A Thesis for the Degree of Ph.D. in Science

Trees and Factors with Bounded Total Excess

February 2012

Graduate School of Science and Technology
Keio University

Yukichika Ohnishi

主　論　文　要　旨

報告番号	甲　乙　第　　　　号	氏　名	大西 幸周

主 論 文 題 目：

Trees and Factors with Bounded Total Excess

（超過数を制限した木および因子）

（内容の要旨）

グラフ理論において，ハミルトン閉路は最もよく研究されてきた対象のひとつである．グラフがハミルトン閉路を含むための十分条件は数多く知られているが，その中に Chvátal(1973) が提唱したタフネスという概念に関するものがある．グラフ $G-S$ の連結成分数を $\omega(G-S)$ で表すとき，$\omega(G-S) \geq 2$ を満たす任意の集合 $S \subset V(G)$ について，$t \cdot \omega(G-S) \leq |S|$ が成り立つならば，G は t-tough であると定義する．ハミルトン閉路を含むグラフが 1-tough であることは簡単に示せるが，Chvátal は逆に，任意の t_0-tough グラフがハミルトン閉路を持つといえるある定数 t_0 が存在すると予想した．この予想の真偽は未だ示されていない．

$k \geq 3$ のとき，グラフが k-tree を含むためのタフネスに関する十分条件は知られている．ここで云う k-tree とは，最大次数が高々k の全域木のことである．グラフが k-tree を含むことは，ハミルトン性のある意味での緩和である．Win は 1989 年に，$V(G)$ の任意の部分集合 S について $\omega(G-S) \leq (k-2)|S|+2$ を満たす連結グラフ G が k-tree を含むことを示した．

本論文では，グラフが種々の全域連結部分グラフを含むためのタフネスに基づいた条件について考え，より詳細な構造に関する研究を行う．このために，超過数という概念を導入する．

第 2 章では，全域木の超過数について議論する．連結グラフの全域木 T において，頂点 v の k-超過を $\max\{0, \deg_T(v)-k\}$ と定義する．総 k-超過 は全頂点における k-超過の総和である．この章では，先に述べた Win の定理の一般化として，総 k-超過を制限した全域木がグラフに含まれるための十分条件を与える．

第 3 章では，再び全域木の超過数について議論する．ここでは特に，t を固定し t-tough グラフを考える．第 2 章の結果を用いると，任意の整数 $k \geq 3$ について，k, t 及び $|V(G)|$ に依存した上界で総 k-超過を制限した全域木を得ることができる．本章ではそれら複数の全域木の関係について議論し，その結果として，すべての k の値に対して総 k-超過 を抑えたある全域木の存在を示す．

第 4 章では，全域連結部分グラフを得るためのさらに一般的な問題について議論する．最初に G の全域非連結部分グラフ F と，どの $v \in V(G)$ についても $\varphi(v) \geq \deg_F(v)$ であるような整数値関数 φ が与えられているとする．この章では，F に辺を加えて作る，総 'φ-超過' を定数で抑えた全域連結部分グラフが存在するための十分条件を与える．

第 5 章では，全域閉歩道の議論を行う．k-walk とは全ての頂点を高々k 回訪れる全域閉歩道である．全域閉歩道の総 k-超過も全域木のそれと同じように定義できる．第 2 章で得られた結果を用いると，$k \geq 3$ のとき，グラフが総 k-超過を制限した全域閉歩道を含むためのタフネス的条件が直ちに得られる．この章では総 2-超過を制限した全域閉歩道についても結果を与える．

SUMMARY OF Ph.D. DISSERTATION

School Fundamental Science and Technology	Student Identification Number 80545043	SURNAME, First name OHNISHI, Yukichika

Title

Trees and Factors with Bounded Total Excess

Abstract

A hamiltonian cycle is one of the most well-studied subjects in graph theory. A lot of sufficient conditions have been considered for a graph to contain a hamiltonian cycle. Among them, Chvátal(1973) introduced the notion of toughness. A graph G is said to be t-tough, if $t \cdot \omega(G-S) \leq |S|$ for every subset $S \subset V(G)$ with $\omega(G-S) \geq 2$, where $\omega(G-S)$ denotes the number of components in the graph $G-S$. It is an easy observation that every graph containing a hamiltonian cycle is 1-tough. Chvátal conjectured that there exists a constant t_0 such that every t_0-tough graph contains a hamiltonian cycle. This conjecture is still open.

For $k \geq 3$, there is a sufficient condition concerning the toughness for a graph to have a k-tree. A k-tree in a graph is a spanning tree with maximum degree at most k. The property of containing a k-tree is a relaxation of the hamiltonian property. Win proved in 1989 that if a connected graph G satisfies $\omega(G-S) \leq (k-2)|S|+2$, for every subset S of $V(G)$, then G contains a k-tree.

In this thesis, we obtain more sophisticated results on spanning connected subgraphs in terms of toughness-like conditions. For this purpose, we introduce the notion of total excess.

In Chapter 2, we consider the total excess of spanning trees. For a spanning tree T of a connected graph, the k-excess of a vertex v is defined to be $\max\{0, \deg_T(v)-k\}$. The total k-excess is the amount of the k-excesses of all vertices. This chapter gives a sufficient condition for a graph to have a spanning tree with bounded total k-excess, which is a generalization of Win's theorem.

In Chapter 3, we discuss total excess of spanning trees again. Especially, we consider a t-tough graph for a fixed t. By using the result in Chapter 2, for each integer $k \geq 3$, we obtain a spanning tree with certain total k-excess upper bound depending on k, t and $|V(G)|$. We discuss the relation between these spanning trees. As a consequence, we prove the existence of 'a universal tree' in a sense.

In Chapter 4, we discuss a more general problem obtaining a spanning connected subgraph. Suppose that we are given a spanning disconnected subgraph F of G, and an integer-valued function φ with $\varphi(v) \geq \deg_F(v)$ for each $v \in V(G)$. We give a sufficient condition to be able to obtain a spanning connected subgraph by adding edges to F such that the total 'φ-excess' is bounded by a prescribed constant.

In Chapter 5, we deal with spanning walks. A k-walk in a graph is a spanning closed walk visiting each vertex at most k times. We can define the total k-excess of a spanning walk similarly. By using the result in Chapter 2, for $k \geq 3$, we immediately obtain a toughness condition for a graph to contain a spanning walk with bounded total k-excess. In this chapter, we also discuss on a spanning walk with bounded total 2-excess.

Acknowledgment

I am very grateful to many people who have contributed to this thesis.

First of all, I would like to express my deep gratitude to Professor Katsuhiro Ota and Professor Hikoe Enomoto for their valuable suggestions, encouragement, careful reading and so on. This dissertation would not have been completed without their advice and help. I would like to thank Professor Masafumi Hagiwara, Professor Akihisa Tamura and Professor Yoshiaki Oda for their kindness and encouragement. I would like to thank them for giving me an opportunity to study under them for nine years. Discussion with them was my pleasure and they had a lot of great influences on me.

I would like to thank Dr. Kenta Ozeki from National Institute of Informatics for some joint work and fruitful discussions. Also thanks to all members of Ota-Oda Laboratory at Keio University. I also thank all staff and colleagues of the Department of Mathematics for their support. I would like to thank Professor Shinobu Ariga of Sagami Women's University for the permission to use his illustration in my presentation slides of the final oral defense of the thesis.

Finally, I would like to thank all of my teachers, friends and my family. They have been a great support to me.

Yukichika Ohnishi
2011

Preface

This thesis is written on the subject "Trees and Factors with Bounded Total Excess" and is to be submitted for the degree of Doctor of Science at Keio University. The basis of this thesis is formed by papers written during these eight years.

The *toughness* of a graph is an invariant introduced by Chvátal [8]. Let G be a graph, and let S be a subset of $V(G)$. The number of components in $G - S$ is denoted by $\omega(G - S)$. For a real number t, if $|S| \geq t \cdot \omega(G - S)$ holds for every $S \subseteq V(G)$ with $\omega(G - S) \geq 2$, then G is called t-tough. The maximum number t for which G is t-tough is the toughness of G. If G is a complete graph, its toughness is defined to be ∞. It is easy to see that every hamiltonian graph is 1-tough. On the other hand, Chvátal conjectured that there exists a constant t_0 such that every t_0-tough graph is hamiltonian.

A *k-walk* in a graph is a spanning closed walk using each vertex at most k times. When $k = 1$, a 1-walk is a hamiltonian cycle, and the above-mentioned conjecture by Chvátal states that any graph with sufficiently large toughness has a 1-walk.

In this thesis, we introduce the notion of *Total Excess*, and show how to handle the concept. We define several variations of the total excess of graphs in each chapter accordingly.

After an introductory chapter, the reader will find four chapters. General terminology and notation in graph theory can be found in Chapter 1. The other chapters can be read independently from one another.

Chapter 2 discusses total excess of spanning trees. Win proved in 1989 that if a connected graph G satisfies

$$\omega(G - S) \le (k-2)|S| + 2, \text{ for every subset } S \text{ of } V(G),$$

then G has a spanning tree with maximum degree at most k.

For a spanning tree T of a connected graph, the k-excess of a vertex v is defined to be $\max\{0, \deg_T(v) - k\}$. The total k-excess $\mathrm{te}(T, k)$ is the summation of the k-excesses of all vertices, namely,

$$\mathrm{te}(T, k) = \sum_{v \in V(T)} \max\{0, \deg_T(v) - k\}.$$

This chapter gives a sufficient condition for a graph to have a spanning tree with bounded total k-excess. Our main result is as follows. Suppose $k \ge 2$, $b \ge 0$, and G is a connected graph satisfying the following condition:

$$\omega(G - S) \le (k-2)|S| + 2 + b, \text{ for every subset } S \text{ of } V(G).$$

Then, G has a spanning tree with total k-excess at most b.

Chapter 3 discusses total excess of spanning trees again. Especially, the relationship of many spanning trees is treated. Win's result implies that for any integer $k \ge 3$ every $\frac{1}{k-2}$-tough graph has a spanning tree with maximum degree at most k. In this chapter, we investigate t-tough graphs including the cases where $t \notin \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, and consider spanning trees in such graphs. Using the notion of total excess, we prove that if G is $\frac{1-\varepsilon}{k-2+\varepsilon}$-tough for an integer $k \ge 2$ and a real number ε with $\frac{2}{|V(G)|} \le \varepsilon < 1$, then G has a spanning tree T such that

$$\mathrm{te}(T, k) \le \varepsilon |V(G)| - 2.$$

We also investigate the relation between spanning trees in a graph obtained by different pairs of parameters (k, ε). As a consequence, we prove the existence of "a universal tree" in a connected t-tough graph G, that is a spanning tree T such that $\mathrm{te}(T,k) \leq \varepsilon|V(G)|-2$ for any integer $k \geq 2$ and real number ε with $\frac{2}{|V(G)|} \leq \varepsilon \leq 1$, which satisfy $t \geq \frac{1-\varepsilon}{k-2+\varepsilon}$.

Chapter 4 discusses total excess of connected factors. For a spanning subgraph H of a graph G, and nonnegative integer-valued function φ on $V(G)$, the total φ-excess $\mathrm{te}(H,\varphi)$ is defined as

$$\mathrm{te}(H,\varphi) = \sum_{v \in V(H)} \max\{0, \deg_H(v) - \varphi(v)\}.$$

Let F be a disconnected spanning subgraph of a connected graph G. Let h be a nonnegative integer-valued function on $V(G)$, and let b be a nonnegative integer. A spanning (F,h,b)-tree H is a spanning connected subgraph of G with $E(H) \supseteq E(F)$ such that $\mathrm{te}(H,\varphi) \leq b$, where $\varphi(v) = \deg_F(v) + h(v)$ for $v \in V(G)$, and that every edge of $E(H) \backslash E(F)$ is a cutedge of H.

Our result in Chapter 4 can be stated as follows. Assume that each component of a spanning subgraph F of G has at least α vertices. We prove that G has a spanning (F,h,b)-tree if for every nonempty $S \subseteq V(G)$ at least one of the following holds:

(i) $\omega(G-S) < \sum_{v \in S} h(v) - 2|S| + 3 + b$; or

(ii) $\alpha \geq 2$ and $\omega(G-S) < \sum_{v \in S} h(v) - |S| + 3 + b$; or

(iii) $\omega(G-S) < \frac{1}{2}\sum_{v \in S} h(v) - \frac{|S|}{\alpha} + 2 + \lfloor \frac{b}{2} \rfloor$.

This result is a total-excess generalization of the result by Ellingham, Nam and Voss [9].

