Forbidden Induced Subgraphs Implying Properties in Graphs

by

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to my beloved wife
Yuliya Malashok

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Summary

In the present work we study the relation between forbidden induced subgraphs and the resulting properties in large enough graph. More formally, we study the following problem. Given a property P on graphs, find all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free graph satisfies P. This problem has been studied before for several properties in particular. For example, Hamiltonian graphs, traceable graphs, graphs containing a 2-factor, pancyclic graphs, Hamilton-connected graphs, cycle extendable graphs, graphs containing a perfect matching, etc.

In this thesis we give a full characterization of all families of forbidden subgraphs for several classes of graphs: claw-free graphs, star-free graphs, graphs having a perfect matching, graphs having a near perfect matching and t-tough graphs. Concretely, for each of these classes, we give a complete characterization of all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free graph is in the desired class.

Papers underlying this thesis

[a] J. Fujisawa, K. Ota, K. Ozeki and G. Sueiro, *Forbidden induced subgraphs for star-free graphs*, Discrete Math. **311** (2011), 2475–2484.

[b] K. Ota, K. Ozeki and G. Sueiro, *Forbidden induced subgraphs for near perfect matchings*, submitted.

[c] K. Ota and G. Sueiro, *Forbidden induced subgraphs for perfect matchings*, To appear in Graphs and Combin.

[d] K. Ota and G. Sueiro, *Forbidden induced subgraphs for toughness*, submitted.

Contents

Chapter 1

Introduction

In this chapter we give an introduction to the problems addressed in this thesis. In Section 1.1 we do a brief introduction to the general topic of forbidden induced subgraphs. In Section 1.2 we present some results on the subject found in previous works. In Section 1.3 we describe the particular problems studied in this thesis and present our main results.

1.1 The forbidden induced subgraph problem

Let G and H be two graphs. G is said to be H*-free* if G does not contain H as an induced subgraph. Let $\mathcal{F}$ be a family of connected graphs. G is said to be $\mathcal{F}$*-free* if G is H-free for all $H \in \mathcal{F}$. In this case, we say that $\mathcal{F}$ is *forbidden* in G.

The *forbidden subgraph theory* studies the relation between the family of forbidden subgraphs $\mathcal{F}$ and resulting properties in the graph G. In this theory, we can think mainly of two problems. Given a family of graphs $\mathcal{F}$, find the properties that $\mathcal{F}$-free graphs have. Or in the opposite direction, given some property, find the families of graphs $\mathcal{F}$ such that the $\mathcal{F}$-free graphs satisfy the desired property. In this thesis, we concentrate in the later.

If we think of the simplest graphs that can be forbidden, we can start from the smallest connected graphs, like P_1, P_2, P_3 (the path on one, two and three vertices, respectively). But P_1-free graph, P_2-free graphs and P_3-graphs are not interesting. There are no P_1-free graphs. P_2-free graphs are just edge-less graphs. And P_3-free graphs are just a collection of complete connected components. None of these classes of graphs have any interesting structural property. Therefore, we will only consider forbidden subgraphs with at least three edges.

There are three non-isomorphic connected graphs with three edges: K_3, P_4 and $K_{1,3}$. One of them, $K_{1,3}$, appears in the theorem that might be considered the most basic result in *forbidden subgraph theory.*

Theorem 1.1 ([32],[24]). *Every $K_{1,3}$-free connected graph of even order has a perfect matching.*

This theorem shows a property of $K_{1,3}$-free graphs. But, as we stated before, we can ask the opposite question. That is, what other forbidden subgraphs imply a perfect matching? In [30], the authors showed that $K_{1,3}$ is essentially the only forbidden subgraph implying a perfect matching, even when allowing a finite number of exceptions.

Theorem 1.2 ([30]). *Let H be a connected graph of order at least three. If there exists a positive constant n_0 such that every H-free connected graph of even order at least n_0 has a perfect matching, then $H = K_{1,2}$ or $H = K_{1,3}$.*

These two theorems provide basic examples of the kind of results found in *forbidden subgraph theory*. But it is possible to consider several variations and restrictions. For example, fixing or limiting the number of forbidden subgraphs, allowing a finite number of exceptions, not allowing exceptions at all, considering only connected forbidden subgraphs, etc. In the following section we show several theorems including some of these variations and restrictions.

