4.2.3) for $n-1$. Lemma 4.1.3 leads to

$$\begin{aligned}
&E\left[\sup_{0\le s\le t}|X_{n+1}(s)-X_n(s)|^p\right]^{\frac{1}{p}}\\
&\le\sum_{k=1}^r E\left[\sup_{0\le s\le t}\left|\int_0^s(\sigma_k(u,X_n(u-))-\sigma_k(u,X_{n-1}(u-)))dZ_k(u)\right|^p\right]^{\frac{1}{p}}\\
&\qquad\qquad+E\left[\sup_{0\le s\le t}\left|\int_0^s(b(u,X_n(u))-b(u,X_{n-1}(u)))du\right|^p\right]^{\frac{1}{p}}\\
&\le C_3(p)\left\{\sum_{k=1}^r E\left[\left(\int_0^t|\sigma_k(u,X_n(u-))-\sigma_k(u,X_{n-1}(u-))|^2d\phi_k(u)\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}\right.\\
&\qquad\qquad\left.+E\left[\left|\int_0^t(b(u,X_n(u))-b(u,X_{n-1}(u)))du\right|^p\right]^{\frac{1}{p}}\right\}\\
&\le C_4(p,K)\left\{\sum_{k=1}^r E\left[\left(\int_0^t|X_n(u-)-X_{n-1}(u-)|^2d\phi_k(u)\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}\right.\\
&\qquad\qquad\left.+E\left[\left(\int_0^t|X_n(u)-X_{n-1}(u)|du\right)^p\right]^{\frac{1}{p}}\right\}\\
&\le C_4(p,K)\left\{\sum_{k=1}^r\left(\int_0^t E[|X_n(u-)-X_{n-1}(u-)|^p]^{\frac{2}{p}}d\phi_k(u)\right)^{\frac{1}{2}}\right.\\
&\qquad\qquad\left.+\int_0^t E[|X_n(u)-X_{n-1}(u)|^p]^{\frac{1}{p}}du\right\}.
\end{aligned}$$

From the assumption of induction we derive

$$\begin{aligned}
&E\left[\sup_{0\le s\le t}|X_{n+1}(s)-X_n(s)|^p\right]^{\frac{1}{p}}\\
&\le\frac{x_0}{2^n}C_4(p,K)\left\{\sum_{k=1}^r\left(\int_0^t\exp\left\{2M\left(u+\sum_{k=1}^r\phi_k(u-)\right)\right\}d\phi_k(u)\right)^{\frac{1}{2}}\right.\\
&\qquad\qquad\left.+\int_0^t\exp\left\{M\left(u+\sum_{l=1}^r\phi_l(u)\right)\right\}du\right\}
\end{aligned}$$

$$\leq \frac{x_0}{2^n} C_4(p,K) \left[\sum_{k=1}^{r} \exp\left\{ M\left(t + \sum_{l \neq k} \phi_l(t-) \right) \right\} \left(\int_0^t \exp\left(2M\phi_k(u-)\right) d\phi_k(u) \right)^{\frac{1}{2}} \right.$$
$$\left. + \exp\left(M \sum_{l=1}^{r} \phi_l(t) \right) \int_0^t e^{Mu} du \right].$$

Since $\phi_k(\phi_k^{-1}(s)-) \leq s$, Lemma 4.1.1 implies

$$\begin{aligned}
\int_0^t \exp\left(2M\phi_k(u-)\right) d\phi_k(u) &= \int_0^{\phi_k(t)} \exp\left(2M\phi_k(\phi_k^{-1}(s)-)\right) ds \\
&\leq \int_0^{\phi_k(t)} \exp(2Ms) ds \\
&\leq \frac{1}{2M} \exp(2M\phi_k(t)).
\end{aligned}$$

Hence it is holds that

$$\begin{aligned}
&E\left[\sup_{0 \leq s \leq t} |X_{n+1}(s) - X_n(s)|^p \right]^{\frac{1}{p}} \\
&\leq \frac{x_0}{2^n} C_4(p,K) \left[\sum_{k=1}^{r} \exp\left\{ M\left(t + \sum_{l \neq k} \phi_l(t-) \right) \right\} \frac{1}{\sqrt{2M}} e^{M\phi_k(t)} \right. \\
&\qquad\qquad \left. + \exp\left(M \sum_{l=1}^{r} \phi_l(t) \right) \frac{1}{M} e^{Mt} \right] \\
&\leq \frac{x_0}{2^n} C_4(p,K) \left\{ \frac{r}{\sqrt{2M}} + \frac{1}{M} \right\} \exp\left\{ M\left(t + \sum_{l=1}^{r} \phi_l(t) \right) \right\}.
\end{aligned}$$

If we choose M sufficiently large such that

$$\left(\frac{r}{\sqrt{2M}} + \frac{1}{M} \right) C_4(p,K) \leq \frac{1}{2},$$

then the inequality (4.2.3) holds for $n+1$. Therefore we complete the induction.

The inequality (4.2.3) leads to

$$E\left[\sup_{0 \leq s \leq t} |X_n(s) - X_m(s)|^p \right]^{\frac{1}{p}} \leq \frac{x_0}{2^m} \exp\left\{ M\left(t + \sum_{k=1}^{r} \phi_k(t) \right) \right\}.$$

for any positive integers n and m such that $n > m$. This inequality implies that $\{X_n\}$ is a Cauchy sequence. Hence there exists an $(\mathscr{F}_t)$-adapted right-continuous process X with left limits such that the discontinuous points of X correspond with the discontinuous points of ϕ almost surely, and

$$\lim_{n \to \infty} E\left[\sup_{0 \leq s \leq t} |X(s) - X_n(s)|^p \right]^{\frac{1}{p}} = 0.$$

It is easily seen that X is a solution of the equation (4.2.1). The estimate of X follows easily.

To prove the uniqueness, let X and Y be two solutions of the equation (4.2.1). A similar discussion as above yields

$$\begin{aligned}
&E\left[\sup_{0\le s\le t}|X(s)-Y(s)|^2\right]\\
&\le C_5(p,r,K)\left\{\sum_{k=1}^r\int_0^t E[|X(s-))-Y(s-)|^2]d\phi_k(s)+\int_0^t E[|X(s))-Y(s)|^2]ds\right\}\\
&\le C_5(p,r,K)\int_0^t E\left[\sup_{0\le u\le s}|X(u-))-Y(u-)|^2\right]d\left(s+\sum_{k=1}^r\phi_k(s)\right).
\end{aligned}$$

Applying Lemma 4.1.4 to this inequality, we have

$$E\left[\sup_{0\le t\le T}|X(t)-Y(t)|^2\right]=0.$$

□

Now we apply Malliavin calculus to the solution $X=(X(t))$ of the equation (4.2.1).

Theorem 4.2.2 *We assume that* $\sigma_k\in C^{0,m}([0,T]\times\mathbf{R}^N;\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ *and* $\nabla\sigma_k\in C_b^{0,m-1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ *for* $k=1,2,\dots,r$, $b\in C^{0,m}([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$, *and* $\nabla b\in C_b^{0,m-1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^N)$. *Then we have* $X(t)\in W^{m,p}(\mathbf{R}^N)$ *for* $t\in[0,T]$, *and there exists a constant* M *depending on* r,p,m *and the bounds of the spatial derivatives of* σ_k *and* b *up to order* m *such that*

$$||X(t)||_{m,p}\le\exp\left\{M\left(t+\sum_{k=1}^r\phi_k(t)\right)\right\},\quad t\in[0,T]. \tag{4.2.4}$$