Chapter 5 discusses total excess of spanning walks. When $k \geq 3$, Jackson and Wormald used a result of Win to show that any graph with sufficiently large toughness has a k-walk. We extend k-walks by introducing the notion of *total k-excess*. We define the total k-excess of a spanning closed walk W as

$$\sum_{v \in V(G)} \max\{\mathrm{visit}_W(v) - k, 0\},$$

where $\mathrm{visit}_W(v)$ is the number of times W visits v. Usually, a spanning closed walk with total k-excess at most b is written for short as a (k,b)-*walk*.

By using the result of Chapter 2, it is easy to show the following statement on the existence of a (k,b)-walk. Suppose $k \geq 2$, $b \geq 0$, and G is a connected graph satisfying the following condition.

$$\text{For every subset } S \text{ of } V(G),\ \omega(G-S) \leq (k-2)|S| + b + 2.$$

Then, G has a spanning walk with total k-excess at most b.

However, when $k = 2$ this does not give a sufficient condition on toughness. Ellingham and Zha [10] proved that all 4-tough graphs have a 2-walk. This chapter gives a sufficient condition for a graph to have a $(2,b)$-walk based on a result of a 2-walk proved by Ellingham and Zha. Our main result is as follows.

Let b be an integer with $b \geq 0$. Suppose that G is a graph, where

$$\omega(G-S) < \begin{cases} \min\{\frac{|S|}{2}, \frac{|S|+3b+9}{4}\} & \text{if } b \text{ is odd} \\ \min\{\frac{|S|}{2}, \frac{|S|+3b+10}{4}\} & \text{if } b \text{ is even} \end{cases}$$

for every subset $S \subseteq V(G)$ with $\omega(G-S) \geq 2$. Then G has a $(2,b)$-walk.

Papers Underlying the Thesis

[1] H. Enomoto, Y. Ohnishi and K. Ota, Spanning Trees with Bounded Total Excess, Ars Combinatoria 102 (2011), 289–295.
[2] Y. Ohnishi, K. Ota and K. Ozeki, A Note on Total Excess of Spanning Trees, AKCE Int. J. Graphs Comb. 8 (2011), 97–103.
[3] Y. Ohnishi and K. Ota, Connected Factors with Bounded Total Excess, submitted.
[4] Y. Ohnishi and K. Ota, Spanning Walks with Bounded Total Excess, preprint.

Contents

Chapter 1

Introduction

1.1 Terminology, Notation and Preliminary

A *graph* $G = (V, E)$ consists of a finite nonempty set V whose elements are called *vertices* and a set E of 2-element subsets of V whose elements are called *edges*. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Let $\binom{V}{2}$ be the set of all 2-element subsets of V, then $E(G) \subseteq \binom{V}{2}$. We denote by $|X|$ the number of elements of a finite set X, called the *cardinality* of X. The *order of a graph* is the number of vertices in the graph, and is written by $|G|$.

The edge $e = \{u, v\}$ is said to *join* the vertices u and v. If $e = \{u, v\}$ is an edge of G, u and v are called *adjacent*, while u and e are *incident*, as are v and e. It is convenient to henceforce denote an edge by uv or vu rather than by $\{u, v\}$. Sometimes, we call u and v *endvertices* of e.

A *loop* is an edge whose endvertices are equal. *Multiple edges* are the edges which have same pair of endvertices. We call a graph which has no loops or multiple edges a *simple* graph, otherwise we call a *multigraph*. Unless otherwise noted, we consider only simple graphs in this thesis.

A graph is *complete* if every two of its vertices are adjacent. We denote a complete graph of order n by K_n. A graph is *bipartite* if its vertex set can be partitioned into subsets X and Y such that each edge joins a vertex of X and a vertex of Y. We denote a bipartite graph G with partition (X, Y) by $G = (X \cup Y, E)$. A graph $G = (X \cup Y, E)$ is *complete bipartite* if $E(G) = \{uv : u \in X, v \in Y\}$. A complete bipartite graph $G = (X \cup Y, E)$ in which $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$.

A graph H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Particularly if $V(H) = V(G)$ then H is called a *spanning subgraph* of G. A spanning subgraph of G is sometimes called a *factor* of G. For $X \subset V(G)$, a graph $G[X]$ is an *induced subgraph* (*the subgraph induced by* X) if $V(G[X]) = X$ and $E(G[X]) = \{uv \in E(G) : u, v \in X\}$.

If $X \subset V(G)$, we denote by $G - X$ the subgraph induced by $V(G) \setminus X$. If $X \subset \binom{V}{2} \setminus E(G)$, we denote $G' = (V, E \cup X)$ by $G + X$. If $X \subset E(G)$, we denote $G'' = (V, E \setminus X)$ by $G - X$. For $v \in V(G)$ and $e \in E(G)$, we denote $G - \{v\}$, $G - \{e\}$ and $G + \{e\}$ simply by $G - v$, $G - e$ and $G + e$ respectively. Furthermore, if H is a subgraph of G, the subgraph $G - V(H)$ is denoted simply by $G - H$.

When $X \subset V(G)$ and $X \neq \emptyset$, if $G[X]$ has no edges, then X is called an *independent* set. We denote the cardinality of a maximum independent set of vertices in G by $\alpha(G)$.

The *neighborhood* $N_G(x)$ of a vertex x in G is the set of all vertices adjacent to x in G. The *degree* of a vertex x, denoted by $\deg_G(x)$, is the cardinality of the neighborhood of x. The *minimum degree* of G is the minimum value of degrees among the vertices of G and is denoted by $\delta(G)$. The *maximum degree* of G is defined similarly and is denoted by $\Delta(G)$. A *k-factor* of G is a spanning subgraph F such that for any vertex $v \in V(G)$, $\deg_F(v) = k$.

A sequence of vertices $W = x_0x_1 \dots x_l$ is called a *walk* (*joining* x_0 *and* x_l) of G if $x_i \in V(G)$ for $0 \leq i \leq l$ and $x_ix_{i+1} \in E(G)$ for $0 \leq i \leq l - 1$. Let $W = x_0x_1 \dots x_l$ be a walk in G. Then l is called the *length* of W and denoted by $l(W)$. A walk is called a *path* if its vertices are distinct. Let $P = y_0y_1 \dots y_m$ be a path in G, then P is called (y_0, y_m)*-path*. A walk $W = x_0x_1 \dots x_l$ is called a *circuit* if $l \geq 3$, the endvertices, namely, x_0 and x_l are the same, and

$x_0x_1, x_1x_2, \ldots, x_{l-1}x_l$ are distinct. A circuit $C = x_0x_1 \ldots x_{l-1}x_0$ is called a *cycle* if $x_0, x_1, \ldots, x_{l-1}$ are distinct.

A graph G is *connected* if any two vertices of G are joined by a path. A maximal connected subgraph is called a *component* of G. We denote the number of components of G by $\omega(G)$. A subset $S \subset V(G)$ is a *cutset* in G if G is connected and $G - S$ is not connected. The cardinality of a minimum cutset in G is called the *connectivity* of G, denoted by $\kappa(G)$. Exceptionally, if $G = K_n$, we define $\kappa(G) = n - 1$. A graph G is called k*-connected* if $k \leq \kappa(G)$.

A circuit containing all edges of a graph is called an *eulerian circuit* in the graph. We say that a graph G is *eulerian* if G has an eulerian circuit. A cycle containing all vertices of a graph is called a *hamiltonian cycle* in the graph. A path containing all vertices of a graph is also called a *hamiltonian path* in the graph. We say that a graph G is *hamiltonian* if G has a hamiltonian cycle.

1.2 Toughness and Spanning Trees

The *toughness* of a graph is an invariant introduced by Chvátal [8]. Let G be a graph, and let S be a subset of $V(G)$. The number of components in $G - S$ is denoted by $\omega(G - S)$. For a real number t, if $|S| \geq t \cdot \omega(G - S)$ holds for every $S \subseteq V(G)$ with $\omega(G - S) \geq 2$, then G is called t*-tough*. The maximum number t for which G is t-tough is the toughness of G which is denoted by $t(G)$. If G is a complete graph, its toughness is defined to be ∞.

The notion of toughness was introduced in the study of hamiltonian cycle. It is clear that 1-tough is a necessary condition for a graph to be hamiltonian.

Proposition 1 *Every hamiltonian graph G is* 1*-tough.*

Proof. Take a hamiltonian cycle C of G. If we remove one vertex from C, then there remains a path. In general, if we remove k vertices from C, then there remain at most k components. Therefore, $\omega(G-S) \leq \omega(C-S) \leq |S|$ for every nonempty subset $S \subseteq V(G)$. From the definition of toughness, we conclude that G is 1-tough. $\Box$

Conversely, Chvátal conjectured as follows.

Conjecture 1 (Chvátal, 1973 [8]) *There exists a constant t_0 such that every t_0-tough graph is hamiltonian.*

Theorem 2 (Enomoto, Jackson, Katerinis and Saito, 1985 [11]) *Let k be a positive integer. If G is a k-tough graph with $k|V(G)|$ even, then G has a k-factor. Moreover, for any $\varepsilon > 0$, there exists a $(k-\varepsilon)$-tough graph G with $k|V(G)|$ even which has no k-factor.*

This implies that for any $\varepsilon > 0$, there exists a $(2-\varepsilon)$-tough graph which has no 2-factor, and hence no hamiltonian cycle. So, it had been believed that every 2-tough graph would be hamiltonian. The following conjecture concerning $K_{1,3}$-free graphs is a special case of 2-tough hamiltonian conjecture, because every 4-connected $K_{1,3}$-free graph is 2-tough, where a $K_{1,3}$-free graph is a graph which does not contain $K_{1,3}$ as an induced subgraph.

Conjecture 2 (Matthews and Sumner, 1984 [16]) *Every 4-connected and $K_{1,3}$-free graph is hamiltonian.*

Although Conjecture 2 still remains open, it has been proved that 2-tough is not a sufficient condition for a graph to be hamiltonian. Bauer, Broersma and Veldman [2] showed that there exists a $(\frac{9}{4}-\varepsilon)$-tough non-hamiltonian graph.

Theorem 3 (Bauer, Broersma and Veldman, 2000 [2]) *For every $\varepsilon > 0$ there exists a $(\frac{9}{4}-\varepsilon)$-tough graph without hamiltonian paths.*

Thus, if Conjecture 1 is true, the value t_0 must be at least $\frac{9}{4}$.

There are only a few structural results on graphs by only assuming certain toughness condition. On the other side, in hamiltonian graph theory, it is known that assuming certain condition on the toughness, sufficient conditions of various theorems about hamiltonicity can be weakened. Let $\sigma_k(G)$ be the minimum degree sum of k vertices taken over all independent set of G. This "degree sum condition" is one of the classic conditions of hamiltonian graph theory.

Theorem 4 (Ore, 1960 [18]) *Let G be a graph on n vertices with $n \geq 3$. If $\sigma_2(G) \geq n$, then G is hamiltonian.*

The $\sigma_2(G)$ condition has been weakened a little by assuming 1-tough, although we have to assume that $|V(G)|$ is large.

Theorem 5 (Jung, 1987 [14]) *Let G be a 1-tough graph on $n \geq 11$ vertices with $\sigma_2(G) \geq n - 4$. Then G is hamiltonian.*

Theorem 6 (Faßbender, 1992 [12]) *Let G be a 1-tough graph on $n \geq 13$ vertices with $\sigma_3(G) \geq \frac{3n-14}{2}$. Then G is hamiltonian.*

Note that, the $\sigma_2(G)$ condition and $\sigma_3(G)$ condition are the best possible in each theorem.

Theorem 7 (Bauer, Chen and Lasser, 1991 [3]) *Let G be a t-tough graph on $n \geq 30$ vertices with $t > 1$. If $\sigma_2(G) \geq n - 7$, then G is hamiltonian.*

About minimum degree condition together with the independence number, the following sharp result is known.

Theorem 8 (Nash-Williams, 1971 [17]) *Let G be a 2-connected graph on n vertices with $\delta(G) \geq \max\{\frac{n+2}{3}, \alpha(G)\}$. Then G is hamiltonian.*

The condition can be also weakened a little for 1-tough graphs.

Theorem 9 (Bigalke and Jung, 1979 [4]) *Let G be a 1-tough graph on $n \geq 3$ vertices with $\delta(G) \geq \max\{\frac{n}{3}, \alpha(G) - 1\}$. Then G is hamiltonian.*

Bondy generalized Theorem 8 in 1980 as follows.

Theorem 10 (Bondy, 1980 [5]) *Let G be a 2-connected graph on n vertices with $\sigma_3(G) \geq \max\{n + 2, 3\alpha(G)\}$. Then G is hamiltonian.*

In the condition using $\sigma_3(G)$ and connectivity, a similar phenomenon is known.

