A Thesis for the Degree of Ph.D. in Science

Initial Boundary Value Problems for Model Equations of Drift Wave Turbulence

August 2011

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Contents

Chapter 1

Introduction

1.1 Physical background

Plasma is a mixture of ions, electrons and neutral particles, collectively electrically neutral, and usually permeated by macroscopic electrical and magnetic fields. In addition to these 'smoothed' or averaged electromagnetic fields, which with laboratory plasma are often imposed from outside the plasma volume, there are the microfields due to the individual particles.

There are four levels of description for a system of particles in magnetoplasma, namely (a) the individual particle orbits, (b) kinetic theory giving their collective behaviour in six-dimensional phase-space, (c) a multi-fluid model in which the separate species are treated as being distinct but interacting continua, and finally (d) a one-fluid description, which lumps the species into a single continuum with averaged properties ([61]).

Kinetic theory describes the behaviour of plasma in terms of the particle motions. Numerical solutions of the Fokker–Planck equation give the slowing down of a beam of particles injected into plasma and the resulting heating of ions and electrons. Approximate forms of the equation allow the calculation of the collisional transport of particles and energy. Magnetohydrodynamics (MHD) is used for the research of plasma equilibrium and MHD instabilities. It is noted that in this model the separate identities of ions and electrons do not appear ([60]).

In the 1940s, nuclear fusion, the process that powers the sun and other stars, was identified as a possible energy source and small groups carried out early experiments. In the 1950s larger organized efforts to explore the possibility of using fusion for peaceful purposes began in secrecy in Europe, the United States and the Soviet Union ([34]). Particularly since the 2nd Atoms for Peace Conference in Geneva in 1958, fusion scientists have shared the information including the details of the Soviet Tokamak designed in 1951 to enable them to learn from each other. However they found that the way to fusion power is more difficult, complex and costly than the first anticipated ([21], [36]). To accomplish the controlled nuclear fusion, we have to research the way to control the high temperature plasma.

The fusion of Deuterium and Tritium nuclei is the lowest ignition temperature, hence is the most suitable for fuel of nuclear fusion (Figure A.1 in Appendix A). In order to obtain the fusion of Deuterium and Tritium nuclei and the corresponding enormous release of energy, high collision speeds are required to overcome the Coulomb repulsion of the positively charged nuclei. One technique for confining the plasma is to use the interaction between the electrically conducting plasma and magnetic fields ([33]). Since plasma consists of positive and negative particles, it may be contained within a region away from the vessel walls by the forces of magnetic fields on the charged particles in the gas ([9]). Fusion machines of various type such as Stellarator, Tokamak and so on, have been operating in the Soviet Union, the US, the UK, Germany, France and Japan, and Tokamak is the most advanced magnetic confinement device until now.

In Tokamak an axisymmetric plasma is confined by a strong magnetic field (toroidal magnetic field) (see, for example, [38], [42] and Figure A.2 in Appendix A). There plasma is surrounded by a vacuum insulation to sustain the sufficiently high temperatures at which thermonuclear reactions take place ([23]). Thereupon plasma exhibits anomalously high levels of particle and energy transport. This enhanced transport of energy out of plasma is deleterious, since it reduces the energy containment time. Thus, understanding and controlling this transport are very important for controlled nuclear fusion ([55]). Moreover the research of turbulence in fusion plasma is important as well as its transport. Indeed, transport of plasma is greatly affected by the presence of plasma turbulence ([20]).

It is said that the drift wave turbulence is important in fusion plasma. High frequency electronic instability is no less dangerous in Tokamak since ions are less sensitive for high frequency electromagnetic fields. On the other hand low frequency instability can cause a loss of ions and are classified into three types: Reyleigh–Taylor instability, current driven instability and drift instability ([4]). In general, if the electric fields are set up in plasma by charge separation, both positive and negative particles obtain drift velocities. It has been well known that the spatial gradients in plasma lead to the drift waves, whose turbulence is a natural cause of anomalous transport from which the dramatic reduction in confinement results ([1], [4], [18]). Experimentally it was found that low frequency fluctuations in Tokamak turbulence plasmas are in the domain of drift waves ([64], [65]). Furthermore, a vast variety of plasma wave phenomena are found in the planet's magnetosphere where the anomalous transport occurs ([2], [7], [53]). Thereby the analysis of such drift wave turbulences is important from various point of view.

Tokamak is subject to a variety of macroscopic instabilities which can be attributed to identifiable MHD modes. The three principal instabilities are Mirnov oscillations, sawtooth oscillations, and disruptions ([60]). Mirnov oscillations are magnetic fluctuations mainly associated with the start-up phase of a Tokamak discharge ([17], Figure A.3 in Appendix A). This activity was first measured by Mirnov and Semenov with magnetic coils around the plasma surface. Sawtooth oscillations are expressed by periodic, coherent magnetic pulses, generated near the interior of the confined plasma (Figure A.4 in Appendix A). The magnetic perturbation is prominently accompanied by local variation in plasma temperature; the name "sawtooth" refers to the temporal shape of the X-ray signal, each period of which consists of a relatively slow rise followed by a very sharp drop. Although the two instabilities described above do not prevent satisfactory operation of Tokamak, disruptions involve a sudden loss of confinement and a rapid decay of the whole current, leading to an end of the discharge (Figure A.5 in Appendix A). Since there is at present no generally satisfactory theory of disruptions, the analysis of experimental results is mainly carried out in terms of the experimental operating conditions. It is considered that according to the experimental operating conditions, with enough space for standard operation of ITER disruption is almost completely avoided ([22]).

In addition to the instabilities predicted by analysis of the ideal or resistive MHD fluid equations, Tokamak plasma is susceptible to a number of other instabilities. Predictions of such instabilities came initially from studies using the Vlasov equation. These instabilities were termed microinstabilities. Tokamak microinstabilities are classified as follows: the drift instability, the trapped electron instability, the micro-tearing instability and the low frequency ion modes. The trapped electron instability relies on the presence of magnetically trapped, banana-orbiting electrons in the equilibrium. It is driven unstable either by an inverse Landau damping mechanism or by collisional dissipation of the trapped electrons. The micro-tearing instability is a form of short wavelength tearing mode. The perturbed magnetic fields associated with micro-tearing instabilities create magnetic islands in the magnetic surface structure of a Tokamak. It is considered that the turbulence must be of microinstability origin since the transport anomaly remains when Tokamaks operate in MHD stable regions ([60]).

In 1977 Hasegawa and Mima proposed the model equation of drift wave turbulence with zero resistivity (Hasegawa–Mima equation) from the one fluid model ([13], [14]). In 1983 Hasegawa and Wakatani proposed the equations (Hasegawa–Wakatani equations) from two fluids model, which describe the resistive drift wave turbulence in Tokamak ([12], [15], [16], [37]). They proposed these model equations based on the prediction that drift instability plays an important role in anomalous transport. It is noted that experimental investigations of the spectra of low-frequency plasma oscillations in American and French Tokamaks indicate that the observed oscillations result from the development of the drift instability ([5]). In 2005 Das, Sen, Kaw, Benkadda and Beyer ([6]) studied the magnetic-curvature-driven Rayleigh–Taylor instability for the plasma density, the electrostatic potential and the vector potential for electromagnetic perturbations and derived the model equations for it. Their model equations are an extended model of Hasegawa–Wakatani equations.

It is noteworthy that Hasegawa–Mima equation has a dipolar vortex solution, which is called modon ([19], [35]). Modon plays an important role in anomalous transport according to the following predictions of Nezlin and Snezhkin based on the experimental results ([41]). First, the drift instability of plasma in magnetic traps (either closed, like Tokamak, or open, like magnetic mirror) will generate monopolar vortical solitons which are larger compared to the characteristic ion Larmor radius and carry trapped particles, as well as dipolar vortices degenerating into monopolar solitons. Second, trapped particles will transfer from one vortex to another during the transition processes of streamline reconnection in the collisions (and merging) of vortical solitons, and also in the stationary regime on the borders between neighboring solitons, to that the particles will be carried over large distances across the strong magnetic field. Such processes can therefore give rise to a very large increase in transverse diffusion and heat transfer in the plasma.

1.2 Model equations of drift wave turbulence

In 1983 Hasegawa and Wakatani proposed the equations (Hasegawa–Wakatani equations) from two fluids model, which describe the resistive drift wave turbulence in Tokamak ([12], [15], [16], [37]). They consist of two nonlinear partial differential equations for the perturbations of plasma density n and electrostatic potential ϕ in the homogeneous strong magnetic field $\mathbf{B} = B_0\vec{e}$ and the inhomogeneous plasma equilibrium density $n^* = n^*(|x'|)$ $(x = (x_1, x_2, x_3) = (x', x_3))$

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) \Delta\phi = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2\Delta^2\phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) (n + \log n^*) = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n). \end{cases} \tag{1.2.1}$$

Here B_0 is the strength of a magnetic field assumed to be a constant, $\vec{e} = (0, 0, 1)$, $c_1 = T_e/(e^2\eta\omega_{ci})$, $c_2 = \mu/(\rho_s^2\omega_{ci})$, $\mu = 3T_i\nu_i/(10m_i\omega_{ci}^2)$ is the kinematic ion-viscosity coefficient, T_e is the electron temperature, T_i is the ion temperature, ν_i is the collision frequency of the ion, e is the elementary charge, η is the resistivity, m_i is the ion mass, $\omega_{ci} = eB_0/m_i$ is the cyclotoron frequency and $\rho_s = \sqrt{T_e}/(\omega_{ci}\sqrt{m_i})$ is the ion Larmor radius. Here for simplicity we assume that c_1 and c_2 are positive constants.

In 2005 Das *et al.* ([6]) studied the magnetic-curvature-driven Rayleigh–Taylor instability for the plasma density, the electrostatic potential and the vector potential for electromagnetic perturbations and derived the model equations for it. When neglecting the effect of electromagnetic perturbations and gravitational drift, the model equations yield

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) \Delta\phi = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2\Delta^2\phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) (n + \log n^*) \\ \qquad\qquad = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) + D\Delta(n + \log n^*). \end{cases} \tag{1.2.2}$$

Here $D = m_eT_e\nu_e/(eB_0)^2$ is the diffusion coefficient, m_e is the electron mass, ν_e is the collision frequency of the electron ([49], [50]). We also assume that D is a positive constant for simplicity.

In advance of Hasegawa–Wakatani equations Hasegawa and Mima in 1977 ([13], [14]) proposed the equation

$$\left(\frac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right)(\Delta\phi - \phi - \log n^*) = 0 \tag{1.2.3}$$

(Hasegawa–Mima equation) from the one fluid model under the same magnetic field and plasma equilibrium state. It is to be noted that Hasegawa–Mima equation has a dipolar vortex solution, which is called modon ([19], [35]). In a study of plasma turbulence, coherent vortex is an important research topic, since plasma turbulence may produce self-organized structures in the form of vortices, and indeed coherent vortices are observed in a variety of contents (see, for example, [25], [45], [54]). It is noteworthy that the same equation can be found in geophysics, Charney–Obukhov equation with respect to the quasi-geostrophic potential vorticity for Rossby wave ([3], [40], [44], [47]).

1.3 Related mathematical results

For Hasegawa–Mima equation we have had some mathematical results. For the initial value problem the temporally local existence and uniqueness of the strong solution and the temporally global existence of the weak solution were proved by Guo and Han [11] and Paumond [46] independently in 2004, and the global existence of a strong solution by Gao and Zhu [8] in 2005. The global in time existence and uniqueness of the solution and the existence of a global attractor to the initial boundary value problem for generalized Hasegawa–Mima equation with periodic boundary condition were proved by Zhang and Guo [62] in two dimensional case and [63] in three dimensional case. Furthermore, we have some stability proofs for modon (a dipolar vortex solution of Hasegawa–Mima equation) ([32], [48], [56]).

Concerning the mathematical issue of Hasegawa–Wakatani equations we have a few results [26], [27], [28]. In [26] we first established the existence and uniqueness of a strong global solution to the initial boundary value problems for (1.2.2), and second the existence and uniqueness of a strong solution to the initial boundary value problems for (1.2.1). In [27] and [28] we proved that the solution of Hasegawa-Wakatani equations converges strongly to that of the model equations of drift wave turbulence with zero resistivity as the resistivity tends to zero.

It is noted that concerning the model equations of Das *et al.* ([6]), it seems to be no mathematical results.

1.4 Formulation of the problems

We consider the initial boundary value problems for (1.2.2) first and for (1.2.1) second in $\Omega \times (0, \infty)$ under the initial and the boundary conditions

$$\begin{cases} \phi(x,0) = \phi_0(x), \quad n(x,0) = n_0(x) \quad \text{for } x \in \Omega, \\ \phi(x,t) = \Delta\phi(x,t) = n(x,t) = 0 \quad \text{for } x \in \Gamma,\ t > 0, \\ \phi,\ n, \quad \text{periodic in the } x_3\text{-direction}. \end{cases} \tag{1.4.1}$$

Here $\Omega = \omega \times (-L, L)$ is a 3-dimensional torus, $\omega = \{x' = (x_1, x_2) \in \mathbf{R}^2 \mid |x'| < R\}$, $\partial\omega = \{x' = (x_1, x_2) \in \mathbf{R}^2 \mid |x'| = R\}$, $\Gamma = \partial\omega \times [-L, L]$, R and L are positive real numbers (see Figure A.6 in Appendix A).

In place of $n(x,t) + \log n^*(|x'|) - \log n^*(R)$ and $n_0(x) + \log n^*(|x'|) - \log n^*(R)$, we use the same letters $n(x,t)$ and $n_0(x)$, respectively. Then equations (1.2.1) and (1.2.2) become

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) \Delta\phi = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2 \Delta^2 \phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) n = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) \quad \text{for } x \in \Omega,\ t > 0 \end{cases} \tag{1.4.2}$$

and

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) \Delta\phi = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2 \Delta^2 \phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) n = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) + D\Delta n \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in \Omega,\ t > 0, \end{cases} \tag{1.4.3}$$

respectively, but (1.4.1) is unchanged.

Next by denoting $\varepsilon = 1/c_1$, (1.4.2) is clearly written as

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla\right)(\Delta\phi^\varepsilon - n^\varepsilon) = c_2\Delta^2\phi^\varepsilon, \\ \varepsilon\left(\dfrac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla\right) n^\varepsilon = -\dfrac{1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\phi^\varepsilon - n^\varepsilon) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in \Omega,\ t > 0, \end{cases} \tag{1.4.4}$$

Let $(\phi^\varepsilon, n^\varepsilon) = (\phi^\varepsilon, n^\varepsilon)(x,t)$ be a solution of the initial boundary value problem (1.4.4) with $\varepsilon > 0$ for $x \in \Omega$, $t > 0$ and the initial-boundary conditions

$$\begin{cases} \phi^\varepsilon(x,0) = \phi_0^\varepsilon(x),\ n^\varepsilon(x,0) = n_0^\varepsilon(x) \quad \text{for } \ x \in \Omega, \\ \phi^\varepsilon(x,t) = \Delta\phi^\varepsilon(x,t) = n^\varepsilon(x,t) = 0 \quad \text{for } \ x \in \Gamma,\ t > 0, \\ \phi^\varepsilon,\ n^\varepsilon, \quad \text{periodic in the } x_3\text{-direction.} \end{cases} \tag{1.4.5}$$

For convenience, we introduce

$$\begin{cases} \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x', x_3)\ \mathrm{d}x_3 = \mathcal{M}f(x) = \overline{f}(x), \\ \widetilde{f}(x) = f(x) - \mathcal{M}f(x) \equiv (\mathcal{I} - \mathcal{M})\, f(x). \end{cases}$$

Then it is easily seen that problem (1.4.4), (1.4.5) is equivalent to the problem

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla\right)(\Delta\phi^\varepsilon - n^\varepsilon) = c_2\Delta^2\phi^\varepsilon, \\ \varepsilon\,(\mathcal{I} - \mathcal{M})\left\{\left(\dfrac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla\right) n^\varepsilon\right\} = -\dfrac{1}{n^*}\dfrac{\partial^2}{\partial x_3^2}\left(\widetilde{\phi^\varepsilon} - \widetilde{n^\varepsilon}\right), \\ \mathcal{M}\left\{\left(\dfrac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla\right) n^\varepsilon\right\} = 0 \quad \text{for } x \in \Omega,\ t > 0, \end{cases}$$

and (1.4.5).

Putting $\varepsilon = 0$ in this problem, we have

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \left(\nabla\phi^0 \times \vec{e}\right) \cdot \nabla\right)(\Delta\phi^0 - n^0) = c_2\Delta^2\phi^0, \\ \dfrac{1}{n^*}\dfrac{\partial^2}{\partial x_3^2}\left(\widetilde{\phi^0} - \widetilde{n^0}\right) = 0, \\ \mathcal{M}\left\{\left(\dfrac{\partial}{\partial t} - \left(\nabla\phi^0 \times \vec{e}\right) \cdot \nabla\right) n^0\right\} = 0 \quad \text{for } x \in \Omega,\ t > 0, \end{cases} \tag{1.4.6}$$

and (1.4.5) with $\varepsilon = 0$.

In Chapter 2 first we establish the existence and uniqueness of a strong global solution to problem (1.4.3), (1.4.1). Second we establish the existence and uniqueness of a strong solution to problem (1.4.2), (1.4.1) (see [26]). In Chapter 3 first we obtain uniform estimates to the solutions for Hasegawa-Wakatani equations with respect to the resistivity. Then we establish the existence and uniqueness of a strong solution to the problem (1.4.6), (1.4.5) with $\varepsilon = 0$. Finally we prove that the solution of Hasegawa-Wakatani equations converges strongly to that of the model equations of drift wave turbulence with zero resistivity as the resistivity tends to zero (see [27]).

1.5 Function spaces

We aim to solve the problems (1.4.3), (1.4.1) and (1.4.2), (1.4.1) and (1.4.6), (1.4.5) with $\varepsilon = 0$ in Sobolev–Slobodetskiĭ spaces. Before describing the main theorem we introduce the function spaces that we use in the sequel.

Let Ω be a domain in $\mathbf{R}^m$ ($m = 1, 2, 3, \ldots$). By $W_2^l(\Omega)$ ($l \in \mathbf{R},\ l \geq 0$) we denote the space of functions $u(x),\ x \in \Omega$, equipped with the finite norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{|\alpha|<l} \|D_x^\alpha u\|^2_{L^2(\Omega)} + \|u\|^2_{\dot{W}_2^l(\Omega)},$$

where

$$\|u\|^2_{\dot{W}_2^l(\Omega)} = \begin{cases} \displaystyle\sum_{|\alpha|=l} \|D_x^\alpha u\|^2_{L^2(\Omega)} & \text{if } l \in \mathbf{Z}, \\ \displaystyle\sum_{|\alpha|=[l]} \int_\Omega \int_\Omega \frac{|D_x^\alpha u(x) - D_y^\alpha u(y)|^2}{|x-y|^{m+2(l-[l])}} \,\mathrm{d}x\,\mathrm{d}y & \text{if } l \notin \mathbf{Z}. \end{cases}$$

Here, $[l]$ is the integral part of l, and $D_x^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}$ is the generalized derivative of order $|\alpha| = \alpha_1 + \ldots + \alpha_m$, and $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a multi-index. For $1 \leq p \leq \infty$, we denote by $\|\cdot\|_{L^p(\Omega)}$ the norm of the Lebesgue space $L^p(\Omega)$.

Similarly, the norm of the space $W_2^l(0,T)$ ($T \in \mathbf{R},\ T>0$) is defined by

$$\|u\|^2_{W_2^l(0,T)} = \begin{cases} \displaystyle\sum_{j=0}^{l} \|D_t^j u\|^2_{L^2(0,T)} \quad \text{if } \ l \in \mathbf{Z}, \\ \displaystyle\sum_{j=0}^{l} \|D_t^j u\|^2_{L^2(0,T)} \\ \qquad \displaystyle + \int_0^T \int_0^T \frac{|D_t^{[l]} u(t) - D_\tau^{[l]} u(\tau)|^2}{|t-\tau|^{1+2(l-[l])}} \, \mathrm{d}t \, \mathrm{d}\tau \quad \text{if } \ l \notin \mathbf{Z}. \end{cases}$$

The anisotropic Sobolev–Slobodetskiĭ space $W_2^{l,l/2}(Q_T)$ ($Q_T \equiv \Omega \times (0,T)$) is defined as $L^2(0,T; W_2^l(\Omega)) \cap L^2(\Omega; W_2^{l/2}(0,T))$, equipped with the finite norm

$$\begin{aligned} \|u\|^2_{W_2^{l,l/2}(Q_T)} &= \|u\|^2_{W_2^{l,0}(Q_T)} + \|u\|^2_{W_2^{0,l/2}(Q_T)} \\ &\equiv \int_0^T \|u(t)\|^2_{W_2^l(\Omega)} \, \mathrm{d}t + \int_\Omega \|u(x)\|^2_{W_2^{l/2}(0,T)} \, \mathrm{d}x. \end{aligned}$$

Chapter 2

Initial boundary value problem for model equations of resistive drift wave turbulence

2.1 Main results

In this chapter the following theorems are proved.

Theorem 2.1.1 *Let* $1 \geq l > 1/2$, $D > 0$, $n^*(|x'|) \in W_2^{1+l}(\omega)$ *and* $n^*(|x'|) \geq n_*$, n_* *is a positive constant. Assume that* $(\phi_0, n_0) \in W_2^{3+l}(\Omega) \times W_2^{1+l}(\Omega)$ *satisfies the compatibility conditions*

$$\begin{cases} \phi_0(x) = \Delta\phi_0(x) = n_0(x) = 0 \quad \text{for } x \in \Gamma, \\ \\ \phi_0, n_0, \quad \text{periodic in the } x_3\text{-direction.} \end{cases} \tag{2.1.1}$$

Then the initial boundary value problem (1.4.3), (1.4.1) *has a unique solution* $(\phi, n) \in \Big(L^2(0,T; W_2^{4+l}(\Omega)) \cap W_2^{1+l/2}(0,T; W_2^2(\Omega))\Big) \times W_2^{2+l,1+l/2}(Q_T)$ *for any*

$T > 0$.

Theorem 2.1.2 *Let $n^*(|x'|) \in W_2^2(\omega)$ and $n^*(|x'|) \geq n_*$, n_* is a positive constant. Assume that $(\phi_0, n_0) \in W_2^4(\Omega) \times W_2^2(\Omega)$ satisfies the compatibility conditions* (2.1.1). *Then there exists a unique solution (ϕ, n) to problem* (1.4.2), (1.4.1) *on some interval $[0, T]$ such that $(\phi, n) \in L^2(0, T; W_2^4(\Omega)) \times W_2^{2,1}(Q_T)$, $\partial\phi/\partial t \in L^2(0, T; W_2^2(\Omega))$.*

In §2.2 we shall prove Theorem 2.1.1. In §2.3 we first derive the uniform estimates of the solution to problem (1.4.3), (1.4.1). Following the arguments due to Kato ([26], [43]) by passing to the limit $D \to 0$, we shall prove Theorem 2.1.2.

As a natural extension of Theorems 2.1.1 and 2.1.2, we can consider the initial boundary value problem with Stepanov-almost-periodic initial data to the magnetic field direction (see [29]).

Throughout this chapter, we denote by c a constant which may differ at each occurrence.

2.2 Existence theorem for model equations of resistive drift wave turbulence

2.2.1 Local in time existence and uniqueness

The following lemmas are well-known (see, for example, [10], [30], [31], [39], [51], [59]).