Proof. It is sufficient to show (4.2.4) in the case $p\ge 2$. We define X_n as in the proof of Theorem 4.2.1. The proof of Theorem 4.2.1 implies that $X_n(t)$ converges to $X(t)$ in L^p for each $t\in[0,T]$. Now we proceed to show that $X_n(t)$ is in $W^{m,p}$ for every $t\in[0,T]$ and all n, and satisfies that the inequality

$$E\left[\sup_{0\le s\le t}|D^jX_n(s)|^p_{\mathscr{L}_2^j(H;\mathbf{R}^N)}\right]^{\frac{1}{p}}\le\exp\left\{M\left(t+\sum_{k=1}^r\phi_k(t)\right)\right\} \tag{4.2.5}$$

with a constant M depending on p,m and the bounds of the spatial derivatives of σ_k and b up to order m such that for $n=0,1,2,\dots,$ $j=0,1,\dots,m$. For it we use the induction on (n,j). By the proof of Theorem 4.2.1, we know that $X_n(t)$ is in L^p for each $t\in[0,T]$

and all n, and there exists a constant M such that (4.2.5) holds for $j=0$. Clearly $X_0(t)$ is in $W^{m,p}$ for each $t\in[0,T]$, and there exists a constant M such that (4.2.5) holds for $n=0$. Let $j_0\le m$. Next, as the hypothesis of the induction we assume that $X_n(t)$ is in $W^{j,p}$ for "each $t\in[0,T]$, all n and $j=0,1,\ldots,j_0-1$" and for "each $t\in[0,T]$, $n=0,1,\ldots,n_0$ and $j=0,1,\ldots,j_0$", and that there exists a constant M satisfying (4.2.5) for "all n and $j=0,1,\ldots,j_0-1$" and for "$n=0,1,\ldots,n_0$ and $j=0,1,\ldots,j_0$". Then we show that $X_{n_0+1}(t)$ is in $W^{j_0,p}$ for each $t\in[0,T]$, and that there exists a constant M satisfying (4.2.5) for n_0+1 and j_0. Proposition 4.1.5 gives the explicit expression of DX_{n_0+1}

$$\begin{aligned} DX_{n_0+1}(t) &= \sum_{k=1}^r\int_0^t \nabla\sigma_k(s,X_{n_0}(s-))DX_{n_0}(s-)dZ_k(s)+\int_0^t\nabla b(s,X_{n_0}(s))DX_{n_0}(s)ds \\ &\quad +\sum_\lambda\sum_{k=1}^r h_k^\lambda\otimes\int_0^{\phi_k(t)}\sigma_k(\phi_k^{-1}(s),X_{n_0}(\phi_k^{-1}(s)))\dot h_k^\lambda(s)ds, \end{aligned}$$

where $\{h_\lambda=(h_1^\lambda,h_2^\lambda,\ldots,h_r^\lambda)\}_\lambda$ is a complete orthonormal normal system of $H=H_1\otimes H_2\otimes\ldots\otimes H_r$. Repeating this procedure, we have

$$\begin{aligned} & D^{j_0}X_{n_0+1}(t) \\ &= \sum_{k=1}^r\int_0^t\Big\{\nabla\sigma_k(s,X_{n_0}(s-))D^{j_0}X_{n_0}(s-) \\ &\qquad\qquad +\sum_{l=1}^{j_0}A_l^k(s,X_{n_0}(s-))Q_l^k(DX_{n_0}(s-),\ldots,D^{j_0-1}X_{n_0}(s-))\Big\}dZ_k(s) \\ &\quad +\int_0^t\Big\{\nabla b_k(s,X_{n_0}(s))D^{j_0}X_{n_0}(s) \\ &\qquad\qquad\qquad +\sum_{l=1}^{j_0}\tilde A_l^k(s,X_{n_0}(s))\tilde Q_l^k(DX_{n_0}(s),\ldots,D^{j_0-1}X_{n_0}(s))\Big\}ds \\ &\quad +\sum_\lambda\sum_{k=1}^r h_k^\lambda\otimes\int_0^{\phi_k(t)}\sum_{l=0}^{j_0-1}\hat A_l^k(\phi_k^{-1}(s),X_{n_0}(\phi_k^{-1}(s))) \\ &\qquad\qquad \times\hat Q_l^k(DX_{n_0}(\phi_k^{-1}(s)),\ldots,D^{j_0-1}X_{n_0}(\phi_k^{-1}(s)))\dot h_k^\lambda(s)ds, \end{aligned} \tag{4.2.6}$$

where $A_l^k,\tilde A_l^k\in C_b^1([0,T]\times\mathbf{R}^N;(\mathbf{R}^N)^{\otimes l}\otimes\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$, $\hat A_l^k\in C^1([0,T]\times\mathbf{R}^N;(\mathbf{R}^N)^{\otimes l}\otimes\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ satisfy that

$$\max_{l,k}|\hat A_l^k(t,x)|\le C_6(\{||\nabla^l\sigma_k||_\infty\}_{1\le l\le m,1\le k\le r})(1+|x|),\quad x\in\mathbf{R}^N,\ t\in[0,T],$$

and $Q_l^k,\tilde Q_l^k,\hat Q_l^k$ are $(\mathbf{R}^N)^{\otimes l}\otimes H^{j_0}$-valued functions whose components are polynomials of order l. Therefore, from Lemma 4.1.3, it follows that

$$
\begin{aligned}
&E\left[\sup_{0\le s\le t}|D^{j_0}X_{n_0+1}(s)|^p_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}\right]^{\frac{1}{p}}\\
&\le C_7(p)E\left[\left(\sum_{k=1}^r\int_0^t|\nabla\sigma_k(s,X_{n_0}(s))D^{j_0}X_{n_0}(s)\right.\right.\\
&\qquad\qquad\left.\left.+\sum_{l=1}^{j_0}A_l^k(s,X_{n_0}(s))Q_l^k(DX_{n_0}(s),\dots,D^{j_0-1}X_{n_0}(s))|^2_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}d\phi_k(s)\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}\\
&\quad+\int_0^t E\left[|\nabla b_k(s,X_{n_0}(s))D^{j_0}X_{n_0}(s)\right.\\
&\qquad\qquad\left.+\sum_{l=1}^{j_0}\tilde{A}_l^k(s,X_{n_0}(s))\tilde{Q}_l^k(DX_{n_0}(s),\dots,D^{j_0-1}X_{n_0}(s))|^p_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}\right]^{\frac{1}{p}}ds\\
&\quad+\sum_{k=1}^r E\left[\left(\int_0^{\phi_k(t)}|\sum_{l=0}^{j_0-1}\hat{A}_l^k(\phi_k^{-1}(s),X_{n_0}(\phi_k^{-1}(s)))\right.\right.\\
&\qquad\qquad\qquad\left.\left.\times\hat{Q}_l^k(DX_{n_0}(\phi_k^{-1}(s)),\dots,D^{j_0-1}X_{n_0}(\phi_k^{-1}(s)))|^2_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}ds\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}.
\end{aligned}
$$

From Lemma 4.1.1 the last term of this inequality equals to

$$
\sum_{k=1}^r E\left[\left(\int_0^t|\sum_{l=0}^{j_0-1}\hat{A}_l^k(s,X_{n_0}(s-))\hat{Q}_l^k(DX_{n_0}(s-),\dots,D^{j_0-1}X_{n_0}(s-))|^2_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}d\phi_k(s)\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}.
$$

On the other hand, the induction assumptions imply that

$$
\begin{aligned}
&E\left[\sup_{0\le s\le t}|D^{j}X_{n_0}(s)|^p_{\mathscr{L}_2^{j}(H;\mathbf{R}^N)}\right]^{\frac{1}{p}}\\
&\le C_8\left(m,p,\{||\nabla^l\sigma_k||_\infty\}_{1\le l\le m,1\le k\le r},\{||\nabla^l b||_\infty\}_{1\le l\le m}\right)\\
&\qquad\times\exp\left\{C_8\left(m,p,\{||\nabla^l\sigma_k||_\infty\}_{1\le l\le m,1\le k\le r},\{||\nabla^l b||_\infty\}_{1\le l\le m}\right)\left(t+\sum_{k=1}^r\phi_k(t)\right)\right\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=0,1,\dots,j_0.
\end{aligned}
$$

Hence, by Hölder's inequality, we have

$$
\begin{aligned}
&E\left[|A_l^k(s,X_{n_0}(s))Q_l^k(DX_{n_0}(s),\dots,D^{j_0-1}X_{n_0}(s))|^p_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}\right]\\
&\le C_9\left(m,p,\{||\nabla^l\sigma_k||_\infty\}_{1\le l\le m,1\le k\le r},\{||\nabla^l b||_\infty\}_{1\le l\le m}\right)\\
&\qquad\times\exp\left\{C_9\left(m,p,\{||\nabla^l\sigma_k||_\infty\}_{1\le l\le m,1\le k\le r},\{||\nabla^l b||_\infty\}_{1\le l\le m}\right)\left(t+\sum_{k=1}^r\phi_k(t)\right)\right\}.
\end{aligned}
$$

The same estimates also hold for $\tilde{A}_l^k \tilde{Q}_l^k$ and $\hat{A}_l^k \hat{Q}_l^k$. Then we can make similar argument to the proof of Theorem 4.2.1, so that $X_{n_0+1}(t) \in W^{j_0,p}$ and (4.2.5) holds for sufficiently large M depending on r, p, m and the bounds of the spatial derivatives of σ_k and b up to order m. Thus we have

$$X_n(t) \longrightarrow X(t) \text{ in } L^p, \quad \sup_n ||X_n(t)||_{m,p} < \infty.$$

In help of Lemma 1.5.3. in [19], we have the conclusion. □

Next we consider the relation between the ellipticity of equations and the non-degeneracy of Malliavin covariance matrices.