Theorem 11 (Bauer, Broersma, Li and Veldman, 1989 [1]) *Let G be a 2-connected graph on n vertices with $\sigma_3(G) \geq n + \kappa(G)$. Then G is hamiltonian.*

Theorem 12 (Wei, 1993 [19]) *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n + \kappa(G) - 2$. Then G is hamiltonian.*

A *k-walk* in a graph is a spanning closed walk of G that visits every vertex of G at most k times. A *k-tree* is a spanning tree whose maximum degree is at most k. Needless to say, a 1-walk is a hamiltonian cycle, and a 2-tree is a hamiltonian path. Jackson and Wormald [13] showed that the existence of a k-walk implies the existence of a $(k + 1)$-tree. And it is easy to see that any graph with a k-tree has a k-walk.

Win [20] gave a sufficient condition for a graph G to contain a k-tree, in terms of $|S|$ and $\omega(G - S)$ with $S \subset V(G)$.

Theorem 13 (Win, 1989 [20]) *Let k be an integer with $k \geq 2$. If G is a connected graph satisfying the following condition:*

For every subset S of $V(G)$, $\omega(G - S) \leq (k - 2)|S| + 2$.

Then, G has a k-tree.

Let h be a positive integer-valued function on $V(G)$. An *h-tree* T is a spanning tree with $\deg_T(v) \leq h(v)$ for every $v \in V(G)$. If $h(v) = k$ for every $v \in V(G)$, an h-tree is nothing but a k-tree.

Theorem 14 (Ellingham and Zha, 2000 [10]) *Let G be a connected graph. If for every $S \subseteq V(G)$,*

$$\omega(G - S) \leq \sum_{v \in S}(h(v) - 2) + 2,$$

then, G has an h-tree.

Theorem 13 implies a sufficient condition of toughness for the existence of a k-tree, and hence a k-walk.

Corollary 15 *For $k \geq 3$, every $\frac{1}{k-2}$-tough graph has a k-tree, and hence has a k-walk.*

However, 1-walk and 2-walk are not so easily obtained by toughness condition. The 1-walk case, that is hamiltonian cycle case, corresponds to Conjecture 1. So it is a difficult problem to find a toughness condition implying the existence of a 1-walk. The 2-walk case was solved by Ellingham and Zha.

Theorem 16 (Ellingham and Zha, 2000 [10]) *Every 4-tough graph has a 2-walk.*

For the lower bound of toughness for the existence of a k-walk, Ellingham and Zha generalized the example of a $(\frac{9}{4} - \varepsilon)$-tough non-hamiltonian graph in Theorem 3, and proved the following theorem.

Theorem 17 (Ellingham and Zha, 2000 [10]) *For every $\varepsilon > 0$ and every $k \geq 1$, there exists a $(\frac{8k+1}{4k(2k-1)} - \varepsilon)$-tough graph with no k-walk.*

1.3 Total Excess

In this thesis, we introduce the notion of *Total Excess*, and show how to handle the concept. We define several variations of the total excess of graphs in each chapter accordingly.

In Chapter 2, for a spanning subgraph H of a connected graph G, we define the k-excess of a vertex v as $\max\{0, \deg_H(v) - k\}$. We define the total k-excess $\mathrm{te}(H, k)$ as follows,

$$\mathrm{te}(H, k) = \sum_{v \in V(H)} \max\{0, \deg_H(v) - k\}.$$

This chapter gives a sufficient condition for a graph to have a spanning subgraph with bounded total k-excess. Our main result is an extension of Theorem 13. Suppose $k \geq 2$, $b \geq 0$, and G is a connected graph satisfying the following condition:

$$\omega(G - S) \leq (k-2)|S| + 2 + b, \text{ for every subset } S \text{ of } V(G).$$

Then, G has a spanning tree with total k-excess at most b.

In Chapter 3, we consider the graphs with toughness of intermediate fractions, other than $1, \frac{1}{2}, \frac{1}{3}, \ldots$, and discuss the spanning trees contained in such graphs. We again consider the k-excess of spanning trees as Chapter 2. However, we estimate the k-excess by a function depending on the order of G. Using the notion of total excess, we prove that if G is $\frac{1-\varepsilon}{k-2+\varepsilon}$-tough for an integer $k \geq 2$ and a real number ε with $\frac{2}{|V(G)|} \leq \varepsilon < 1$, then G has a spanning tree T such that

$$\mathrm{te}(T, k) \leq \varepsilon |V(G)| - 2.$$

We also investigate the relation between spanning trees in a graph obtained by different pairs of parameters (k, ε). As a consequence, we prove the existence of "a universal tree" in a connected t-tough graph G.

In Chapter 4, we consider total excess from a given factor. Let φ be a nonnegative integer-valued function on $V(G)$. For a spanning subgraph H

of G, we define the *φ-excess* of a vertex v as $\max\{0, \deg_H(v) - \varphi(v)\}$. We define the *total φ-excess* $\mathrm{te}(H,\varphi)$ to be the summation of the φ-excesses of all vertices, namely,

$$\mathrm{te}(H,\varphi) = \sum_{v\in V(H)} \max\{0, \deg_H(v) - \varphi(v)\}.$$

Let F be a disconnected spanning subgraph of a connected graph G. Let h be a nonnegative integer-valued function on $V(G)$, and b be a nonnegative integer. A spanning (F,h,b)-tree H is a spanning connected subgraph of G with $E(H) \supseteq E(F)$ such that $\mathrm{te}(H,\varphi) \leq b$, where $\varphi(v) = \deg_F(v) + h(v)$ for $v \in V(G)$, and that every edge of $E(H)\backslash E(F)$ is a cutedge of H.

Our result is a total-excess generalization of the result by Ellingham, Nam and Voss [9].

In Chapter 5, we introduce total k-excess of spanning closed walks. We may generalize the idea of a hamiltonian cycle to that of a *k-walk*; a closed walk that visits every vertex of a graph at most k times. We extend k-walks by introducing the notion of *total k-excess*. We define the total k-excess of a spanning closed walk W as

$$\sum_{v\in V(G)} \max\{\mathrm{visit}_W(v) - k, 0\},$$

where $\mathrm{visit}_W(v)$ is the number of times W visits v. Usually, a spanning closed walk with total k-excess at most b is written for short as a *(k,b)-walk*.

When $k \geq 3$, it is easy to show the existence of a (k,b)-walk by the result of Chapter 2.

Suppose $k \geq 2$, $b \geq 0$, and G is a connected graph satisfying the following condition.

For every subset S of $V(G)$, $\omega(G - S) \leq (k-2)|S| + b + 2$.

Then, G has a spanning walk with total k-excess at most b.

However, when $k = 2$ this does not give a sufficient condition on toughness. Ellingham and Zha [10] proved that all 4-tough graphs have a 2-walk.

This chapter gives a sufficient condition for a graph to have a $(2, b)$-walk based on a result of a 2-walk proved by Ellingham and Zha.

Chapter 2

Spanning Trees with Total Excess

2.1 Total Excess of Trees

In this chapter, we consider what kind of spanning trees we can get if we replace the constant term in the inequality of the condition in Win's theorem (Theorem 13). We give one answer to this problem, based on another proof of Win's theorem by Ellingham and Zha [10]. We introduce the following notion.

Definition 1 *For a spanning subgraph H of a connected graph, we define the k-excess of a vertex v as* $\max\{0, \deg_H(v) - k\}$. *We define the total k-excess* $\mathrm{te}(H, k)$ *as follows.*

$$\mathrm{te}(H, k) = \sum_{v \in V(H)} \max\{0, \deg_H(v) - k\}$$

The main result in this chapter is the following.

Theorem 18 *Suppose $k \geq 2$, $b \geq 0$, and G is a connected graph satisfying the following condition.*

For every subset S of $V(G)$, $\omega(G-S) \leq (k-2)|S| + b + 2$.

Then, G has a spanning tree with total k-excess at most b.

2.2 Proof of Theorem 18

In the proof, we need a notion of bridge.

Definition 2 *For $S \subseteq V(G)$, an S-bridge of G is*

- *a subgraph consisting of an edge both of whose ends are contained in S, or*
- *a subgraph consisting of a component C of $G-S$ together with the edges joining S and C.*

A k-forest of G is a spanning subgraph of G which is a forest with maximum degree at most k. Take a k-forest F of G with the smallest number of components. Let r be the number of components in F.

Let $\mathcal{F}$ be the set of k-forests in G such that the vertex sets of the components coincide with the ones of F. For $S \subseteq V(G)$, let $\mathcal{F}(S)$ be the set of k-forests $F' \in \mathcal{F}$ such that the vertex sets of the S-bridges of F' coincide with those of the S-bridges of F. Let A_0 be the set of vertices which have degree k in all k-forests in $\mathcal{F}$. Let A_1 be the set of vertices which have degree k in all k-forests in $\mathcal{F}(A_0)$. In every forest in $\mathcal{F}(A_0)$, the degree of vertices in A_0 is k, therefore $A_0 \subseteq A_1$.

Claim 1. Each edge of G which connects different components of $F - A_0$ has an end vertex in A_1.

Proof of Claim 1. Let $uv \in E(G)$ be an edge which connects different components of $F - A_0$. Then, for every $F' \in \mathcal{F}(A_0)$, u and v are contained in different components of $F' - A_0$. Suppose $u \notin A_1$ and $v \notin A_1$. Then, there exist $F_1, F_2 \in \mathcal{F}(A_0)$ satisfying $\deg_{F_1}(u) < k$ and $\deg_{F_2}(v) < k$. By replacing the A_0-bridge in F_1 that contains v with the A_0-bridge in F_2 that contains v, we get another k-forest $F_3 \in \mathcal{F}(A_0)$ such that the degrees of u and v are less than k.

If there does not exist a (u, v)-path in F_3, $F_3 + uv$ is a k-forest of G with less number of components than F. This contradicts the minimality of F.

If there exists a (u, v)-path $F_3(u, v)$ in F_3, the path contains a vertex w of A_0. By adding uv, and removing one of the edges in $F_3(u, v)$ incident with w, we obtain a k-forest in $\mathcal{F}$ such that the degree of w is less than k. This contradicts the fact that $w \in A_0$. Therefore, we establish $u \in A_1$ or $v \in A_1$. Thus the proof of Claim 1 is completed.

To continue this inductively, we define A_{j+1} as the set of vertices which have degree k in all forests in $\mathcal{F}(A_j)$. Then we can show the following claim by the same argument as in Claim 1.

Claim 2. Each edge connecting different components of $F - A_j$ has an end vertex in A_{j+1}.

Proof of Claim 2. Let $uv \in E(G)$ be an edge which connects different components of $F - A_j$. Then, for every $F' \in \mathcal{F}(A_j)$, u and v are contained in different components of $F' - A_j$. Suppose $u \notin A_{j+1}$ and $v \notin A_{j+1}$. Then, there exist $F_1, F_2 \in \mathcal{F}(A_j)$ satisfying $\deg_{F_1}(u) < k$ and $\deg_{F_2}(v) < k$. By replacing the A_j-bridge in F_1 that contains v with the A_j-bridge in F_2 that contains v, we get another k-forest $F_3 \in \mathcal{F}(A_j)$ such that the degrees of u and v are less than k.

There exists a (u, v)-path $F_3(u, v)$ in F_3, and the path contains a vertex w of A_j. If $w \in A_{j-1}$, then uv is an edge connecting different components of $F - A_{j-1}$. By the induction hypothesis of Claim 2 (or by Claim 1), $u \in A_j$ or $v \in A_j$. This contradicts the fact that u and v are in $F - A_j$. Thus, $w \in A_j \setminus A_{j-1}$. By adding the edge uv, and removing one of the edges in $F_3(u, v)$ incident with w, we obtain a k-forest $F_3' \in \mathcal{F}(A_{j-1})$ such that the degree of w is less than k. This contradicts the fact $w \in A_j$. Therefore, we establish $u \in A_{j+1}$ or $v \in A_{j+1}$.

Any vertex in A_j keeps degree k in $\mathcal{F}(A_j)$, therefore $A_j \subseteq A_{j+1}$. Therefore, we get the following progression, where $V_k(F)$ is the set of all vertices whose degree is k in F.

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_j \subseteq \cdots \subseteq V_k(F)$$

Because $V_k(F)$ is a finite set, we get $A_m = A_{m+1}$ at some integer m. Then, by Claim 2, A_m has the property that any edge connecting different components of $F - A_m$ has an end vertex in A_m. In other words, there is no edge of G connecting different components of $F - A_m$. This implies that for $S = A_m$, we have $\omega(G - S) = \omega(F - S)$.