Lemma 2.2.1 *Let* $l \geq 0$, $D > 0$, $n^*(|x'|) \in W_2^{1+l}(\omega)$, $n^*(|x'|) \geq n_*$, n_* *is a positive constant and* $0 < T < \infty$. *Assume that* $(\psi_0, n_0) \in W_2^{1+l}(\Omega) \times W_2^{1+l}(\Omega)$ *satisfies the compatibility conditions up to order* $\max\{[l - 3/2], 0\}$. *Then for any* $(f, g) \in W_2^{l,l/2}(Q_T) \times W_2^{l,l/2}(Q_T)$, *there exists a unique solution* $(\psi, n) \in W_2^{2+l,1+l/2}(Q_T) \times W_2^{2+l,1+l/2}(Q_T)$ *to the problem*

$$\begin{cases} \dfrac{\partial \psi}{\partial t} - c_2 \Delta \psi - \dfrac{c_1}{n^*} \dfrac{\partial^2 n}{\partial x_3^2} = f \quad \textit{for} \;\; x \in \Omega, \;\; t > 0, \\ \dfrac{\partial n}{\partial t} - \dfrac{c_1}{n^*} \dfrac{\partial^2 n}{\partial x_3^2} - D \Delta n = g \quad \textit{for} \;\; x \in \Omega, \;\; t > 0, \\ \psi(x, 0) = \psi_0(x), \quad n(x, 0) = n_0(x) \quad \textit{for} \;\; x \in \Omega, \\ \psi(x, t) = n(x, t) = 0 \quad \textit{for} \; x \in \Gamma, \;\; t > 0, \\ \psi, n, \quad \textit{periodic in the } x_3\textit{-direction.} \end{cases}$$

Moreover, this solution satisfies the inequality

$$\begin{aligned} &\|\psi\|_{W_2^{2+l,1+l/2}(Q_T)} + \|n\|_{W_2^{2+l,1+l/2}(Q_T)} \\ &\quad \leq c \left(\|\psi_0\|_{W_2^{1+l}(\Omega)} + \|n_0\|_{W_2^{1+l}(\Omega)} + \|f\|_{W_2^{l,l/2}(Q_T)} + \|g\|_{W_2^{l,l/2}(Q_T)} \right). \end{aligned}$$

Lemma 2.2.2 *Assume that* $\psi \in W_2^{2+l,1+l/2}(Q_T),\ l \geq 0$. *Then the problem*

$$\begin{cases} \Delta\phi = \psi \quad \textit{for } x \in \Omega,\ t > 0, \\ \phi(x,t) = 0 \quad \textit{for } x \in \Gamma,\ t > 0, \\ \phi, \quad \textit{periodic in the } x_3\textit{-direction} \end{cases}$$

has a unique solution $\phi \in L^2(0,T; W_2^{4+l}(\Omega)) \cap W_2^{1+l/2}(0,T; W_2^2(\Omega))$, *which satisfies the inequality*

$$\|\phi\|_{L^2(0,T;W_2^{4+l}(\Omega))} + \|\phi\|_{W_2^{1+l/2}(0,T;W_2^2(\Omega))} \leq c\|\psi\|_{W_2^{2+l,1+l/2}(Q_T)}.$$

Let us reduce problem (1.4.3), (1.4.1) to the problem with zero initial data. According to Lemmas 2.2.1 and 2.2.2, there exist $\phi^\star \in L^2(0,T; W_2^{4+l}(\Omega)) \cap W_2^{1+l/2}(0,T; W_2^2(\Omega))$, $n^\star \in W_2^{2+l,1+l/2}(Q_T)$ satisfying

$$\begin{cases} \dfrac{\partial \Delta\phi^\star}{\partial t} - c_2\Delta^2\phi^\star - \dfrac{c_1}{n^*}\dfrac{\partial^2 n^\star}{\partial x_3^2} = 0 \quad \text{for } x \in \Omega,\ t > 0, \\ \dfrac{\partial n^\star}{\partial t} - \dfrac{c_1}{n^*}\dfrac{\partial^2 n^\star}{\partial x_3^2} - D\Delta n^\star = 0 \quad \text{for } x \in \Omega,\ t > 0, \\ \phi^\star(x,0) = \phi_0(x), \quad n^\star(x,0) = n_0(x) \quad \text{for } x \in \Omega, \\ \phi^\star(x,t) = \Delta\phi^\star(x,t) = n^\star(x,t) = 0 \quad \text{for } x \in \Gamma,\ t > 0, \\ \phi^\star, n^\star, \quad \text{periodic in the } x_3\text{-direction} \end{cases}$$

and the inequality

$$\begin{aligned} &\|\phi^\star\|_{L^2(0,T;W_2^{4+l}(\Omega))} + \|\phi^\star\|_{W_2^{1+l/2}(0,T;W_2^2(\Omega))} + \|n^\star\|_{W_2^{2+l,1+/l/2}(Q_T)} \\ &\quad \leq c\left(\|\phi_0\|_{W_2^{3+l}(\Omega)} + \|n_0\|_{W_2^{1+l}(\Omega)}\right). \end{aligned} \tag{2.2.1}$$

By putting $\Phi \equiv \phi - \phi^\star$, $N \equiv n - n^\star$, the problem (1.4.3), (1.4.1) is equivalent to the problem

$$\begin{cases} \dfrac{\partial \Delta\Phi}{\partial t} - c_2\Delta^2\Phi - \dfrac{c_1}{n^*}\dfrac{\partial^2 N}{\partial x_3^2} = -\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\Phi + \phi^\star) \\ \qquad +(\nabla(\Phi+\phi^\star)\times\vec{e})\cdot\nabla\Delta(\Phi+\phi^\star) \equiv F(\Phi), \\ \dfrac{\partial N}{\partial t} - \dfrac{c_1}{n^*}\dfrac{\partial^2 N}{\partial x_3^2} - D\Delta N = -\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\Phi+\phi^\star) \\ \qquad +(\nabla(\Phi+\phi^\star)\times\vec{e})\cdot\nabla(N+n^\star) \equiv G(\Phi, N) \quad \text{for } x\in\Omega,\ t>0, \end{cases} \tag{2.2.2}$$

$$\begin{cases} \Phi(x,0) = N(x,0) = 0 \quad \text{for } x\in\Omega, \\ \Phi(x,t) = \Delta\Phi(x,t) = N(x,t) = 0 \quad \text{for } x\in\Gamma,\ t>0, \\ \Phi, N, \quad \text{periodic in the } x_3\text{-direction.} \end{cases} \tag{2.2.3}$$

We solve problem (2.2.2), (2.2.3) by the method of successive approximations. Let $\left(\Phi^{(0)}, N^{(0)}\right) = (0,0)$ and $\left(\Phi^{(m+1)}, N^{(m+1)}\right)$ $(m = 0,1,2,\ldots)$ be a solution of the initial boundary value problem

$$\begin{cases} \dfrac{\partial\Delta\Phi^{(m+1)}}{\partial t} - c_2\Delta^2\Phi^{(m+1)} - \dfrac{c_1}{n^*}\dfrac{\partial^2 N^{(m+1)}}{\partial x_3^2} = F(\Phi^{(m)}), \\ \dfrac{\partial N^{(m+1)}}{\partial t} - \dfrac{c_1}{n^*}\dfrac{\partial^2 N^{(m+1)}}{\partial x_3^2} - D\Delta N^{(m+1)} = G(\Phi^{(m)}, N^{(m)}) \\ \qquad\qquad \text{for } x\in\Omega,\ t>0, \end{cases} \tag{2.2.4}$$

$$\begin{cases} \Phi^{(m+1)}(x,0) = N^{(m+1)}(x,0) = 0 \quad \text{for } x\in\Omega, \\ \Phi^{(m+1)}(x,t) = \Delta\Phi^{(m+1)}(x,t) = N^{(m+1)}(x,t) = 0 \\ \qquad\qquad \text{for } x\in\Gamma,\ t>0, \\ \Phi^{(m+1)}, N^{(m+1)}, \quad \text{periodic in the } x_3\text{-direction,} \end{cases} \tag{2.2.5}$$

where $(\Phi^{(m)}, N^{(m)}) \in \left(L^2(0,T;W_2^{4+l}(\Omega)) \cap W_2^{1+l/2}(0,T;W_2^2(\Omega))\right) \times W_2^{2+l,1+l/2}(Q_T)$ is given.

It is easily seen that problem (2.2.4), (2.2.5) is equivalent to the problem

$$\begin{cases} \Delta\Phi^{(m+1)} = \Psi^{(m+1)}, \\ \dfrac{\partial\Psi^{(m+1)}}{\partial t} - c_2\Delta\Psi^{(m+1)} - \dfrac{c_1}{n^*}\dfrac{\partial^2 N^{(m+1)}}{\partial x_3^2} = F(\Phi^{(m)}, \Psi^{(m)}), \\ \dfrac{\partial N^{(m+1)}}{\partial t} - \dfrac{c_1}{n^*}\dfrac{\partial^2 N^{(m+1)}}{\partial x_3^2} - D\Delta N^{(m+1)} = G(\Phi^{(m)}, N^{(m)}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in \Omega,\ t > 0, \end{cases} \tag{2.2.6}$$

$$\begin{cases} \Phi^{(m+1)}(x,0) = \Psi^{(m+1)}(x,0) = N^{(m+1)}(x,0) = 0 \quad \text{for } x \in \Omega, \\ \Phi^{(m+1)}(x,t) = \Psi^{(m+1)}(x,t) = N^{(m+1)}(x,t) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in \Gamma,\ t > 0, \\ \Phi^{(m+1)}, \Psi^{(m+1)}, N^{(m+1)}, \quad \text{periodic in the } x_3\text{-direction}, \end{cases} \tag{2.2.7}$$

where $(\Phi^{(m)}, \Psi^{(m)}, N^{(m)}) \in \left(L^2(0,T;W_2^{4+l}(\Omega)) \cap W_2^{1+l/2}(0,T;W_2^2(\Omega))\right) \times W_2^{2+l,1+l/2}(Q_T) \times W_2^{2+l,1+l/2}(Q_T)$ is given and

$$F(\Phi,\Psi) \equiv -\frac{c_1}{n^*}\frac{\partial^2(\Phi+\phi^\star)}{\partial x_3^2} + (\nabla(\Phi+\phi^\star)\times\vec{e})\cdot\nabla(\Psi+\Delta\phi^\star).$$

By applying Lemmas 2.2.1 and 2.2.2, a unique solution $(\Phi^{(m+1)}, \Psi^{(m+1)}, N^{(m+1)})$ exists and satisfies the inequality

$$\begin{aligned} z^{(m+1)}(T) \ &\equiv\ \|\Phi^{(m+1)}\|_{L^2(0,T;W_2^{4+l}(\Omega))} + \|\Phi^{(m+1)}\|_{W_2^{1+l/2}(0,T;W_2^2(\Omega))} \\ &\quad + \|\Psi^{(m+1)}\|_{W_2^{2+l,1+l/2}(Q_T)} + \|N^{(m+1)}\|_{W_2^{2+l,1+l/2}(Q_T)} \\ &\leq\ c\left(\|F(\Phi^{(m)},\Psi^{(m)})\|_{W_2^{l,l/2}(Q_T)} + \|G(\Phi^{(m)},N^{(m)})\|_{W_2^{l,l/2}(Q_T)}\right). \end{aligned}$$

In order to estimate each term in the right hand side of the above inequality we use the following well-known lemma (see, for example, [52]).

Lemma 2.2.3 *Let $l > 1/2$ and Ω be a bounded domain in $\mathbf{R}^3$. Then the following estimates hold.*

$$\|fg\|_{W_2^l(\Omega)} \le c\|f\|_{W_2^l(\Omega)}\|g\|_{W_2^{1+l}(\Omega)} \quad \textit{for } f \in W_2^l(\Omega),\ g \in W_2^{1+l}(\Omega),$$

$$\|f\nabla g\|_{L^2(\Omega)} \le c\|f\|_{L^6(\Omega)}\|\nabla g\|_{L^3(\Omega)}$$

$$\le c\|f\|_{W_2^1(\Omega)}\|g\|_{W_2^2(\Omega)}^{3/4}\|g\|_{L^2(\Omega)}^{1/4} \quad \textit{for } f \in W_2^1(\Omega),\ g \in W_2^2(\Omega).$$

From this lemma and the interpolation and Young's inequalities, it easily follows

$$\|F(\Phi^{(m)}, \Psi^{(m)})\|_{W_2^{l,l/2}(Q_T)}$$

$$\le c\left\|\frac{\partial^2\phi^\star}{\partial x_3^2}\right\|_{W_2^{l,l/2}(Q_T)} + \delta\left(\|\Phi^{(m)}\|_{W_2^{4+l,0}(Q_T)} + \|\Phi^{(m)}\|_{W_2^{1+l/2}(0,T;W_2^2(\Omega))}\right)$$

$$+C(\delta)T\left(\|\Phi^{(m)}\|_{W_2^{0,1}(Q_T)} + \|\Phi^{(m)}\|_{W_2^1(0,T;W_2^2(\Omega))}\right)$$

$$+cT^{3/4}\|\Phi^{(m)} + \phi^\star\|_{W_2^1(0,T;W_2^{2+l}(\Omega))}\|\Psi^{(m)} + \Delta\phi^\star\|_{W_2^1(0,T;W_2^l(\Omega))}^{1/4}$$

$$\times\|\Psi^{(m)} + \Delta\phi^\star\|_{W_2^{2+l,0}(Q_T)}^{3/4}$$

$$+cT^{3/4}\|\Phi^{(m)} + \phi^\star\|_{W_2^{1+l/2}(0,T;W_2^2(\Omega))}\|\Psi^{(m)} + \Delta\phi^\star\|_{W_2^{1+l/2}(0,T;L^2(\Omega))}^{1/4}$$

$$\times\|\Psi^{(m)} + \Delta\phi^\star\|_{W_2^{l/2}(0,T;W_2^2(\Omega))}^{3/4}$$

for any $\delta > 0$. For $\|G(\Phi^{(m)}, N^{(m)})\|_{W_2^{l,l/2}(Q_T)}^2$ one can get the similar estimate.

Consequently from these inequalities and (2.2.1), we obtain

$$z^{(m+1)}(T) \le cE_l + cE_l^2T^{3/4} + \left(\delta + C(\delta)T + cE_lT^{3/4}\right)z^{(m)}(T) + cT^{3/4}z^{(m)}(T)^2,$$

where $E_l \equiv \|\phi_0\|_{W_2^{3+l}(\Omega)} + \|n_0\|_{W_2^{1+l}(\Omega)}$. We choose first a positive constant M in such a way that $M > cE_l$, second a positive constant δ so that $\delta M < M - cE_l$ and finally a positive constant T' so that $cE_l^2T'^{3/4} +$

$\left(C(\delta)T' + cE_l T'^{3/4}\right) M + cT'^{3/4}M^2 < M - cE_l - \delta M$. Then we conclude that $z^{(m)}(T') < M$ implies $z^{(m+1)}(T') < M$. This means that the sequence $\left\{\left(\Phi^{(m)}, \Psi^{(m)}, N^{(m)}\right)\right\}_{m=0}^{\infty}$ is well-defined on $(0, T')$ and $z^{(m)}(T') < M$ for all m.

Now let us verify the convergence of $\left\{\left(\Phi^{(m)}, \Psi^{(m)}, N^{(m)}\right)\right\}_{m=0}^{\infty}$. Subtract from (2.2.6) the similar equations for $(\Phi^{(m)}, \Psi^{(m)}, N^{(m)})$ and set $\Phi^{*(m+1)} \equiv \Phi^{(m+1)} - \Phi^{(m)}$, $\Psi^{*(m+1)} \equiv \Psi^{(m+1)} - \Psi^{(m)}$, $N^{*(m+1)} \equiv N^{(m+1)} - N^{(m)}$. Then we have

$$\begin{cases} \Delta\Phi^{*(m+1)} = \Psi^{*(m+1)}, \\ \dfrac{\partial\Psi^{*(m+1)}}{\partial t} - c_2\Delta\Psi^{*(m+1)} - \dfrac{c_1}{\overline{n}}\dfrac{\partial^2 N^{*(m+1)}}{\partial x_3^2} \\ \qquad\qquad = F(\Phi^{(m)}, \Psi^{(m)}) - F(\Phi^{(m-1)}, \Psi^{(m-1)}), \\ \dfrac{\partial N^{*(m+1)}}{\partial t} - \dfrac{c_1}{\overline{n}}\dfrac{\partial^2 N^{*(m+1)}}{\partial x_3^2} - D\Delta N^{*(m+1)} \\ \qquad = G(\Phi^{(m)}, N^{(m)}) - G(\Phi^{(m-1)}, N^{(m-1)}) \quad \text{for } x \in \Omega,\ T' > t > 0, \end{cases}$$

$$\begin{cases} \Phi^{*(m+1)}(x,0) = \Psi^{*(m+1)}(x,0) = N^{*(m+1)}(x,0) = 0 \quad \text{for } x \in \Omega, \\ \Phi^{*(m+1)}(x,t) = \Psi^{*(m+1)}(x,t) = N^{*(m+1)}(x,t) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in \Gamma,\ T' > t > 0, \\ \Phi^{*(m+1)}, \Psi^{*(m+1)}, N^{*(m+1)}, \quad \text{periodic in the } x_3\text{-direction.} \end{cases}$$

By the same way as above, we can deduce

$$\begin{aligned} Z^{(m+1)}(t) &\equiv \|\Phi^{*(m+1)}\|_{L^2(0,t;W_2^{4+l}(\Omega))} + \|\Phi^{*(m+1)}\|_{W_2^{1+l/2}(0,t;W_2^2(\Omega))} \\ &\quad + \|\Psi^{*(m+1)}\|_{W_2^{2+l,1+l/2}(Q_t)} + \|N^{*(m+1)}\|_{W_2^{2+l,1+l/2}(Q_t)} \\ &\leq \left(\delta + C(\delta)t + c\left(E_l + z^{(m)}(t)\right) t^{3/4}\right) Z^{(m)}(t) \end{aligned} \tag{2.2.8}$$

for any $\delta > 0$ and any $t \in [0, T']$. Since we can find positive constants δ and $T''(\leq T')$ satisfying $\left(\delta + C(\delta)T'' + c\,(E_l + M)\,T''^{3/4}\right) < 1$, the sequence $\left\{\left(\Phi^{(m)}, \Psi^{(m)}, N^{(m)}\right)\right\}$ converges uniformly on $[0, T'']$ to (Φ, Ψ, N) as $m \to \infty$.

It is clear that $(\Phi, \Psi, N) \in \Big(L^2(0,T''; W_2^{4+l}(\Omega)) \cap W_2^{1+\frac{l}{2}}(0,T''; W_2^2(\Omega))\Big) \times W_2^{2+l,1+\frac{l}{2}}(Q_{T''}) \times W_2^{2+l,1+\frac{l}{2}}(Q_{T''})$ and (Φ, N) is a solution of problem (2.2.2), (2.2.3).

The uniqueness of such a solution can be easily proved by making use of the estimate analogous to (2.2.8).

2.2.2 *A priori* estimates

In this subsection we proceed to get *a priori* estimates of the solution (ϕ, n) established in §2.2.1. Let $l \in (0,1]$ and (ϕ, n) be a solution of (1.4.3), (1.4.1) belonging to $\Big(L^2(0,T; W_2^{4+l}(\Omega)) \cap W_2^{1+\frac{l}{2}}(0,T; W_2^2(\Omega))\Big) \times W_2^{2+l,1+\frac{l}{2}}(Q_T)$ for any $T > 0$. By $\|\cdot\|$ we denote the $L^2(\Omega)$-norm. Since the regularity of the solution is not sufficient, the following arguments are formal. However, one can justify them by using the method of difference quotients or mollifiers. It is to be noted that the estimates obtained in this subsection are improved from [26].

Lemma 2.2.4 *For any* $t \in [0,T]$

$$\|\nabla\phi(t)\|^2 + \|\Delta\phi(t)\|^2 + \|n(t)\|^2 \le E_0^2 \mathrm{e}^{-c^\sharp t}, \tag{2.2.9}$$

$$\int_0^t \left(\|\nabla\Delta\phi(\tau)\|^2 + \left\| \frac{\partial n}{\partial x_3}(\tau) \right\|^2 \right) \mathrm{d}\tau \le c E_0^2. \tag{2.2.10}$$

Here c *is a constant independent of* D *and* $c^\sharp \equiv \min\{c_2, D\}$.

Proof. Multiplying the first equation of (1.4.3) by ϕ and integrating over Ω, we have, by virtue of the integration by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi(t)\|^2 + c_2\|\Delta\phi(t)\|^2 + \int_\Omega \frac{c_1}{n^*}\frac{\partial(\phi - n)}{\partial x_3}\frac{\partial\phi}{\partial x_3}\ \mathrm{d}x = 0. \tag{2.2.11}$$

Multiplying the second equations of (1.4.3) by n and integrating over Ω, we have, by virtue of the integration by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n(t)\|^2 + D\,\|\nabla n(t)\|^2 - \int_\Omega \frac{c_1}{n^*}\frac{\partial(\phi - n)}{\partial x_3}\frac{\partial n}{\partial x_3}\ \mathrm{d}x = 0. \tag{2.2.12}$$

Adding (2.2.12) and (2.2.13) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\phi(t)\|^2+\|n(t)\|^2\right)+c_2\|\Delta\phi(t)\|^2+D\,\|\nabla n(t)\|^2$$

$$+\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial(\phi-n)}{\partial x_3}(t)\right\|^2=0. \tag{2.2.13}$$

In the similar way, multiplying the first equation of (1.4.3) by $\Delta\phi$ and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\phi(t)\|^2+c_2\|\nabla\Delta\phi(t)\|^2$$

$$\le\frac{c_2}{2}\left\|\frac{\partial\Delta\phi}{\partial x_3}(t)\right\|^2+\frac{c}{c_2}\left\|\frac{\partial(\phi-n)}{\partial x_3}(t)\right\|^2. \tag{2.2.14}$$

Adding (2.2.13) and (2.2.14) multiplied by ε yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\phi(t)\|^2+\varepsilon\|\Delta\phi(t)\|^2+\|n(t)\|^2\right)+c_2\left(\|\Delta\phi(t)\|^2+\varepsilon\|\nabla\Delta\phi(t)\|^2\right)$$

$$+D\,\|\nabla n(t)\|^2+\left(1-\varepsilon\frac{c_2}{2}\right)\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial(\phi-n)}{\partial x_3}(t)\right\|^2\le 0. \tag{2.2.15}$$

Here we take ε small enough to hold the inequality $1-\varepsilon c_2/2>0$.

Put

$$S_1(t)\equiv\|\nabla\phi(t)\|^2+\varepsilon\|\Delta\phi(t)\|^2+\|n(t)\|^2.$$

Then $S_1(t)$ satisfies the differential inequality

$$\frac{\mathrm{d}S_1(t)}{\mathrm{d}t}+c^\sharp S_1(t)\le 0.$$

From this one can derive (2.2.9). Integrating (2.2.15) over $[0,t]$ with the help of (2.2.9), we have (2.2.10). □

Lemma 2.2.5 *For any* $t\in[0,T]$

$$\|\nabla\Delta\phi(t)\|^2+\|\nabla n(t)\|^2+\int_0^t\left(\left\|\Delta^2\phi(\tau)\right\|^2+\left\|\frac{\partial\Delta\phi}{\partial\tau}(\tau)\right\|^2\right.$$

$$+D\left\|\Delta n(\tau)\right\|^2+\left\|\frac{\partial\nabla n}{\partial x_3}(\tau)\right\|^2+\left\|\frac{\partial n}{\partial\tau}(\tau)\right\|^2\Bigg)\,\mathrm{d}\tau$$

$$\leq C(D)\left(E_0^2+E_0^{18}\right). \tag{2.2.16}$$

Here $C(D)$ is a constant depending on D.