Theorem 4.2.3 *We assume that* $\sigma_k \in C^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^{d_k} \otimes \mathbf{R}^N)$ *and* $\nabla \sigma_k$ *is bounded for* $k = 1, 2, \ldots, r$, $b \in C^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$, ∇b *is bounded, and that there exists a positive constant* ε *such that*

$$\sum_{k=1}^r \sigma_k(0, x_0)\ {}^t\sigma_k(0, x_0) \geq \varepsilon.$$

Then, Malliavin covariance matrix $\Delta(t) = ((DX^i(t), DX^j(t))_{H^*})_{ij}$ *is invertible, and there exists a constant* $C = C(x_0, N, p, \varepsilon, r, \{||\nabla \sigma_k||_\infty\}_{1 \leq k \leq r}, ||\nabla b||_\infty)$ *such that for all* $p > 1$

$$E[\det(\Delta(t))^{-p}] \leq C \min\{\phi_i(t); i = 1, 2, \ldots, r\}^{-Np} \exp\left[C(t + \max\{\phi_i(t); i = 1, 2, \ldots, r\})\right]. \tag{4.2.7}$$

Moreover, if there exists a positive constant ε *and* t_0 *such that*

$$\sum_{k=1}^r \sigma_k(t, x)\ {}^t\sigma_k(t, x) \geq \varepsilon, \quad t \in [0, t_0],\ x \in \mathbf{R}^N,$$

then we can choose a constant $C = C(t_0, N, p, \varepsilon, r, \{||\nabla \sigma_k||_\infty\}_{1 \leq k \leq r}, ||\nabla b||_\infty)$ *satisfying (4.2.7).*

Proof. Let

$$A_k(t) := [B_k(\phi_k(\cdot)), B_k(\phi_k(\cdot))](t), \quad t \in [0,T],\ k = 1, 2, \ldots, r.$$

We define two $N \times N$-matrix-valued processes J_1 and J_2 as the solutions of the following stochastic differential equations, respectively.

$$\begin{cases} dJ_1(t) = \displaystyle\sum_{k=1}^r \nabla \sigma_k(t, X(t-)) J_1(t-) dZ_k(t) + \nabla b(t, X(t-)) J_1(t-) dt, \\ J_1(0) = I, \end{cases}$$

$$\begin{cases} dJ_2(t) = -\sum_{k=1}^{r} J_2(t-)\nabla\sigma_k(t,X(t-))dZ_k(t) - J_2(t-)\nabla b(t,X(t-))dt \\ \qquad\qquad\qquad + \sum_{k=1}^{r} J_2(t-)\nabla\sigma_k(t,X(t-))\nabla\sigma_k(t,X(t-))dA_k(t), \\ J_2(0) = I. \end{cases}$$

Corollary 2 and Theorem 29 of Section 6 of Chapter II in [20] imply that $J_1(t)J_2(t) = I$ for all $t \in [0,T]$, from which it follows that $J_1(t) = J_2(t)^{-1}$ and

$$J_2(t)DX(t)[h] = \sum_{k=1}^{r} \int_0^t J_2(s-)\sigma_k(s,X(s-))dh_k(\phi_k(s)),$$

with $h = (h_1, h_2, \ldots, h_r) \in H$. By virtue of Lemma 4.1.1 this can be expressed as

$$J_2(t)DX(t)[h] = \sum_{k=1}^{r} \int_0^{\phi_k(t)} J_2(\phi_k^{-1}(u)-)\sigma_k(\phi_k^{-1}(u), X(\phi_k^{-1}(u)-))\dot{h}_k(u)du.$$

Hence, for a complete orthonormal normal system $\{h^\lambda\}$ of H we have

$$\begin{aligned} \Delta(t) &= J_1(t)\sum_{\lambda}\sum_{k=1}^{r} \int_0^{\phi_k(t)} J_2(\phi_k^{-1}(u)-)\sigma_k(\phi_k^{-1}(u), X(\phi_k^{-1}(u)-))\dot{h}_k^\lambda(u)du \\ &\qquad \times \int_0^{\phi_k(t)} {}^t[J_2(\phi_k^{-1}(u)-)\sigma_k(\phi_k^{-1}(u), X(\phi_k^{-1}(u)-))]\dot{h}_k^\lambda(u)du\, {}^tJ_1(t) \\ &= J_1(t)\sum_{k=1}^{r} \int_0^{\phi_k(t)} J_2(\phi_k^{-1}(u)-)\sigma_k(\phi_k^{-1}(u), X(\phi_k^{-1}(u)-)) \\ &\qquad \times {}^t\sigma_k(\phi_k^{-1}(u), X(\phi_k^{-1}(u)-))\, {}^tJ_2(\phi_k^{-1}(u)-)du\, {}^tJ_1(t) \\ &= J_1(t)\sum_{k=1}^{r} \int_0^t J_2(s-)\sigma_k(s,X(s-))\, {}^t\sigma_k(s,X(s-))\, {}^tJ_2(s-)d\phi_k(s)\, {}^tJ_1(t). \end{aligned}$$

From this one can derive

$$\det(\Delta(t)) = \det(J_1(t))^2 \det\left(\sum_{k=1}^{r} \int_0^t J_2(s-)\sigma_k(s,X(s-))\, {}^t\sigma_k(s,X(s-))\, {}^tJ_2(s-)d\phi_k(s)\right). \tag{4.2.8}$$

For the estimate of $\det(J_1(t))$, the following lemma holds.

Lemma 4.2.4

$$\begin{aligned} &E[|\det(J_2(t))|^p] \\ &\le C_{10}(p,N,r,\{||\nabla\sigma_k||_\infty\}_{1\le k\le r}, ||\nabla b||_\infty) \\ &\qquad \times \exp\left[C_{10}(p,N,r,\{||\nabla\sigma_k||_\infty\}_{1\le k\le r}, ||\nabla b||_\infty)(t + \max\{\phi_i(t); i = 1,2,\ldots,r\})\right]. \end{aligned}$$

Proof of Lemma 4.2.4. Lemma 4.1.3 enable us to make similar discussion in the proof of Theorem 4.2.1, and it follows

$$\begin{aligned}
&\max_{i,j} E\left[\sup_{0\le s\le t}|(J_2(s))_{ij}|^p\right]^{\frac{1}{p}} \\
&\le C_{11}(p,r,\{||\nabla\sigma_k||_\infty\}_{1\le k\le r},||\nabla b||_\infty) \\
&\quad\times\int_0^t \max_{i,j} E\left[\sup_{0\le u\le s}|(J_2(u))_{ij}|^p\right]^{\frac{1}{p}} d\left(s+\sum_{k=1}^r \phi_k(s)\right).
\end{aligned}$$

Lemma 4.1.4 yields that

$$\begin{aligned}
\max_{i,j} E[|(J_2(t))_{ij}|^p]^{\frac{1}{p}} &\le C_{11}(p,r,\{||\nabla\sigma_k||_\infty\}_{1\le k\le r},||\nabla b||_\infty) \\
&\quad\times\exp\left[C_{11}(p,r,\{||\nabla\sigma_k||_\infty\}_{1\le k\le r},||\nabla b||_\infty)\right. \\
&\quad\left.\times(t+\max\{\phi_i(t); i=1,2,\dots,r\})\right].
\end{aligned}$$

By Hölder's inequality, we have

$$E[|\det(J_2(t))|^p] \le N!\max_{i,j} E[|(J_2(t))_{ij}|^{Np}]^{\frac{1}{Np}}.$$

Therefore we have the conclusion of Lemma 4.2.4.