Let $r = \omega(F)$, and let s be the number of components in F which does not contain a vertex of S. If $r = 1$, then F is a desired k-tree. Assume $r \geq 2$. Then, since G is connected, we have $S \neq \emptyset$. Thus, we have $s + 1 \leq r$.

We shall construct a spanning tree of G by adding edges to F. At first, we add edges connecting a component C containing no vertices of S with another component C'. Note that C' must contain a vertex of S, because $\omega(G - S) = \omega(F - S)$. At this point, the total k-excess increases by at most 1 for adding one edge. We repeat this procedure until there is no component containing no vertices of S. Then the total k-excess increases by at most s. Next, we add edges between the components until only one component remains. The total k-excess increases by at most 2 for adding one edge. So, this operation increases the total k-excess by at most $2(r - s - 1)$. Therefore, the total k-excess of the resulting spanning tree T is at most $2(r - 1) - s$.

On the other hand, we can evaluate $\omega(F - S)$ as follows. At first, the number of components in F is r. For each component of F containing a vertex of S, when we remove the first vertex in S, the number of components increases $k - 1$, since the degree of this vertex is k. Then we remove vertices of S according to the distance from the first vertex. If the removing vertex is adjacent to the vertex already removed, then the number of components increases by $k-2$. Otherwise, the removal increases the number of components by $k-1$. Taking sum of them, we have

$$\omega(F-S) \geq r + (k-2)|S| + r - s = (k-2)|S| + 2r - s.$$

By the condition of this theorem $\omega(G-S) \leq (k-2)|S| + b + 2$, we obtain

$$(k-2)|S| + 2r - s \leq \omega(F-S) = \omega(G-S) \leq (k-2)|S| + b + 2.$$

So we have $2r - s \leq b + 2$. Thus the total k-excess of T is at most $2(r-1) - s \leq b$. □

2.3 Remarks

When the constant term b in the condition of Theorem 18 is negative, what kind of spanning trees does the graph contain? In [9], Ellingham, Nam and Voss proved the following result, which is also a generalization of Win's theorem.

Theorem 19 ([9]) *Let G be a connected graph, and let h be a positive integer-valued function on $V(G)$. Then, G has a spanning tree T with* $\deg_T(v) \leq h(v)$ *for every $v \in V(G)$, if for every $S \subseteq V(G)$*

$$\omega(G-S) \leq \sum_{v \in S} (h(v) - 2) + 2.$$

For a given subset $X \subseteq V(G)$ with $|X| = b$, define

$$h(v) = \begin{cases} k-1, & v \in X \\ k, & v \in V(G) \setminus X. \end{cases}$$

Suppose that G satisfies the following condition; for every nonempty subset $S \subseteq V(G)$,

$$\omega(G-S) \leq (k-2)|S| + 2 - b.$$

Then,

$$\begin{aligned}\omega(G-S) &\leq (k-2)|S|+2-|X| \\ &\leq (k-2)|S|+2-|S\cap X| \\ &= \sum_{v\in S}(h(v)-2)+2.\end{aligned}$$

Thus, by Theorem 19, G has a k-tree in which the vertices in X have degree strictly less than k.

Similarly, for a subset $X \subseteq V(G)$ with $|X| = b$, we can consider the following function;

$$h(v) = \begin{cases} k+1, & v \in X \\ k, & v \in V(G) \setminus X. \end{cases}$$

By Theorem 19, if for every subset $S \subseteq V(G)$,

$$\omega(G-S) \leq (k-2)|S|+2+|S\cap X|,$$

then G has a spanning $(k+1)$-tree T such that $\deg_T(x) \leq k$ for $x \in V(G)\setminus X$. In particular, G has a spanning tree T with $\mathrm{te}(T,k) \leq b$. However, this condition is slightly stronger than the one in Theorem 18. Thus, Theorem 19 does not imply Theorem 18.

Chapter 3

Excess Depending on the Order of the Graph

3.1 Many Spanning Trees of Graphs

As a corollary to Theorem 13, we can easily see that every $\frac{1}{k-2}$-tough graph has a k-tree for any integer $k \geq 3$. In this chapter, we consider the graphs with toughness of intermediate fractions, other than $1, \frac{1}{2}, \frac{1}{3}, \ldots$, and discuss spanning trees contained in such graphs.

Recall that for a spanning subgraph H of G and an integer k, the total k-excess of H is

$$\mathrm{te}(H,k) = \sum_{v \in V(H)} \max\{0, \deg_H(v) - k\}.$$

We proved the following theorem in Chapter 2, which gives a sufficient condition for a graph to have a spanning tree with bounded total excess.

Theorem 20 (Chapter 2, Theorem 18) *Suppose that $k \geq 2$, $b \geq 0$, and G is a connected graph satisfying the following condition.*

For any subset S of $V(G)$, $\omega(G - S) \leq (k-2)|S| + b + 2$.

Then, G has a spanning tree T with $\mathrm{te}(T,k) \le b$.

Using this theorem, we can easily prove the following corollary.

Corollary 21 *Let G be a connected graph, $k \ge 2$ be an integer and ε be a real number with $\frac{2}{|V(G)|} \le \varepsilon \le 1$. If G is $\frac{1-\varepsilon}{k-2+\varepsilon}$-tough, then there exists a spanning tree T such that*

$$\mathrm{te}(T,k) \le \varepsilon|V(G)| - 2.$$

Proof of Corollary 21. Let S be a nonempty subset of $V(G)$. If $\omega(G-S) \ge 2$, then since G is $\frac{1-\varepsilon}{k-2+\varepsilon}$-tough, we obtain

$$|S| \ge \frac{1-\varepsilon}{k-2+\varepsilon}\omega(G-S),$$

or

$$(k-2+\varepsilon)|S| \ge (1-\varepsilon)\omega(G-S).$$

Since each component of $G-S$ has at least one vertex, we have $|S| + \omega(G-S) \le |V(G)|$. Thus, by the above inequality,

$$\begin{aligned} \omega(G-S) &\le (k-2)|S| + \varepsilon(|S| + \omega(G-S)) \\ &\le (k-2)|S| + (\varepsilon|V(G)| - 2) + 2. \end{aligned}$$

The last inequality holds even when $\omega(G-S) = 1$. Thus, it follows from Theorem 20 that there exists a spanning tree T with $\mathrm{te}(T,k) \le \varepsilon|V(G)| - 2$. □

For a given graph G, there are many pairs (k,ε) which satisfy the assumption of Corollary 21. Therefore, we obtain a lot of spanning trees from such pairs by applying Corollary 21. Needless to say, they are not necessarily the same tree. But sometimes, one spanning tree may satisfy the conclusion of Corollary 21 for many distinct pairs (k,ε). In the next section, we discuss the relation of the conclusions of Corollary 21 for distinct pairs (k,ε).

3.2 A Relation Between Spanning Trees

We obtained a lot of spanning trees by applying Corollary 21. In this section, we compare these spanning trees.

Formally, for an integer k and for positive real numbers ε_1 and ε_2, we set

$$\frac{1-\varepsilon_1}{k-2+\varepsilon_1} = \frac{1-\varepsilon_2}{(k+1)-2+\varepsilon_2} \tag{3.1}$$

and suppose that G is a connected graph satisfying $\frac{|S|}{\omega(G-S)} \geq \frac{1-\varepsilon_1}{k-2+\varepsilon_1} = \frac{1-\varepsilon_2}{(k+1)-2+\varepsilon_2}$ for any nonempty subset S of $V(G)$. And suppose $\varepsilon_1 \geq \frac{2}{|V(G)|}$ and $\varepsilon_2 \geq \frac{2}{|V(G)|}$. Note that by (3.1), we get

$$\varepsilon_2 = \frac{k\varepsilon_1 - 1}{k-1}.$$

By applying Corollary 21 to the pairs (k, ε_1) and $(k+1, \varepsilon_2)$, we obtain two spanning trees T_1 and T_2 with

$$\mathrm{te}(T_1, k) \leq \varepsilon_1|V(G)| - 2$$

and

$$\mathrm{te}(T_2, k+1) \leq \varepsilon_2|V(G)| - 2 = \frac{k\varepsilon_1 - 1}{k-1}|V(G)| - 2,$$

respectively. We shall show that T_2 can play the same role as T_1.

Let $V_p(T_2) = \{v \in V(G) | \deg_{T_2}(v) = p\}$. We shall estimate $\mathrm{te}(T_2, k)$. Since $\mathrm{te}(T_2, k+1) \leq \varepsilon_2|V(G)| - 2$, we have

$$\frac{k\varepsilon_1 - 1}{k-1}|V(G)| - 2 \geq \sum_{l\geq 1}(l-1)|V_{k+l}(T_2)|. \tag{3.2}$$

On the other hand, since $|E(T_2)| = |V(G)| - 1$ and $2|E(T_2)| = \sum_{p\geq 1} p|V_p(T_2)|$, we have

$$2|V(G)| - 2 = \sum_{p\geq 1} p|V_p(T_2)|,$$

and hence

$$|V(G)| - 2 = \sum_{p\geq 1}(p-1)|V_p(T_2)| \geq \sum_{l\geq 1}(k+l-1)|V_{k+l}(T_2)|. \tag{3.3}$$

By computing (3.2) $\times \frac{k-1}{k}$ + (3.3) $\times \frac{1}{k}$, we deduce

$$\varepsilon_1|V(G)| - 2 \geq \sum_{l \geq 1} l|V_{k+l}(T_2)| = \mathrm{te}(T_2, k).$$

Thus, T_2 has the same bound on the total k-excess as T_1.

Applying the above argument repeatedly, we obtain the following theorem, in which a spanning tree T of G is said to be *good at* a pair (k, ε), if T satisfies the conclusion of Corollary 21, namely $\mathrm{te}(T, k) \leq \varepsilon|V(G)| - 2$.

Theorem 22 *Let $\frac{2}{|V(G)|} \leq \varepsilon_0 \leq 1$ and $k_0 \geq 2$. If a spanning tree T of G is good at (k_0, ε_0), then T is also good at all pairs (k, ε) such that $2 \leq k \leq k_0$ and $\frac{1-\varepsilon}{k-2+\varepsilon} = \frac{1-\varepsilon_0}{k_0-2+\varepsilon_0}$.*

3.3 A Universal Tree

In this section, we shall prove the existence of a universal tree, that is a spanning tree which is good at any pair (k, ε) satisfying the assumption of Corollary 21.

Theorem 23 *Let G be a connected graph and let $t = t(G)$. Then there is a spanning tree T of G such that $\mathrm{te}(T, k) \leq \varepsilon|V(G)| - 2$ for any integer $k \geq 2$ and real number $\frac{2}{|V(G)|} \leq \varepsilon \leq 1$, which satisfy $t \geq \frac{1-\varepsilon}{k-2+\varepsilon}$.*

Proof of Theorem 23. Consider all pairs (k, ε) satisfying $\frac{1-\varepsilon}{k-2+\varepsilon} = t$. Among them, let $k \geq 2$ be the maximum integer such that the corresponding ε satisfies $\varepsilon \geq \frac{2}{|V(G)|}$, equivalently $\frac{1-t(k-2)}{1+t} \geq \frac{2}{|V(G)|}$.

Claim 1. G has a $(k+1)$-tree.

Proof of Claim 1. Let ε' be the real number corresponding to $k+1$, namely $\frac{1-\varepsilon'}{(k+1)-2+\varepsilon'} = t$. By the definition of k, we have $\varepsilon' < \frac{2}{|V(G)|}$. Let $\varepsilon_0 = \frac{2}{|V(G)|}$ so that $\varepsilon_0 > \varepsilon'$. Then,

$$t = \frac{1-\varepsilon'}{(k+1)-2+\varepsilon'} > \frac{1-\varepsilon_0}{(k+1)-2+\varepsilon_0},$$

and hence by Corollary 21, G has a spanning tree T such that $\mathrm{te}(T, k+1) \leq \varepsilon_0|V(G)| - 2 = 0$. Thus, T is a $(k+1)$-tree.

Let T be a $(k+1)$-tree of G such that $|V_{k+1}(T)|$ is as small as possible. The most important property of T is the following claim, which is on the total k-excess of T.

Claim 2. $\mathrm{te}(T, k) \leq \varepsilon|V(G)| - 2$, where ε is the real number satisfying $\frac{1-\varepsilon}{k-2+\varepsilon} = t$.

We first finish the proof of Theorem 23 by using Claim 2. We shall prove that T is a desired spanning tree of G, that is T is good at any pair (k', ε') such that k' is an integer at least 2 and $\frac{2}{|V(G)|} \leq \varepsilon' \leq 1$ satisfying $t \geq \frac{1-\varepsilon'}{k'-2+\varepsilon'}$.