Proof. Multiplying the first equation of (1.4.3) by $\Delta^2\phi$ and integrating over Ω, we have, by virtue of the integration by parts and Schwarz' inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\phi(t)\|^2+c_2\|\Delta^2\phi(t)\|^2$$

$$\leq\frac{c_2}{4}\|\Delta^2\phi(t)\|^2+\frac{c}{c_2}\left\|\nabla\Delta\phi(t)\right\|^2_{L^4(\Omega)}\left\|\nabla\phi(t)\right\|^2_{L^4(\Omega)}$$

$$+\frac{c}{c_2n_*^2}\left\|\frac{\partial^2\phi}{\partial x_3^2}(t)\right\|^2+\frac{c}{c_2n_*}\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial^2 n}{\partial x_3^2}(t)\right\|^2.$$

Applying the Gagliardo-Nirenberg and Young inequalities to the second term of the right hand side of the above inequality, we obtain with the help of (2.2.9)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\phi(t)\|^2+c_2\left\|\Delta^2\phi(t)\right\|^2$$

$$\leq\frac{c}{c_2}\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial^2 n}{\partial x_3^2}(t)\right\|^2+\frac{c}{c_2}\left(E_0^2+E_0^{18}\right)\mathrm{e}^{-c^\sharp t}. \tag{2.2.17}$$

Similarly, multiplying the second equation of (1.4.3) by Δn and integrating over Ω, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n(t)\|^2+D\left\|\Delta n(t)\right\|^2+\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla n}{\partial x_3}(t)\right\|^2$$

$$\leq\left\|\nabla\phi(t)\right\|_{L^4(\Omega)}\left\|\nabla n(t)\right\|_{L^4(\Omega)}\left\|\Delta n(t)\right\|+\frac{c_1}{n_*}\left\|\Delta n(t)\right\|\left\|\frac{\partial^2\phi}{\partial x_3^2}(t)\right\|$$

$$+c\sup_{x'\in\omega}n^*(|x'|)^7\left\|\nabla\frac{c_1}{n^*}\right\|^8_{L^4(\omega)}\left\|\frac{\partial n}{\partial x_3}(t)\right\|^2+\frac{1}{2}\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla n}{\partial x_3}(t)\right\|^2.$$

Here we used the inequality

$$\left|\int_\Omega \nabla\frac{c_1}{n^*}\cdot\frac{\partial\nabla n}{\partial x_3}(t)\frac{\partial n}{\partial x_3}(t)\ \mathrm{d}x\right| \le c\left\|\nabla\frac{c_1}{n^*}\right\|_{L^4(\omega)}\left\|\frac{\partial n}{\partial x_3}(t)\right\|_{L^4(\Omega)}\left\|\frac{\partial\nabla n}{\partial x_3}(t)\right\|$$

$$\le c\left\|\nabla\frac{c_1}{n^*}\right\|_{L^4(\omega)}\left\|\frac{\partial n}{\partial x_3}(t)\right\|^{\frac14}\left\|\frac{\partial\nabla n}{\partial x_3}(t)\right\|^{\frac34}\left\|\frac{\partial\nabla n}{\partial x_3}(t)\right\|$$

$$\le c\sup_{x'\in\omega} n^*(|x'|)^7\left\|\nabla\frac{c_1}{n^*}\right\|^8_{L^4(\omega)}\left\|\frac{\partial n}{\partial x_3}(t)\right\|^2+\frac12\left\|\left(\frac{c_1}{n^*}\right)^{\frac12}\frac{\partial\nabla n}{\partial x_3}(t)\right\|^2.$$

For the first term of the right hand side we again use Gagliardo–Nirenberg and Young's inequalities. Then we have with the help of (2.2.9)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n(t)\|^2+D\,\|\Delta n(t)\|^2+\left\|\left(\frac{c_1}{\bar n}\right)^{\frac12}\frac{\partial\nabla n}{\partial x_3}(t)\right\|^2$$

$$\le c\left\|\frac{\partial n}{\partial x_3}(t)\right\|^2+C(D)\left(E_0^2+E_0^{18}\right)\mathrm{e}^{-c^\sharp t}. \qquad (2.2.18)$$

Adding (2.2.17) and (2.2.18) multiplied by $\dfrac{2c}{c_2}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\Delta\phi(t)\|^2+\frac{2c}{c_2}\|\nabla n(t)\|^2\right)+c_2\|\Delta^2\phi(t)\|^2+\frac{2cD}{c_2}\,\|\Delta n(t)\|^2$$

$$+\frac{c}{c_2}\left\|\left(\frac{c_1}{n^*}\right)^{\frac12}\frac{\partial\nabla n}{\partial x_3}(t)\right\|^2\le\frac{2c^2}{c_2}\left\|\frac{\partial n}{\partial x_3}(t)\right\|^2+\frac{2c}{c_2}C(D)\left(E_0^2+E_0^{18}\right)\mathrm{e}^{-c^\sharp t}.$$

Integrating this over $[0,T]$, we obtain (2.2.16) with the help of (2.2.10). The estimates of the derivatives of $\Delta\phi$ and n with respect t are easily derived from the estimates above and equation (1.4.3). □

Lemma 2.2.6 *For any* $t\in[0,T]$

$$\left\|\Delta^2\phi(t)\right\|^2+\|\Delta n(t)\|^2$$

$$+\int_0^t\left(\left\|\nabla\Delta^2\phi(\tau)\right\|^2+D\,\|\nabla\Delta n(\tau)\|^2+\left\|\frac{\partial\Delta n}{\partial x_3}(\tau)\right\|^2\right)\mathrm{d}\tau$$

$$\le cE_1^2+C(D)\left((E_0^2+E_0^{18})^2+E_0^2+E_0^{18}\right)(1+t)\,. \qquad (2.2.19)$$

Here $C(D)$ is a constant depending on D.

Proof. The boundary conditions on Γ in (1.4.1) yield that $\partial^2/\partial x_3^2$ and $(\nabla\phi(x,t)\times\vec{e})\cdot\nabla$ are tangential derivatives on Γ, and hence

$$\Delta^2\phi(x,t)=\Delta n(x,t)=0 \quad \text{for } x\in\Gamma,\ t>0. \tag{2.2.20}$$

Applying the Laplacian Δ to the first equation of (1.4.3), multiplying it by $\Delta^2\phi$ and integrating over Ω, we have, by virtue of the integration by parts and Schwarz' and Young's inequalities,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^2\phi(t)\|^2+c_2\|\nabla\Delta^2\phi(t)\|^2\\
&=-\int_\Omega\nabla\left((\nabla\phi\times\vec{e})\cdot\nabla\Delta\phi\right)\cdot\nabla\Delta^2\phi\ \mathrm{d}x+\int_\Omega\nabla\left(\frac{c_1}{n^*}\frac{\partial^2\phi}{\partial x_3^2}\right)\cdot\nabla\Delta^2\phi\ \mathrm{d}x\\
&\quad+\int_\Omega\left(\frac{c_1}{n^*}\frac{\partial^2\Delta n}{\partial x_3^2}+2\nabla\left(\frac{c_1}{n^*}\right)\cdot\frac{\partial^2\nabla n}{\partial x_3^2}+\Delta\left(\frac{c_1}{n^*}\right)\frac{\partial^2 n}{\partial x_3^2}\right)\Delta^2\phi\ \mathrm{d}x\\
&\le\frac{c_2}{4}\|\nabla\Delta^2\phi(t)\|^2+\frac{c}{c_2}\Big(\|\Delta\phi(t)\|^2_{L^4(\Omega)}\|\nabla\Delta\phi(t)\|^2_{L^4(\Omega)}\\
&\quad+\|\nabla\phi(t)\|^2_{L^\infty(\Omega)}\left\|\Delta^2\phi(t)\right\|^2\Big)+c\left\{\left(\left\|\nabla\frac{c_1}{n^*}\right\|\left\|\frac{\partial^2\phi}{\partial x_3^2}(t)\right\|_{L^\infty(\Omega)}\right.\right.\\
&\quad\left.+\frac{c_1}{n_*}\left\|\frac{\partial^2\nabla\phi}{\partial x_3^2}(t)\right\|\right)\left\|\nabla\Delta^2\phi(t)\right\|+\frac{c_1}{n_*}\left\|\frac{\partial\Delta n}{\partial x_3}(t)\right\|\left\|\frac{\partial\Delta^2\phi}{\partial x_3}(t)\right\|\\
&\quad+\left\|\nabla\frac{c_1}{n^*}\right\|_{L^4(\omega)}\left(\left\|\frac{\partial^2\nabla n}{\partial x_3^2}(t)\right\|\left\|\Delta^2\phi(t)\right\|_{L^4(\Omega)}\right.\\
&\quad\left.\left.+\left\|\frac{\partial^2 n}{\partial x_3^2}(t)\right\|_{L^4(\Omega)}\left\|\nabla\Delta^2\phi(t)\right\|\right)\right\}.
\end{aligned}$$

Then we have, with the help of Gagliardo–Nirenberg and Young's inequalities, Sobolev imbedding theorem and (2.2.16),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^2\phi(t)\|^2+c_2\|\nabla\Delta^2\phi(t)\|^2$$

$$\leq \Big(C(D)\left(E_0^2+E_0^{18}\right)+c\Big)\|\Delta^2\phi(t)\|^2+c\left\|\frac{\partial^2 n}{\partial x_3^2}(t)\right\|^2$$

$$+\frac{c}{c_2}\left(\left\|\frac{\partial^2\nabla n}{\partial x_3^2}(t)\right\|^2+\left\|\frac{\partial\Delta n}{\partial x_3}(t)\right\|^2\right)+C(D)(E_0^2+E_0^{18})^2. \quad (2.2.21)$$

In the similar way as in (2.2.21), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n(t)\|^2+D\,\|\nabla\Delta n(t)\|^2+\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\Delta n}{\partial x_3}(t)\right\|^2$$

$$\leq \|\nabla\Delta\phi(t)\|\|\Delta n(t)\|\,\|\nabla n(t)\|_{L^\infty(\Omega)}$$

$$+\|\Delta\phi(t)\|_{L^4(\Omega)}\|\Delta n(t)\|_{L^4(\Omega)}\|\Delta n(t)\|$$

$$+c\Bigg\{\left(\left\|\nabla\frac{c_1}{n^*}\right\|_{L^4(\omega)}\left\|\frac{\partial^2\phi}{\partial x_3^2}(t)\right\|_{L^4(\Omega)}+\frac{c_1}{n^*}\left\|\frac{\partial^2\nabla\phi}{\partial x_3^2}(t)\right\|\right)\|\nabla\Delta n(t)\|$$

$$+\left\|\nabla\frac{c_1}{n^*}\right\|_{L^4(\omega)}\left(\left\|\frac{\partial^2 n}{\partial x_3^2}(t)\right\|_{L^4(\Omega)}\|\nabla\Delta n(t)\|\right.$$

$$\left.+\left\|\frac{\partial^2\nabla n}{\partial x_3^2}(t)\right\|\|\Delta n(t)\|_{L^4(\Omega)}\right)\Bigg\}.$$

Then we obtain, with the help of Gagliardo–Nirenberg and Young's inequalities, Sobolev imbedding theorem and (2.2.16),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n(t)\|^2+D\,\|\nabla\Delta n(t)\|^2+\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\Delta n}{\partial x_3}(t)\right\|^2$$

$$\leq C(D)(E_0^2+E_0^{18})+\Big(C(D)(E_0^2+E_0^{18})+c\Big)\|\Delta n(t)\|^2. \quad (2.2.22)$$

Adding (2.2.21) and (2.2.22) multiplied by $\dfrac{3c}{Dc_2}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta^2\phi(t)\|^2+\frac{3c}{Dc_2}\|\Delta n(t)\|^2\right)+c_2\|\nabla\Delta^2\phi(t)\|^2$$

$$+\frac{c}{c_2}\|\nabla\Delta n(t)\|^2+\frac{3c}{Dc_2}\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\Delta n}{\partial x_3}(t)\right\|^2$$

$$\le \Big(C(D)(E_0^2+E_0^{18})+c\Big)\Big(\|\Delta^2\phi(t)\|^2+\frac{3c}{Dc_2}\|\Delta n(t)\|^2\Big)$$

$$+c\left\|\frac{\partial^2 n}{\partial {x_3}^2}(t)\right\|^2+C(D)\Big((E_0^2+E_0^{18})^2+E_0^2+E_0^{18}\Big)\Big(1+\frac{3c}{Dc_2}\Big).$$

Integrating this over $[0,T]$, we have (2.2.19) by (2.2.16). $\square$

2.2.3 Proof of Theorem 2.1.1

By the standard arguments with the help of Lemmas 2.2.4-2.2.6 the solution (ϕ,n) established in §2.2.1 can be extended to any time interval $[0,T]$. Thus the proof of Theorem 2.1.1 is complete.

2.3 Existence theorem for Hasegawa–Wakatani equations

2.3.1 Uniform estimates

Note that the estimates in Lemma 2.2.4 hold uniformly in D. The aim of this subsection is to get the D-independent version of Lemma 2.2.5 and the uniform estimates for n in Lemma 2.2.6. For that again the regularity of the solution is not sufficient, so that the following arguments are formal. However, one can justify them by using the method of difference quotients or mollifiers.

Lemma 2.3.1 *There exists a positive constant T^* independent of D such that the estimate*

$$\|\nabla\Delta\phi(t)\|^2 + \|\nabla n(t)\|^2 + \int_0^t \left(\|\Delta^2\phi(\tau)\|^2 + D\,\|\Delta n(\tau)\|^2 \right.$$

$$\left. + \left\|\frac{\partial \nabla n}{\partial x_3}(\tau)\right\|^2\right) \mathrm{d}\tau \leq \frac{C_1(0)C_1(t)}{C_1(0) - c(1 + C_1(0))C_1(t)t} \qquad (2.3.1)$$

holds for any $t \in [0, T^*)$. Here c is a constant independent of D and

$$C_1(t) \equiv c\left(E_0^2 + E_0^{18}\right)(1+t).$$

Proof. Multiplying the second equation of (1.4.3) by Δn and integrating over Ω, we have, by virtue of the integration by parts and (2.2.11) (cf. (2.2.18)),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n(t)\|^2 + D\,\|\Delta n(t)\|^2 + \left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}} \frac{\partial \nabla n}{\partial x_3}(t)\right\|^2$$

$$\leq \varepsilon\left(\|\nabla\Delta\phi(t)\|^2 + \left\|\Delta^2\phi(t)\right\|^2\right)$$

$$+ C_\varepsilon\left(\|\nabla n(t)\|^2 + \|\nabla n(t)\|^4\right) + cE_0^2. \qquad (2.3.2)$$

Here we used Schwarz' and Young's inequalities and the Sobolev imbedding theorem.

Adding (2.2.17) and (2.3.2) multiplied by $\dfrac{c_2}{2\varepsilon}$ yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\Delta\phi(t)\|^2 + \frac{c_2}{2\varepsilon}\|\nabla n(t)\|^2\right) + \frac{c_2}{2}\|\Delta^2\phi(t)\|^2 + \frac{c_2 D}{2\varepsilon}\|\Delta n(t)\|^2 \\
&\quad + \frac{c_2^2 - 2\varepsilon c}{2\varepsilon c_2}\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla n}{\partial x_3}(t)\right\|^2 \\
&\le \frac{c_2}{2}\|\nabla\Delta\phi(t)\|^2 + \frac{c_2 C_\varepsilon}{2\varepsilon}\left(\|\nabla n(t)\|^2 + \|\nabla n(t)\|^4\right) \\
&\quad + \left(\frac{c}{c_2} + \frac{c c_2}{2\varepsilon}\right)\left(E_0^2 + E_0^{18}\right). \qquad (2.3.3)
\end{aligned}
$$

Here we take ε small enough to hold the inequality $c_2^2 - 2\varepsilon c > 0$. Integrating this over $[0,t]$, we get with the help of (2.2.10)

$$
\begin{aligned}
\|\nabla n\|^2 &\le c\int_0^t \left(\|\nabla n(\tau)\|^4 + \|\nabla n(\tau)\|^2\right)\mathrm{d}\tau + c\left(E_0^2 + E_0^{18}\right)(1+t) \\
&\equiv c\int_0^t \left(\|\nabla n(\tau)\|^4 + \|\nabla n(\tau)\|^2\right)\mathrm{d}\tau + C_1(t) \equiv S_2(t).
\end{aligned}
$$

Differentiating $S_2(t)$ with respect to t, we have

$$
\begin{aligned}
\frac{\mathrm{d}S_2(t)}{\mathrm{d}t} &= c\left(\|\nabla n(t)\|^2 + \|\nabla n(t)\|^4\right) + \frac{\mathrm{d}C_1(t)}{\mathrm{d}t} \\
&\le c\left(S_2(t) + S_2(t)^2\right) + \frac{\mathrm{d}C_1(t)}{\mathrm{d}t}.
\end{aligned}
$$

Since $C_1(t)$ is increasing, one can derive from this inequality,

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{S_2(t)}\right) &\le c\left(\frac{1}{S_2(t)} + 1\right) + \frac{1}{S_2(t)^2}\frac{\mathrm{d}C_1(t)}{\mathrm{d}t} \\
&\le c\left(\frac{1}{C_1(0)} + 1\right) + \frac{1}{C_1(t)^2}\frac{\mathrm{d}C_1(t)}{\mathrm{d}t}.
\end{aligned}
$$

Integrating this inequality over $[0,t]$, we have

$$
-\frac{1}{S_2(t)} + \frac{1}{S_2(0)} \le c\left(\frac{1}{C_1(0)} + 1\right)t - \frac{1}{C_1(t)} + \frac{1}{C_1(0)},
$$

and hence

$$S_2(t) \le \frac{C_1(0)C_1(t)}{C_1(0) - c(1 + C_1(0))C_1(t)t}.$$

Then we choose $T^* > 0$ such that $C_1(0) - c(1 + C_1(0))C_1(T^*)T^* = 0$. This and the integral of (2.3.3) over $[0, t]$ lead to the inequality (2.3.1). □

Lemmas 2.2.4 and 2.3.1 imply that there exists a subsequence of $\{(\phi^D, n^D)\}_{D>0}$ converging to some function (ϕ, n) as $D \to 0$ weakly in $(L^2(0, T^*; W_2^4(\Omega)) \cap W_2^1(0, T^*; W_2^2(\Omega))) \times W_2^{2,1}(Q_{T^*})$. In order to prove the convergence of the full sequence $\{(\phi^D, n^D)\}_{D>0}$, we prepare the following lemma.

Lemma 2.3.2 *Let $l = 1$. Then there exists a positive constant T^{**} independent of D such that the estimate*

$$\|\Delta n(t)\|^2 + \int_0^t \left(D \|\nabla \Delta n(\tau)\|^2 + \left\| \frac{\partial \Delta n}{\partial x_3}(\tau) \right\|^2 \right) \mathrm{d}\tau$$

$$\le \frac{C_2(0)C_2(t)}{C_2(0) - c(1 + C_2(0))C_2(t)t} \qquad (2.3.4)$$

holds for any $t \in [0, T^{**})$. Here c is a constant independent of D and

$$C_2(t) \equiv E_1^2 + \frac{C_1(0)C_1(t)(1 + E_0^2)}{C_1(0) - c(1 + C_1(0))C_1(t)t}.$$

Proof. We can prove this lemma in almost the same way as that in Lemma 2.3.1. Indeed, apply the Laplacian Δ to the second equation of (1.4.3), multiply it by Δn and integrate over Ω. Then we have, by virtue of the integration by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n(t)\|^2 + D\|\nabla \Delta n(t)\|^2 + \left\| \left(\frac{c_1}{n^*}\right)^{\frac{1}{2}} \frac{\partial \Delta n}{\partial x_3}(t) \right\|^2$$

$$\le \|\nabla \Delta \phi(t)\|_{L^4(\Omega)} \|\nabla n(t)\|_{L^4(\Omega)} \|\Delta n(t)\| + \sum_{|\alpha|=2} \|D_x^\alpha \phi(t)\|_{L^\infty(\Omega)} \|\Delta n(t)\|^2$$

$$+c\Bigg\{ \left\|\Delta \frac{c_1}{n^*}\right\| \left\| \frac{\partial \phi}{\partial x_3}(t) \right\|_{L^\infty(\Omega)} + \left\| \nabla \frac{c_1}{n^*} \right\|_{L^4(\omega)} \left\| \frac{\partial \nabla \phi}{\partial x_3}(t) \right\|_{L^4(\Omega)}$$

$$+\frac{c_1}{n_*}\left\|\frac{\partial\Delta\phi}{\partial x_3}(t)\right\| + \left\|\Delta\frac{c_1}{n^*}\right\|\left\|\frac{\partial n}{\partial x_3}(t)\right\|_{L^\infty(\Omega)}$$

$$+\left\|\nabla\frac{c_1}{n^*}\right\|_{L^4(\omega)}\left\|\frac{\partial\nabla n}{\partial x_3}(t)\right\|_{L^4(\Omega)}\Bigg\}\left\|\frac{\partial\Delta n}{\partial x_3}(t)\right\|.$$

Here we used the condition (2.2.20). From this we obtain, with the help of Gagliardo–Nirenberg and Young's inequalities, Sobolev imbedding theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n(t)\|^2 + D\,\|\nabla\Delta n(t)\|^2 + \left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\Delta n}{\partial x_3}(t)\right\|^2$$

$$\leq c\Big(\|\Delta n(t)\|^4 + \|\Delta n(t)\|^2 + \|\nabla n(t)\|^2$$

$$+\|\Delta^2\phi(t)\|^2 + \|\nabla\Delta\phi(t)\|^4 + \|\nabla\Delta\phi(t)\|^2\Big). \quad (2.3.5)$$

Here we used the inequalities

$$\|\phi\|_{W_2^2(\Omega)} \leq c\,\|\Delta\phi\|\,, \quad \|\Delta\phi\|_{W_2^2(\Omega)} \leq c\|\Delta^2\phi\|.$$

Integrating this over $[0,t]$, we get with the help of (2.2.10), (2.3.1)

$$\|\Delta n(t)\|^2 \;\leq\; E_1^2 + c\int_0^t(\|\Delta n(\tau)\|^4 + \|\Delta n(\tau)\|^2)\,\mathrm{d}\tau$$

$$+\frac{C_1(0)C_1(t)(1+E_0^2)}{C_1(0) - c(1+C_1(0))C_1(t)t}$$

$$\equiv\; c\int_0^t(\|\Delta n(\tau)\|^4 + \|\Delta n(\tau)\|^2)\,\mathrm{d}\tau + C_2(t) \equiv S_3(t).$$

Differentiating $S_3(t)$ with respect to t, we have

$$\frac{\mathrm{d}S_3(t)}{\mathrm{d}t} \;=\; c\left(\|\Delta n(t)\|^2 + \|\Delta n(t)\|^4\right) + \frac{\mathrm{d}C_2(t)}{\mathrm{d}t}$$

$$\leq\; c\left(S_3(t) + S_3(t)^2\right) + \frac{\mathrm{d}C_2(t)}{\mathrm{d}t}.$$

Since $C_2(t)$ is increasing, one can derive from this inequality

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{S_3(t)}\right) \;\leq\; c\left(\frac{1}{S_3(t)} + 1\right) + \frac{1}{S_3^2(t)}\frac{\mathrm{d}C_2(t)}{\mathrm{d}t}$$

$$\leq \; c\left(\frac{1}{C_2(0)}+1\right)+\frac{1}{C_2(t)^2}\frac{\mathrm{d}C_2(t)}{\mathrm{d}t}.$$

Integrating this inequality over $[0,t]$, we have

$$-\frac{1}{S_3(t)}+\frac{1}{S_3(0)}\leq c\left(\frac{1}{C_2(0)}+1\right)t-\frac{1}{C_2(t)}+\frac{1}{C_2(0)},$$

and hence

$$S_3(t)\leq\frac{C_2(0)C_2(t)}{C_2(0)-c(1+C_2(0))C_2(t)t}.$$

Then we choose $T^{**}>0$ such that $C_2(0)-c(1+C_2(0))C_2(T^{**})T^{**}=0$. This and the integral of (2.3.5) over $[0,t]$ lead to the inequality (2.3.4). □

2.3.2 Proof of Theorem 2.1.2

By virtue of Lemmas 2.2.4, 2.3.1 and 2.3.2, we prove in this subsection that the sequence $\{(\phi^D,n^D)\}_{D>0}$ is a Cauchy sequence in $\Big(L^2(0,T^{**};W_2^4(\Omega))\cap W_2^1(0,T^{**};W_2^2(\Omega))\Big)\times W_2^{2,1}(Q_{T^{**}})$. Subtracting (1.4.3) with $D=D'$ from those with $D=D''$ $(0<D''<D'\leq 1)$ and denoting by $\overline{\Phi}\equiv\phi^{D'}-\phi^{D''}$, $\overline{N}\equiv n^{D'}-n^{D''}$, we have

$$\begin{cases}\dfrac{\partial\Delta\overline{\Phi}}{\partial t}-\Big(\nabla\phi^{D'}\times\vec{e}\Big)\cdot\nabla\Delta\overline{\Phi}-\Big(\nabla\overline{\Phi}\times\vec{e}\Big)\cdot\nabla\Delta\phi^{D''}\\ \qquad\qquad =-\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\overline{\Phi}-\overline{N})+c_2\Delta^2\overline{\Phi},\\ \dfrac{\partial\overline{N}}{\partial t}-\Big(\nabla\phi^{D'}\times\vec{e}\Big)\cdot\nabla\overline{N}-\Big(\nabla\overline{\Phi}\times\vec{e}\Big)\cdot\nabla n^{D''}\\ \qquad\qquad =-\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\overline{\Phi}-\overline{N})+D''\Delta\overline{N}+(D'-D'')\,\Delta n^{D'}\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } x\in\Omega,\ t>0,\end{cases}\tag{2.3.6}$$

$$\begin{cases}\overline{\Phi}(x,0)=\overline{N}(x,0)=0\quad\text{for } x\in\Omega,\\ \overline{\Phi}(x,t)=\Delta\overline{\Phi}(x,t)=\overline{N}(x,t)=0\quad\text{for } x\in\Gamma,\ T^{**}>t>0,\\ \overline{\Phi},\overline{N},\quad\text{periodic in the } x_3 \text{ direction.}\end{cases}$$

We separate five steps for the proof. In the rest of this subsection, we denote inessential functions determined from Lemmas 2.2.4, 2.3.1 and 2.3.2 by the same symbol $C(t)$ which is independent of D', D''.