The estimate of Lemma 4.2.4 is sufficient for the part $\det(J_1(t))$. We estimate the other part. Let $\xi\in S^{N-1}$ where S^{N-1} is the $(N-1)$-dimensional sphere centered at 0. From the assumption of ellipticity and the compactness of S^{N-1}, we can choose $n\in\mathbf{N}$, open sets G_i in S^{N-1}, and $k_i=1,2,\dots,r$, for $i=1,2,\dots,n$ such that

$$\bigcup_{i=1}^n G_i = S^{N-1},$$

$${}^t\xi\sigma_{k_i}(0,x_0)\,{}^t\sigma_{k_i}(0,x_0)\xi > \frac{\varepsilon}{2r},\quad \xi\in G_i,\ i=1,2,\dots,n.$$

By the continuity of $\{\sigma_k\}$, there exist $R_i>0$ and $t_i\in(0,T]$ such that

$${}^t\xi\sigma_{k_i}(s,x)\,{}^t\sigma_{k_i}(s,x)\xi > \frac{\varepsilon}{3r},\quad x\in B(x_0,R_i),\ s\in[0,t_i]\ ,\xi\in G_i$$

for $i=1,2,\dots,n$. Let $R:=\min_i R_i$ and $t_0:=\min_i t_i$. We define a stopping time ζ by

$$\zeta:=\inf\{t\in[0,T]; |X(t)-x_0|>R \text{ or } |J_1(t)-I|>\delta\}\wedge T,$$

where $\delta\in(0,t_0)$ is chosen so small that

$$\begin{aligned}
&{}^t\xi J_2(s)\sigma_{k_i}(s,x)\,{}^t\sigma_{k_i}(s,x)\,{}^tJ_2(s)\xi \ge \frac{\varepsilon}{4r}, \\
&\qquad\qquad x\in B(x_0,R),\ s\in[0,\zeta),\ \xi\in S^{N-1}.
\end{aligned}$$

To simplify the notation, we denote $\min\{\phi_i(t); i = 1, 2, \dots, r\}$ by $\eta(t)$. We note that η is also a right-continuous increasing function on $[0, T]$. Since Lemma 4.1.1 yields that for $i = 1, 2, \dots, n$ and $\xi \in G_i$

$$\begin{aligned} & {}^t\xi \left(\sum_{k=1}^{r} \int_0^t J_2(s-)\sigma_k(s, X(s-))\, {}^t\sigma_k(s, X(s-))\, {}^tJ_2(s-) d\phi_k(s) \right) \xi \\ & \geq \int_0^{t\wedge\zeta} {}^t\xi J_2(s-)\sigma_{k_i}(s, X(s-))\, {}^t\sigma_{k_i}(s, X(s-))\, {}^tJ_2(s-)\xi d\phi_{k_i}(s) \\ & \geq \frac{\varepsilon}{4r}\eta(t\wedge\zeta), \end{aligned}$$

we have

$$\begin{aligned} & \det \left(\sum_{k=1}^{r} \int_0^t J_2(s-)\sigma_k(s, X(s-))\, {}^t\sigma_k(s, X(s-))\, {}^tJ_2(s-) d\phi_k(s) \right) \\ & \geq 4^{-N} r^{-N} \varepsilon^N \eta(t\wedge\zeta)^N. \end{aligned}$$

Hence

$$\begin{aligned} & E\left[\det \left(\sum_{k=1}^{r} \int_0^t J_2(s-)\sigma_k(s, X(s-))\, {}^t\sigma_k(s, X(s-))\, {}^tJ_2(s-) d\phi_k(s) \right)^{-p} \right] \\ & \leq 4^{Np} r^{Np} \varepsilon^{-Np} E[\eta(t\wedge\zeta)^{-Np}] \\ & = 4^{Np} r^{Np} \varepsilon^{-Np} E[\eta(t)^{-Np}; \zeta \geq t] + 4^{Np} r^{Np} \varepsilon^{-Np} E[\eta(\zeta)^{-Np}; \zeta < t]. \end{aligned}$$

Since $\eta(\eta^{-1}(u)-) \leq u$, from Lemma 4.1.1 we have

$$\begin{aligned} \eta(\zeta)^{-Np} - \eta(t)^{-Np} &= Np \int_{\eta(\zeta)}^{\eta(t)} u^{-Np-1} du \\ &\leq Np \int_{\eta(\zeta)}^{\eta(t)} \eta(\eta^{-1}(u)-)^{-Np-1} du \\ &= Np \int_{\zeta}^{t} \eta(s-)^{-Np-1} d\eta(s). \end{aligned}$$

Hence we have

$$\begin{aligned} & E\left[\det \left(\sum_{k=1}^{r} \int_0^t J_2(s-)\sigma_k(s, X(s-))\, {}^t\sigma_k(s, X(s-))\, {}^tJ_2(s-) d\phi_k(s) \right)^{-p} \right] \\ & \leq 4^{Np} r^{Np} \varepsilon^{-Np} \eta(t)^{-Np} P(\zeta \geq t) \\ & \qquad + 4^{Np} r^{Np} \varepsilon^{-Np} E\left[Np \int_\zeta^t \eta(s-)^{-Np-1} d\eta(s) + \eta(t)^{-Np}; \zeta < t \right] \\ & = 4^{Np} r^{Np} \varepsilon^{-Np} \eta(t)^{-Np} \end{aligned}$$

$$
\begin{aligned}
&\quad +4^{Np}r^{Np}\varepsilon^{-Np}NpE\left[\int_0^t \mathbf{1}_{(\zeta,t]}(s)\eta(s-)^{-Np-1}d\eta(s);\zeta<t\right]\\
&= 4^{Np}r^{Np}\varepsilon^{-Np}\eta(t)^{-Np}\\
&\quad +4^{Np}r^{Np}\varepsilon^{-Np}Np\int_0^t \eta(s-)^{-Np-1}E[\mathbf{1}_{(\zeta,t]}(s);\zeta<t]d\eta(s)\\
&= 4^{Np}r^{Np}\varepsilon^{-Np}\eta(t)^{-Np}+4^{Np}r^{Np}\varepsilon^{-Np}Np\int_0^t \eta(s-)^{-Np-1}P(\zeta<s)d\eta(s). \quad (4.2.9)
\end{aligned}
$$

On the other hand, we have the following estimate for ζ.

Lemma 4.2.5

$$
P(\zeta\le t)\le 2Nr\exp\{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)\eta(t)^{-1}\}.
$$

Proof of Lemma 4.2.5. Note that it is sufficient to prove the estimate for small t. Set

$$
\begin{aligned}
\zeta_1 &= \inf\{t\in[0,T];|X(t)-x_0|>R\}\wedge T,\\
\zeta_2 &= \inf\{t\in[0,T];|J_1(t)-I|>\delta\}\wedge T.
\end{aligned}
$$

Then it is sufficient to prove the same estimate for ζ_1 and ζ_2. Since the estimates for ζ_1 and ζ_2 are proved similarly, we prove the estimate only for ζ_2. We define continuous martingale $(M_k(t))$ by

$$
M_k(t):=\int_0^t \nabla\sigma_k(\phi_k^{-1}(s),X(\phi_k^{-1}(s)-))J_1(\phi_k^{-1}(s)-)dB_k(s),\quad k=1,2,\ldots,r.
$$

Denoting $\sum_{i,j=1}^N\langle (M_k)_{ij}\rangle$ by $\langle M_k\rangle$ for $k=,1,2,\ldots,r$, we have

$$
\begin{aligned}
\langle M_k\rangle(\phi_k(t\wedge\zeta_2)) &= \int_0^{\phi_k(t\wedge\zeta_2)} |\nabla\sigma_k(\phi_k^{-1}(s),X(\phi_k^{-1}(s)-))J_1(\phi_k^{-1}(s)-)|^2ds\\
&\le C_{13}(\delta)\phi_k(t\wedge\zeta_2).
\end{aligned}
$$

From Lemma 4.1.2, it follows that

$$
\sup_{s\in[0,t]}|J_1(s\wedge\zeta_2)-I|\le\sum_{k=1}^r\sup_{s\in[0,\phi_k(t\wedge\zeta_2)]}|M_k(s)|+C_{14}(||\nabla b||_\infty)t.
$$

Therefore, if $t\le\frac{\delta}{2C_{14}(||\nabla b||_\infty)}$, then by Proposition 6.8 of [24], we have

$$
\begin{aligned}
P(\zeta_2\le t) &\le P\left(\sup_{s\in[0,t]}|J_1(s\wedge\zeta_2)-I|\ge\delta\right)\\
&\le P\left(\sum_{k=1}^r\sup_{s\in[0,\phi_k(t\wedge\zeta_2)]}|M_k(s)|\ge\frac{\delta}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{k=1}^{r} P\left(\sup_{s\in[0,\phi_k(t\wedge\zeta_2)]} |M_k(s)| \ge \frac{\delta}{2r}\right) \\
&= \sum_{k=1}^{r} P\left(\sup_{s\in[0,\phi_k(t\wedge\zeta_2)]} |M_k(s)| \ge \frac{\delta}{2r},\ \langle M_k\rangle(\phi_k(t\wedge\zeta_2)) \le C_{13}(\delta)\phi_k(t\wedge\zeta_2)\right) \\
&= 2N\sum_{k=1}^{r} \exp\left(-\frac{\delta^2}{8N^2r^2C_{13}(\delta)\phi_k(t)}\right).
\end{aligned}
$$

This completes the proof of Lemma 4.2.5.