Suppose that $2 \leq k' \leq k$. Let ε' be a real number satisfying $\frac{2}{|V(G)|} \leq \varepsilon' \leq 1$ and $t \geq \frac{1-\varepsilon'}{k'-2+\varepsilon'}$, namely $\max\{\frac{1-t(k'-2)}{1+t}, \frac{2}{|V(G)|}\} \leq \varepsilon' \leq 1$. Since the value $\varepsilon'|V(G)| - 2$ is monotone increasing of ε', it suffices to prove that T is good at (k', ε') for $\varepsilon' = \frac{1-t(k'-2)}{1+t}$, namely, $\frac{1-\varepsilon'}{k'-2+\varepsilon'} = t$. If $k' = k$, then the assertion is equivalent to Claim 2. Moreover, by using Theorem 22, we can verify that T is good at any pair (k', ε') with $2 \leq k' \leq k$ and $\frac{1-\varepsilon'}{k'-2+\varepsilon'} = t$.

Suppose that $k' \geq k+1$. Since T is a $(k+1)$-tree, we have $\mathrm{te}(T, k') = 0$, and hence T is good at $(k', \frac{2}{|V(G)|})$. We can easily verify that T is good at any pair (k', ε') with $k' \geq k+1$ and $\frac{2}{|V(G)|} \leq \varepsilon' \leq 1$.

Thus, T is good at any pair (k', ε') satisfying $k' \geq 2$, $\frac{2}{|V(G)|} \leq \varepsilon' \leq 1$ and $t \geq \frac{1-\varepsilon'}{k'-2+\varepsilon'}$.

In the rest of this chapter, we shall prove Claim 2. In order to prove Claim 2, we use the notion of a bridge. Recall that for $S \subseteq V(G)$, an S-bridge of G is a subgraph consisting of a component C of $G - S$ together with the edges joining S and C, or an edge both of whose ends are contained in S.

Proof of Claim 2. For $S \subseteq V(G)$, let $\mathcal{T}(S)$ denote the set of $(k+1)$-trees T' of G such that $V_{k+1}(T') = V_{k+1}(T)$ and the vertex sets of the S-bridges of

T' coincide with those of the S-bridges of T. Let $A_0 = V_{k+1}(T)$. Note that $\mathrm{te}(T, k) = |A_0|$ since T is a $(k+1)$-tree. If $A_0 = \emptyset$, then $\mathrm{te}(T, k) = 0$, which means T is a desired tree. Thus, we may assume $A_0 \neq \emptyset$.

Let $A_1 = A_0 \cup \{x \in V(G) | \deg_{T'}(x) = k \text{ for all } T' \in \mathcal{T}(A_0)\}$.

Subclaim 1. Each edge of G which connects different components of $T - A_0$ has an end vertex in A_1.

Proof of Subclaim 1. Let $uv \in E(G)$ be an edge which connects different components of $T - A_0$. Then, for every $T' \in \mathcal{T}(A_0)$, u and v are contained in different components of $T' - A_0$. Suppose $u \notin A_1$ and $v \notin A_1$. Then, there exist $T_1, T_2 \in \mathcal{T}(A_0)$ satisfying $\deg_{T_1}(u) < k$ and $\deg_{T_2}(v) < k$. By replacing the A_0-bridge in T_1 that contains v with the A_0-bridge in T_2 that contains v, we get another $(k+1)$-tree $T_3 \in \mathcal{T}(A_0)$ such that the degrees of u and v are less than k.

There exists a (u, v)-path $T_3(u, v)$ in T_3, and the path contains a vertex w of A_0. By adding the edge uv, and removing one of the edges in $T_3(u, v)$ incident with w, we obtain a $(k+1)$-tree T_3' such that $V_{k+1}(T_3') \subseteq V_{k+1}(T) \setminus \{w\}$ since the degree of w is less than $k+1$. This contradicts the minimality of $|V_{k+1}(T)|$. Therefore, we establish $u \in A_1$ or $v \in A_1$.

To continue this inductively, we define $A_{j+1} = A_j \cup \{x \in V(G) | \deg_{T'}(x) = k \text{ for all } T' \in \mathcal{T}(A_j)\}$. Then we can show the following subclaim by the same argument as in Subclaim 1.

Subclaim 2. Each edge connecting different components of $T - A_j$ has an end vertex in A_{j+1}.

Proof of Subclaim 2. Let $uv \in E(G)$ be an edge which connects different components of $T - A_j$. Then, for every $T' \in \mathcal{T}(A_j)$, u and v are contained in different components of $T' - A_j$. Suppose $u \notin A_{j+1}$ and $v \notin A_{j+1}$. Then, there exist $T_1, T_2 \in \mathcal{T}(A_j)$ satisfying $\deg_{T_1}(u) < k$ and $\deg_{T_2}(v) < k$. By replacing the A_j-bridge in T_1 that contains v with the A_j-bridge in T_2 that contains v, we get another $(k+1)$-tree $T_3 \in \mathcal{T}(A_j)$ such that the degrees of u and v are less than k.

There exists a (u,v)-path $T_3(u,v)$ in T_3, and the path contains a vertex w of A_j. If $w \in A_{j-1}$, then uv is an edge connecting different components of $T - A_{j-1}$. By the induction hypothesis of Subclaim 2 (or by Subclaim 1), $u \in A_j$ or $v \in A_j$. This contradicts the fact that u and v are in $T - A_j$. Thus, $w \in A_j \setminus A_{j-1}$. By adding the edge uv, and removing one of the edges in $T_3(u,v)$ incident with w, we obtain a $(k+1)$-tree $T_3' \in \mathcal{T}(A_{j-1})$ such that the degree of w is less than k. This contradicts the fact $w \in A_j$. Therefore, we establish $u \in A_{j+1}$ or $v \in A_{j+1}$.

We get the following sequence of vertex sets, where $V_{\geq k}(T)$ is the set of vertices whose degree is at least k in T.

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_j \subseteq \cdots \subseteq V_{\geq k}(T).$$

Because $V_{\geq k}(T)$ is a finite set, we get $A_m = A_{m+1}$ at some integer m. Then, by Subclaim 2, A_m has the property that any edge connecting different components of $T - A_m$ has an end vertex in A_m. In other words, there is no edge of G connecting different components of $T - A_m$. This implies that for $S = A_m$, we have $\omega(G-S) = \omega(T-S)$.

Let $B = S \setminus A_0$. Then,

$$\omega(G-S) = \omega(T-S) \geq 2 + (k-1)|A_0| + (k-2)|B|. \tag{3.4}$$

In particular, we have $\omega(G-S) \geq 2$ by (3.4). Since $\frac{|S|}{\omega(G-S)} \geq t = \frac{1-\varepsilon}{k-2+\varepsilon}$,

$$(1-\varepsilon)\omega(G-S) \leq (k-2+\varepsilon)|S|.$$

$$\begin{aligned} \omega(G-S) &\leq (k-2)|S| + \varepsilon(|S| + \omega(G-S)) \\ &\leq (k-2)(|A_0| + |B|) + \varepsilon|V(G)|. \end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we have $\varepsilon|V(G)| - 2 \geq |A_0| = \mathrm{te}(T,k)$. This completes the proof of Claim 2. $\square$

3.4 An Example of Theorem 23

In this section, we present a theorem which is a special case of Theorem 23.

First, this is a corollary of Theorem 13.

Corollary 24 *Let G be a connected graph. If for any nonempty subset S of $V(G)$,*

$$\frac{|S|}{\omega(G-S)} \geq \frac{1}{k-1}$$

holds, then there exists a $(k+1)$-tree T.

On the other hand, when we substitute ε for $\frac{1}{k}$ in Corollary 21, we get the following result.

Corollary 25 *Let G be a connected graph with $|V(G)| \geq 2k$. If for any nonempty subset S of $V(G)$,*

$$\frac{|S|}{\omega(G-S)} \geq \frac{1-\frac{1}{k}}{k-2+\frac{1}{k}} = \frac{1}{k-1}$$

holds, then there exists a spanning tree T such that

$$te(T,k) \leq \frac{1}{k}|V(G)| - 2.$$

Note that for any $(k+1)$-tree T of G, $\mathrm{te}(T,k)$ can be bounded as the following proposition.

Proposition 26 *If T is a $(k+1)$-tree of G, then $te(T,k) \leq \frac{1}{k}|V(G)| - \frac{2}{k}$.*

Proof. Recall that $V_i(G)$ is the vertex set of G such that its degree is i.

$$|V(G)| = |V_1(G)| + |V_2(G)| + \cdots + |V_k(G)| + |V_{k+1}(G)|.$$

Counting the number of edges of T, we obtain

$$\begin{aligned} 2|E(T)| &= |V_1(T)| + 2|V_2(T)| + \cdots + k|V_k(T)| + (k+1)|V_{k+1}(T)| \\ &= |V(G)| + |V_2(T)| + \cdots + (k-1)|V_k(T)| + k|V_{k+1}(T)| \\ &\geq |V(G)| + k|V_{k+1}(T)|. \end{aligned}$$

Because T is a tree, $|E(T)| = |V(T)| - 1$.

$$\begin{aligned} k|V_{k+1}(T)| &\leq |V(T)| - 2 \\ |V_{k+1}(T)| &\leq \frac{1}{k}|V(G)| - \frac{2}{k}.\square \end{aligned}$$

Thus, the assumptions of Corollary 24 and 25 are same, but their conclusions are different. However, we can prove the existence of a $(k+1)$-tree which satisfies the conclusion of Corollary 25.

Theorem 27 *Let G be a connected graph with $|V(G)| \geq 2k$. If for any nonempty subset S of $V(G)$,*

$$\frac{|S|}{\omega(G-S)} \geq \frac{1}{k-1}$$

holds, then there exists a $(k+1)$-tree T such that

$$te(T,k) \leq \frac{1}{k}|V(G)| - 2.$$

Proof of Theorem 27. By the assumption, the toughness t of G is at least $\frac{1}{k-1}$. Note that the pairs $(k+1, \frac{2}{|V(G)|})$ and $(k, \frac{1}{k})$ satisfy the condition of Theorem 23, because

$$\frac{1}{k-1} \geq \frac{1 - \frac{2}{|V(G)|}}{k - 1 + \frac{2}{|V(G)|}},$$

and

$$\frac{1}{k-1} = \frac{1 - \frac{1}{k}}{k - 2 + \frac{1}{k}}.$$

Thus the Theorem 23, there exists a spanning tree T such that

$$\mathrm{te}(T, k+1) \leq \frac{2}{|V(G)|}|V(G)| - 2,$$

and

$$\mathrm{te}(T,k) \leq \frac{1}{k}|V(G)| - 2.$$

That is, T is a $(k+1)$-tree such that $\mathrm{te}(T,k) \leq \frac{1}{k}|V(G)| - 2$. $\square$

Chapter 4

Connected Factors and Total Excess

4.1 Total Excess of Connected Factors

Let G be a graph, and let φ be a nonnegative integer-valued function on $V(G)$. For a spanning subgraph H of G, we define the *φ-excess* of a vertex v as $\max\{0, \deg_H(v) - \varphi(v)\}$. We define the *total φ-excess* $\mathrm{te}(H,\varphi)$ to be the summation of the φ-excesses of all vertices, namely,

$$\mathrm{te}(H,\varphi) = \sum_{v \in V(H)} \max\{0, \deg_H(v) - \varphi(v)\}.$$

For $S \subseteq V(G)$, we denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with their incident edges. We denote by $\omega(G)$ the number of components of G. A *cutedge* of a graph is an edge whose deletion increases the number of components.

Before stating our results precisely, some further definitions from the paper [9] are required. Let F be a factor of G. An *F-forest* is a subgraph H of G such that every component of F is either contained in or vertex-disjoint from H, and that every edge of $E(H)\backslash E(F)$ is a cutedge of H. A connected

F-forest is called an F-*tree*. Loosely, an F-forest consists of some components of F joined together in a forest-like way, without creating any new cycles. Given a nonnegative integer-valued function h on $V(G)$, we define an (F,h)-*forest* to be an F-forest H with $\deg_H(v) \le \deg_F(v) + h(v)$ for all $v \in V(G)$. A connected (F,h)-forest is called an (F,h)-*tree*.

Ellingham, Nam and Voss [9] proved the following theorem.

Theorem 28 ([9]) *Let G be a connected graph, and h be a nonnegative integer-valued function on $V(G)$. Assume that G has a factor F in which each component has at least α vertices. Then G has an (F,h)-tree if for every nonempty $S \subseteq V(G)$ at least one of the following holds:*

(i) $\omega(G-S) < \sum_{v\in S} h(v) - 2|S| + 3$*; or*

(ii) $\alpha \ge 2$ *and* $\omega(G-S) < \sum_{v\in S} h(v) - |S| + 3$*; or*

(iii) $\omega(G-S) < \lceil \frac{1}{2}\sum_{v\in S} h(v) - \frac{|S|}{\alpha} + 2 \rceil$.