(i) *Estimate of* $\left\|\nabla\overline{\Phi}(t)\right\|^2 + \left\|\overline{N}(t)\right\|^2 + \int_0^t \left\|\Delta\overline{\Phi}(\tau)\right\|^2 \mathrm{d}\tau.$

Multiplying the first equation of (2.3.6) by $\overline{\Phi}$ and integrating over Ω, we have, by virtue of the integration by parts, Schwarz' inequality and the Sobolev imbedding theorem,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\overline{\Phi}(t)\right\|^2 + c_2\left\|\Delta\overline{\Phi}(t)\right\|^2 + \left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\overline{\Phi}}{\partial x_3}(t)\right\|^2$$

$$\leq \frac{c_2}{2}\left\|\Delta\overline{\Phi}(t)\right\|^2 + \frac{c}{c_2}\left\|\nabla\phi^{D'}(t)\right\|^2_{L^\infty(\Omega)}\left\|\nabla\overline{\Phi}(t)\right\|^2 + \frac{c}{c_2}\frac{c_1^2}{n_*^2}\left\|\overline{N}(t)\right\|^2,$$

from which, together with (2.2.9) and (2.3.1), it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\overline{\Phi}(t)\right\|^2 + c_2\|\Delta\overline{\Phi}(t)\|^2 + \left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\overline{\Phi}}{\partial x_3}(t)\right\|^2$$

$$\leq C(t)\|\nabla\overline{\Phi}(t)\|^2 + c\|\overline{N}(t)\|^2. \tag{2.3.7}$$

Similarly, multiplying the second equation of (2.3.6) by $\overline{N}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{N}(t)\|^2 + D''\left\|\nabla\overline{N}(t)\right\|^2 + \left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\overline{N}}{\partial x_3}(t)\right\|^2$$

$$\leq c\left\|\nabla n^{D''}(t)\right\|_{L^4(\Omega)}\left\|\nabla\overline{\Phi}(t)\right\|_{L^4(\Omega)}\left\|\overline{N}(t)\right\|$$

$$+\frac{c_2}{4}\left\|\Delta\overline{\Phi}(t)\right\|^2 + \left(\frac{c}{c_2}\frac{c_1^2}{n_*^2} + \frac{1}{2}\right)\|\overline{N}(t)\|^2 + \frac{D'^2}{2}\left\|\Delta n^{D'}(t)\right\|^2.$$

Here we used the inequality

$$\left\|\frac{\partial^2\overline{\Phi}}{\partial x_3^2}(t)\right\| \leq c\left\|\Delta\overline{\Phi}(t)\right\|.$$

Thus we get, by virtue of (2.2.9) and (2.3.4),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{N}(t)\|^2 + D''\left\|\nabla\overline{N}(t)\right\|^2 + \left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\overline{N}}{\partial x_3}(t)\right\|^2$$

$$\leq D'C(t) + c\|\overline{N}(t)\|^2 + C(t)\|\nabla\overline{\Phi}(t)\|^2 + \frac{c_2}{2}\|\Delta\overline{\Phi}(t)\|^2. \quad (2.3.8)$$

Adding (2.3.7) and (2.3.8) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\overline{\Phi}(t)\|^2 + \|\overline{N}(t)\|^2\right) + c_2\|\Delta\overline{\Phi}(t)\|^2 + D''\left\|\nabla\overline{N}(t)\right\|^2$$

$$+\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\overline{\Phi}}{\partial x_3}(t)\right\|^2 + \left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\overline{N}}{\partial x_3}(t)\right\|^2$$

$$\leq D'C(t) + C(t)\left(\|\nabla\overline{\Phi}(t)\|^2 + \|\overline{N}(t)\|^2\right).$$

Then Gronwall's lemma leads to

$$\|\nabla\overline{\Phi}(t)\|^2 + \|\overline{N}(t)\|^2 + \int_0^t \left(\|\Delta\overline{\Phi}(\tau)\|^2 + D''\left\|\nabla\overline{N}(\tau)\right\|^2\right.$$

$$\left. + \left\|\frac{\partial\overline{\Phi}}{\partial x_3}(\tau)\right\|^2 + \left\|\frac{\partial\overline{N}}{\partial x_3}(\tau)\right\|^2\right)\mathrm{d}\tau \leq D'C(t).$$

(ii) *Estimate of* $\left\|\Delta\overline{\Phi}(t)\right\|^2 + \int_0^t \left\|\nabla\Delta\overline{\Phi}(\tau)\right\|^2 \mathrm{d}\tau$.

Similarly as in (i), multiplying the first equation of (2.3.6) by $\Delta\overline{\Phi}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\overline{\Phi}(t)\|^2 + c_2\|\nabla\Delta\overline{\Phi}(t)\|^2$$

$$\leq c\left\|\nabla\Delta\phi^{D''}(t)\right\|\left\|\nabla\overline{\Phi}(t)\right\|_{L^\infty(\Omega)}\left\|\Delta\overline{\Phi}(t)\right\|$$

$$+\frac{c_2}{4}\left\|\frac{\partial\Delta\overline{\Phi}}{\partial x_3}(t)\right\|^2 + \frac{1}{c_2}\frac{c_1^2}{n_*^2}\left\|\frac{\partial\overline{N}}{\partial x_3}(t)\right\|^2 + \frac{c_1}{n_*}\|\Delta\overline{\Phi}(t)\|^2.$$

This inequality, together with (2.3.1) and (i), implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\overline{\Phi}(t)\|^2 + c_2\|\nabla\Delta\overline{\Phi}(t)\|^2 \leq c\left\|\frac{\partial\overline{N}}{\partial x_3}(t)\right\|^2 + C(t)\|\Delta\overline{\Phi}(t)\|^2.$$

Applying Gronwall's lemma and the estimate in (i), we obtain

$$\|\Delta\overline{\Phi}(t)\|^2 + c_2\int_0^t \|\nabla\Delta\overline{\Phi}(\tau)\|^2 \mathrm{d}\tau \le D'C(t).$$

(iii) *Estimate of* $\left\|\nabla\Delta\overline{\Phi}(t)\right\|^2 + \left\|\nabla\overline{N}(t)\right\|^2 + \int_0^t \left\|\Delta^2\overline{\Phi}(\tau)\right\|^2 \mathrm{d}\tau.$

Multiplying the first equation of (2.3.6) by $\Delta^2\overline{\Phi}$ and integrating over Ω, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\overline{\Phi}(t)\|^2 + c_2\|\Delta^2\overline{\Phi}(t)\|^2\\ &\quad\le c\left\|\nabla\phi^{D'}(t)\right\|^2_{L^\infty(\Omega)}\left\|\nabla\Delta\overline{\Phi}(t)\right\|^2 + c\left\|\nabla\Delta\phi^{D''}(t)\right\|^2\left\|\nabla\overline{\Phi}(t)\right\|^2_{L^\infty(\Omega)}\\ &\qquad\qquad + \frac{c_2}{4}\|\Delta^2\overline{\Phi}(t)\|^2 + \frac{c}{c_2}\frac{1}{n_*}\left\|\frac{\partial^2\overline{\Phi}}{\partial x_3^2}(t)\right\|^2 + \frac{c}{c_2}\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial^2\overline{N}}{\partial x_3^2}(t)\right\|^2.\end{aligned}$$

From this, (2.2.9), (2.3.1) and (i) it follows

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\overline{\Phi}(t)\|^2 + c_2\|\Delta^2\overline{\Phi}(t)\|^2\\ &\quad\le D'C(t) + C(t)\|\nabla\Delta\overline{\Phi}(t)\|^2 + \frac{c}{c_2}\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial^2\overline{N}}{\partial x_3^2}(t)\right\|^2. \quad (2.3.9)\end{aligned}$$

Multiplying the second equation of (2.3.6) by $\Delta\overline{N}$ and integrating over Ω, we have, by virtue of the inequality similar to the one preceding (2.2.18),

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\overline{N}(t)\|^2 + D''\left\|\Delta\overline{N}(t)\right\|^2 + \left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla\overline{N}}{\partial x_3}(t)\right\|^2\\ &\le c\Bigg\{\sum_{|\alpha|=2}\left\|D_x^\alpha\phi^{D'}(t)\right\|_{L^\infty(\Omega)}\left\|\nabla\overline{N}(t)\right\|^2 + \Bigg(\left\|\nabla n^{D''}(t)\right\|_{L^4(\Omega)}\left\|\Delta\overline{\Phi}(t)\right\|_{L^4(\Omega)}\\ &\qquad + \left\|\Delta n^{D''}(t)\right\|\left\|\nabla\overline{\Phi}(t)\right\|_{L^\infty(\Omega)} + \left\|\nabla\frac{c_1}{n^*}\right\|_{L^4(\omega)}\left\|\frac{\partial^2\overline{\Phi}}{\partial x_3^2}(t)\right\|_{L^4(\Omega)}\\ &\qquad + \frac{c_1}{n_*}\left\|\frac{\partial^2\nabla\overline{\Phi}}{\partial x_3^2}(t)\right\|\Bigg)\left\|\nabla\overline{N}(t)\right\|\end{aligned}$$

$$+\sup_{x'\in\omega} n^*(|x'|)^7 \left\|\nabla\frac{c_1}{n^*}\right\|^8_{L^4(\omega)} \left\|\frac{\partial\overline{N}}{\partial x_3}(t)\right\|^2 \Bigg\}$$

$$+\frac{1}{2}\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla\overline{N}}{\partial x_3}(t)\right\|^2$$

$$+D'\left\|\Delta n^{D'}(t)\right\|\left(\left\|\Delta n^{D'}(t)\right\|+\left\|\Delta n^{D''}(t)\right\|\right).$$

Then we have, with the help of (2.2.9), (2.3.1), (2.3.4) and (i),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\overline{N}(t)\|^2 + D''\left\|\Delta\overline{N}(t)\right\|^2 + \left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla\overline{N}}{\partial x_3}(t)\right\|^2$$

$$\leq D'C(t) + C(t)\left(1+\left\|\Delta^2\phi^{D'}(t)\right\|\right)\|\nabla\overline{N}(t)\|^2 + c\|\nabla\Delta\overline{\Phi}(t)\|^2. \quad (2.3.10)$$

Here we used the inequality

$$\left\|\Delta\phi^{D'}(t)\right\|_{W_2^2(\Omega)} \leq c\left\|\Delta^2\phi^{D'}(t)\right\|.$$

Adding (2.3.9) and (2.3.10) multiplied by $\frac{2c}{c_2}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\Delta\overline{\Phi}(t)\|^2 + \frac{c}{c_2}\|\nabla\overline{N}(t)\|^2\right) + c_2\|\Delta^2\overline{\Phi}(t)\|^2 + \frac{2cD''}{c_2}\left\|\Delta\overline{N}(t)\right\|^2$$

$$+\frac{c}{c_2}\left\|\left(\frac{c_1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla\overline{N}}{\partial x_3}(t)\right\|^2$$

$$\leq D'C(t) + C(t)\left(1+\left\|\Delta^2\phi^{D'}(t)\right\|\right)\left(\|\nabla\Delta\overline{\Phi}(t)\|^2 + c\|\nabla\overline{N}(t)\|^2\right).$$

Then Gronwall's lemma and (2.3.1) lead to

$$\|\nabla\Delta\overline{\Phi}(t)\|^2 + \|\nabla\overline{N}(t)\|^2 + \int_0^t \Bigg(\|\Delta^2\overline{\Phi}(\tau)\|^2 + D''\left\|\Delta\overline{N}(\tau)\right\|^2$$

$$+\left\|\frac{\partial\nabla\overline{N}}{\partial x_3}(\tau)\right\|^2\Bigg)\mathrm{d}\tau \leq D'C(t).$$

(iv) *Estimate of* $\int_0^t\left(\left\|\frac{\partial\Delta\overline{\Phi}(\tau)}{\partial\tau}\right\|^2 + \left\|\frac{\partial\overline{N}(\tau)}{\partial\tau}\right\|^2\right)\mathrm{d}\tau.$

From the above estimates and (2.3.6) one can deduce

$$\int_0^t \left(\left\| \frac{\partial \Delta \overline{\Phi}(\tau)}{\partial \tau} \right\|^2 + \left\| \frac{\partial \overline{N}(\tau)}{\partial \tau} \right\|^2 \right) \mathrm{d}\tau \le D' C(t).$$

(v) *Passage to the limit* $D \to 0$.

From the estimates in (i)–(iv) it is easy to see that the sequence $\{(\phi^D, n^D)\}_{D>0}$ is a Cauchy sequence in $\left(L^2(0, T^{**}; W_2^4(\Omega)) \cap W_2^1(0, T^{**}; W_2^2(\Omega))\right) \times W_2^{2,1}(Q_{T^{**}})$. Hence, $(\phi, n)(x,t) = \lim\limits_{D\to 0}(\phi^D, n^D)(x,t)$ exists in $(L^2(0, T^{**}; W_2^4(\Omega)) \cap W_2^1(0, T^{**}; W_2^2(\Omega))) \times W_2^{2,1}(Q_{T^{**}})$ and (ϕ, n) is our desired solution to problem (1.4.2), (1.4.1).

The uniqueness of such a solution can be easily proved by making use of the estimate analogous to the estimates in (i)–(iv).

Thus the proof of Theorem 2.1.2 is complete.

Chapter 3

Hasegawa–Wakatani equations with vanishing resistivity

3.1 Main results

First main theorem is concerned with the problem (1.4.4), (1.4.5).

Theorem 3.1.1 *Let $\varepsilon > 0$ and $n^*(|x'|) \in W_2^2(\omega)$ satisfy $n^*(|x'|) \geq n_*$ with a positive constant n_*. Assume that $(\phi_0^\varepsilon, n_0^\varepsilon) \in W_2^4(\Omega) \times W_2^2(\Omega)$ satisfies the compatibility conditions*

$$\begin{cases} \phi_0^\varepsilon(x) = \Delta\phi_0^\varepsilon(x) = n_0^\varepsilon(x) = 0 \quad \text{for } \ x \in \Gamma, \\ \phi_0^\varepsilon, \ n_0^\varepsilon, \quad \text{periodic in the } x_3\text{-direction.} \end{cases} \tag{3.1.1}$$

Then there exists a unique solution $(\phi^\varepsilon, n^\varepsilon)$ to problem (1.4.4), (1.4.5) *on some interval $[0, T]$ such that $(\phi^\varepsilon, n^\varepsilon) \in L^2(0, T; W_2^4(\Omega)) \times W_2^{2,1}(Q_T)$, $\partial\phi^\varepsilon/\partial t \in L^2(0, T; W_2^2(\Omega))$. Here T is a constant independent of ε.*

To obtain the existence theorem to the problem (1.4.6), (1.4.5) with $\varepsilon = 0$, we rewrite this problem as follows: It is clear that the second equation of

(1.4.6) implies $\widetilde{\phi^0} - \widetilde{n^0} = 0$ by virtue of the periodicity condition in x_3 and $\mathcal{M}\widetilde{\phi^0} = \mathcal{M}\widetilde{n^0} = 0$. This yields $\mathcal{M}\{(\nabla\widetilde{\phi^0} \times \vec{e}) \cdot \nabla\widetilde{n^0}\} = 0$. Hence the third equation of (1.4.6) implies

$$\left(\frac{\partial}{\partial t} - \left(\nabla\overline{\phi^0} \times \vec{e}\right) \cdot \nabla\right)\overline{n^0} = 0.$$

Subtracting this from the first equation of (1.4.6), we have, with the help of $\widetilde{\phi^0} - \widetilde{n^0} = 0$,

$$\left(\frac{\partial}{\partial t} - \left(\nabla\phi^0 \times \vec{e}\right) \cdot \nabla\right)\left(\Delta\phi^0 - \widetilde{\phi^0}\right) + \left(\nabla\widetilde{\phi^0} \times \vec{e}\right) \cdot \nabla\overline{n^0} = c_2\Delta^2\phi^0.$$

Hence we can rewrite problem (1.4.6), (1.4.5) with $\varepsilon = 0$ into the form

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \left(\nabla\phi^0 \times \vec{e}\right) \cdot \nabla\right)\left(\Delta\phi^0 - \widetilde{\phi^0}\right) + \left(\nabla\widetilde{\phi^0} \times \vec{e}\right) \cdot \nabla\overline{n^0} = c_2\Delta^2\phi^0, \\ \left(\dfrac{\partial}{\partial t} - \left(\nabla\overline{\phi^0} \times \vec{e}\right) \cdot \nabla\right)\overline{n^0} = 0 \quad \text{for } x \in \Omega,\ t > 0, \\ \phi^0(x,0) = \phi^0_0(x) \quad \text{for} \quad x \in \Omega, \\ \overline{n^0}(x',0) = \overline{n^0_0}(x') \quad \text{for} \quad x' \in \omega, \\ \phi^0(x,t) = \Delta\phi^0(x,t) = 0 \quad \text{for} \quad x \in \Gamma,\ t > 0, \\ \overline{n^0}(x',t) = 0 \quad \text{for} \quad x' \in \partial\omega,\ t > 0, \\ \phi^0, \quad \text{periodic in the } x_3\text{-direction}. \end{cases} \tag{3.1.2}$$

It is to be noted that if we impose an additional condition $\overline{n^0_0}(x') = 0$, the equations of (3.1.2) is similar to the Hasegawa–Mima equation (1.2.3) with an higher order correction term. In [28] we establish the unique existence of a strong solution to the problem (1.4.4), (1.4.5) with $\varepsilon = 0$ satisfying $\overline{n^0}(x',t) = 0$, and the convergence of $(\phi^\varepsilon, n^\varepsilon)$ to (ϕ^0, n^0) as ε tends to zero on some interval $[0,T]$, which corresponds to the vanishing resistivity of Hasegawa–Wakatani equations.

Second main existence theorem is to the problem (3.1.2).

Theorem 3.1.2 *Assume that* $\left(\phi_0^0, \overline{n_0^0}\right) \in W_2^4(\Omega)\times W_2^3(\omega)$ *satisfies the compatibility conditions* (3.1.1) *with* $\varepsilon = 0$. *Then there exists a unique solution* $\left(\phi^0, \overline{n^0}\right)$ *to the problem* (3.1.2) *on some interval* $[0, T^*]$ *such that* $\left(\phi^0, \overline{n^0}\right) \in L^2(0, T^*; W_2^4(\Omega))\times L^\infty(0, T^*; W_2^3(\omega))$, $\partial\phi^0/\partial t \in L^2(0, T^*; W_2^2(\Omega))$, $\partial\overline{n^0}/\partial t \in L^2(0, T^*; W_2^2(\omega))$.

For this solution $\left(\phi^0, \overline{n^0}\right)$ let $\widetilde{n^0}(x,t) = \widetilde{\phi^0}(x,t)$, and $\widetilde{n_0^0}(x) = \widetilde{\phi_0^0}(x)$. Then it is easily seen that (ϕ^0, n^0) satisfy (1.4.6), (1.4.5).

Our final main theorem is the following.

Theorem 3.1.3 *Let* $(\phi^\varepsilon, n^\varepsilon)$ *and* (ϕ^0, n^0) *be the solutions established in Theorems* 3.1.1 *and* 3.1.2, *respectively. If the initial data* $(\phi_0^\varepsilon, n_0^\varepsilon) \to (\phi_0^0, n_0^0)$ *as* $\varepsilon \to 0$ *in* $W_2^3(\Omega) \times W_2^2(\Omega)$, *then* $(\phi^\varepsilon, n^\varepsilon) \to (\phi^0, n^0)$ *in* $L^2\left(0, T^\sharp; W_2^4(\Omega)\right) \times W_2^{2,0}(Q_{T^\sharp})$ *and* $\Delta\phi^\varepsilon - n^\varepsilon \to \Delta\phi^0 - n^0$ *in* $W_2^{0,1}(Q_{T^\sharp})$ *and* $\overline{n^\varepsilon} \to \overline{n^0}$ *in* $W_2^{0,1}(\omega_{T^\sharp})$, $\left(\omega_{T^\sharp} \equiv \omega \times \left(0, T^\sharp\right)\right)$ *as* $\varepsilon \to 0$ *on the some time interval* $[0, T^\sharp]$ *which is determined from Theorems* 3.1.1 *and* 3.1.2.

This chapter organized as follows. In §3.2 we prove Theorem 3.1.1 from *a priori* estimates for problem (1.4.4), (1.4.5). In §3.3 Theorem 3.1.2 is proved through the local in time existence and *a priori* estimates. In §3.4 we give a proof of Theorem 3.1.3 by virtue of *a priori* estimates, Theorems 3.1.1 and 3.1.2. Throughout this chapter, we denote by c a constant which may differ.

3.2 Uniform estimates for resistivity

3.2.1 *A priori* estimates

We denote the solution (ϕ, n) established in Theorem 2.1.2 in the case of $c_1 = 1/\varepsilon$ by $(\phi^\varepsilon, n^\varepsilon)$. Then it is easy to see that $(\phi^\varepsilon, n^\varepsilon)$ is also the solution of problem (1.4.4), (1.4.5). Since T^{**} in Theorem 2.1.2 may depend on ε, to complete the proof of Theorem 3.1.1, it is sufficient to show that T^{**} can be taken independent of ε.

We proceed to get *a priori* estimates of the solution $(\phi^\varepsilon, n^\varepsilon)$. Let it belong to $(L^2(0,T;W_2^4(\Omega)) \cap W_2^1(0,T;W_2^2(\Omega))) \times W_2^{2,1}(Q_T)$ for $T > 0$.

First we prove

Lemma 3.2.1 *For any* $t \geq 0$

$$\|\nabla\phi^\varepsilon(t)\|^2 + \|n^\varepsilon(t)\|^2 + c_2 \int_0^t \|\Delta\phi^\varepsilon(\tau)\|^2 \,\mathrm{d}\tau \leq \|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2, \tag{3.2.1}$$

$$\int_0^t \left\| \frac{\partial(\phi^\varepsilon - n^\varepsilon)}{\partial x_3}(\tau) \right\|^2 \mathrm{d}\tau \leq \varepsilon \left(\|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2 \right). \tag{3.2.2}$$

Proof. Multiplying the first equation of (1.4.4) by ϕ^ε and integrating over Ω, we have, by virtue of integration by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi^\varepsilon(t)\|^2 + c_2\|\Delta\phi^\varepsilon(t)\|^2 = -\int_\Omega \frac{\partial n^\varepsilon}{\partial t}\phi^\varepsilon \,\mathrm{d}x. \tag{3.2.3}$$

Multiplying the second equation of (1.4.4) by $\phi^\varepsilon - n^\varepsilon$ and integrating over Ω, we have

$$\varepsilon\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n^\varepsilon(t)\|^2 + \left\| \left(\frac{1}{n^*}\right)^{\frac{1}{2}} \frac{\partial(\phi^\varepsilon - n^\varepsilon)}{\partial x_3}(t) \right\|^2 = \varepsilon \int_\Omega \frac{\partial n^\varepsilon}{\partial t}\phi^\varepsilon \,\mathrm{d}x. \tag{3.2.4}$$

Adding (3.2.4) and (3.2.3) multiplied by ε yields

$$\varepsilon\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\phi^\varepsilon(t)\|^2 + \|n^\varepsilon(t)\|^2\right) + c_2\|\Delta\phi^\varepsilon(t)\|^2\right) + \left\| \left(\frac{1}{n^*}\right)^{\frac{1}{2}} \frac{\partial(\phi^\varepsilon - n^\varepsilon)}{\partial x_3}(t) \right\|^2 = 0.$$

Integrating this over $[0,t]$, we have (3.2.1) and (3.2.2). □

Next we prove

Lemma 3.2.2 *There exists a positive constant T independent of ε such that the estimate*

$$\varepsilon\Big(\|\Delta\phi^\varepsilon(t)\|^2+\|\nabla n^\varepsilon(t)\|^2+\|\nabla\Delta\phi^\varepsilon(t)\|^2+\|\Delta n^\varepsilon(t)\|^2$$

$$+c_2\int_0^t\left(\|\nabla\Delta\phi^\varepsilon(\tau)\|^2+\|\Delta^2\phi^\varepsilon(\tau)\|^2\right)\,\mathrm{d}\tau\Big)$$

$$+\int_0^t\left(\left\|\frac{\partial\nabla(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(\tau)\right\|^2+\left\|\frac{\partial\Delta(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(\tau)\right\|^2\right)\,\mathrm{d}\tau$$

$$\leq\varepsilon\frac{C_3(t)}{1-cC_3(t)t} \tag{3.2.5}$$

holds for any $t\in[0,T)$. Here $C_3(t)=\|\Delta\phi_0^\varepsilon\|^2+\|\nabla n_0^\varepsilon\|^2+\|\nabla\Delta\phi_0^\varepsilon\|^2+\|\Delta n_0^\varepsilon\|^2$ $+c\left(\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2\right)+c\left(\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2\right)^2t$, and c is a constant independent of ε.