From Lemma 4.2.5 it holds that

$$
P(\zeta < t) \le 2Nr\exp\{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)\eta(t-)^{-1}\}.
$$

Therefore, by (4.2.9) we have

$$
\begin{aligned}
&E\left[\det\left(\sum_{k=1}^{r}\int_0^t J_2(s-)\sigma_k(s,X(s-))\,{}^t\sigma_k(s,X(s-))\,{}^tJ_2(s-)d\phi_k(s)\right)^{-p}\right] \\
&\le 2^{2Np}r^{Np}\varepsilon^{-Np}\eta(t)^{-Np} \\
&\quad +2^{2Np+1}r^{Np+1}\varepsilon^{-Np}N^2p\int_0^t \eta(s-)^{-Np-1}\exp\{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)\eta(s-)^{-1}\}d\eta(s).
\end{aligned}
$$

Now we estimate the second term in the right hand side of above inequality. Let

$$
f(x) := x^{-Np-1}e^{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)x^{-1}}.
$$

It is easily seen that f is a positive, bounded, and concave function on $(0,\infty)$, and the maximum is attained at $x = \frac{C_{12}(N,r,R,\delta,||\nabla b||_\infty)}{Np+1}$. We denote the maximum jump of $(\eta(t); t\in[0,T])$ by J. Since $u - J \le \eta(\eta^{-1}(u)-) \le u$, the following inequalities hold by Lemma 4.1.1

$$
\begin{aligned}
&\int_0^t \eta(s-)^{-Np-1}e^{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)\eta(s-)^{-1}}d\eta(s) \\
&= \int_0^{\eta(t)} \eta(\eta^{-1}(u)-)^{-Np-1}e^{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)\eta(\eta^{-1}(u)-)^{-1}}du \\
&\le \int_0^{J+\frac{C_{12}(N,r,R,\delta,||\nabla b||_\infty)}{Np+1}} \eta(\eta^{-1}(u)-)^{-Np-1}e^{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)\eta(\eta^{-1}(u)-)^{-1}}du \\
&\qquad + \int_{J+\frac{C_{12}(N,r,R,\delta,||\nabla b||_\infty)}{Np+1}}^{\infty} \eta(\eta^{-1}(u)-)^{-Np-1}e^{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)\eta(\eta^{-1}(u)-)^{-1}}du \\
&\le ||f||_\infty\left(J+\frac{C_{12}(N,r,R,\delta,||\nabla b||_\infty)}{Np+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_{J+\frac{1}{Np+1}}^{\infty} (u-J)^{-Np-1} e^{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)(u-J)^{-1}} du \\
&\leq ||f||_\infty \left(J + \frac{C_{12}(N,r,R,\delta,||\nabla b||_\infty)}{Np+1} \right. \\
&\quad \left. + \int_0^{\infty} u^{-Np-1} e^{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)u^{-1}} du. \right.
\end{aligned}
$$

Changing variables in the integral of the right hand side, we have

$$
\int_0^{\infty} u^{-Np-1} e^{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)u^{-1}} du = C_{12}(N,r,R,\delta,||\nabla b||_\infty)^{-Np}\Gamma(Np).
$$

Here Γ is the gamma function. This leads to the inequality

$$
\int_0^t \eta(s-)^{-Np-1} e^{-C_{12}(N,r,R,\delta,||\nabla b||_\infty)\eta(s-)^{-1}} d\eta(s) \leq C_{15}(N,p,r,R,\delta,||\nabla b||_\infty)(1+J).
$$

Therefore, we can conclude that for all $t \in [0,T]$

$$
\begin{aligned}
&E\left[\det\left(\sum_{k=1}^{r} \int_0^t J_2(s-)\sigma_k(s,X(s-))\, {}^t\sigma_k(s,X(s-))\, {}^tJ_2(s-) d\phi_k(s) \right)^{-p} \right] \\
&\leq 2^{2Np} r^{Np} \varepsilon^{-Np} \eta(t)^{-Np} + 2^{2Np+1} r^{Np+1} \varepsilon^{-Np} N^2 p C_{15}(N,p,r,R,\delta,||\nabla b||_\infty)(1+J) \\
&\leq C_{16}(N,p,r,\varepsilon,R,\delta,||\nabla b||_\infty)(1+J+\eta(t)^{-Np}).
\end{aligned}
$$

Note that R and δ are determined by $\{||\nabla\sigma_k||_\infty\}_{1\leq k\leq r}, x_0$, and ε. From (4.2.8), this estimate, and Lemma 4.2.4, the first assertion follows. Since the condition of the second assertion implies that the constants for the estimates can be chosen independently of x_0 but dependently on t_0, the second assertion is derived. □

Theorems 4.2.2 and 4.2.3 enable us to apply Sobolev's inequality with respect to H-derivative to the solution of the stochastic differential equation (4.2.1), and the following theorem follows.

Theorem 4.2.6 *For the stochastic differential equation* (4.2.1)*, we assume that* $\sigma_k \in C^{0,m+2}([0,T]\times\mathbf{R}^N;\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$*,* $\nabla\sigma_k \in C_b^{0,m+1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ *for* $k = 1,2,\ldots,r$*,* $b \in C^{0,m+2}([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$*,* $\nabla b \in C_b^{0,m+1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^N)$*, and there exists a positive constant* ε *such that*

$$
\sum_{k=1}^{r} \sigma_k(0,x_0)\ {}^t\sigma_k(0,x_0) \geq \varepsilon.
$$

Then, the law $P(t, x_0, dy)$ of $X(t, x_0)$ is absolutely continuous with respect to the Lebesgue measure and its density function $p(t, x, y)$ is estimated as

$$\max_{0 \le l \le m} \sup_{y \in \mathbf{R}^d} \left| \nabla_y^l p(t, x_0, y) \right| \le c_1 \min\{\phi_i(t); i = 1, 2, \ldots, r\}^{-c_3} \exp \left\{ c_2 \left(t + \sum_{k=1}^r \phi_k(t) \right) \right\}. \tag{4.2.10}$$

with positive constants c_1, c_2, c_3. Moreover, if there exist positive constants ε and t_0 such that

$$\sum_{k=1}^r \sigma_k(t, x)\ {}^t\sigma_k(t, x) \ge \varepsilon, \quad t \in [0, t_0],\ x \in \mathbf{R}^N,$$

then the constants c_1, c_2, c_3 in (4.2.10) can be chosen independently of x_0 but dependently on t_0.

Proof. The conclusion follows from Theorems 4.2.3, 4.2.2, and Theorem 5.9 of [24]. □

4.3 Regularity properties of conditional probabilities

In this section, we consider the inheritance of regularity of densities from those of conditional probabilities.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and $\mathscr{G}$ a sub-σ-field of $\mathscr{F}$. We assume that there exists a regular conditional probability $p(\omega, d\omega')$ of P with respect to $\mathscr{G}$. First we discuss the absolute continuity.

Theorem 4.3.1 *If the regular conditional probability $p(\omega, d\omega')$ is absolutely continuous with respect to a measure ν on $(\Omega, \mathscr{F})$ for almost all ω, then P is also absolutely continuous with respect to ν.*

Proof. Let $A \in \mathscr{F}$ be a ν-null set. Since A is also $p(\omega, d\omega')$-null set for almost all ω,

$$\int_\Omega \mathbf{1}_A(\omega) P(d\omega) = \int_\Omega \int_\Omega \mathbf{1}_A(\omega') p(\omega, d\omega') P(d\omega) = 0.$$

Thus we have the conclusion. □

Next we consider the regularity in case of $\Omega := \mathbf{R}^N$ and $\mathscr{F} := \mathscr{B}(\mathbf{R}^N)$. Assume that the regular conditional probability $p(\omega, d\omega')$ has the density function $p(\omega, y)$ for almost all ω.