We also consider the total excess in this chapter. Let F be a factor of G, h be a nonnegative integer-valued function on $V(G)$, and b be a nonnegative integer. An (F,h,b)-*forest* is an F-forest H with $\mathrm{te}(H,\varphi) \le b$, where $\varphi(v) = \deg_F(v)+h(v)$ for $v \in V(G)$. A connected (F,h,b)-forest is called an (F,h,b)-*tree*. We give sufficient conditions for a connected graph to contain a spanning (F,h,b)-tree, corresponding to the conditions in Theorem 28. The following theorem is our main result.

Theorem 29 *Let G be a connected graph, b be a nonnegative integer, and h be a nonnegative integer-valued function on $V(G)$. Assume that G has a factor F in which each component has at least α vertices. Then G has a spanning (F,h,b)-tree if for every nonempty $S \subseteq V(G)$ at least one of the following holds:*

(i) $\omega(G-S) < \sum_{v\in S} h(v) - 2|S| + 3 + b$*; or*

(ii) $\alpha \geq 2$ *and* $\omega(G-S) < \sum_{v\in S} h(v) - |S| + 3 + b$*; or*

(iii) $\omega(G-S) < \frac{1}{2}\sum_{v\in S} h(v) - \frac{|S|}{\alpha} + 2 + \lfloor \frac{b}{2} \rfloor$.

In particular, let F be a totally disconnected spanning subgraph, namely, $E(F) = \emptyset$, of a connected graph G. Then, by Theorem 29 with h being constant, we get the following theorem.

Theorem 30 (Chapter 2, Theorem 18) *Suppose* $h \geq 2$*,* $b \geq 0$*, and* G *is a connected graph satisfying the following condition.*

For every subset S *of* $V(G)$*,* $\omega(G-S) \leq (h-2)|S| + b + 2$.

Then, G *has a spanning tree* T *with* $te(T,h) \leq b$.

Moreover, we derive some corollaries of Theorem 29.

Corollary 31 *Let* G *be an* m*-edge-connected graph,* $m \geq 1$*,* h *be a nonnegative integer-valued function on* $V(G)$*, and*

$$b' = \sum_{v\in V(G)} \max\{\deg_G(v) - m(h(v)-2), 0\}.$$

If $b = \max\{\lceil \frac{b'}{m} \rceil - 2, 0\}$*, then* G *has a spanning tree* T *with* $te(T,h) \leq b$.

This is a generalization of Theorem 32.

Theorem 32 ([9],Theorem 20) *If* G *is an* m*-edge-connected graph,* $m \geq 1$*, then* G *has a spanning tree* T *such that*

$$\deg_T(v) \leq 2 + \left\lceil \frac{\deg_G(v)}{m} \right\rceil$$

for every vertex $v \in V(G)$.

Corollary 33 *Let* G *be an* m*-edge-connected graph,* $m \geq 1$*,* b *be a positive integer. If* $\sum_{v\in V(G)}(\deg_G(v) - m) \leq m(b+2)$*, then* G *has a spanning tree* T *with* $te(T,3) \leq b$.

This is a generalization of Corollary 34.

Corollary 34 ([9],Corollary 22) *Every m-edge-connected m-regular graph has a 3-tree.*

In section 4.2, we prove Theorem 29. In section 4.3, we prove these corollaries of Theorem 29.

4.2 Proof of Theorem 29

In this section we prove Theorem 29. Following [9], we will give a few preliminary definitions and lemmas.

Let G be a graph and let v be a vertex of G. Let $\Omega(G, v)$ denote the component of G containing a vertex v. Given a set of edges A, $V(A)$ will denote the set of ends of the edges in A. Given $u, v \in V(G)$, we say that an edge e (necessarily a cut edge of G) *separates u and v in G* if $\Omega(G, u) = \Omega(G, v)$ but $\Omega(G - e, u) \neq \Omega(G - e, v)$.

In the subsequent argument, we fix a factor F of a connected graph G. We define M to be the set of edges in G joining different components of F. Let H be an F-forest. Note that $E(H) \setminus E(F) = E(H) \cap M$. Given two vertices u and v in the same component of H, there is a unique set of edges of $E(H) \cap M$ each of which separates u and v in H, which we denote by $P_H(u, v)$. (Note that if $\Omega(H, u) = \Omega(H, v)$ then $P_H(u, v) = \emptyset$ by definition.)

Given a graph H, an induced subgraph J of H, and a graph K with $V(J) = V(K)$, the graph $H - E(J) + E(K)$ will be called *H with J replaced by K* and denoted by $H[J \to K]$. The following lemma from [9] is easy to verify.

Lemma 35 *If H is an F-forest and J,K are F-trees with $J \subseteq H$ and $V(J) = V(K)$, then $H[J \to K]$ is also an F-forest consisting of the same vertex sets of components as H.*

The following lemma corresponds to Theorem 1 in [9], but it gives a little more information. The proof is essentially same as the one in [9].

Lemma 36 *Let G be a connected graph, F be a factor of G, and h be a non-negative integer-valued function on $V(G)$. If G does not contain an (F,h)-tree, then there exists a disconnected (F,h)-forest H and a nonempty subset $S \subseteq V(G)$ such that*

(a) $\deg_H(v) = \deg_F(v) + h(v)$ *for every* $v \in S$*;*

(b) *for each* $u \in V(G) \setminus S$*, there exists an* (F,h)*-tree* L_u *with* $V(L_u) = V(\Omega(H,u))$ *such that* $\deg_{L_u}(u) < \deg_F(u) + h(u)$*; and*

(c) *if* R_G *is the set of edges in* G *with at least one end in* S *and with ends in different components of* F*, and* $R_H = R_G \cap E(H)$*, then every edge of* G *joining two components of* $H - R_H$ *belongs to* R_G*.*

Proof of Lemma 36. Let H be an (F,h)-forest of G with the least number of components. By the assumption, H is disconnected.

Now, we consider a set of vertices $T \subseteq V(G)$, and subgraphs J_v and K_v for each $v \in T$ satisfying the following properties (1), (2) and (3).

(1) J_v is an F-tree with $J_v \subseteq \Omega(H * T, v)$ (and hence, J_v is an induced subgraph of H).

(2) K_v is an F-tree of G such that $V(K_v) = V(J_v)$.

(3) $H' = H[J_v \to K_v]$ is an (F,h)-forest such that $\deg_{H'}(v) < \deg_F(v) + h(v)$.

Let $T_0 = \{v \in V(G) | \deg_H(v) < \deg_F(v) + h(v)\}$. Then,

$$T = T_0, \quad J_v = K_v = \Omega(H * T, v) \text{ for } v \in T$$

obviously satisfy the above properties. We choose a maximal subset $T \subseteq V(G)$ with $T_0 \subseteq T$ such that we can take J_v and K_v for each $v \in T$ satisfying the properties (1), (2) and (3).

Claim 1. There exists no edge $wx \in E(G)$ with $w, x \in T$ such that w and x belong to different components of H.

Proof. Suppose that there exists an edge wx with $w, x \in T$ and $\Omega(H, w) \neq \Omega(H, x)$. Let $H' = (H[J_w \to K_w][J_x \to K_x]) + wx$. Then, H' is an (F, h)-forest H' with one fewer components by properties (1), (2), (3) and Lemma 1. This contradicts the choice of H. Thus, Claim 1 holds.

Claim 2. There exists no edge $wx \in E(G)$ with $w, x \in T$ such that w and x belong to different components of $H * T$.

Proof. Suppose that there exists an edge $wx \in E(G)$ with $w, x \in T$ and $\Omega(H * T, w) \neq \Omega(H * T, x)$. By Claim 1, w and x are in the same component in H. Let $\Delta T = V(P_H(w, x)) \setminus T$ and $T' = T \cup \Delta T$. Since w and x are in different components of $H * T$, there exists an edge of $P_H(w, x)$ which is not in $H * T$. This means that at least one end of this edge does not belong to T. Hence, $\Delta T \neq \emptyset$. Remark that for each $t \in T$, F-trees J_t and K_t satisfy (1), (2) and (3) for T', because $J_t \subseteq \Omega(H * T, t) \subseteq \Omega(H * T', t)$. This means that we can employ the same F-trees J_t and K_t also for T'.

For each $v \in \Delta T$, we define $J_v = \Omega(H * T', v)$ and $K_v = (J_v[J_w \to K_w][J_x \to K_x]) - uv + wx$, where $uv \in P_H(w, x)$ for some u (possibly $u = w$ or $u = x$). Then, J_v is an F-tree because it is a component of the F-forest $H * T'$. We must show that K_v is an F-tree and that properties (1), (2) and (3) are satisfied. Since $uv \in P_H(w, x)$, it follows that uv separates w and x in J_v. Let A_w and A_x be the components of $J_v - uv$ such that $w \in V(A_w)$ and $x \in V(A_x)$. Note that A_w and A_x are also F-trees. Since $v \notin T$, uv is not an edge of $H * T$, and hence is not an edge of J_w or J_x. Thus, we have $J_w \subseteq A_w$ and $J_x \subseteq A_x$. Now, it is easy to see that $K_v = A_w[J_w \to K_w] \cup A_x[J_x \to K_x] + uv$ and it is an F-tree, as required.

Properties (1) and (2) now hold trivially. To verify property (3), let $H^* = H[J_w \to K_w][J_x \to K_x]$, and note that $H[J_v \to K_v] = H^* - uv + wx$. Now, H^* has degrees equal to those of $H[J_w \to K_w]$ for vertices of $V(G) \setminus V(J_x)$, and equal to those of $H[J_x \to K_x]$ for vertices of $V(J_x) \subseteq V(G) \setminus V(J_w)$. Thus, H^* is an (F, h)-forest, and by property (3) for w and x, we have

$$\deg_{H^*}(z) < \deg_F(z) + h(z), \text{ for } z \in \{w, x\}.$$

Consequently, $H' = H^* - uv + wx$ is an (F, h)-forest such that $\deg_{H'}(v) < \deg_F(v) + h(v)$.

Thus, T' and J_v,K_v for $v \in T'$ satisfy the properties (1), (2) and (3), which contradicts the maximality of T. Thus, Claim 2 holds.

Now we define $S = V(G) \setminus T$. We shall show that H and S satisfy the conditions (a), (b) and (c).

Since $T_0 \subseteq T$, we have $S \subseteq V(G) \setminus T_0$, and so, the condition (a) holds. Let $u \in V(H) \setminus S$ be an arbitrary vertex. Since $u \in T$, there are F-trees J_u and K_u satisfying (1), (2) and (3). Let $L_u = \Omega(H, u)[J_u \to K_u]$. Then, $H[\Omega(H, u) \to L_u] = H[J_u \to K_u]$, and the degree of u in this graph is less than $\deg_F(u) + h(u)$ by (3). This shows the condition (b).

Now $H - R_H = H * T$. By Claim 2, there exists no edge wx with w and x both in T and in different components of $H * T = H - R_H$. So every edge of G with ends in two components of $H - R_H$ has at least one end in S. Thus, condition (c) holds.

Since G is connected but H is disconnected, G has an edge joining two components of H. By Claim 1, at least one end of the edge is not in T, and hence is in S. Thus, $S \neq \emptyset$. □

Proof of Theorem 29. Suppose that G does not have a spanning (F, h, b)-tree. Then, in particular, G does not have a spanning (F, h)-tree. Thus, there exist H, S, R_H, and R_G as in Lemma 36.

Claim 1. If K and K' are components of H such that $K \cap S = \emptyset$ and $K' \cap S = \emptyset$, then there is no edge of G joining K and K'.

Proof. If such an edge exists, it joins two components of $H - R_H$ but does not belong to R_G, which contradicts the condition Lemma 36 (c). Thus, Claim 1 holds.

Let $c = \omega(H)$ and let d be the number of components of H containing a vertex of S.

Claim 2. There exists a spanning $(F, h, c + d - 2)$-tree in G.