Proof. Multiplying the first equation of (1.4.4) by $\Delta\phi^\varepsilon$ and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\phi^\varepsilon(t)\|^2+c_2\|\nabla\Delta\phi^\varepsilon(t)\|^2-\int_\Omega\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n^\varepsilon\Delta\phi^\varepsilon\,\mathrm{d}x$$

$$=0. \tag{3.2.6}$$

Multiplying the second equation of (1.4.4) by $\Delta(\phi^\varepsilon-n^\varepsilon)$ and integrating over Ω, we have

$$\varepsilon\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n^\varepsilon(t)\|^2+\int_\Omega\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n^\varepsilon\Delta\phi^\varepsilon\,\mathrm{d}x\right)$$

$$+\frac{1}{2}\left\|\left(\frac{1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2$$

$$\leq\varepsilon\left(\frac{c_2}{4}\left(\|\nabla\Delta\phi^\varepsilon(t)\|^2+\|\Delta\phi^\varepsilon(t)\|^2\right)+c\left(\|\Delta n^\varepsilon(t)\|^4+\|\nabla n^\varepsilon(t)\|^4\right)\right)$$

$$+c\left\|\frac{\partial(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2. \tag{3.2.7}$$

Adding (3.2.7) and (3.2.6) multiplied by ε yields

$$\varepsilon\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta\phi^\varepsilon(t)\|^2+\|\nabla n^\varepsilon(t)\|^2\right)+\frac{3c_2}{4}\|\nabla\Delta\phi^\varepsilon(t)\|^2\right)$$
$$+\frac{1}{2}\left\|\left(\frac{1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2$$
$$\leq\varepsilon\left(\frac{c_2}{4}\|\Delta\phi^\varepsilon(t)\|^2+c\left(\|\Delta n^\varepsilon(t)\|^4+\|\nabla n^\varepsilon(t)\|^4\right)\right)$$
$$+c\left\|\frac{\partial(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2. \tag{3.2.8}$$

Multiplying the first equation of (1.4.4) by $\Delta^2\phi^\varepsilon$ and integrating over Ω, we have, by virtue of Sobolev imbedding theorem and Poincaré and Young's inequalities and (3.2.1),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\phi^\varepsilon(t)\|^2+\frac{3c_2}{4}\|\Delta^2\phi^\varepsilon(t)\|^2+\int_\Omega\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n^\varepsilon\Delta^2\phi^\varepsilon\,\mathrm{d}x$$
$$\leq c\|\nabla\Delta\phi^\varepsilon(t)\|^4+c(\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2)^2. \tag{3.2.9}$$

Applying the Laplacian Δ to the second equation of (1.4.4), multiplying it by $\Delta(\phi^\varepsilon-n^\varepsilon)$ and integrating over Ω, we have

$$\varepsilon\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n^\varepsilon(t)\|^2-\int_\Omega\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n^\varepsilon\Delta^2\phi^\varepsilon\,\mathrm{d}x\right)$$
$$-\int_\Omega\Delta\left(\frac{1}{n^*}\frac{\partial^2(\phi^\varepsilon-n^\varepsilon)}{\partial x_3^2}\right)\Delta(\phi^\varepsilon-n^\varepsilon)\,\mathrm{d}x$$
$$\leq\varepsilon\Bigg(\frac{c_2}{4}\left(\|\nabla\Delta\phi^\varepsilon(t)\|^2+\|\Delta^2\phi^\varepsilon(t)\|^2\right)$$
$$+c\left(\|\nabla\Delta\phi^\varepsilon(t)\|^4+\|\Delta n^\varepsilon(t)\|^4+\|\nabla n^\varepsilon(t)\|^4\right)\Bigg).$$

Hence we get

$$\varepsilon\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n^\varepsilon(t)\|^2-\int_\Omega\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n^\varepsilon\Delta^2\phi^\varepsilon\,\mathrm{d}x\right)$$

$$+\frac{1}{4}\left\|\left(\frac{1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\Delta(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2$$

$$\leq \varepsilon\Big(\frac{c_2}{4}\left(\|\nabla\Delta\phi^\varepsilon(t)\|^2+\|\Delta^2\phi^\varepsilon(t)\|^2\right)$$

$$+c\Big(\|\nabla\Delta\phi^\varepsilon(t)\|^4+\|\Delta n^\varepsilon(t)\|^4+\|\nabla n^\varepsilon(t)\|^4\Big)\Big)$$

$$+c\left\|\left(\frac{1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2+c\left\|\frac{\partial(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2. \qquad (3.2.10)$$

Adding (3.2.10) and (3.2.9) multiplied by ε yields

$$\varepsilon\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\Delta\phi^\varepsilon(t)\|^2+\|\Delta n^\varepsilon(t)\|^2\right)+\frac{c_2}{2}\|\Delta^2\phi^\varepsilon(t)\|^2\right)$$

$$+\frac{1}{4}\left\|\left(\frac{1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\Delta(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2$$

$$\leq \varepsilon\left(\frac{c_2}{4}\|\nabla\Delta\phi^\varepsilon(t)\|^2+c(\|\nabla\Delta\phi^\varepsilon(t)\|^4+\|\Delta n^\varepsilon(t)\|^4+\|\nabla n^\varepsilon(t)\|^4)\right)$$

$$+c\left\|\left(\frac{1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2+c\left\|\frac{\partial(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2$$

$$+c\left(\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2\right)^2. \qquad (3.2.11)$$

Adding (3.2.11) and (3.2.8) multiplied by c yields

$$\varepsilon\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(c\|\Delta\phi^\varepsilon(t)\|^2+c\|\nabla n^\varepsilon(t)\|^2+\|\nabla\Delta\phi^\varepsilon(t)\|^2+\|\Delta n^\varepsilon(t)\|^2\right)\right.$$

$$\left.+\frac{c_2}{2}\left(c\|\nabla\Delta\phi^\varepsilon(t)\|^2+\|\Delta^2\phi^\varepsilon(t)\|^2\right)\right)$$

$$+c\left\|\left(\frac{1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\nabla(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2+\frac{1}{4}\left\|\left(\frac{1}{n^*}\right)^{\frac{1}{2}}\frac{\partial\Delta(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\right\|^2$$

$$\leq \varepsilon c\Big(\|\nabla\Delta\phi^\varepsilon(t)\|^4 + \|\Delta n^\varepsilon(t)\|^4 + \|\nabla n^\varepsilon(t)\|^4 + (\|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2)^2\Big)$$

$$+c\left\|\frac{\partial(\phi^\varepsilon - n^\varepsilon)}{\partial x_3}(t)\right\|^2 + \varepsilon c\|\Delta\phi^\varepsilon(t)\|^2. \qquad (3.2.12)$$

Putting

$$S_4(t) \equiv \|\Delta\phi^\varepsilon(t)\|^2 + \|\nabla n^\varepsilon(t)\|^2 + \|\nabla\Delta\phi^\varepsilon(t)\|^2 + \|\Delta n^\varepsilon(t)\|^2$$

and integrating (3.2.12) over $[0,t]$, we have with the help of (3.2.1) and (3.2.2)

$$S_4(t) \leq c\int_0^t S_4(\tau)^2\,\mathrm{d}\tau + S_4(0) + c\left(\|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2\right)$$

$$+c\left(\|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2\right)^2 t \equiv c\int_0^t S_4(\tau)^2\,\mathrm{d}\tau + C_3(t) \equiv S_4^*(t).$$

Then $S_4^*(t)$ satisfis the differential inequality

$$\frac{\mathrm{d}S_4^*(t)}{\mathrm{d}t} \leq cS_4^*(t)^2 + \frac{\mathrm{d}C_3(t)}{\mathrm{d}t},$$

Since $C_3(t)$ is increasing, one can derive from this inequality,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{S_4^*(t)}\right) \leq c + \frac{1}{S_4^*(t)^2}\frac{\mathrm{d}C_3(t)}{\mathrm{d}t} \leq c + \frac{1}{C_3(t)^2}\frac{\mathrm{d}C_3(t)}{\mathrm{d}t}.$$

Integrating this inequality over $[0,t]$ and noting $S_4^*(0) = C_3(0)$, we have

$$S_4^*(t) \leq \frac{C_3(t)}{1 - cC_3(t)t}.$$

Then we choose $T > 0$ such that $C_3(0) - c(1 + C_3(0))C_3(T)T = 0$. This and the integral of (3.2.12) over $[0,t]$ lead to the inequality (3.2.5). □

3.2.2 Proof of Theorem 3.1.1

By the standard arguments based upon the *a priori* estimates in Lemmas 3.2.1 and 3.2.2 the solution can be extended up to T indicated in the proof of Lemma 3.2.2. Thus the proof of Theorem 3.1.1 is complete.

3.3 Existence theorem for model equations of drift wave turbulence with zero resistivity

The proof of Theorem 3.1.2 is divided into two parts. First we prove the local in time existence by the method of characteristics (see [57]) and the method of successive approximations in §3.3.1. Second we prove the following theorem with the help of *a priori* estimates in §3.3.2.

3.3.1 Local in time existence and uniqueness

Rewrite the first equation of (3.1.2) as a system of equations for $(\mathcal{I}-\mathcal{M})\,(\Delta\phi^0-\phi^0)$ and $\mathcal{M}\,(\Delta\phi^0-\phi^0)$,

$$\begin{cases} \left(\dfrac{\partial}{\partial t}-(\nabla\overline{\phi^0}\times\vec{e})\cdot\nabla\right)\left(\Delta\widetilde{\phi^0}-\widetilde{\phi^0}\right)-\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\left(\Delta\overline{\phi^0}-\overline{n^0}\right) \\ \qquad -(\mathcal{I}-\mathcal{M})\left\{\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\Delta\widetilde{\phi^0}\right\}=c_2\Delta^2\widetilde{\phi^0}, \\ \left(\dfrac{\partial}{\partial t}-(\nabla\overline{\phi^0}\times\vec{e})\cdot\nabla\right)\Delta\overline{\phi^0}-\mathcal{M}\left\{\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\Delta\widetilde{\phi^0}\right\}=c_2\Delta^2\overline{\phi^0}. \end{cases}$$

Furthermore, denoting by $\vec{v}(x',t)\equiv-\nabla\overline{\phi^0}(x',t)\times\vec{e}$ and $V(x',t)\equiv\Delta\overline{\phi^0}$, we rewrite problem (3.1.2) into the form

$$
\begin{cases}
\left(\dfrac{\partial}{\partial t}+\vec{v}\cdot\nabla-c_2\Delta\right)\left(\Delta\widetilde{\phi^0}-\widetilde{\phi^0}\right)=\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\left(V-\overline{n^0}\right) \\
\quad+(\mathcal{I}-\mathcal{M})\left\{\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\Delta\widetilde{\phi^0}\right\}+c_2\Delta\widetilde{\phi^0}\quad\text{for } x\in\Omega,\ t>0, \\
\left(\dfrac{\partial}{\partial t}+\vec{v}\cdot\nabla-c_2\Delta\right)V=\mathcal{M}\left\{\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\Delta\widetilde{\phi^0}\right\}, \\
\left(\dfrac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)\overline{n^0}=0, \\
\nabla\cdot(\vec{v}\times\vec{e})=V,\quad \nabla\times(\vec{v}\times\vec{e})=0\quad\text{for } x'\in\omega,\ t>0, \\
\widetilde{\phi^0}(x,0)=\widetilde{\phi^0_0}(x)\quad\text{for }\ x\in\Omega, \\
V(x',0)=\Delta\overline{\phi^0_0}(x'),\quad \overline{n^0}(x',0)=\overline{n^0_0}(x')\quad\text{for }\ x'\in\omega, \\
\widetilde{\phi^0}(x,t)=\Delta\widetilde{\phi^0}(x,t)=0\quad\text{for }\ x\in\Gamma,\ t>0, \\
V(x',t)=\overline{n^0}(x',t)=0,\quad \vec{v}(x',t)=\vec{0}\quad\text{for }\ x'\in\partial\omega,\ t>0, \\
\widetilde{\phi^0},\quad\text{periodic in the } x_3\text{-direction.}
\end{cases}
$$

It is easily seen that this problem is equivalent to the problem

$$
\left\{
\begin{aligned}
&\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla - c_2\Delta\right)\psi = \left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\left(V - \overline{n^0}\right) \\
&\qquad\qquad + (\mathcal{I}-\mathcal{M})\left\{\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\psi\right\} + c_2\left(\widetilde{\phi^0}+\psi\right), \\
&\Delta\widetilde{\phi^0} - \widetilde{\phi^0} = \psi \quad \text{for } x\in\Omega,\ t>0, \\
&\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla - c_2\Delta\right)V = \mathcal{M}\left\{\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\psi\right\}, \\
&\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\overline{n^0} = 0, \\
&\nabla\cdot(\vec{v}\times\vec{e}) = V, \quad \nabla\times(\vec{v}\times\vec{e}) = 0 \quad \text{for } x'\in\omega,\ t>0, \\
&\psi(x,0) = \Delta\widetilde{\phi_0^0}(x) - \widetilde{\phi_0^0}(x) \quad \text{for } \ x\in\Omega, \\
&V(x',0) = \Delta\overline{\phi_0^0}(x'), \quad \overline{n^0}(x',0) = \overline{n_0^0}(x') \quad \text{for } \ x'\in\omega, \\
&\psi(x,t) = \widetilde{\phi^0}(x,t) = 0 \quad \text{for } \ x\in\Gamma,\ t>0, \\
&V(x',t) = \overline{n^0}(x',t) = 0, \quad \vec{v}(x',t) = \vec{0} \quad \text{for } \ x'\in\partial\omega,\ t>0, \\
&\psi,\ \widetilde{\phi^0}, \quad \text{periodic in the } x_3\text{-direction.}
\end{aligned}
\right.
\tag{3.3.1}
$$

The following lemma is well-known (see, for example, [30], [51]).

Lemma 3.3.1 *Let $l \geq 0$. Assume that $\psi_0 \in W_2^{1+l}(\Omega)$ satisfies the compatibility conditions up to order* $\max\{[l-3/2],0\}$. *Then for any $f \in W_2^{l,l/2}(Q_T)$ there exists a unique solution $\psi \in W_2^{2+l,1+l/2}(Q_T)$ to problem*

$$
\left\{
\begin{aligned}
&\frac{\partial\psi}{\partial t} - c_2\Delta\psi = f \quad \textit{for } x\in\Omega,\ t>0, \\
&\psi(x,0) = \psi_0(x) \quad \textit{for } x\in\Omega, \\
&\psi(x,t) = 0 \quad \textit{for } x\in\Gamma,\ t>0, \\
&\psi, \quad \textit{periodic in the } x_3\textit{-direction.}
\end{aligned}
\right.
$$

Moreover, this solution satisfies the inequality

$$\|\psi\|_{W_2^{2+l,1+l/2}(Q_T)} \leq c \left(\|\psi_0\|_{W_2^{1+l}(\Omega)} + \|f\|_{W_2^{l,l/2}(Q_T)} \right).$$

Let us reduce problem (3.3.1) to the problem with zero initial data for ψ and V. According to Lemma 3.3.1, there exists $(\psi^\star, V^\star) \in W_2^{3,3/2}(Q_T) \times W_2^{3,3/2}(\omega_T)$ satisfying the equations

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - c_2\Delta\right)\psi^\star = 0 \quad \text{for } x \in \Omega,\ t > 0, \\ \left(\dfrac{\partial}{\partial t} - c_2\Delta\right)V^\star = 0 \quad \text{for } x' \in \omega,\ t > 0, \\ \psi^\star(x,0) = \Delta\widetilde{\phi_0^0}(x) - \widetilde{\phi_0^0}(x) \quad \text{for } x \in \Omega, \\ V^\star(x',0) = \Delta\overline{\phi_0^0}(x') \quad \text{for } x' \in \omega, \\ \psi^\star(x,t) = 0 \quad \text{for } x \in \Gamma,\ t > 0, \\ V^\star(x',t) = 0 \quad \text{for } x' \in \partial\omega,\ t > 0, \\ \psi^\star, \quad \text{periodic in the } x_3\text{-direction}, \end{cases}$$

and the inequalities

$$\begin{cases} \|\psi^\star\|_{W_2^{3,3/2}(Q_T)} \leq c \left\|\Delta\widetilde{\phi_0^0} - \widetilde{\phi_0^0}\right\|_{W_2^2(\Omega)}, \\ \|V^\star\|_{W_2^{3,3/2}(\omega_T)} \leq c \left\|\Delta\overline{\phi_0^0}\right\|_{W_2^2(\omega)}. \end{cases} \tag{3.3.2}$$

By putting $\psi^\sharp \equiv \psi - \psi^\star$ and $V^\sharp \equiv V - V^\star$, the problem (3.3.1) is equivalent to the problem

$$
\left\{
\begin{aligned}
&\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla - c_2\Delta\right)\psi^\sharp = \left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\left(V^\sharp + V^\star - \overline{n^0}\right)\\
&\qquad + (\mathcal{I} - \mathcal{M})\left\{\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\left(\psi^\sharp + \psi^\star\right)\right\} - \vec{v}\cdot\nabla\psi^\star\\
&\qquad + c_2\left(\widetilde{\phi^0} + \psi^\sharp + \psi^\star\right),\\
&\Delta\widetilde{\phi^0} - \widetilde{\phi^0} = \psi^\sharp + \psi^\star \quad \text{for } x\in\Omega,\ t>0,\\
&\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla - c_2\Delta\right)V^\sharp = \mathcal{M}\left\{\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\left(\psi^\sharp + \psi^\star\right)\right\}\\
&\qquad\qquad\qquad\qquad\qquad - \vec{v}\cdot\nabla V^\star,\\
&\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\overline{n^0} = 0,\\
&\nabla\cdot(\vec{v}\times\vec{e}) = V^\sharp + V^\star, \quad \nabla\times(\vec{v}\times\vec{e}) = 0 \quad \text{for } x'\in\omega,\ t>0,\\
&\psi^\sharp(x,0) = 0 \quad \text{for } \ x\in\Omega,\\
&V^\sharp(x',0) = 0, \quad \overline{n^0}(x',0) = \overline{n^0_0}(x') \quad \text{for } \ x'\in\omega,\\
&\psi^\sharp(x,t) = \widetilde{\phi^0}(x,t) = 0 \quad \text{for } \ x\in\Gamma,\ t>0,\\
&V^\sharp(x',t) = \overline{n^0}(x',t) = 0, \quad \vec{v}(x',t) = \vec{0} \quad \text{for } \ x'\in\partial\omega,\ t>0,\\
&\psi^\sharp,\ \widetilde{\phi^0} \quad \text{periodic in the } x_3\text{-direction.}
\end{aligned}
\right.
\tag{3.3.3}
$$

Now we transform the problem (3.3.3) by the method of characteristics. As usual, we introduce the characteristic transformation $\Pi^{x'}_{\xi'} : x' \longmapsto \xi' = (\xi_1, \xi_2) \equiv X(0; x', t)$, where $X(\tau; x', t)$ is the solution curve of the ordinary differential equations

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}X(\tau;x',t) = \vec{v}(X(\tau;x',t),\tau), \quad X(t;x',t) = x' \quad (0\le\tau\le t). \tag{3.3.4}
$$

The unique existence of such a solution curve $X(\tau; x', t)$ $(x' \in \Omega,\ 0 \le \tau \le t)$ of (3.3.4) is due to the fundamental existence theorem of ordinary differential equations provided that v is suitably smooth. Let $x' = X^{-1}(t; \xi', 0)$ be the inverse of $X(0;x',t)=\xi'$. Then (3.3.4) implies that X^{-1} is a solution curve of

$$\frac{\mathrm{d}}{\mathrm{d}\tau}X^{-1}(\tau;\xi',0)=\vec{u}(\xi',\tau),\quad X^{-1}(0;\xi',0)=\xi'\quad(0\le\tau\le t)\tag{3.3.5}$$

with $\vec{u}(\xi',t)\equiv\vec{v}(X^{-1}(t;\xi',0),t)=\vec{v}(x',t)$, whose solution is expressed by

$$x'=X^{-1}(t;\xi',0)=\xi'+\int_0^t\vec{u}(\xi',\tau)d\tau\equiv X_u(\xi',t).\tag{3.3.6}$$

According to the condition $\vec{v}=\vec{0}$ on $\partial\omega$, $\Pi_{\xi'}^{x'}$ is a one-to-one mapping from $\overline{\omega}$ and $\partial\omega$ onto $\overline{\omega}$ and $\partial\omega$, respectively for each $t>0$.

The fourth equation of (3.3.3) yields

$$\partial\overline{\rho}/\partial t=0\quad\text{for }\overline{\rho}(\xi',t)\equiv\overline{n^0}(X_u(\xi',t),t),$$

which is easily solved as

$$\overline{\rho}(\xi',t)=\overline{\rho}(\xi',0)=\overline{n_0^0}\,(X_u(\xi',0))=\overline{n_0^0}(\xi').\tag{3.3.7}$$

For simplicity the transformation from x_3 to $\xi_3=x_3$ is denoted by $\Pi_{\xi_3}^{x_3}$ and $\Pi_{\xi',\xi_3}^{x',x_3}\equiv\Pi_{\xi'}^{x'}\Pi_{\xi_3}^{x_3}$. Transform the problem (3.3.3) by $\Pi_{\xi',\xi_3}^{x',x_3}$ and replace $\overline{\rho}$ with (3.3.7). Then we obtain the following problem for

$$\begin{cases}\left(\widetilde{\psi},\psi^{\star(u)},\widetilde{\varphi}\right)(\xi,t)\equiv\Pi_{\xi',\xi_3}^{x',x_3}\left(\psi^{\sharp},\psi^{\star},\widetilde{\phi^0}\right)(x,t),\\ \left(U,V^{\star(u)}\right)(\xi',t)\equiv\Pi_{\xi'}^{x'}\left(V^{\sharp},V^{\star}\right)(x',t)\quad(\xi=(\xi',\xi_3),\,\xi_3=x_3):\end{cases}$$

$$
\begin{cases}
\left(\frac{\partial}{\partial t}-c_2\Delta_\xi\right)\widetilde{\psi}=\left(\nabla_u\widetilde{\varphi}\times\vec{e}\right)\cdot\nabla_u\left(U+V^{\star(u)}-\overline{n_0^0}\right) \\
\qquad\qquad +\left(\mathcal{I}-\mathcal{M}\right)\left\{\left(\nabla_u\widetilde{\varphi}\times\vec{e}\right)\cdot\nabla_u\left(\widetilde{\psi}+\psi^{\star(u)}\right)\right\}-\vec{u}\cdot\nabla_u\psi^{\star(u)} \\
\qquad\qquad +c_2\left(\widetilde{\varphi}+\widetilde{\psi}+\psi^{\star(u)}\right)+c_2\left(\Delta_u-\Delta_\xi\right)\widetilde{\psi}\equiv F\left(\widetilde{\psi},\widetilde{\varphi},U,\vec{u}\right), \\
\Delta_\xi\widetilde{\varphi}-\widetilde{\varphi}=\widetilde{\psi}+\psi^{\star(u)}-\left(\Delta_u-\Delta_\xi\right)\widetilde{\varphi}\equiv G\left(\widetilde{\psi},\widetilde{\varphi},\vec{u}\right)\quad \text{for } \xi\in\Omega,\ t>0, \\
\left(\frac{\partial}{\partial t}-c_2\Delta_\xi\right)U=\mathcal{M}\left\{\left(\nabla_u\widetilde{\varphi}\times\vec{e}\right)\cdot\nabla_u\left(\widetilde{\psi}+\psi^{\star(u)}\right)\right\} \\
\qquad\qquad -\vec{u}\cdot\nabla_u V^{\star(u)}+c_2\left(\Delta_u-\Delta_\xi\right)U\equiv H\left(\widetilde{\psi},\widetilde{\varphi},U,\vec{u}\right), \\
\nabla_\xi\cdot\left(\vec{u}\times\vec{e}\right)=U+V^{\star(u)}-\left(\nabla_u-\nabla_\xi\right)\cdot\left(\vec{u}\times\vec{e}\right)\equiv I\left(U,\vec{u}\right), \\
\nabla_\xi\times\left(\vec{u}\times\vec{e}\right)=-\left(\nabla_u-\nabla_\xi\right)\times\left(\vec{u}\times\vec{e}\right)\equiv\vec{J}\left(\vec{u}\right)\quad \text{for } \xi'\in\omega,\ t>0, \\
\widetilde{\psi}(\xi,0)=0\quad \text{for}\ \ \xi\in\Omega, \\
U(\xi',0)=0\quad \text{for}\ \ \xi'\in\omega, \\
\widetilde{\psi}(\xi,t)=\widetilde{\varphi}(\xi,t)=0\quad \text{for}\ \ \xi\in\Gamma,\ t>0, \\
U(\xi',t)=0,\quad \vec{u}(\xi',t)=\vec{0}\quad \text{for}\ \ \xi'\in\partial\omega,\ t>0, \\
\widetilde{\psi},\ \widetilde{\varphi},\quad \text{periodic in the } \xi_3\text{-direction},
\end{cases}
\tag{3.3.8}
$$

where $\nabla_u \equiv (\nabla_{u,1},\nabla_{u,2},\nabla_{u,3}) = \left(\mathcal{G}\nabla_{\xi'},\frac{\partial}{\partial\xi_3}\right)$, $\nabla_{\xi'} = \left(\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_2}\right)$, $\mathcal{G} \equiv \mathcal{G}^{[u]} = \left(\frac{\partial X_u}{\partial\xi'}\right)^{-\mathrm{T}} = \left(\delta_{jk}+\int_0^t\frac{\partial u_j}{\partial\xi_k}d\tau\right)^{-\mathrm{T}}$ (the superscript $^{-\mathrm{T}}$ stands for the transposition and inverse), $\Delta_u=\nabla_u\cdot\nabla_u$, $\nabla_\xi=\left(\nabla_{\xi'},\frac{\partial}{\partial\xi_3}\right)$, $\Delta_\xi=\nabla_\xi\cdot\nabla_\xi$.