Theorem 4.3.2 *Let $p(\omega,\cdot) \in C_b^n(\mathbf{R}^N)$ for almost all ω and there exists a positive random variable Y such that $E[Y] < \infty$ and for almost all ω*

$$||\partial^\alpha p(\omega,\cdot)||_\infty \leq Y(\omega), \quad |\alpha| \leq n.$$

Then P has its density function $q \in C_b^n(\mathbf{R}^N)$.

Proof. Theorem 4.3.1 implies that P has its density function q. For $f \in C_0^\infty(\mathbf{R}^N)$ and multi-index α, $|\alpha| \leq n$ we obtain

$$\begin{aligned}
\left| \int_{\mathbf{R}^N} (\partial^\alpha f)(x) q(x) dx \right| &= \left| \int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} (\partial^\alpha f)(y) p(\omega, y) dy \right) P(d\omega) \right| \\
&= \left| \int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} f(y) (\partial_y^\alpha p)(\omega, y) dy \right) P(d\omega) \right| \\
&\leq \int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} |f(y)| |(\partial_y^\alpha p)(\omega, y)| dy \right) P(d\omega) \\
&\leq \int_{\mathbf{R}^N} |f(y)| dy \int_{\mathbf{R}^N} Y(\omega) P(d\omega).
\end{aligned}$$

From this $q \in W^{n,\infty}(\mathbf{R}^N, dx)$ follows, and hence the assertion by Theorem 2 of Chapter V in [25]. □

4.4 Regularity properties of solutions of stochastic differential equations driven by subordinated Brownian motions

In this section, we combine the results of Sections 4.2 and 4.3.

Let r be a positive integer, $d_1, d_2, \ldots, d_k$ positive integers, $(\Omega, \mathscr{F}, P)$ a probability space, and $Z_k(t)$ a d_k-dimensional right continuous process on $[0,T]$ with left limits for $k = 1, 2, \ldots, r$, where $\{Z_k; k = 1, 2, \ldots, r\}$ are independent totally. Let $(\mathscr{F}_t)$ be a σ-field generated by $\{Z_k(s); 0 \leq s \leq t, k = 1, 2, \ldots, r\}$.

We assume that $(Z_k(t))$ can be expressed as $(B_k(\tau_k(t)))$ for $k = 1, 2, \ldots, r$, where $(B_k(t))$ is a d_k-dimensional Brownian motion for $k = 1, 2, \ldots, r$, $\{\tau_k; k = 1, 2, \ldots, r\}$ are one-dimensional right continuous increasing processes starting at 0, and $\{B_k; k = 1, 2, \ldots, r\}$ and $\{\tau_k; k = 1, 2, \ldots, r\}$ are independent totally. We call such a process $Z_k(t)$ a subordinated Brownian motion and τ_k a subordinator. Let Z_k^J be the jump part of Z_k for $k = 1, 2, \ldots, r$. We define a Poisson point process p_k by

$$p_k(t) := Z_k^J(t) - Z_k^J(t-),$$

and decompose the counting measure $N_{p_k}(dtdx)$ on $[0,T]\times\mathbf{R}^{d_k}$ of p_k as

$$N_{p_k}(dtdx)=\mathbf{1}_D(x)N_{p_k}(dtdx)+\mathbf{1}_{D^c}(x)N_{p_k}(dtdx),$$

where D is a unit ball with center 0 in $\mathbf{R}^{d_k}$. Then it holds that

$$Z_k^J(t)=\int_0^{t+}\int_{\mathbf{R}^{d_k}}x\mathbf{1}_D(x)N_{p_k}(dsdx)+\int_0^{t+}\int_{\mathbf{R}^{d_k}}x\mathbf{1}_{D^c}(x)N_{p_k}(dsdx).$$

In additional, we assume that the first term of the right-hand side is square integrable, and the second term is a function of bounded variation with respect to t. Note that the first term is martingale. This assumption implies that we can define the stochastic integrals by $\{Z_k^J\}$ and therefore by $\{Z_k\}$. The detail of the definition can be found in [9].

We consider the following N-dimensional stochastic differential equation:

$$\begin{cases} dX(t)=\displaystyle\sum_{k=1}^r\sigma_k(t,X(t-))dZ_k(t)+b(t,X(t))dt, \\ X(0)=x_0, \end{cases} \tag{4.4.1}$$

where $\sigma_k\in C([0,T]\times\mathbf{R}^N;\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ for $k=1,2,\dots,r$, and $b\in C([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$.

It is known that the solution of the stochastic differential equation has pathwise uniqueness when the coefficients are Lipschitz continuous (c.f. Section 9 of Chapter IV in [9]).

We denote the σ-field generated by $\{\tau_k;k=1,2,\dots,r\}$ by $\mathscr{F}^\tau$. Then, the argument in Section 4.2 is available when we consider the equation (4.4.1) on $(\Omega,\mathscr{F},P(\cdot|\mathscr{F}^\tau))$, and the argument in Section 4.3 yields the following theorem.

Theorem 4.4.1 *Assume that* $\sigma_k\in C^{0,1}([0,T]\times\mathbf{R}^N;\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$*,* $\nabla\sigma_k$ *is bounded for* $k=1,2,\dots,r$*,* $b\in C^{0,1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$*,* ∇b *is bounded, and there exists a positive constant* ε *such that*

$$\sum_{k=1}^r\sigma_k(0,x_0)\ {}^t\sigma_k(0,x_0)\geq\varepsilon.$$

Then the equation (4.4.1) *has the unique solution* $(X(t))$*, and the distribution of* $X(t)$ *has its density for* $t\in(0,T]$*.*

Proof. Under the probability $P(\cdot|\mathscr{F}^\tau)$ we can apply Theorems 4.2.2, 4.2.3, and Theorem 1 of Chapter VIII in [25], so that $P(\cdot|\mathscr{F}^\tau)$ is absolutely continuous with respect to the N-dimensional Lebesgue measure almost surely. Thus the conclusion follows from Theorem 4.3.1 $\square$

For the regularity of the density function of the solution, we have the following theorem.

Theorem 4.4.2 *Assume that* $\sigma_k \in C^{0,m+2}([0,T]\times\mathbf{R}^N;\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ *and* $\nabla\sigma_k \in C_b^{0,m+1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ *for* $k=1,2,\dots,r$, $b \in C^{0,m+2}([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$, *and* $\nabla b \in C_b^{0,m+1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^N)$, *and there exists a positive constant* ε *such that*

$$\sigma(0,x_0)\ {}^t\sigma(0,x_0) \geq \varepsilon.$$

Moreover, we assume that

$$\sum_{k=1}^{r} E\left[(\tau_k(T))^{-A}\exp\left(A\tau_k(T)\right)\right] < \infty, \quad \textit{for all } A \in [0,\infty).$$

Let $(X(t))$ *be the solution of the stochastic differential equation* (4.4.1). *Then the distribution of* $X(T)$ *has its density* $q(x)$, *and* $q \in C_b^m(\mathbf{R}^N)$.

Proof. Under the probability $P(\cdot|\mathscr{F}^\tau)$ we can use Theorem 4.2.6. Therefore we have the conclusion by Theorem 4.3.2. □

4.5 Regularity properties of solutions of stochastic differential equations driven by stable processes

In this section, we consider special, but the most interesting, case of above results, that is, stochastic differential equations driven by stable processes.

Let r be a positive integer, $d_1, d_2, \dots, d_k$ positive integers, $(\Omega, \mathscr{F}, P)$ a probability space, $\alpha_k \in (0,2]$ for $k=1,2,\dots,r$, and $Z_k(t)$ a d_k-dimensional rotation-invariant α_k-stable process for $k=1,2,\dots,r$, where $\{Z_k; k=1,2,\dots,r\}$ are independent. Let $(\mathscr{F}_t)$ be a σ-field generated by $\{Z_k(s); 0\leq s\leq t, k=1,2,\dots,r\}$. We consider the following N-dimensional stochastic differential equation:

$$\begin{cases} dX(t) = \displaystyle\sum_{k=1}^{r}\sigma_k(t,X(t-))dZ_k(t) + b(t,X(t))dt, \\ X(0) = x_0, \end{cases} \tag{4.5.1}$$

where $\sigma_k \in C([0,T]\times\mathbf{R}^N;\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ for $k=1,2,\dots,r$, and $b \in C([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$. The stochastic integrals are defined as in the previous section.