Proof. Let $K_1, K_2, \dots, K_{c-d}$ be the components of H containing no vertex of S. For each $i \in \{1, 2, \dots, c - d\}$, we choose an edge $s_i t_i \in E(G)$ with $s_i \in V(K_i)$ and $t_i \notin V(K_i)$. Since $s_i \notin S$, by the condition of Lemma 36 (b), we have an (F, h)-tree L_i with $V(L_i) = V(K_i)$ such that $\deg_{L_i}(s_i) < \deg_F(s_i) + h(s_i)$. Note that by Claim 1, t_i is not contained in $K_1 \cup \cdots \cup K_{c-d}$. Thus $H' = H[K_1 \to L_1] + \{s_i t_i | 1 \le i \le c - d\}$ is a spanning F-forest whose total $(\deg_F + h)$-excess is no more than $c - d$, since $\deg_{H'}(s_i) \le \deg_F(s_i) + h(s_i)$ for $1 \le i \le c - d$. The graph H' consists of d components. We add $d - 1$ suitable edges to H' so that we obtain a spanning connected subgraph whose total $(\deg_F + h)$-excess is at most $(c - d) + 2(d - 1) = c + d - 2$. Thus, Claim 2 holds.

Since G does not have a spanning (F, h, b)-tree, by Claim 2, we have

$$c + d \ge b + 3. \tag{4.1}$$

We call a component of $H - R_H$ *bad* if all its vertices belong to S, and *good* otherwise. Since $G - R_G$ and $H - R_H$ have the same vertex sets of components and $G - S = (G - R_G) - S$, corresponding to each good component of $H - R_H$, we obtain at least one component of $G - S$. So, in order to estimate $\omega(G - S)$, we only need to estimate the number of good components of $H - R_H$.

Let C be a component of H. Let $S_C = S \cap V(C)$ and $R_C = R_H \cap E(C)$. Note that each edge of R_C is a cutedge of C. Let good(C) be the number of good components of $C - R_C$, and let bad(C) be the number of bad components of $C-R_C$. Then $\omega(G-S) \geq \sum_C \mathrm{good}(C)$. For $i=1,2$, let r_i be the number of edges of R_C with i ends in S. By the condition Lemma 36 (a), we have $\sum_{v\in S_C} h(v) = r_1+2r_2$, and hence, $|R_C| = r_1+r_2 = (r_1+2r_2)-r_2 = \sum_{v\in S_C} h(v) - r_2$.

If $S_C = \emptyset$, then $\mathrm{good}(C) = 1$.

Suppose that $S_C \neq \emptyset$. Let ε be the number of components of $C - R_C$ that contain at least one vertex of S. Since each bad component of $C - R_C$ contains α vertices of S, we have $\varepsilon \leq |S_C| - (\alpha-1)\mathrm{bad}(C)$. And then $r_2 \leq \varepsilon - 1 \leq |S_C| - (\alpha-1)\mathrm{bad}(C) - 1$. Therefore, $|R_C| = \sum_{v\in S_C} h(v) - r_2 \geq \sum_{v\in S_C} h(v) - |S_C| + (\alpha-1)\mathrm{bad}(C) + 1$.

Among the $|R_C|+1$ components of $C-R_C$, the number of good ones is

$$\mathrm{good}(C) = |R_C| + 1 - \mathrm{bad}(C) \geq \sum_{v\in S_C} h(v) - |S_C| + (\alpha-2)\mathrm{bad}(C) + 2. \quad (4.2)$$

Since $\alpha \geq 1$ and $\mathrm{bad}(C) \leq |S_C|$, it follows from (4.2) that

$$\begin{aligned} \mathrm{good}(C) &\geq \sum_{v\in S_C} h(v) - |S_C| - \mathrm{bad}(C) + 2 \\ &\geq \sum_{v\in S_C} h(v) - 2|S_C| + 2. \end{aligned}$$

Thus,

$$\begin{aligned} \omega(G-S) &\geq \sum_{C:S_C\neq\emptyset} \mathrm{good}(C) + \sum_{C:S_C=\emptyset} \mathrm{good}(C) \\ &\geq \sum_{C:S_C\neq\emptyset} \Big(\sum_{v\in S_C} h(v) - 2|S_C| + 2\Big) + \sum_{C:S_C=\emptyset} 1 \\ &= \sum_{v\in S} h(v) - 2|S| + 2d + (c-d) \\ &= \sum_{v\in S} h(v) - 2|S| + c + d \\ &\geq \sum_{v\in S} h(v) - 2|S| + b + 3, \quad \text{(by (4.1))} \end{aligned}$$

a contradiction to the hypothesis (i) of Theorem 29.

If $\alpha \geq 2$, then for each component C of H with $S_C \neq \emptyset$, by (4.2),

$$\text{good}(C) \geq \sum_{v \in S_C} h(v) - |S_C| + 2.$$

Thus,

$$\begin{aligned}
\omega(G-S) &\geq \sum_{C:S_C \neq \emptyset} \text{good}(C) + \sum_{C:S_C=\emptyset} \text{good}(C) \\
&\geq \sum_{C:S_C \neq \emptyset} (\sum_{v \in S_C} h(v) - |S_C| + 2) + \sum_{C:S_C=\emptyset} 1 \\
&= \sum_{v \in S} h(v) - |S| + 2d + (c-d) \\
&\geq \sum_{v \in S} h(v) - |S| + b + 3, \quad \text{(by (4.1))}
\end{aligned}$$

a contradiction to the hypothesis (ii) of Theorem 29.

Since $\text{bad}(C) \leq |S_C|/\alpha$ and $|R_C| = r_1 + r_2 \geq \frac{1}{2}(r_1 + 2r_2) \geq \frac{1}{2}\sum_{v \in S_C} h(v)$, we have

$$\text{good}(C) = |R_C| + 1 - \text{bad}(C) \geq \frac{1}{2}\sum_{v \in S_C} h(v) - |S_C|/\alpha + 1.$$

Hence,

$$\begin{aligned}
\omega(G-S) &\geq \sum_{C:C \neq \emptyset} \text{good}(C) + \sum_{C:S_C=\emptyset} \text{good}(C) \\
&\geq \sum_{C:S_C \neq \emptyset} (\frac{1}{2}\sum_{v \in S_C} h(v) - \frac{|S_C|}{\alpha} + 1) + \sum_{C:S_C=\emptyset} 1 \\
&= \frac{1}{2}\sum_{v \in S} h(v) - \frac{|S|}{\alpha} + d + (c-d) \\
&\geq \frac{1}{2}\sum_{v \in S} h(v) - \frac{|S|}{\alpha} + c.
\end{aligned}$$

Since $c \geq d$, we have $c \geq \lceil \frac{1}{2}(c+d) \rceil \geq \lceil \frac{1}{2}(b+3) \rceil = 2 + \lfloor \frac{b}{2} \rfloor$ by (1). Thus,

$$\omega(G-S) \geq \frac{1}{2}\sum_{v \in S} h(v) - \frac{|S|}{\alpha} + 2 + \lfloor \frac{b}{2} \rfloor,$$

a contradiction to the hypothesis (iii) of Theorem 29.

Therefore, G certainly has a spanning (F, h, b)-tree. □

4.3 Proof of Corollaries

Proof of Corollary 31. Let S be a nonempty set of vertices of G. Since each component of $G - S$ has at least m edges leaving it,

$$m\omega(G - S) \leq \sum_{v \in S} \deg_G(v).$$

Note that

$$\begin{aligned} b' &= \sum_{v \in V(G)} \max\{\deg_G(v) - m(h(v) - 2), 0\} \\ &\geq \sum_{v \in S} \max\{\deg_G(v) - m(h(v) - 2), 0\} \\ &\geq \sum_{v \in S} (\deg_G(v) - m(h(v) - 2)) \\ &= \sum_{v \in S} \deg_G(v) - \sum_{v \in S} m(h(v) - 2). \end{aligned}$$

Then,

$$\omega(G - S) \leq \sum_{v \in S} \frac{\deg_G(v)}{m} \leq \sum_{v \in S} (h(v) - 2) + \frac{b'}{m} \leq \sum_{v \in S} h(v) - 2|S| + b + 2.$$

Taking F to be the totally disconnected spanning subgraph of G, by Theorem 29(i), we conclude that G contains a spanning tree T with $\mathrm{te}(T, h) \leq b$. $\square$

Proof of Corollary 33. Consider a constant function $h \equiv 3$ in Corollary 31. $\square$

Chapter 5

Spanning Walks with Total Excess

5.1 Total Excess of Walks

We introduce the notion of *total k-excess* for spanning closed walks. Define the total k-excess of a spanning closed walk W as

$$\sum_{v \in V(G)} \max\{\mathrm{visit}_W(v) - k, 0\},$$

where $\mathrm{visit}_W(v)$ is the number of times W visits v. Usually, a spanning closed walk with total k-excess at most b is written for short as a *(k, b)-walk*.

Jackson and Wormald [13, Lemma 2.2] observed that the existence of a k-tree implies the existence of a k-walk. In a similar way, we can obtain a spanning walk with total k-excess at most b from a spanning tree with total k-excess at most b. Thus, Theorem 18 implies the following corollary.

Corollary 37 *Suppose $k \geq 2$, $b \geq 0$, and G is a connected graph satisfying the following condition.*

For every subset S of $V(G)$, $\omega(G - S) \leq (k-2)|S| + b + 2$.

Then, G has a spanning walk with total k-excess at most b.

This corollary shows that $\frac{1}{k-2}$-tough or a little weaker condition is sufficient for the existence of a (k, b)-walk when $k \geq 3$. However, when $k = 2$ this does not give a sufficient condition on toughness. Ellingham and Zha [10] proved that all 4-tough graphs have a 2-walk. In this chapter, we discuss the existence of a $(2, b)$-walk based on a result of a 2-walk proved by Ellingham and Zha.

5.2 Toughness and F-trees

In this section we present a sufficient toughness-like condition for the existence of a spanning connected subgraph obtained from a given spanning subgraph of G by adding some edges in G with some restriction on the number of new edges incident with each vertex.

Given a graph G, fix a (usually disconnected) spanning subgraph F. Color the edges of G as follows: all edges joining two vertices of the same component of F are *red*, and all edges joining two vertices in different components of F are *green*. An *F-forest Q (derived from F)* is a subgraph of G that has m components and that is the union of i components of F and $i-m$ green edges whose ends lie in those components, for some $i \geq 1$. Loosely, Q is a subgraph of G obtained by joining some (not necessarily all) of the components of F together in a forest structure using green edges. Especially, when $m = 1$ we call an F-forest an *F-tree*. Given $k \geq 1$, an *(F, k)-forest H* is an F-forest of G in which every vertex is incident with at most k green edges.

Ellingham and Zha proved the following theorem in [10].

Theorem 38 (Ellingham and Zha, 2000 [10]) *Suppose that g and k are positive integers with $g + k \geq 3$. Suppose further that G is a connected graph with a spanning subgraph F, each component of which has order at least g,* *and that for every $S \subseteq V(G)$ we have*

$$\omega(G-S) < \begin{cases} \frac{(g-2)|S|+4g-2}{2g-2} & \text{if } k=1 \text{ and } g \geq 2, \text{ or} \\ (k-2)|S|+3 & \text{if } k \geq 2 \text{ and } g=1, \text{ or} \\ (k-1)|S|+3 & \text{if } k \geq 2 \text{ and } g \geq 2. \end{cases}$$

Then, G has a spanning (F,k)-tree.

We will generalize this theorem with bounded "total excess." The main idea is same as the proof of Theorem 18, but the details are more complicated, because we cannot delete and replace edges of F in trying to extend our tree structure, and the argument counting the components varies according to the degree condition and the order of the components of F.

We now state our generalization of Theorem 18. Theorem 18 is just the case of our result when F is the spanning subgraph of G with no edges, so that $g=1$. For $k \geq 3$, this theorem implies that every $\frac{1}{k-2}$-tough graph has a k-tree.

We need some more notions. For any $v \in V(H)$, $\mathrm{qdeg}_H(v)$ is the number of green edges of H incident with v. For an F-tree H of G, we define the total 1-excess of H (or, simply, the total excess of H) as

$$\sum_{v \in V(G)} \max\{\mathrm{qdeg}_H(v)-1, 0\}.$$

An $(F,1,b)$-tree is an F-tree with $\sum_{v \in V(G)} \max\{\mathrm{qdeg}_H(v)-1,0\} \leq b$. In a similar way, we can define the total excess of an F-forest.

Theorem 39 *Suppose that g is a positive integer with $g \geq 2$, and b is an integer with $b \geq 0$. Suppose further that G is a connected graph with a spanning subgraph F, each component of which has order at least g, and that for every $S \subseteq V(G)$ we have*

$$\omega(G-S) < \begin{cases} \frac{(g-2)|S|+(b+3)g}{2g-2} & \text{if } b \text{ is odd} \\ \frac{(g-2)|S|+(b+4)g-2}{2g-2} & \text{if } b \text{ is even.} \end{cases}$$

Then, G has a spanning $(F,1,b)$-tree.