Notice that from the definition, it follows

$$
\widetilde{\varphi}(\xi,0)=\widetilde{\phi_0^0}\left(\xi\right),\quad \vec{u}(\xi',0)=-\nabla_\xi\overline{\phi_0^0}\left(\xi'\right)\times\vec{e}. \tag{3.3.9}
$$

Now we solve problem (3.3.8) by the method of successive approximations. Let $\left(\widetilde{\psi}^{(0)},\widetilde{\varphi}^{(0)},U^{(0)},\vec{u}^{(0)}\right) = (0,0,0,\vec{0})$ and $\left(\widetilde{\psi}^{(m+1)},\widetilde{\varphi}^{(m+1)},U^{(m+1)},\vec{u}^{(m+1)}\right)$

$(m = 0, 1, 2, \ldots)$ be a solution of the initial boundary value problem

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - c_2\Delta_\xi\right)\widetilde{\psi}^{(m+1)} = F\left(\widetilde{\psi}^{(m)}, \widetilde{\varphi}^{(m)}, U^{(m)}, \vec{u}^{(m)}\right) \equiv F^{(m)}, \\ \Delta_\xi\widetilde{\varphi}^{(m+1)} - \widetilde{\varphi}^{(m+1)} = G\left(\widetilde{\psi}^{(m)}, \widetilde{\varphi}^{(m)}, \vec{u}^{(m)}\right) \equiv G^{(m)} \\ \qquad\qquad \text{for } \xi \in \Omega,\ t > 0, \\ \left(\dfrac{\partial}{\partial t} - c_2\Delta_\xi\right)U^{(m+1)} = H\left(\widetilde{\psi}^{(m)}, \widetilde{\varphi}^{(m)}, U^{(m)}, \vec{u}^{(m)}\right) \equiv H^{(m)}, \\ \nabla_\xi \cdot \left(\vec{u}^{(m+1)} \times \vec{e}\right) = I\left(U^{(m)}, \vec{u}^{(m)}\right) \equiv I^{(m)}, \\ \nabla_\xi \times \left(\vec{u}^{(m+1)} \times \vec{e}\right) = \vec{J}\left(\vec{u}^{(m)}\right) \equiv \vec{J}^{(m)} \quad \text{for } \xi' \in \omega,\ t > 0, \\ \widetilde{\psi}^{(m+1)}(\xi, 0) = 0 \quad \text{for } \ \xi \in \Omega, \\ U^{(m+1)}(\xi', 0) = 0 \quad \text{for } \ \xi' \in \omega, \\ \widetilde{\psi}^{(m+1)}(\xi, t) = \widetilde{\varphi}^{(m+1)}(\xi, t) = 0 \quad \text{for } \ \xi \in \Gamma,\ t > 0, \\ U^{(m+1)}(\xi', t) = 0, \quad \vec{u}^{(m+1)}(\xi', t) = \vec{0} \quad \text{for } \ \xi' \in \partial\omega,\ t > 0, \\ \widetilde{\psi}^{(m+1)}, \quad \widetilde{\varphi}^{(m+1)}, \quad \text{periodic in the } \xi_3\text{-direction,} \end{cases} \tag{3.3.10}$$

where $\nabla_m = \nabla_{u^{(m)}}$, $\Delta_m = \nabla_m \cdot \nabla_m$ and $\left(\widetilde{\psi}^{(m)}, \widetilde{\varphi}^{(m)}, U^{(m)}, \vec{u}^{(m)}\right) \in W_2^{2,1}(Q_T) \times (L^2(0, T; W_2^4(\Omega)) \cap W_2^1\,(0, T; W_2^2(\Omega))) \times W_2^{2,1}(\omega_T) \times (L^2\,(0, T; W_2^3(\omega)) \cap W_2^1(0, T; W_2^1(\omega)))$ is given in such a way that

$$T^{1/2}\left\|\vec{u}^{(m)}\right\|_{L^2\left(0,T;W_2^3(\omega)\right)} \leq \delta \tag{3.3.11}$$

with some positive constant δ and satisfy

$$\widetilde{\varphi}^{(m)}(\xi, 0) = \widetilde{\phi_0^0}\,(\xi)\,, \quad \vec{u}^{(m)}(\xi', 0) = -\nabla_\xi\overline{\phi_0^0}\,(\xi') \times \vec{e}. \tag{3.3.12}$$

For convenience, we denote

$$\psi^{\star(m)}(\xi, t) = \psi^{\star\left[u^{(m)}\right]}(\xi, t), \quad V^{\star(m)}(\xi', t) = V^{\star\left[u^{(m)}\right]}(\xi', t), \quad \mathcal{G}^{(m)} = \mathcal{G}^{\left[u^{(m)}\right]}.$$

The following lemma is well-known (see, for example, [10], [31]).

Lemma 3.3.2 *Assume that $\psi \in W_2^{2,1}(Q_T)$. Then problem*

$$\begin{cases} \Delta\phi - \phi = \psi \quad \text{for } x \in \Omega,\ t > 0, \\ \phi(x,t) = 0 \quad \text{for } x \in \Gamma,\ t > 0, \\ \phi, \quad \text{periodic in the } x_3\text{-direction} \end{cases}$$

has a unique solution $\phi \in L^2(0,T;W_2^4(\Omega)) \cap W_2^1(0,T;W_2^2(\Omega))$, which satisfies the inequality

$$\|\phi\|_{L^2(0,T;W_2^4(\Omega))} + \|\phi\|_{W_2^1(0,T;W_2^2(\Omega))} \le c\|\psi\|_{W_2^{2,1}(Q_T)}.$$

Since the fourth and fifth equations of (3.3.10) constitute Cauchy–Riemann equations, they have a unique solution $\vec{u}^{(m+1)} \times \vec{e}$. Applying Lemmas 3.3.1 and 3.3.2, one can find a unique solution $\left(\widetilde{\psi}^{(m+1)}, \widetilde{\varphi}^{(m+1)}, U^{(m+1)}, \vec{u}^{(m+1)}\right)$ to the problem (3.3.10) satisfying the inequality

$$\begin{aligned} z^{(m+1)}(T) \quad &\equiv \left\|\widetilde{\psi}^{(m+1)}\right\|_{W_2^{2,1}(Q_T)} + \left\|\widetilde{\varphi}^{(m+1)}\right\|_{L^2\left(0,T;W_2^4(\Omega)\right)} \\ &\quad + \left\|\widetilde{\varphi}^{(m+1)}\right\|_{W_2^1\left(0,T;W_2^2(\Omega)\right)} + \left\|U^{(m+1)}\right\|_{W_2^{2,1}(\omega_T)} \\ &\quad + \left\|\vec{u}^{(m+1)}\right\|_{L^2\left(0,T;W_2^3(\omega)\right)} + \left\|\vec{u}^{(m+1)}\right\|_{W_2^1\left(0,T;W_2^1(\omega)\right)} \\ &\le c\Big(\left\|F^{(m)}\right\|_{L^2(Q_T)} + \left\|G^{(m)}\right\|_{W_2^{2,1}(Q_T)} + \left\|H^{(m)}\right\|_{L^2(\omega_T)} \\ &\qquad\qquad + \left\|I^{(m)}\right\|_{W_2^{2,1}(\omega_T)} + \left\|\vec{J}^{(m)}\right\|_{W_2^{2,1}(\omega_T)} \Big). \end{aligned}$$

In order to estimate each term on the right hand side of the above inequality we use the following lemmas (see, for example, [52], [58]).

Lemma 3.3.3 ([52]) *Let Ω be a bounded domain in $\mathbf{R}^3$. Then the following estimates hold.*

$$\|fg\|_{L^2(\Omega)} \le c\|f\|_{L^4(\Omega)}\|g\|_{L^4(\Omega)} \le c\|f\|_{W_2^1(\Omega)}\|g\|_{W_2^1(\Omega)}$$

$$for\ f \in W_2^1(\Omega),\ \ g \in W_2^1(\Omega),$$

$$\|f \cdot \nabla g\|_{L^2(\Omega)} \le c\|f\|_{L^6(\Omega)}\|\nabla g\|_{L^3(\Omega)} \le c\|f\|_{W_2^1(\Omega)}\|g\|_{W_2^2(\Omega)}^{3/4}\|g\|_{L^2(\Omega)}^{1/4}$$

$$for\ f \in W_2^1(\Omega),\ \ g \in W_2^2(\Omega).$$

Lemma 3.3.4 ([58]) *Let ω be a bounded domain in $\mathbf{R}^2$. For any $\vec{u}, \vec{u}' \in (L^2(0,T;W_2^3(\omega)) \cap W_2^1(0,T;W_2^1(\omega)))$ satisfying* (3.3.11) *and for any $t \le T$, the following inequality holds.*

$$\left\|\mathcal{G}^{[u]} - \mathcal{G}^{[u']}\right\|_{W_2^2(\omega)} \le C(t,\delta)t^{1/2}\left\|\vec{u} - \vec{u}'\right\|_{L^2\left(0,t;W_2^3(\omega)\right)},$$

where $C(t,\delta)$ is a positive constant depending increasingly on both arguments.

Lemma 3.3.5 *Let Ω and ω be a bounded domain in $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively. Assume that $f \in (L^2(0,T;W_2^3(\Omega)) \cap W_2^1(0,T;W_2^1(\Omega)))$ and $\vec{u}, \vec{u}' \in (L^2(0,T;W_2^3(\omega)) \cap W_2^1(0,T;W_2^1(\omega)))$ satisfy* (3.3.11). *Then for any $t \le T$, the following inequalities hold.*

$$\left\|f^{[u]}\right\|_{L^2(Q_t)} \le C(\delta)t^{1/2}\left\|f\right\|_{L^\infty(0,t;L^2(\Omega))},$$

$$\left\|f^{[u]}\right\|_{W_2^{2,0}(Q_t)} \le C(\delta)\left\|f\right\|_{W_2^{2,0}(Q_t)}\left(1 + t^{1/2}\left\|\vec{u}\right\|_{W_2^{3,0}(\omega_t)}\right),$$

$$\left\|f^{[u]} - f^{[u']}\right\|_{L^2(Q_t)} \le C(\delta)t^{1/2}\left\|f\right\|_{L^2\left(0,t;W_2^1(\Omega)\right)}\left\|\vec{u} - \vec{u}'\right\|_{L^2\left(0,t;W_2^2(\omega)\right)},$$

$$\left\|f^{[u]} - f^{[u']}\right\|_{L^2\left(0,t;W_2^2(\Omega)\right)} \le C(\delta)t^{1/2}\left\|f\right\|_{L^2\left(0,t;W_2^3(\Omega)\right)}\left\|\vec{u} - \vec{u}'\right\|_{L^2\left(0,t;W_2^2(\omega)\right)},$$

where $C(\delta)$ is a positive constant depending on δ.

Now we evaluate $\left\|F^{(m)}\right\|_{L^2(Q_T)}$. From (3.3.2), (3.3.12) and Lemmas 3.3.3-3.3.5 with $\vec{u} = \vec{u}^{(m)}$, $\vec{u}' \equiv \vec{0}$, it follows the inequalities

$$\left\|\left(\nabla_m\widetilde{\varphi}^{(m)} \times \vec{e}\right) \cdot \nabla_m\left(U^{(m)} + V^{\star(m)} - \overline{n_0^0}\right)\right\|_{L^2(Q_T)}$$

$$\leq c\int_0^t \Big(\left\|\nabla_m\widetilde{\varphi}^{(m)}\right\|_{L^6(\Omega)}^2 \left\|\nabla_m\left(U^{(m)}-\overline{n_0^0}\right)\right\|_{L^3(\Omega)}^2$$

$$+\left\|\nabla_m\widetilde{\varphi}^{(m)}\right\|_{L^4(\Omega)}^2 \left\|\nabla_m V^{\star(m)}\right\|_{L^4(\Omega)}^2 \Big)\mathrm{d}\tau$$

$$\leq c\left(\left\|\widetilde{\phi_0^0}\right\|_{W_2^2(\Omega)}+T^{1/2}z^{(m)}(T)\right)\Big(T^{1/2}z^{(m)}(T)$$

$$+C(\delta)\left\|\overline{\phi_0^0}\right\|_{W_2^4(\omega)}\left(1+T^{1/2}z^{(m)}(T)\right)+T^{1/2}\left\|\overline{n_0^0}\right\|_{W_2^2(\omega)}\Big),$$

$$\left\|(\mathcal{I}-\mathcal{M})\left\{\left(\nabla_m\widetilde{\varphi}^{(m)}\times\vec{e}\right)\cdot\nabla_m\left(\widetilde{\psi}^{(m)}+\psi^{\star(m)}\right)\right\}\right\|_{L^2(Q_T)}$$

$$\leq c\int_0^t \Big(\left\|\nabla_m\widetilde{\varphi}^{(m)}\right\|_{L^6(\Omega)}^2 \left\|\nabla_m\widetilde{\psi}^{(m)}\right\|_{L^3(\Omega)}^2$$

$$+\left\|\nabla_m\widetilde{\varphi}^{(m)}\right\|_{L^4(\Omega)}^2 \left\|\nabla_m\psi^{\star(m)}\right\|_{L^4(\Omega)}^2 \Big)\mathrm{d}\tau$$

$$\leq c\left(\left\|\widetilde{\phi_0^0}\right\|_{W_2^2(\Omega)}+T^{1/2}z^{(m)}(T)\right)$$

$$\times\left(T^{1/2}z^{(m)}(T)+C(\delta)\left\|\widetilde{\phi_0^0}\right\|_{W_2^4(\Omega)}\left(1+T^{1/2}z^{(m)}(T)\right)\right),$$

$$\left\|\vec{u}^{(m)}\cdot\nabla_m\psi^{\star(m)}\right\|_{L^2(Q_T)}\leq C(\delta)\left(\left\|\overline{\phi_0^0}\right\|_{W_2^2(\omega)}+T^{1/2}z^{(m)}(T)\right)\left\|\widetilde{\phi_0^0}\right\|_{W_2^4(\Omega)}$$

$$\times\left(1+T^{1/2}z^{(m)}(T)\right),$$

$$\left\|\widetilde{\varphi}^{(m)}+\widetilde{\psi}^{(m)}+\psi^{\star(m)}\right\|_{L^2(Q_T)}$$

$$\leq c\left\|\widetilde{\phi_0^0}\right\|_{L^2(\Omega)}+cT^{1/2}\left(C(\delta)\left\|\widetilde{\phi_0^0}\right\|_{W_2^3(\Omega)}+z^{(m)}(T)\right).$$

Lemmas 3.3.3 and 3.3.4 with $\vec{u}=\vec{u}^{(m)}$, $\vec{u}'\equiv\vec{0}$, yield the inequality

$$\left\|\left(\Delta_\xi-\Delta_m\right)\widetilde{\psi}^{(m)}\right\|_{L^2(Q_T)}$$

$$=\Big\|\Big[\left(\mathcal{G}^{(m)\mathrm{T}}-I\right)\left(\mathcal{G}^{(m)}-I\right)+\left(\mathcal{G}^{(m)}-I\right)+\left(\mathcal{G}^{(m)\mathrm{T}}-I\right)\Big]\nabla_\xi\cdot\nabla_\xi\widetilde{\psi}^{(m)}$$

$$+\Big[\left(\mathcal{G}^{(m)}\nabla_\xi\right)^{\mathrm{T}}\cdot\mathcal{G}^{(m)}\Big]\cdot\nabla_\xi\widetilde{\psi}^{(m)}\Big\|_{L^2(Q_T)}$$

$$\le C_1(T,\delta)z^{(m)}(T)\left(1+z^{(m)}(T)\right), \tag{3.3.13}$$

where $C_1(T,\delta)$ is a positive constant depending increasingly on both arguments and $C_1\downarrow 0$ as $T\downarrow 0$. In what follows, we denote by $C_i(T,\delta)$ positive constants having the same property as $C_1(T,\delta)$. Therefore, we obtain

$$\left\|F^{(m)}\right\|_{L^2(Q_T)}\le C(\delta)+C_2(T,\delta)\left(1+z^{(m)}(T)+z^{(m)}(T)^2\right),$$

where $C(\delta)$ is a positive increasing function of δ and $C_2(T,\delta)$ is a positive increasing function of each argument.

Similar estimates for $\left\|G^{(m)}\right\|_{W_2^{2,1}(Q_T)}$, $\left\|H^{(m)}\right\|_{L^2(\omega_T)}$, $\left\|I^{(m)}\right\|_{W_2^{2,1}(\omega_T)}$, and $\left\|\vec{J}^{(m)}\right\|_{W_2^{2,1}(\omega_T)}$ one can get.

Finally we have

$$z^{(m+1)}(T)\le C'(\delta)+C_3(T,\delta)\left(1+z^{(m)}(T)+z^{(m)}(T)^2\right),$$

where $C'(\delta)$ is a positive increasing function of δ and $C_3(T,\delta)$ is a positive increasing function of each argument.

We choose first a positive constant M in such a way that

$$M>C'(\delta),$$

where δ is the constant appeared in (3.3.11), and second a positive constant T' ($<\delta^2/M^2$) so that

$$C_3(T',\delta)\left(1+M+M^2\right)<M-C'(\delta).$$

Thus we conclude that $z^{(m)}(T')<M$ implies $z^{(m+1)}(T')<M$. By induction we see that the sequence $\left\{\left(\widetilde{\psi}^{(m)},\widetilde{\varphi}^{(m)},U^{(m)},\vec{u}^{(m)}\right)\right\}_{m=0}^{\infty}$ is well-defined on $(0,T')$ and $z^{(m)}(T')<M$ for all m.

Now we verify the convergence of $\left\{\left(\widetilde{\psi}^{(m)},\widetilde{\varphi}^{(m)},U^{(m)},\vec{u}^{(m)}\right)\right\}_{m=0}^{\infty}$. Subtracting from equations in (3.3.10) the similar equations for $\Big(\widetilde{\psi}^{(m)},\widetilde{\varphi}^{(m)},U^{(m)},$ $\vec{u}^{(m)}\Big)$ and setting $\left(\widetilde{\psi}^{*(m+1)},\widetilde{\varphi}^{*(m+1)},U^{*(m+1)},\vec{u}^{*(m+1)}\right)\equiv\Big(\widetilde{\psi}^{(m+1)}-\widetilde{\psi}^{(m)},$

$\widetilde{\varphi}^{(m+1)} - \widetilde{\varphi}^{(m)}, U^{(m+1)} - U^{(m)}, \vec{u}^{(m+1)} - \vec{u}^{(m)}\Big)$, we obtain

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - c_2\Delta_\xi\right)\widetilde{\psi}^{*(m+1)} = F^{(m)} - F^{(m-1)}, \\ \Delta_\xi \widetilde{\varphi}^{*(m+1)} - \widetilde{\varphi}^{*(m+1)} = G^{(m)} - G^{(m-1)} \quad \text{for } \xi \in \Omega,\ 0 < t < T', \\ \left(\dfrac{\partial}{\partial t} - c_2\Delta_\xi\right)U^{*(m+1)} = H^{(m)} - H^{(m-1)}, \\ \nabla_\xi \cdot \left(\vec{u}^{*(m+1)} \times \vec{e}\right) = I^{(m)} - I^{(m-1)}, \\ \nabla_\xi \times \left(\vec{u}^{*(m+1)} \times \vec{e}\right) = \vec{J}^{(m)} - \vec{J}^{(m-1)} \quad \text{for } \xi' \in \omega,\ 0 < t < T', \\ \widetilde{\psi}^{*(m+1)}(\xi, 0) = 0 \quad \text{for } \ \xi \in \Omega, \\ U^{*(m+1)}(\xi', 0) = 0 \quad \text{for } \ \xi' \in \omega, \\ \widetilde{\psi}^{*(m+1)}(\xi, t) = \widetilde{\varphi}^{*(m+1)}(\xi, t) = 0 \quad \text{for } \ \xi \in \Gamma,\ 0 < t < T', \\ U^{*(m+1)}(\xi', t) = 0, \quad \vec{u}^{*(m+1)}(\xi', t) = \vec{0} \quad \text{for } \ \xi' \in \partial\omega,\ 0 < t < T', \\ \widetilde{\psi}^{*(m+1)}, \quad \widetilde{\varphi}^{*(m+1)}, \quad \text{periodic in the } \xi_3\text{-direction.} \end{cases}$$

From (3.3.12), it obviously follows that

$$\widetilde{\varphi}^{*(m)}(\xi, 0) = 0, \quad \vec{u}^{*(m)}(\xi', 0) = 0.$$

In the same way as for (3.3.10), we can derive

$$\begin{aligned} Z^{(m+1)}(t) \ &\equiv \left\|\widetilde{\psi}^{*(m+1)}\right\|_{W_2^{2,1}(Q_t)} + \left\|\widetilde{\varphi}^{*(m+1)}\right\|_{L^2\left(0,t;W_2^4(\Omega)\right)} \\ &\quad + \left\|\widetilde{\varphi}^{*(m+1)}\right\|_{W_2^1\left(0,t;W_2^2(\Omega)\right)} + \left\|U^{*(m+1)}\right\|_{W_2^{2,1}(\omega_t)} \\ &\quad + \left\|\vec{u}^{*(m+1)}\right\|_{L^2\left(0,t;W_2^3(\omega)\right)} + \left\|\vec{u}^{*(m+1)}\right\|_{W_2^1\left(0,t;W_2^1(\omega)\right)} \\ &\le C\left(t, \delta, z^{(m)}\right) Z^{(m)}(t) \end{aligned} \tag{3.3.14}$$

for any $t \in [0, T']$ where $C\,(t, \delta, z)$ is a constant depending increasingly on each argument and $C \to 0$ as $t \to 0$. Since we can find a positive constant $T''(\le T')$ satisfying

$$C\,(T'', \delta, M) < 1,$$

the sequence $\left\{\left(\widetilde{\psi}^{(m)}, \widetilde{\varphi}^{(m)}, U^{(m)}, \vec{u}^{(m)}\right)\right\}_{m=0}^{\infty}$ converges uniformly on $[0, T'']$ to $\left(\widetilde{\psi}, \widetilde{\varphi}, U, \vec{u}\right)$ as $m \to \infty$. It is clear that $\left(\widetilde{\psi}, \widetilde{\varphi}, U, \vec{u}\right) \in W_2^{2,1}(Q_{T''}) \times (L^2(0, T''; W_2^4(\Omega)) \cap W_2^1(0, T''; W_2^2(\Omega))) \times W_2^{2,1}(\omega_{T''}) \times (L^2(0, T''; W_2^3(\omega)) \cap W_2^1(0, T''; W_2^1(\omega)))$ and $\left(\widetilde{\psi}, \widetilde{\varphi}, U, \vec{u}\right)$ is a solution of problem (3.3.8).