Now we use subordination. $(Z_k(t))$ can be expressed as $(B_k(\tau_k(t)))$ for $k=1,2,\dots,r$, where B_k is a d_k-dimensional Brownian motion for $k=1,2,\dots,r$, τ_k is a one-sided $\alpha_k/2$-stable process if $\alpha_k \neq 2$ and $\tau_k(t)=t$ if $\alpha_k=2$ for $k=1,2,\dots,r$, and $\{B_k; k=1,2,\dots,r\}$ and $\{\tau_k; k=1,2,\dots,r\}$ are totally independent. If necessary, we extend the probability space $(\Omega, \mathscr{F}, P)$. Note that the assumptions of Section 4.4 are satisfied. We denote the σ-field generated by $\{\tau_k; k = 1, 2, \ldots, r\}$ by $\mathscr{F}^\tau$. Then, by Theorem 4.4.1, we have the following theorem.

Theorem 4.5.1 *Assume that* $\sigma_k \in C^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^{d_k} \otimes \mathbf{R}^N)$*,* $\nabla\sigma_k$ *is bounded for* $k = 1, 2, \ldots, r$*,* $b \in C^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$*,* ∇b *is bounded, and there exists a positive constant* ε *such that*

$$\sum_{k=1}^{r} \sigma_k(0, x_0)\ {}^t\sigma_k(0, x_0) \geq \varepsilon.$$

Then the equation (4.5.1) *has the unique solution* $(X(t))$*, and the distribution of* $X(t)$ *has its density for* $t \in (0, T]$*.*

Finally we consider the regularity of the density function of the solution. Theorem 4.4.2 is not available, because the condition of the expectation of exponential function does not hold. However, in the case that $r = 1$, the following theorem holds.

Theorem 4.5.2 *Assume that* $r = 1$*,* $\sigma \in C^{0,m+2}([0,T]\times\mathbf{R}^N; \mathbf{R}^d\otimes\mathbf{R}^N)$*,* $\nabla\sigma_k \in C_b^{0,m+1}([0,T]\times \mathbf{R}^N; \mathbf{R}^N \otimes \mathbf{R}^d \otimes \mathbf{R}^N)$*,* $b \in C^{0,m+2}([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$*,* $\nabla b \in C_b^{0,m+1}([0,T] \times \mathbf{R}^N; \mathbf{R}^N \otimes \mathbf{R}^N)$*, and there exists a positive constant* ε *such that*

$$\sigma(t,x)\ {}^t\sigma(t,x) \geq \varepsilon, \quad t \in [0,T],\ x \in \mathbf{R}^N.$$

Let $(X(t))$ *be the solution of the stochastic differential equation* (4.5.1)*. Then, the distribution of* $X(T)$ *has its density* $q \in C_b^m(\mathbf{R}^N)$*.*

Proof. Fix $T_0 > 0$. We define an $\mathscr{F}^\tau$-measurable random time ρ by

$$\rho := \sup\{t > 0; \tau(T) - \tau(t) > T_0\}.$$

Let $\mathscr{F}^{\tau,T_0}$ be the σ-field generated by τ and $(B(t); 0 \leq t \leq (\tau(T) - T_0) \vee 0)$. Consider the following stochastic differential equation on $[0, T - \rho]$ under $P(\cdot|\mathscr{F}^{\tau,T_0})$:

$$\begin{cases} d\tilde{X}(t) = \sigma(\rho + t, \tilde{X}(t-))d\tilde{Z}(t) + b(\rho + t, \tilde{X}(t))dt, \\ \tilde{X}(0) = \xi_0 + \xi, \end{cases} \tag{4.5.2}$$

where $\tilde{Z}(t) := B(\tau(\rho + t)) - B(\tau(\rho))$,

$$\begin{aligned} \xi_0 &:= X(\rho-) + \sigma(\rho, X(\rho-))(B(\tau(T) - T_0) - B(\tau(\rho-))) + b(\rho, X(\rho-))(\tau(\rho) - \tau(\rho-)), \\ \xi &:= \sigma(\rho, X(\rho-))(B(\tau(\rho)) - B(\tau(T) - T_0)). \end{aligned}$$

Note that $(\tilde{Z}(t))$ is a Brownian motion with deterministic time change and ξ_0 is a constant under $P(\cdot|\mathscr{F}^{\tau,T_0})$. According to Theorem 4.2.1, the equation (4.5.2) has the unique solution $\tilde{X}$ on $(\Omega, \mathscr{F}, P(\cdot|\mathscr{F}^{\tau,T_0}))$, and it holds that

$$\tilde{X}(t) = X(\rho + t) \text{ for } t \in [0, T-\rho], \quad P(\cdot|\mathscr{F}^{\tau,T_0})\text{-a.s.} \tag{4.5.3}$$

On the other hand, if (W, H, μ) is the Wiener space generated by $(B(t); \tau(T) - T_0 \le t \le \tau(T))$, then Malliavin calculus is available for ξ and $(\tilde{X}(t))$ under $P(\cdot|\mathscr{F}^{\tau,T_0})$. It is easy to see that $|D\xi|_H \le ||\sigma||_\infty$ and that $D^k\xi = 0$ for $k \ge 2$. By the similar discussion to that in the proof of Theorem 4.2.2, for all $p > 0$ there exists a constant M such that

$$\sum_{k=1}^{m+2} E^{P(\cdot|\mathscr{F}^{\tau,T_0})}\left[|D^k\tilde{X}(\rho)|^p_{\mathscr{L}_2^k(H;\mathbf{R}^N)}\right] \le M\exp\{M(T + \tau(T) - \tau(\rho))\}.$$

Since $\tau(T) - \tau(\rho) \le T_0$, we have

$$\sum_{k=1}^{m+2} E^{P(\cdot|\mathscr{F}^{\tau,T_0})}\left[|D^k\tilde{X}(\rho)|^p_{\mathscr{L}_2^k(H;\mathbf{R}^N)}\right] \le M\exp\{M(T + T_0)\}. \tag{4.5.4}$$

Now we consider the case that $\tau(T) > T_0$. From the equation (4.5.2) and Proposition 4.1.5, it is clearly derived that for $h \in H$

$$\begin{aligned}
&D\tilde{X}(T-\rho)[h] \\
&= D\xi[h] + \int_0^{T-\rho} \nabla\sigma(\rho + t, \tilde{X}(t-))D\tilde{X}(t-)[h]d\tilde{Z}(t) \\
&\quad + \int_0^{T-\rho} \sigma(\rho + t, \tilde{X}(t-))dh(\tau(\rho + t)) + \int_0^{T-\rho} \nabla b(\rho + t, \tilde{X}(t))D\tilde{X}(t)[h]dt.
\end{aligned}$$

We make similar discussion to that in the proof of Theorem 4.2.3. Let

$$\tilde{A}(t) := [\tilde{Z}, \tilde{Z}](t), \quad t \in [0, T-\rho].$$

We define two $N \times N$-matrix-valued processes $\tilde{J}_1$ and $\tilde{J}_2$ on $[0, T-\rho]$ by the solutions of the following stochastic differential equations, respectively.

$$\begin{cases}
d\tilde{J}_1(t) = \nabla\sigma(\rho + t, \tilde{X}(t-))\tilde{J}_1(t-)d\tilde{Z}(t) + \nabla b(\rho + t, \tilde{X}(t))\tilde{J}_1(t-)dt, \\
\tilde{J}_1(0) = I,
\end{cases}$$

$$\begin{cases}
d\tilde{J}_2(t) = -\tilde{J}_2(t-)\nabla\sigma(\rho + t, \tilde{X}(t-))d\tilde{Z}(t) - \tilde{J}_2(t-)\nabla b(\rho + t, \tilde{X}(t))dt \\
\qquad\qquad + \tilde{J}_2(t-)\nabla\sigma(\rho + t, \tilde{X}(t-))\nabla\sigma(\rho + t, \tilde{X}(t-))d\tilde{A}(t), \\
\tilde{J}_2(0) = I.
\end{cases}$$