To prove Theorem 39, we will give a few preliminary definitions and lemmas, which already appeared in Chapter 4.

Let G be a graph. Let $\Omega(G,v)$ denote the component of G containing a vertex v. Given $u,v \in V(G)$, we say that an edge e (necessarily a cut edge of G) *separates u and v in G* if $\Omega(G,u) = \Omega(G,v)$ but $\Omega(G-e,u) \neq \Omega(G-e,v)$.

In the subsequent argument, we fix a factor F of a connected graph G. We define M to be the set of edges in G joining different components of F. Let H be a spanning F-forest. Note that $E(H) \setminus E(F) = E(H) \cap M$. Given two vertices u and v in the same component of H, there is a unique set of edges of $E(H) \cap M$ each of which separates u and v in H, which we denote by $P_H(u,v)$. (Note that if $\Omega(F,u) = \Omega(F,v)$ then $P_H(u,v) = \emptyset$ by definition.)

Lemma 40 (Chapter 4, Lemma 36) *Let G be a connected graph, F be a factor of G, and k be a nonnegative integer. If G does not contain a spanning (F,k)-tree, then there exists a disconnected spanning (F,k)-forest H and a nonempty subset $S \subseteq V(G)$ such that*

(a) $\deg_H(v) = \deg_F(v) + k$ *for every $v \in S$;*

(b) *for each $u \in V(G) \setminus S$, there exists an (F,k)-tree L_u with $V(L_u) = V(\Omega(H,u))$ such that $\deg_{L_u}(u) < \deg_F(u) + k$; and*

(c) *if R_G is the set of edges in G with at least one end in S and with ends in different components of F, and $R_H = R_G \cap E(H)$, then every edge of G joining two components of $H - R_H$ belongs to R_G.*

The following lemma will be used in one of our counting arguments.

Lemma 41 ([10, Lemma 3.2]) *Let G, F and M be as described earlier, and let H be a spanning (F,k)-tree. For any $q \geq 2$ and $R \subseteq M \cap E(H)$, the number of components of $H - R$ incident with fewer than q edges of R is at least $(q-2)|R|/(q-1) + q/(q-1)$.*

Proof of Theorem 39. Suppose that G does not have a spanning $(F, 1, b)$-tree with total excess b. Then, in particular, G does not have a spanning $(F, 1)$-tree. Thus, there exist H, S, R_H, and R_G as in Lemma 40.

Claim 1. If K and K' are components of H such that $K \cap S = \emptyset$ and $K' \cap S = \emptyset$, then there is no edge of G joining K and K'.

Proof. If such an edge exists, it joins two components of $H - R_H$ but does not belong to R_G, which contradicts the condition Lemma 40 (c). Thus, Claim 1 holds.

Let $c = \omega(H)$ and let d be the number of components of H containing a vertex of S.

Claim 2. There exists a spanning $(F, 1, c + d - 2)$-tree in G.

Proof. Let $K_1, K_2, \ldots, K_{c-d}$ be the components of H containing no vertex of S. For each $i \in \{1, 2, \ldots, c - d\}$, we choose an edge $s_i t_i \in E(G)$ with $s_i \in V(K_i)$ and $t_i \notin V(K_i)$. Since $s_i \notin S$, by the condition of Lemma 40(b), we have an F-tree L_i with $V(L_i) = V(K_i)$ such that $\deg_{L_i}(s_i) < \deg_F(s_i) + 1$. Note that by Claim 1, t_i is not contained in $K_1 \cup \cdots \cup K_{c-d}$. Thus $H' = H[K_1 \to L_1][K_2 \to L_2] \cdots [K_{c-d} \to L_{c-d}] + \{s_i t_i | 1 \leq i \leq c-d\}$ is a spanning F-forest whose total excess is no more than $c-d$, since $\deg_{H'}(s_i) \leq \deg_F(s_i) + 1$ for $1 \leq i \leq c - d$. The graph H' consists of d components. We add $d-1$ suitable edges to H' so that we obtain an F-tree whose total excess is at most $(c - d) + 2(d - 1) = c + d - 2$. Thus, Claim 2 holds.

Since G does not have a spanning $(F, 1, b)$-tree, by Claim 2, we have

$$c + d \geq b + 3. \tag{5.1}$$

Moreover, since $c \geq d$, we have

$$c \geq \left\lceil \frac{b+3}{2} \right\rceil. \tag{5.2}$$

We call a component of $H-R_H$ *bad* if all its vertices belong to S, and *good* otherwise. Since $G-R_G$ and $H-R_H$ have the same vertex sets of components and $G-S=(G-R_G)-S$, corresponding to each good component of $H-R_H$, we obtain at least one component of $G-S$. So, in order to estimate $\omega(G-S)$, we only need to estimate the number of good components of $H-R_H$.

By the assumption, suppose that $g \geq 2$. Let C be a component of H and good(C) be the number of good components of $C-R_C$. Note that good(C) is at least the number of components that are incident with fewer than g edges of R_C, which by Lemma 41 is

$$\mathrm{good}(C) \geq \frac{g-2}{g-1}|R_C| + \frac{g}{g-1}.$$

Hence, by using (3.1) and (3.2),

$$\begin{aligned}
\omega(G-S) &\geq \sum_{C:S_C\neq\emptyset} \mathrm{good}(C) + \sum_{C:S_C=\emptyset} \mathrm{good}(C)\\
&\geq \sum_{C\subset H, S_C\neq\emptyset} \Big(\frac{g-2}{g-1}|R_C| + \frac{g}{g-1}\Big) + \sum_{C\subset H, S_C=\emptyset} 1\\
&= \frac{1}{2}\Big(\frac{g-2}{g-1}\Big)|S| + \Big(\frac{g}{g-1}\Big)d + (c-d)\\
&= \frac{1}{2}\Big(\frac{g-2}{g-1}\Big)|S| + \Big(\frac{1}{g-1}\Big)(c+d) + \frac{g-2}{g-1}c\\
&\geq \frac{1}{2}\Big(\frac{g-2}{g-1}\Big)|S| + \Big(\frac{1}{g-1}\Big)(b+3) + \frac{g-2}{g-1}\left\lceil\frac{b+3}{2}\right\rceil.
\end{aligned}$$

If b is odd, then

$$\begin{aligned}
\omega(G-S) &\geq \frac{1}{2}\Big(\frac{g-2}{g-1}\Big)|S| + \Big(\frac{1}{g-1}\Big)(b+3) + \Big(\frac{g-2}{g-1}\Big)\Big(\frac{b+3}{2}\Big)\\
&= \frac{1}{2}\Big(\frac{g-2}{g-1}\Big)|S| + \frac{(b+3)g}{2g-2}.
\end{aligned}$$

If b is even, then

$$\begin{aligned}
\omega(G-S) &\geq \frac{1}{2}\Big(\frac{g-2}{g-1}\Big)|S| + \Big(\frac{1}{g-1}\Big)(b+3) + \Big(\frac{g-2}{g-1}\Big)\Big(\frac{b+4}{2}\Big)\\
&= \frac{1}{2}\Big(\frac{g-2}{g-1}\Big)|S| + \frac{(b+4)g-2}{2g-2}.
\end{aligned}$$

These contradict the hypothesis of Theorem 39.

Therefore, G certainly has a spanning $(F, 1, b)$-tree. □

5.3 Toughness and $(2, b)$-Walks

In this section we apply the main result of Section 5.2. We use a spanning $(F, 1, b)$-tree derived from a 2-factor to establish the existence of a $(2, b)$-walk.

Theorem 42 *Let b be an integer with $b \geq 0$. Suppose that G is a graph, where*

$$\omega(G - S) < \begin{cases} \min\{\frac{|S|}{2}, \frac{|S|+3b+9}{4}\} & \text{if } b \text{ is odd} \\ \min\{\frac{|S|}{2}, \frac{|S|+3b+10}{4}\} & \text{if } b \text{ is even} \end{cases}$$

for every subset $S \subseteq V(G)$ with $\omega(G - S) \geq 2$. Then G has a $(2, b)$-walk.

This theorem is a generalization of the following theorem. The outline of this proof is also similar.

Theorem 43 ([10, Theorem 4.1]) *Suppose that G is a graph, where $\omega(G - S) \leq \min\{|S|/2, (|S| + 9)/4\}$ for every subset $S \subseteq V(G)$ with $\omega(G - S) \geq 2$. Then G has a 2-walk.*

Proof of Theorem 42. The given condition implies that G is 2-tough and hence connected; moreover, since Enomoto, Jackson, Katernis, and Saito [11] proved that every k-tough graph has a k-factor, G has a 2-factor F. Now, since

$$\omega(G - S) < \begin{cases} \frac{|S|+3b+9}{4} & \text{if } b \text{ is odd} \\ \frac{|S|+3b+10}{4} & \text{if } b \text{ is even,} \end{cases}$$

it follows from Theorem 39 with $g = 3$ that G has a spanning $(F, 1, b)$-tree H. Replacing each green edge of H by two multiple edges creates an eulerian multigraph, and an eulerian circuit in this multigraph corresponds to a $(2, b)$-walk in H and hence on G. □

If we know that G has a 2-factor in which every component is of length at least g, where $g \geq 4$, then we can improve our argument in an obvious way.

Theorem 44 *Let b be an integer with $b \geq 0$. Suppose that G is a connected graph with a 2-factor F in which every component has length at least g, $g \geq 3$. Suppose further that*

$$\omega(G-S) < \begin{cases} \frac{(g-2)|S|+(b+3)g}{2g-2} & \textit{if } b \textit{ is odd} \\ \frac{(g-2)|S|+(b+4)g-2}{2g-2} & \textit{if } b \textit{ is even} \end{cases}$$

for all $S \subseteq V(G)$ with $\omega(G-S) \geq 2$. Then G has a spanning $(F,1,b)$-tree, and hence, a $(2,b)$-walk.

Bibliography

[1] D. Bauer, H. J. Broersma, R. Li and H. J. Veldman, A generalization of a result of Häggkvist and Nicoghossian, J. Combin. Theory Ser. B 47 (1989), 237–243.

[2] D. Bauer, H. J. Broersma and H. J. Veldman, Not every 2-tough graph is hamiltonian, Discrete Appl. Math. 99 (2000), 317–321.

[3] D. Bauer, G. Chen and L. Lasser, A degree condition for hamilton cycles in t-tough graphs with $t > 1$, Advances in Graph Theory. Vishwa Int. Publ. (1991), 20–33.

[4] A. Bigalke and H. A. Jung, Über Hamiltonische Kreise und unabhängige Ecken in Graphen. Monatsh Math. 88 (1979), 195–210.

[5] J. A. Bondy, Longest paths and cycles in graphs of high degree, Research Report CORR 80-16, University of Waterloo, Waterloo, Ontario (1980).

[6] M. Cai, Connected $[k, k+1]$-factors of graphs, Discrete Math. 169 (1997), 1–16.

[7] C. P. Chen, Toughness of graphs and $[2, b]$-factors, Graphs Combin. 10 (1994), 97–100.

[8] V. Chvátal, Tough graphs and hamiltonian circuits, Discrete Math. 5 (1973), 215–228.

[9] M. N. Ellingham, Y. Nam and H.-J. Voss, Connected (g, f)-factors, J. Graph Theory 39 (2002), 62–75.

[10] M. N. Ellingham and X. Zha, Toughness, trees, and walks, J. Graph Theory 33 (2000), 125–137.

[11] H. Enomoto, B. Jackson, P. Katernis and A. Saito, Toughness and the existence of k-factors, J. Graph Theory 9 (1985), 87–95.

[12] B. Faßbender, A sufficient condition on degree sums of independent triples for hamiltonian cycles in 1-tough graphs, Ars Combinatoria 33 (1992), 300–304.

[13] B. Jackson and N. C. Wormald, k-walks of graphs, Australas. J. Combin. 2 (1990), 135–146.

[14] H. A. Jung, On maximal circuits infinitegraphs, Ann. Discrete Math. 3 (1987), 129–144.

[15] P. Katerinis, Toughness of graphs and the existence of factors, Discrete Math. 80 (1990), 81–92.

[16] M. M. Matthews and D. P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory 8 (1984), 139-146.

[17] C. St. J. A. Nash-Williams, Edge-disjoint hamiltonian circuits in graphs with vertices of large valency, Studies in Pure Mathematics. Academic Press, London (1971), 157–183.

[18] O. Ore, Note on hamiltonian circuits, Amer. Math. Monthly 67 (1960), 55.

[19] B. Wei, A generalization of a result of Bauer and Schmeichel, Graphs Combin. 9 (1993), 383–389.

[20] S. Win, On a connection between the existence of k-trees and the toughness of a graph, Graphs Combin. 5 (1989), 201–205.