The uniqueness of such a solution can be easily proved by making use of the estimate analogous to (3.3.14).

From (3.3.7), we find that $\overline{\rho} \in L^\infty\left(0, T; W_2^3(\omega)\right)$.

Now we respectively define $x'$and $\left(\psi^\sharp, \psi^\star, \widetilde{\phi^0}, V^\sharp, V^\star, \vec{v}, \overline{n^0}\right)$ by (3.3.6) and

$$\begin{cases} \left(\psi^\sharp, \psi^\star, \widetilde{\phi^0}\right)(x, t) \equiv \Pi_{x', x_3}^{\xi', \xi_3}\left(\widetilde{\psi}, \psi^{\star(u)}, \widetilde{\varphi}\right)(\xi, t), \\ \left(V^\sharp, V^\star, \vec{v}, \overline{n^0}\right)(x', t) \equiv \Pi_{x'}^{\xi'}\left(U, V^{\star(u)}, \vec{u}, \overline{\rho}\right)(\xi', t). \end{cases}$$

Here the transformations from ξ' and ξ_3 to x' and x_3 are denoted by $\Pi_{x'}^{\xi'}$ and $\Pi_{x_3}^{\xi_3}$ respectively, and $\Pi_{x', x_3}^{\xi', \xi_3} \equiv \Pi_{x'}^{\xi'}\Pi_{x_3}^{\xi_3}$. Then we find that $\left(\psi^\sharp, \widetilde{\phi^0}, V^\sharp, \vec{v}, \overline{n^0}\right)$ is a unique solution of the problem (3.3.3) belonging to $W_2^{2,1}(Q_{T''}) \times (L^2(0, T''; W_2^4(\Omega)) \cap W_2^1(0, T''; W_2^2(\Omega))) \times W_2^{2,1}(\omega_{T''}) \times (L^2(0, T''; W_2^3(\omega)) \cap W_2^1(0, T''; W_2^1(\omega))) \times (L^\infty(0, T''; W_2^3(\omega)) \cap W_2^1(0, T''; W_2^2(\omega)))$.

Now we define $\phi^0 = \widetilde{\phi^0} + \overline{\phi^0}$, where $\overline{\phi^0}$ is determined by $\Delta\overline{\phi^0} = V^\sharp + V^\star$ with $\overline{\phi^0}|_{\partial\omega} = 0$. Then it is easily seen that there exists a unique solution $\left(\phi^0, \overline{n^0}\right)$ to the problem (3.1.2) on some interval $[0, T'']$ such that $\left(\phi^0, \overline{n^0}\right) \in L^2(0, T''; W_2^4(\Omega)) \times L^\infty(0, T''; W_2^3(\omega))$, $\partial\phi^0/\partial t \in L^2(0, T''; W_2^2(\Omega))$, $\partial\overline{n^0}/\partial t \in L^2(0, T''; W_2^2(\omega))$.

Thus the proof of local in time existence and uniqueness is complete.

3.3.2 Proof of Theorem 3.1.2

In this subsection we proceed to get *a priori* estimates of the solution $\left(\phi^0, \overline{n^0}\right)$ established in §3.3.1. Let T be an arbitrary positive number and $\left(\phi^0, \overline{n^0}\right)$ be a solution of problem (3.1.2) belonging to $(L^2(0, T; W_2^4(\Omega)) \cap W_2^1(0, T; W_2^2(\Omega))) \times (L^\infty\left(0, T; W_2^3(\omega)\right) \cap W_2^1(0, T; W_2^2(\omega)))$.

Since the regularity of the solution is not sufficient, the following arguments are formal. However, one can justify them by using the method of difference quotients or mollifiers. Throughout this subsection, we denote by c a constant which may differ at each occurrence.

Lemma 3.3.6 *For any* $t \in [0, T]$

$$\left\|\overline{n^0}(t)\right\|^2_{L^\infty(\omega)} = \left\|\overline{n^0_0}\right\|^2_{L^\infty(\omega)}, \tag{3.3.15}$$

$$\begin{aligned}&\left\|\nabla\phi^0(t)\right\|^2 + \left\|\widetilde{\phi^0}(t)\right\|^2 + 2c_2\int_0^t \left\|\Delta\phi^0(\tau)\right\|^2 \,\mathrm{d}\tau \\ &\qquad = \left\|\nabla\phi^0_0\right\|^2 + \left\|\widetilde{\phi^0_0}\right\|^2 \equiv c^*,\end{aligned} \tag{3.3.16}$$

$$\begin{aligned}&\left\|\Delta\phi^0(t)\right\|^2 + \left\|\nabla\widetilde{\phi^0}(t)\right\|^2 + \int_0^t \left\|\nabla\Delta\phi^0(\tau)\right\|^2 \,\mathrm{d}\tau \\ &\quad \le \left\|\Delta\phi^0_0\right\|^2 + \left\|\nabla\widetilde{\phi^0_0}\right\|^2 + cc^*\left(1 + \left(\left\|\overline{n^0_0}\right\|^2_{L^\infty(\omega)} + c^{*4}\right)t\right) \\ &\quad \equiv C^{**}(t).\end{aligned} \tag{3.3.17}$$

Proof. The solution of the second equation of (3.1.2) is given by the formula (3.3.7), which yields (3.3.15).

Multiplying the first equation of (3.1.2) by ϕ^0, and integrating over Ω, we have, by virtue of integration by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla\phi^0(t)\right\|^2 + \left\|\widetilde{\phi^0}(t)\right\|^2\right) + c_2\|\Delta\phi^0(t)\|^2 = 0.$$

Integrating this over $[0, t]$, we obtain (3.3.16).

Multiplying the first equation of (3.1.2) by $\Delta\phi^0$ and integrating over Ω, we have, by virtue of integration by parts, the Gagliardo–Nirenberg, Schwarz's and Young's inequalities and the Sobolev imbedding theorem

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\Delta\phi^0(t)\right\|^2 + \left\|\nabla\widetilde{\phi^0}(t)\right\|^2\right) + c_2\left\|\nabla\Delta\phi^0(t)\right\|^2 \\ &\quad = \int_\Omega \left(\nabla\widetilde{\phi^0} \times \vec{e}\right)\cdot\nabla\overline{n^0}\,\Delta\phi^0\,\mathrm{d}x - \int_\Omega\left(\nabla\phi^0 \times \vec{e}\right)\cdot\nabla\widetilde{\phi^0}\,\Delta\phi^0\,\mathrm{d}x \\ &\quad \le c\left\|\nabla\widetilde{\phi^0}(t)\right\|\left\|\overline{n^0}(t)\right\|_{L^\infty(\omega)}\left\|\nabla\Delta\phi^0(t)\right\| \\ &\qquad\quad + c\left\|\nabla\Delta\phi^0(t)\right\|\left\|\nabla\phi^0(t)\right\|_{L^4(\Omega)}\left\|\widetilde{\phi^0}(t)\right\|_{L^4(\Omega)} \\ &\quad \le \frac{c_2}{2}\left\|\nabla\Delta\phi^0(t)\right\|^2 + \frac{c}{c_2}\left(c^*\left\|\overline{n^0_0}\right\|^2_{L^\infty(\omega)} + c^{*\frac{5}{4}}\left\|\Delta\phi^0(t)\right\|^{\frac{3}{2}}\right),\end{aligned}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\Delta\phi^0(t)\right\|^2+\left\|\nabla\tilde{\phi}^0(t)\right\|^2\right)+c_2\left\|\nabla\Delta\phi^0(t)\right\|^2$$

$$\leq\frac{c}{c_2}\left(c^*\left\|\overline{n_0^0}\right\|^2_{L^\infty(\omega)}+c^{*5}+\left\|\Delta\phi^0(t)\right\|^2\right),$$

Here we used the estimates (3.3.15), (3.3.16) and the inequality

$$\left\|\nabla\widetilde{\phi^0}\right\|\leq\left\|\nabla\phi^0\right\|+\left\|\nabla\overline{\phi^0}\right\|\leq 2\left\|\nabla\phi^0\right\|.$$

Integrating this over $[0,t]$, we obtain (3.3.17) with the help of (3.3.15) and (3.3.16). □

Next we prove

Lemma 3.3.7 *There exists a positive constant T^* such that the estimates*

$$\left\|\nabla\Delta\phi^0(t)\right\|^2+\left\|\Delta\widetilde{\phi^0}(t)\right\|^2+\left\|\nabla\overline{n^0}(t)\right\|^2_{L^2(\omega)}+c_2\int_0^t\left\|\Delta^2\phi^0(\tau)\right\|^2\mathrm{d}\tau$$

$$\leq\frac{C_4(t)}{1-cC_4(t)t},\tag{3.3.18}$$

$$\left\|D^\alpha_{x'}\overline{n^0}(t)\right\|^2_{L^2(\omega)}\leq\left(\sum_{|\alpha'|\leq|\alpha|}\left\|D^{\alpha'}_{x'}\overline{n^0_0}\right\|^2_{L^2(\omega)}\right)C_\alpha(t)\equiv C^*_\alpha(t),$$

$$|\alpha|=1,\ 2,\ 3,\tag{3.3.19}$$

$$\left\|\frac{\partial D^\alpha_{x'}\overline{n^0}}{\partial t}(t)\right\|^2_{L^2(\omega)}\leq\sqrt{C_0^{***}+c^{***}t}\sqrt{C^*_{\alpha+1}(t)},\quad|\alpha|=0,\ 1,\ 2,\tag{3.3.20}$$

hold for any $t\in[0,T^*)$. Here

$$C_4(t)=\left\|\nabla\Delta\phi^0_0\right\|^2+\left\|\Delta\widetilde{\phi^0_0}\right\|^2+\left\|\nabla\overline{n^0_0}\right\|^2_{L^2(\omega)}$$

$$+c\left(C^{**}(t)+c^*C^{**}(t)^4+c^{*2}+c^{*\frac{1}{2}}C^{**}(t)^{\frac{3}{2}}\right)$$

and $C_\alpha(\cdot)$ is a monotonically increasing function ($|\alpha|=1,\ 2,\ 3$).

Proof. Multiplying the first equation of (3.1.2) by $\Delta^2\phi^0$ and integrating over Ω, we have, by virtue of the Gagliardo–Nirenberg and Young's inequalities,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla\Delta\phi^0(t)\right\|^2+\left\|\Delta\widetilde{\phi^0}(t)\right\|^2\right)+c_2\left\|\Delta^2\phi^0(t)\right\|^2$$

$$=-\int_\Omega\left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\overline{n^0}\,\Delta^2\phi^0\,\mathrm{d}x$$

$$-\int_\Omega\left(\nabla\phi^0\times\vec{e}\right)\cdot\nabla\left(\Delta\phi^0-\widetilde{\phi^0}\right)\,\Delta^2\phi^0\,\mathrm{d}x$$

$$\leq c\Big(\left\|\nabla\widetilde{\phi^0}(t)\right\|_{L^\infty(\Omega)}\left\|\nabla\overline{n^0}(t)\right\|_{L^2(\omega)}+\left\|\nabla\phi^0(t)\right\|_{L^4(\Omega)}\left\|\nabla\Delta\phi^0(t)\right\|_{L^4(\Omega)}$$

$$+\left\|\nabla\phi^0(t)\right\|_{L^4(\Omega)}\left\|\nabla\widetilde{\phi^0}\right\|_{L^4(\Omega)}\Big)\left\|\Delta^2\phi^0(t)\right\|$$

$$\leq\frac{c_2}{2}\left\|\Delta^2\phi(t)\right\|^2+\frac{c}{c_2}\Big(\left(\left\|\nabla\Delta\widetilde{\phi^0}(t)\right\|^2+\left\|\nabla\widetilde{\phi^0}(t)\right\|^2\right)\left\|\nabla\overline{n^0}(t)\right\|^2_{L^2(\omega)}$$

$$+\left\|\nabla\phi^0(t)\right\|^2\left\|\Delta\phi^0(t)\right\|^6\left\|\nabla\Delta\phi^0(t)\right\|^2+\left\|\nabla\phi^0(t)\right\|\left\|\Delta\widetilde{\phi^0}(t)\right\|^3$$

$$+\left\|\nabla\widetilde{\phi^0}(t)\right\|\left\|\Delta\phi^0(t)\right\|^3\Big),$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla\Delta\phi^0(t)\right\|^2+\left\|\Delta\widetilde{\phi^0}(t)\right\|^2\right)+c_2\left\|\Delta^2\phi^0(t)\right\|^2$$

$$\leq c\Big(\left\|\nabla\Delta\phi^0(t)\right\|^4+\left\|\nabla\overline{n^0}(t)\right\|^4_{L^2(\omega)}+c^*C^{**}(t)^3\left\|\nabla\Delta\phi^0(t)\right\|^2$$

$$+c^{*2}+c^{*\frac{1}{2}}C^{**}(t)^{\frac{3}{2}}\Big). \tag{3.3.21}$$

Here we used the estimates (3.3.16), (3.3.17) and the inequality

$$\left\|D_x^\alpha\widetilde{\phi^0}\right\|\leq\left\|D_x^\alpha\phi^0\right\|+\left\|D_{x'}^\alpha\overline{\phi^0}\right\|\leq 2\left\|D_x^\alpha\phi^0\right\|,\quad|\alpha|=1,2.$$

Taking the gradient of the second equation of (3.1.2), multiplying it by $\nabla\overline{n^0}$, and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\overline{n^0}(t)\right\|^2_{L^2(\omega)}\leq c\left\|\nabla\nabla\overline{\phi^0}(t)\right\|_{L^\infty(\omega)}\left\|\nabla\overline{n^0}(t)\right\|^2_{L^2(\omega)}$$

$$\le \frac{c_2}{2}\left(\left\|\Delta^2\phi^0(t)\right\|^2 + \left\|\nabla\Delta\phi^0(t)\right\|^2\right) + \frac{c}{c_2}\left\|\nabla\overline{n^0}(t)\right\|^4_{L^2(\omega)}. \tag{3.3.22}$$

Adding (3.3.21) and (3.3.22), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla\Delta\phi^0(t)\right\|^2 + \left\|\Delta\widetilde{\phi^0}(t)\right\|^2 + \left\|\nabla\overline{n^0}(t)\right\|^2_{L^2(\omega)}\right) + c_2\left\|\Delta^2\phi^0(t)\right\|^2$$

$$\le c\bigg(\left\|\nabla\Delta\phi^0(t)\right\|^4 + \left\|\nabla\overline{n^0}(t)\right\|^4_{L^2(\omega)} + \left(1 + c^*C^{**}(t)^3\right)\left\|\nabla\Delta\phi^0(t)\right\|^2$$

$$+c^{*2} + c^{*\frac{1}{2}}C^{**}(t)^{\frac{3}{2}}\bigg). \tag{3.3.23}$$

Putting

$$S_5(t) \equiv \left\|\nabla\Delta\phi^0(t)\right\|^2 + \left\|\Delta\widetilde{\phi^0}(t)\right\|^2 + \left\|\nabla\overline{n^0}(t)\right\|^2_{L^2(\omega)}$$

and integrating (3.3.23) over $[0, t]$, we have with the help of (3.3.17)

$$\begin{aligned} S_5(t) \;&\le\; c\int_0^t S_5(\tau)^2\,\mathrm{d}\tau + S_5(0) + c\left(C^{**}(t) + c^*C^{**}(t)^4 + c^{*2} + c^{*\frac{1}{2}}C^{**}(t)^{\frac{3}{2}}\right) \\ &\equiv\; c\int_0^t S_5(\tau)^2\,\mathrm{d}\tau + C_4(t) \equiv S_5^*(t). \end{aligned}$$

Then $S_5^*(t)$ satisfis the differential inequality

$$\frac{\mathrm{d}S_5^*(t)}{\mathrm{d}t} \le cS_5^*(t)^2 + \frac{\mathrm{d}C_4(t)}{\mathrm{d}t},$$

Since $C_4(t)$ is increasing, one can derive from this inequality,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{S_5^*(t)}\right) \le c + \frac{1}{S_5^*(t)^2}\frac{\mathrm{d}C_4(t)}{\mathrm{d}t} \le c + \frac{1}{C_4(t)^2}\frac{\mathrm{d}C_4(t)}{\mathrm{d}t}.$$

Integrating this inequality over $[0, t]$ and noting $S_5^*(0) = C_4(0)$, we have

$$S_5^*(t) \le \frac{C_4(t)}{1 - cC_4(t)t}.$$

Then we choose $T^* > 0$ such that $C_4(0) - c(1 + C_4(0))C_4(T^*)T^* = 0$. This and the integral of (3.3.23) over $[0, t]$ lead to the inequality (3.3.18).

For $|\alpha| = 1, 2, 3$, applying $D^\alpha_{x'}$ to the second equation of (3.1.2), multiplying it by $D^\alpha_{x'}\overline{n^0}$, integrating over ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^\alpha_{x'}\overline{n^0}(t)\right\|^2_{L^2(\omega)} \le c \sum_{|\alpha'|\le|\alpha|}\left|\int_\omega \left(D^{\alpha'}_{x'}\left(\nabla\overline{\phi^0}\times\vec{e}\right)\cdot\nabla D^{\alpha-\alpha'}_{x'}\overline{n^0}\right) D^\alpha_{x'}\overline{n^0}\,\mathrm{d}x'\right|.$$

Then Gronwall's lemma and (3.3.18) lead to (3.3.19).

From (3.3.18), (3.3.19) and the second equation of (3.1.2), one can easily deduce (3.3.20). □

By the standard arguments based upon the *a priori* estimates in Lemmas 3.3.6 and 3.3.7 the solution $(\phi^0, \overline{n^0})$ established in §3.3.1 can be extended up to T^* indicated in the proof of Lemma 3.3.7. Thus the proof of Theorem 3.1.2 is complete.

3.4 Hasegawa–Wakatani equations with vanishing resistivity : Proof of Theorem 3.1.3

Denoting $T^\sharp = \min(T, T^*)$, where T and T^* are indicated in Theorems 3.1.1 and 3.1.2, and subtracting (1.4.6) from (1.4.4) and denoting by $\phi \equiv \phi^\varepsilon - \phi^0$, $n \equiv n^\varepsilon - n^0$, we have

$$
\begin{cases}
\left(\dfrac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla\right)(\Delta\phi - n) - (\nabla\phi \times \vec{e}) \cdot \nabla\left(\Delta\phi^0 - n^0\right) \\
\qquad\qquad = c_2 \Delta^2 \phi, \\
\varepsilon\left(\dfrac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla\right) n \\
\qquad\qquad = -\varepsilon\,(I - \mathcal{M})\left\{\left(\dfrac{\partial}{\partial t} - \left(\nabla\phi^0 \times \vec{e}\right) \cdot \nabla\right) n^0\right\} \\
\qquad\quad -\dfrac{1}{n^*}\dfrac{\partial^2(\phi - n)}{\partial x_3^2} + \varepsilon(\nabla\phi \times \vec{e}) \cdot \nabla n^0 \quad \text{for } x \in \Omega,\ 0 < t < T^\sharp, \\
\phi(x,0) = \phi_0^\varepsilon - \phi_0^0, \quad n(x,0) = n_0^\varepsilon - n_0^0 \quad \text{for } x \in \Omega, \\
\phi(x,t) = \Delta\phi(x,t) = n(x,t) = 0 \quad \text{for } x \in \Gamma,\ 0 < t < T^\sharp, \\
\phi,\ n,\ \text{periodic in the } x_3\text{-direction.}
\end{cases}
\tag{3.4.1}
$$

By virtue of Lemmas 3.2.1, 3.2.2, 3.3.6 and 3.3.7, we prove in this section four lemmas necessary for the proof of Theorem 3.1.3. Again as in §3.3.2 the following arguments are formal, however they are justified by the method of difference quotients or mollifiers. We denote by c a constant independent of t and by $C(t)$ a constant dependent on both t and the bounds of $\phi^\varepsilon,\ n^\varepsilon,\ \phi^0,\ n^0$, which may differ at each occurrence.

Lemma 3.4.1 *For any* $t \in [0, T^\sharp]$

$$
\varepsilon\left(\|\nabla\phi(t)\|^2 + \|n(t)\|^2 + \int_0^t \|\Delta\phi(\tau)\|^2\,\mathrm{d}\tau\right) + \int_0^t \left\|\frac{\partial(\phi - n)}{\partial x_3}(\tau)\right\|^2 \mathrm{d}\tau
$$
$$
\le \varepsilon C(t)\left(\|\nabla\phi(0)\|^2 + \|n(0)\|^2 + \varepsilon\right). \tag{3.4.2}
$$

Proof. Multiplying the first equation of (3.4.1) by ϕ and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi(t)\|^2 + c_2\|\Delta\phi(t)\|^2 + \int_\Omega \left\{ \left(\frac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla \right) n \right\} \phi \,\mathrm{d}x$$

$$= \int_\Omega (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla\phi \,\Delta\phi \,\mathrm{d}x$$

$$\leq \frac{c_2}{2}\|\Delta\phi(t)\|^2 + c\|\nabla\phi^\varepsilon(t)\|^2_{L^\infty(\Omega)}\|\nabla\phi(t)\|^2. \tag{3.4.3}$$

Here we used the integration by parts and Schwarz' inequality.