By Corollary 2 and Theorem 29 in Section 6 of Chapter II in [20], $\tilde{J}_1(t)\tilde{J}_2(t) = I$ holds for all $t \in [0, T-\rho]$, and hence it follows that $\tilde{J}_1(t) = \tilde{J}_2(t)^{-1}$. To simplify the notation, let $\tilde{J}_2(t) = I$ for $t < 0$. By Corollary 2 and Theorem 29 in Section 6 of Chapter 2 in [20] again, we have

$$\tilde{J}_2(T-\rho)D\tilde{X}(T-\rho)[h] = D\xi[h] + \int_0^{T-\rho} \tilde{J}_2(t-)\sigma(\rho+t, \tilde{X}(t-))dh(\tau(\rho+t)).$$

From this and (4.5.3) one can derive

$$\tilde{J}_2(T-\rho)D\tilde{X}(T-\rho)[h] = D\xi[h] + \int_\rho^T \tilde{J}_2((t-\rho)-)\sigma(t, X(t-))dh(\tau(t)).$$

Lemma 4.1.1 and the definition of ξ yield

$$\begin{aligned}
&\tilde{J}_2(T-\rho)D\tilde{X}(T-\rho)[h] \\
&= \sigma(\rho, X(\rho-))(h(\tau(\rho)) - h(\tau(T)-T_0)) \\
&\qquad\qquad + \int_{\tau(\rho)}^{\tau(T)} \tilde{J}_2((\tau^{-1}(t)-\rho)-)\sigma(\tau^{-1}(t), X(\tau^{-1}(t)-))\dot{h}(\tau^{-1}(t))dt \\
&= \int_{\tau(T)-T_0}^{\tau(\rho)} \sigma(\rho, X(\rho-))dh(t) \\
&\qquad\qquad + \int_{\tau(\rho)}^{\tau(T)} \tilde{J}_2((\tau^{-1}(t)-\rho)-)\sigma(\tau^{-1}(t), X(\tau^{-1}(t)-))\dot{h}(\tau^{-1}(t))dt \\
&= \int_{\tau(T)-T_0}^{\tau(T)} \tilde{J}_2((\tau^{-1}(t)-\rho)-)\sigma(\tau^{-1}(t), X(\tau^{-1}(t)-))\dot{h}(\tau^{-1}(t))dt.
\end{aligned}$$

Hence, if we denote the Malliavin covariance matrix $((D\tilde{X}^i(t), D\tilde{X}^j(t))_{H^*})_{ij}$ by $\tilde{\Delta}(t)$, then

$$\begin{aligned}
\tilde{\Delta}(T-\rho) \;=\; & \tilde{J}_1(T-\rho)\int_{\tau(T)-T_0}^{\tau(T)} \tilde{J}_2((\tau^{-1}(t)-\rho)-)\sigma(\tau^{-1}(t), X(\tau^{-1}(t)-)) \\
& \times {}^t\sigma(\tau^{-1}(t), X(\tau^{-1}(t)-))\, {}^t\tilde{J}_2((\tau^{-1}(t)-\rho)-)dt\, {}^t\tilde{J}_1(T-\rho),
\end{aligned}$$

and hence

$$\begin{aligned}
&\det(\tilde{\Delta}(T-\rho)) \\
&= \det(\tilde{J}_1(T-\rho))^2 \det\Bigg(\int_{\tau(T)-T_0}^{\tau(T)} \tilde{J}_2((\tau^{-1}(t)-\rho)-)\sigma(\tau^{-1}(t), X(\tau^{-1}(t)-)) \\
&\qquad\qquad\qquad \times {}^t\sigma(\tau^{-1}(t), X(\tau^{-1}(t)-))\, {}^t\tilde{J}_2((\tau^{-1}(t)-\rho)-)dt\Bigg).
\end{aligned}$$

Smilarly to the proof of Theorem 4.2.3, we conclude that for $p \geq 1$ and $\tau(T) > T_0$

$$\begin{aligned}
&E^{P(\cdot|\mathscr{F}^{\tau,T_0})}[\det(\tilde{\Delta}(T-\rho))^{-p}] \\
&\leq C_{16}(N, p, \varepsilon, r, \{||\nabla\sigma_k||_\infty\}_{1\leq k\leq r}, ||\nabla b||_\infty)T_0^{-Np} \\
&\qquad\qquad \times \exp\left[C_{16}(N, p, \varepsilon, r, \{||\nabla\sigma_k||_\infty\}_{1\leq k\leq r}, ||\nabla b||_\infty)(T+T_0)\right].
\end{aligned}$$

In the case $\tau(T) \leq T_0$ it holds that $\rho = 0$. Theorem 4.2.3 implies that for $p \geq 1$ and $\tau(T) \leq T_0$

$$\begin{aligned} E^{P(\cdot|\mathscr{F}^{\tau,T_0})}[\det(\tilde{\Delta}(T))^{-p}] &\leq C_{17}(N,p,\varepsilon,r,\{||\nabla\sigma_k||_\infty\}_{1\leq k\leq r},||\nabla b||_\infty)\tau(T)^{-Np} \\ &\quad \times \exp\left[C_{17}(N,p,\varepsilon,r,\{||\nabla\sigma_k||_\infty\}_{1\leq k\leq r},||\nabla b||_\infty)(T+T_0)\right]. \end{aligned}$$

For $p \geq 1$ and for all τ we have

$$\begin{aligned} E^{P(\cdot|\mathscr{F}^{\tau,T_0})}[\det(\tilde{\Delta}(T))^{-p}] &\leq C_{18}(N,p,\varepsilon,r,\{||\nabla\sigma_k||_\infty\}_{1\leq k\leq r},||\nabla b||_\infty)(T_0\wedge\tau(T))^{-Np} \\ &\quad \times \exp\left[C_{18}(N,p,\varepsilon,r,\{||\nabla\sigma_k||_\infty\}_{1\leq k\leq r},||\nabla b||_\infty)(T+T_0)\right]. \end{aligned}$$

Therefore, by (4.5.4) and Theorem 5.9 in [24], the law of $\tilde{X}(\rho)$ under $P(\cdot|\mathscr{F}^{\tau,T_0})$ has its density function p_{τ,T_0} belonging to $C_b^m(\mathbf{R}^N)$ P-almost surely and satisfying

$$\max_{0\leq l\leq m}\sup_{y\in\mathbf{R}^d}\left|\nabla_y^l p_{\tau,T_0}(T-\rho,\xi_0+\xi,y)\right| \leq c_1(T_0\wedge\tau(T))^{-c_3}\exp\{c_2(T+T_0)\}, \tag{4.5.5}$$

for some positive constants c_1, c_2, c_3 independent of ξ

By (4.5.3), (4.5.5), and the Markov property of X under $P(\cdot|\mathscr{F}^{\tau,T_0})$, we have that for $f \in C_b^\infty(\mathbf{R}^N)$ and a multi-index $\beta = (\beta_1,\beta_2,\ldots,\beta_d)$, $|\beta| \leq m$,

$$\begin{aligned} \left|E^P[\partial^\beta f(X(T))]\right| &= \left|E^P[E^P[\partial^\beta f(X(T))|\mathscr{F}^{\tau,T_0}]]\right| \\ &= \left|E^P\left[E^{P(\cdot|\mathscr{F}^{\tau,T_0})}[\partial^\beta f(\tilde{X}^{\tau,T_0}(T-\rho))]\right]\right| \\ &= \left|E^P\left[\int_{\mathbf{R}^N}\partial^\beta f(y)p_{\tau,T_0}(T-\rho,\xi_0+\xi,y)dy\right]\right| \\ &\leq E^P\left[\int_{\mathbf{R}^N}|f(y)|\,|\partial_y^\beta p_{\tau,T_0}(T-\rho,\xi_0+\xi,y)|dy\right] \\ &\leq c_1E^p[(T_0\wedge\tau(T))^{-c_3}]||f||_{L^1(\mathbf{R}^N)}\exp\{c_2(T+T_0)\}. \end{aligned}$$

Since τ is a one-sided $\alpha/2$-stable process and

$$E^P[\tau(T)^{-n}] = \int_0^\infty\int_{\eta_n}^\infty\cdots\int_{\eta_3}^\infty\int_{\eta_2}^\infty E^P[\exp(-\eta_1\tau(T))]d\eta_1d\eta_2\ldots d\eta_{n-1}d\eta_n,$$

we have

$$E^P[(\tau(T))^{-c_3}] < \infty.$$

Therefore applying Theorem 4.3.2 leads to the assertion of the theorem. $\square$

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