Similarly, multiplying the second equation of (3.4.1) by $\phi - n$ and integrating over Ω, we have

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n(t)\|^2 + \left\| \left(\frac{1}{n^*}\right)^{\frac{1}{2}} \frac{\partial(\phi - n)}{\partial x_3}(t) \right\|^2$$

$$-\varepsilon \int_\Omega \left\{ \left(\frac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla \right) n \right\} \phi \,\mathrm{d}x$$

$$= -\varepsilon \int_\Omega (I - \mathcal{M}) \left\{ \left(\frac{\partial}{\partial t} - \left(\nabla\phi^0 \times \vec{e}\right) \cdot \nabla \right) n^0 \right\} (\phi - n) \,\mathrm{d}x$$

$$+\varepsilon \int_\Omega (\nabla\phi \times \vec{e}) \cdot \nabla n^0 \,(\phi - n) \,\mathrm{d}x$$

$$\leq \varepsilon^2 c \left\| \int_{-L}^{x_3} (I - \mathcal{M}) \left\{ \left(\frac{\partial}{\partial t} - \left(\nabla\phi^0 \times \vec{e}\right) \cdot \nabla \right) n^0 \right\} \,\mathrm{d}x_3 \right\|^2$$

$$+\frac{1}{2} \left\| \left(\frac{1}{n^*}\right)^{1/2} \frac{\partial(\phi - n)}{\partial x_3}(t) \right\|^2$$

$$+\varepsilon\|\nabla\phi(t)\|\|\nabla n^0(t)\|_{L^\infty(\Omega)}\|n(t)\|. \tag{3.4.4}$$

Adding (3.4.4) and (3.4.3) multiplied by ε yields

$$\varepsilon \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\|\nabla\phi(t)\|^2 + \|n(t)\|^2 \right) + c_2\|\Delta\phi(t)\|^2 \right) + \left\| \left(\frac{1}{n^*}\right)^{1/2} \frac{\partial(\phi - n)}{\partial x_3}(t) \right\|^2$$

$$\leq \varepsilon c\left(\|\nabla\phi^\varepsilon(t)\|^2_{L^\infty(\Omega)} + \|\nabla n^0(t)\|_{L^\infty(\Omega)}\right)\left(\|\nabla\phi(t)\|^2 + \|n(t)\|^2\right)$$

$$+\varepsilon^2 c\left\|(I-\mathcal{M})\left\{\left(\frac{\partial}{\partial t} - \left(\nabla\phi^0\times\vec{e}\right)\cdot\nabla\right)n^0\right\}\right\|^2.$$

Then from Gronwall's lemma (3.4.2) follows. □

Lemma 3.4.2 *For any* $t\in[0,T^\sharp]$

$$\varepsilon\left(\|\Delta\phi(t)\|^2 + \|\nabla n(t)\|^2 + \int_0^t \|\nabla\Delta\phi(\tau)\|^2\,\mathrm{d}\tau\right)$$

$$+\int_0^t \left\|\frac{\partial\nabla(\phi-n)}{\partial x_3}(\tau)\right\|^2\,\mathrm{d}\tau$$

$$\leq \varepsilon C(t)\left(\|\nabla\phi(0)\|^2_{W_2^1(\Omega)} + \|n(0)\|^2_{W_2^1(\Omega)} + \varepsilon\right). \qquad (3.4.5)$$

Proof. In the similar way to Lemma 3.4.1, multiply the first equation of (3.4.1) by $\Delta\phi$ and integrate over Ω. Then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\phi(t)\|^2 + c_2\|\nabla\Delta\phi(t)\|^2 - \int_\Omega\left\{\left(\frac{\partial}{\partial t} - (\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n\right\}\Delta\phi\,\mathrm{d}x$$

$$\leq \|\nabla\phi(t)\|_{L^4(\Omega)}\left\|(\Delta\phi^0 - n^0)(t)\right\|_{L^4(\Omega)}\|\nabla\Delta\phi(t)\|$$

$$\leq \frac{c_2}{2}\|\nabla\Delta\phi(t)\|^2 + c\left\|\nabla(\Delta\phi^0-n^0)(t)\right\|^2\|\Delta\phi(t)\|^2$$

$$+c\left\|(\Delta\phi^0-n^0)(t)\right\|^2\|\nabla\phi(t)\|^2,$$

and hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\phi(t)\|^2 + \frac{c_2}{2}\|\nabla\Delta\phi(t)\|^2 - \int_\Omega\left\{\left(\frac{\partial}{\partial t} - (\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n\right\}\Delta\phi\,\mathrm{d}x$$

$$\leq c\left\|\nabla(\Delta\phi^0-n^0)(t)\right\|^2\|\Delta\phi(t)\|^2$$

$$+c\left\|(\Delta\phi^0-n^0)(t)\right\|^2\|\nabla\phi(t)\|^2. \qquad (3.4.6)$$

Multiplying the second equation of (3.4.1) by $\Delta(\phi - n)$ and integrating over Ω, we have

$$
\begin{aligned}
&\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n(t)\|^2 + \left\|\left(\frac{1}{n^*}\right)^{1/2}\frac{\partial\nabla(\phi-n)}{\partial x_3}(t)\right\|^2 \\
&\qquad +\varepsilon\int_\Omega\left\{\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n\right\}\Delta\phi\,\mathrm{d}x \\
&\le \varepsilon\left\|\int_{-L}^{x_3}\nabla\left(I-\mathcal{M}\right)\left\{\left(\frac{\partial}{\partial t}-\left(\nabla\phi^0\times\vec{e}\right)\cdot\nabla\right)n^0\right\}\,\mathrm{d}x_3\right\|\left\|\frac{\partial\nabla(\phi-n)}{\partial x_3}(t)\right\| \\
&\quad +\frac{1}{2}\left\|n^{*1/2}\nabla\frac{1}{n^*}\frac{\partial(\phi-n)}{\partial x_3}(t)\right\|^2+\frac{1}{2}\left\|\left(\frac{1}{n^*}\right)^{1/2}\frac{\partial\nabla(\phi-n)}{\partial x_3}(t)\right\|^2 \\
&\quad +\varepsilon c\sum_{|\alpha|=2}\|D_x^\alpha\phi^\varepsilon(t)\|_{L^\infty(\Omega)}\|\nabla n(t)\|^2 \\
&\quad +\varepsilon\left(\|\Delta\phi\|\left\|\nabla n^0(t)\right\|_{L^\infty(\Omega)}+\|\nabla\phi\|_{L^4(\Omega)}\left\|\Delta n^0(t)\right\|_{L^4(\Omega)}\right)\|\nabla(\phi-n)\|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n(t)\|^2+\left\|\left(\frac{1}{n^*}\right)^{1/2}\frac{\partial\nabla(\phi-n)}{\partial x_3}(t)\right\|^2 \\
&\qquad +\varepsilon\int_\Omega\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n\Delta\phi\,\mathrm{d}x \\
&\le\varepsilon^2c\left\|\nabla\left(I-\mathcal{M}\right)\left\{\left(\frac{\partial}{\partial t}-\left(\nabla\phi^0\times\vec{e}\right)\cdot\nabla\right)n^0\right\}\right\|^2 \\
&\quad +C(t)\left\|\frac{\partial(\phi-n)}{\partial x_3}(t)\right\|^2+\varepsilon c\sum_{|\alpha|=2}\|D_x^\alpha\phi^\varepsilon(t)\|_{L^\infty(\Omega)}\|\nabla n(t)\|^2 \\
&\quad +\varepsilon\left(\|\Delta\phi\|\left\|\nabla n^0(t)\right\|_{L^\infty(\Omega)}+\|\nabla\phi\|_{L^4(\Omega)}\left\|\Delta n^0(t)\right\|_{L^4(\Omega)}\right) \\
&\quad \times\|\nabla(\phi-n)\|. \qquad (3.4.7)
\end{aligned}
$$

Adding (3.4.7) and (3.4.6) multiplied by ε yields

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta\phi(t)\|^2+\|\nabla n(t)\|^2\right)+\varepsilon c_2\|\nabla\Delta\phi(t)\|^2+\left\|\left(\frac{1}{n^*}\right)^{1/2}\frac{\partial\nabla(\phi-n)}{\partial x_3}(t)\right\|^2$$

$$\le \varepsilon c\Bigg(\left\|\nabla(\Delta\phi^0-n^0)(t)\right\|+\sum_{|\alpha|=2}\|D_x^\alpha\phi^\varepsilon(t)\|_{L^\infty(\Omega)}+\left\|\nabla n^0(t)\right\|^2_{L^\infty(\Omega)}$$

$$+\left\|\Delta n^0(t)\right\|^{\frac{8}{3}}_{L^4(\Omega)}+1\Bigg)\left(\|\Delta\phi(t)\|^2+\|\nabla n(t)\|^2\right)+\varepsilon c\|\nabla\phi(t)\|^2$$

$$+\varepsilon^2 c\left\|\nabla\left(I-\mathcal{M}\right)\left\{\left(\frac{\partial}{\partial t}-\left(\nabla\phi^0\times\vec{e}\right)\cdot\nabla\right)n^0\right\}\right\|^2+C(t)\left\|\frac{\partial(\phi-n)}{\partial x_3}(t)\right\|^2.$$

Here we used the estimate (3.3.19). Then Gronwall's lemma and Lemma 3.4.1 lead to (3.4.5). □

Lemma 3.4.3 *For any* $t\in[0,T^\sharp]$

$$\varepsilon\left(\|\nabla\Delta\phi(t)\|^2+\|\Delta n(t)\|^2+\int_0^t\|\Delta^2\phi(\tau)\|^2\,\mathrm{d}\tau\right)$$

$$+\int_0^t\left\|\frac{\partial\Delta(\phi-n)}{\partial x_3}(\tau)\right\|^2\mathrm{d}\tau$$

$$\le\varepsilon C(t)\left(\|\nabla\phi(0)\|^2_{W_2^2(\Omega)}+\|n(0)\|^2_{W_2^2(\Omega)}+\varepsilon\right).\quad(3.4.8)$$

Proof. Multiplying the first equation of (3.4.1) by $\Delta^2\phi$ and integrating over Ω, we have,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\phi(t)\|^2+c_2\|\Delta^2\phi(t)\|^2+\int_\Omega\left\{\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n\right\}\Delta^2\phi\,\mathrm{d}x$$

$$\le\|\nabla\phi^\varepsilon(t)\|_{L^\infty(\Omega)}\|\Delta^2\phi(t)\|\|\nabla\Delta\phi(t)\|$$

$$+\left\|\nabla(\Delta\phi^0-n^0)(t)\right\|\|\nabla\phi(t)\|_{L^\infty(\Omega)}\|\Delta^2\phi(t)\|$$

$$\le c\left(\|\nabla\phi^\varepsilon(t)\|^2_{L^\infty(\Omega)}+\left\|\nabla(\Delta\phi^0-n^0)(t)\right\|^2\right)\|\nabla\Delta\phi(t)\|^2+\frac{c_2}{2}\|\Delta^2\phi(t)\|^2$$

$$+c\left\|\nabla(\Delta\phi^0-n^0)(t)\right\|^2\|\nabla\phi(t)\|^2.$$

Therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\phi(t)\|^2+\frac{c_2}{2}\|\Delta^2\phi(t)\|^2+\int_\Omega\left\{\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n\right\}\Delta^2\phi\,\mathrm{d}x$$

$$\le c\left(\|\nabla\phi^\varepsilon(t)\|^2_{L^\infty(\Omega)}+\left\|\nabla(\Delta\phi^0-n^0)(t)\right\|^2\right)\|\nabla\Delta\phi(t)\|^2$$

$$+c\left\|\nabla(\Delta\phi^0-n^0)(t)\right\|^2\|\nabla\phi(t)\|^2. \tag{3.4.9}$$

Multiplying the second equation of (3.4.1) by $\Delta^2(\phi-n)$ and integrating over Ω, we have,

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n(t)\|^2+\left\|\left(\frac{1}{n^*}\right)^{1/2}\frac{\partial\Delta(\phi-n)}{\partial x_3}(t)\right\|^2$$

$$-\varepsilon\int_\Omega\left\{\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n\right\}\Delta^2\phi\,\mathrm{d}x$$

$$\le\varepsilon\left\|\int_{-L}^{x_3}\Delta\left(I-\mathcal{M}\right)\left\{\left(\frac{\partial}{\partial t}-\left(\nabla\phi^0\times\vec{e}\right)\cdot\nabla\right)n^0\right\}\,\mathrm{d}x_3\right\|\left\|\frac{\partial\Delta(\phi-n)}{\partial x_3}(t)\right\|$$

$$+\frac{1}{2}\left\|\left(\frac{1}{n^*}\right)^{1/2}\frac{\partial\Delta(\phi-n)}{\partial x_3}(t)\right\|^2+4\left\|n^{*1/2}\nabla\frac{1}{n^*}\cdot\frac{\partial\nabla(\phi-n)}{\partial x_3}(t)\right\|^2$$

$$+\left\|n^{*1/2}\Delta\frac{1}{n^*}\frac{\partial(\phi-n)}{\partial x_3}(t)\right\|^2+c\varepsilon\sum_{|\alpha|=2}\|D_x^\alpha\phi^\varepsilon(t)\|_{L^\infty(\Omega)}\|\Delta n(t)\|^2$$

$$+\varepsilon\|\nabla\Delta\phi^\varepsilon(t)\|_{L^4(\Omega)}\|\nabla n(t)\|_{L^4(\Omega)}\|\Delta n(t)\|+\varepsilon\Big(\|\nabla\Delta\phi\|\left\|\nabla n^0(t)\right\|_{L^\infty(\Omega)}$$

$$+\|\Delta\phi\|_{L^4(\Omega)}\left\|\Delta n^0(t)\right\|_{L^4(\Omega)}+\|\nabla\phi(t)\|_{L^\infty(\Omega)}\|\nabla\Delta n^0\|\Big)\|\Delta(\phi-n)\|,$$

and hence

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n(t)\|^2+\frac{1}{2}\left\|\left(\frac{1}{n^*}\right)^{1/2}\frac{\partial\Delta(\phi-n)}{\partial x_3}(t)\right\|^2$$

$$-\varepsilon\int_\Omega\left\{\left(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\vec{e})\cdot\nabla\right)n\right\}\Delta^2\phi\,\mathrm{d}x$$

$$
\begin{aligned}
&\leq \varepsilon^2 c \left\| \Delta \left(I - \mathcal{M}\right) \left\{ \left(\frac{\partial}{\partial t} - \left(\nabla \phi^0 \times \vec{e}\right) \cdot \nabla \right) n^0 \right\} \right\|^2 \\
&\quad + C(t) \left(\left\| \frac{\partial \nabla (\phi - n)}{\partial x_3}(t) \right\|^2 + \left\| \frac{\partial (\phi - n)}{\partial x_3}(t) \right\|^2 \right) \\
&\quad + \varepsilon \left\| \nabla \Delta \phi^\varepsilon(t) \right\|_{L^4(\Omega)} \left\| \nabla n(t) \right\|^2 \\
&\quad + \varepsilon c \Bigg(\sum_{|\alpha|=2} \left\| D_x^\alpha \phi^\varepsilon(t) \right\|_{L^\infty(\Omega)} + \left\| \nabla \Delta \phi^\varepsilon(t) \right\|_{L^4(\Omega)} \Bigg) \left\| \Delta n(t) \right\|^2 \\
&\quad + \varepsilon \Big(\left\| \nabla \Delta \phi \right\| \left\| \nabla n^0(t) \right\|_{L^\infty(\Omega)} + \left\| \Delta \phi \right\|_{L^4(\Omega)} \left\| \Delta n^0(t) \right\|_{L^4(\Omega)} \\
&\quad + \left\| \nabla \phi(t) \right\|_{L^\infty(\Omega)} \left\| \nabla \Delta n^0 \right\| \Big) \left\| \Delta (\phi - n) \right\|. \qquad (3.4.10)
\end{aligned}
$$

Adding (3.4.10) and (3.4.9) multiplied by ε yields

$$
\begin{aligned}
&\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \left(\left\| \nabla \Delta \phi(t) \right\|^2 + \left\| \Delta n(t) \right\|^2 \right) + \varepsilon c_2 \left\| \Delta^2 \phi(t) \right\|^2 \\
&\qquad + \left\| \left(\frac{1}{n^*} \right)^{1/2} \frac{\partial \Delta (\phi - n)}{\partial x_3}(t) \right\|^2 \\
&\leq \varepsilon c \Bigg(\left\| \nabla \phi^\varepsilon(t) \right\|_{L^\infty(\Omega)}^2 + \left\| \nabla (\Delta \phi^0 - n^0)(t) \right\|^2 + \sum_{|\alpha|=2} \left\| D_x^\alpha \phi^\varepsilon(t) \right\|_{L^\infty(\Omega)} \\
&\quad + \left\| \nabla \Delta \phi^\varepsilon(t) \right\|_{L^4(\Omega)} + \left\| \nabla n^0(t) \right\|_{L^\infty(\Omega)}^2 + \left\| \Delta n^0(t) \right\|_{L^4(\Omega)} + \left\| \nabla \Delta n^0(t) \right\| \Bigg) \\
&\qquad \times \left(\left\| \nabla \Delta \phi(t) \right\|^2 + \left\| \Delta n(t) \right\|^2 \right) \\
&\quad + \varepsilon^2 c \left\| \Delta \left(I - \mathcal{M}\right) \left\{ \left(\frac{\partial}{\partial t} - \left(\nabla \phi^0 \times \vec{e}\right) \cdot \nabla \right) n^0 \right\} \right\|^2 \\
&\quad + C(t) \left(\left\| \frac{\partial \nabla (\phi - n)}{\partial x_3}(t) \right\|^2 + \left\| \frac{\partial (\phi - n)}{\partial x_3}(t) \right\|^2 \right) \\
&\quad + \varepsilon \left\| \nabla \Delta \phi^\varepsilon(t) \right\|_{L^4(\Omega)} \left\| \nabla n(t) \right\|^2
\end{aligned}
$$

$$+\varepsilon\left(\left\|\nabla n^0(t)\right\|_{L^\infty(\Omega)}+\left\|\Delta n^0(t)\right\|_{L^4(\Omega)}+\left\|\nabla\Delta n^0(t)\right\|\right)\|\Delta\phi(t)\|^2$$

$$+\varepsilon\left(\left\|\nabla\Delta n^0(t)\right\|+c\left\|\nabla(\Delta\phi^0-n^0)(t)\right\|^2\right)\|\nabla\phi(t)\|^2.$$

Then applying Gronwall's lemma and Lemmas 3.4.1, 3.4.2, we obtain (3.4.8).
□

Lemma 3.4.4 *For any* $t\in[0,T^\sharp]$

$$\int_0^t\left\|\frac{\partial\left(\Delta\phi-n\right)}{\partial\tau}(\tau)\right\|^2\mathrm{d}\tau$$

$$\leq C(t)\left(\|\nabla\phi(0)\|^2_{W_2^2(\Omega)}+\|n(0)\|^2_{W_2^2(\Omega)}+\varepsilon\right),\quad(3.4.11)$$

$$\int_0^t\left\|\frac{\partial\mathcal{M}n}{\partial\tau}(\tau)\right\|^2_{L^2(\omega)}\mathrm{d}\tau$$

$$\leq C(t)\left(\|\nabla\phi(0)\|^2_{W_2^1(\Omega)}+\|n(0)\|^2_{W_2^1(\Omega)}+\varepsilon\right).\quad(3.4.12)$$

Proof. Multiplying the first equation of (3.4.1) by $\partial\left(\Delta\phi-n\right)/\partial t$ and integrating over Ω, we have

$$\left\|\frac{\partial\left(\Delta\phi-n\right)}{\partial t}(t)\right\|^2\leq c\Big(\|\nabla\phi^\varepsilon(t)\|^2_{L^\infty(\Omega)}\|\nabla\left(\Delta\phi-n\right)(t)\|^2$$

$$+\|\nabla\phi(t)\|^2_{L^\infty(\Omega)}\left\|\nabla\left(\Delta\phi^0-n^0\right)(t)\right\|^2+c_2\left\|\Delta^2\phi(t)\right\|^2\Big).$$

Integrating this over $[0,t]$, we obtain (3.4.11) with the help of (3.2.1), (3.2.5), (3.3.17), (3.3.19), (3.4.2), (3.4.5) and (3.4.8).

Applying the mean operator $\mathcal{M}$ to the second equation of (3.4.1) and multiplying it by $\partial\mathcal{M}n/\partial t$ and integrating over ω, we have

$$\left\|\frac{\partial\mathcal{M}n}{\partial t}(t)\right\|^2_{L^2(\omega)}\leq c\Bigg(\|\nabla\phi^\varepsilon(t)\|^2_{L^\infty(\Omega)}\|\nabla n(t)\|^2$$

$$+\|\nabla\phi(t)\|^2\left\|\nabla n^0(t)\right\|^2_{L^\infty(\Omega)}\Bigg).$$

Integrating this over $[0, t]$, we obtain (3.4.12) with the help of (3.2.1), (3.2.5), (3.3.19), (3.4.2) and (3.4.5). □

From Lemmas 3.4.1-3.4.4, it is easy to see that if the initial data $(\phi_0^\varepsilon, n_0^\varepsilon) \to (\phi_0^0, n_0^0)$ as $\varepsilon \to 0$ in $W_2^3(\Omega) \times W_2^2(\Omega)$, then $(\phi^\varepsilon, n^\varepsilon) \to (\phi^0, n^0)$ as $\varepsilon \to 0$ in $L^2(0, T^\sharp; W_2^4(\Omega)) \times W_2^{2,0}(\omega_{T^\sharp})$ and $\Delta\phi^\varepsilon - n^\varepsilon \to \Delta\phi^0 - n^0$ as $\varepsilon \to 0$ in $W_2^{0,1}(Q_{T^\sharp})$ and $\overline{n^\varepsilon} \to \overline{n^0}$ as $\varepsilon \to 0$ in $W_2^{0,1}(\omega_{T^\sharp})$. Thus the proof of Theorem 3.1.3 is complete.

Acknowledgement

First of all I would like to show my gratitude to my supervisor Professor Atusi Tani. Since he continually provided guidance for my research, my academic life became fruitful.

I express appreciation to Professors Tatsuo Iguchi, Ippei Ishii, Akiyoshi Hatayama of Keio University and Akitaka Matsumura of Osaka University for helpful comments in the thesis examination. Their comments make my thesis more valuable and expand possibility of my research.

I express appreciation to Assistant Professor Masahiro Takayama for helpful advice on my research.

Finally I express appreciation to my parents who supported and encouraged me.

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Appendix A

Picture

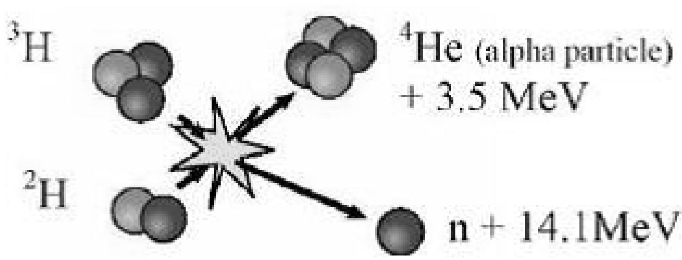

Figure A.1: Nuclear fusion reaction of Deuterium and Tritium ([9])

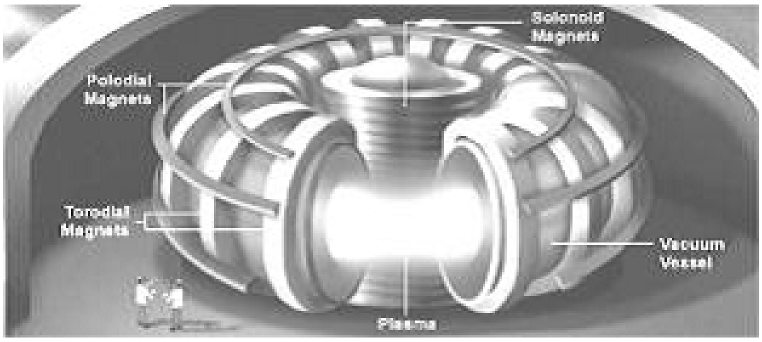

Figure A.2: Tokamak (website of Schlumberger)

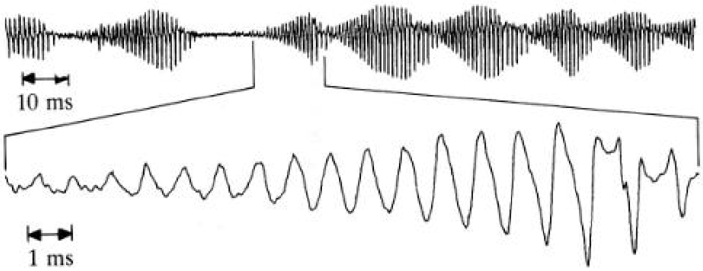

Figure A.3: Mirnov oscillations ([60])

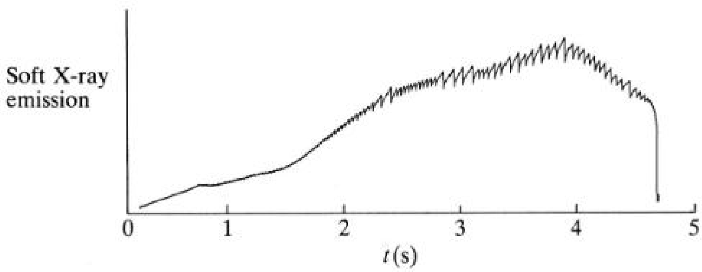

Figure A.4: Sawtooth oscillations ([60])

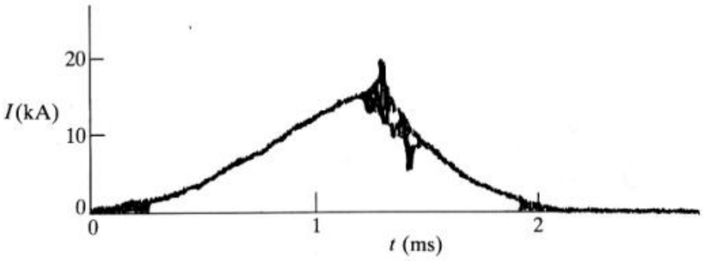

Figure A.5: Disruption ([60])

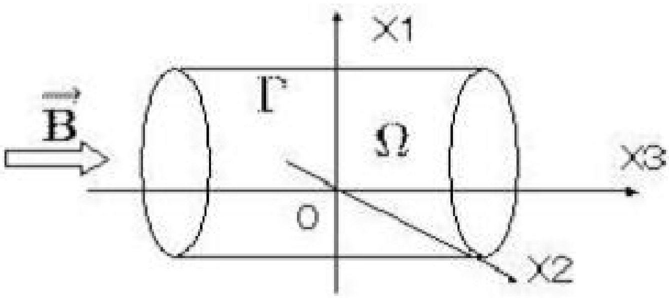

Figure A.6: Domain