In Section 1.3 and Chapter 5, we show and prove generalizations of Theorems 1.1 and 1.2 by allowing more than one forbidden subgraph.

1.2 Background

In this section we present some results on forbidden induced subgraphs found in previous works.

When we state results concerning forbidden induced subgraphs, the following definition is very useful to compare families of graphs. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two families of graphs, we say that $\mathcal{F}_1 \leq \mathcal{F}_2$ if for each $H_2 \in \mathcal{F}_2$, there is an $H_1 \in \mathcal{F}_1$ such that H_1 is an induced subgraph of H_2. It is easy to see that the relation "$\leq$" defines a quasi-order (reflexive and transitive). Furthermore, if $\mathcal{F}_1 \leq \mathcal{F}_2$ then any $\mathcal{F}_1$-free graph is also an $\mathcal{F}_2$-free graph (see for example Lemma 3 of [18]).

A graph is said to be *Hamiltonian* if there is a cycle passing through all its vertices. Such a cycle is called a *Hamiltonian cycle*. During the 1980's, a number of results were proved showing that forbidden some subgraphs implies the existence of a Hamiltonian cycle in a 2-connected graph. The most basic and fundamental of them is the following result.

Theorem 1.3 ([11]). *Every 2-connected $\{K_{1,3}, N\}$-free graph is Hamiltonian.*

Where $N = N_{1,1,1}$, being $N_{i,j,k}$ the graph consisting of K_3 and three vertex disjoint paths of lengths i, j and k rooted at its vertices. Define also $W = N_{2,1,0}$ and $Z_i = N_{i,0,0}$.

Later, some additional results showing other forbidden pairs were proved.

Theorem 1.4 ([6]). *Every 2-connected $\{K_{1,3}, P_6\}$-free graph is Hamiltonian.*

Theorem 1.5 ([3]). *Every 2-connected $\{K_{1,3}, W\}$-free graph is Hamiltonian.*

Finally, Bedrossian[3] showed that these are essentially all possible pairs.

Theorem 1.6 ([3]). *Let R and S be connected graphs. Then every 2-connected $\{R, S\}$-free graph is Hamiltonian if and only if R is an induced subgraph of $K_{1,3}$ and S is an induced subgraph of one of P_6, N and W.*

Faudree et al.[13] considered a variation allowing a finite number of exceptions.

Theorem 1.7 ([13]). *Every 2-connected $\{K_{1,3}, Z_3\}$-free graph of order at least 10 is Hamiltonian.*

Theorem 1.8 ([13]). *Let R and S be connected graphs. Then every 2-connected $\{R, S\}$-free graph of order at least 10 is Hamiltonian if and only if R is an induced subgraph of $K_{1,3}$ and S is an induced subgraph of one of P_6, N, W and Z_3.*

These results include all variations concerning a pair of forbidden subgraphs for Hamiltonian graphs. Several other works show similar characterizations for triples of forbidden subgraphs implying the existence of a Hamiltonian cycle in all graphs ([7, 14]), and in graphs of sufficiently large order ([16, 15]). See also [19] for a survey on Hamiltonian graphs including some of these results.

A graph is said to be *traceable* if there is a path passing through all its vertices. Such path is called a *Hamiltonian path*.

Theorem 1.9 ([11]). *Every $\{K_{1,3}, N\}$-free connected graph is traceable.*

Theorem 1.9 shows an example of a pair of forbidden subgraph implying the existence of a Hamiltonian path. Faudree et al.[13] showed that such pair is essentially the only one.

Theorem 1.10 ([13]). *Let R and S be connected graphs. Then every $\{R, S\}$-free connected graph is traceable if and only if $\{R, S\} \leq \{K_{1,3}, N\}$.*

In [20, 21], the authors considered three forbidden subgraphs. In [22], they completed the characterization of triples of forbidden subgraphs implying the existence of a Hamiltonian path in large enough graphs.

We now present some other results connecting forbidden subgraphs and Hamiltonian properties like cycle extendability, pancyclability and Hamiltonian-connectedness.

A graph is said to be *cycle extendable* if any non-Hamiltonian cycle can be extended to a cycle containing exactly one more vertex.

Theorem 1.11 ([13]). *Let R and S be connected graphs.Then every 2-connected $\{R, S\}$-free graph of order at least 10 is cycle extendable if and only if $\{R, S\} \leq \{K_{1,3}, Z_2\}$.*

A graph of order n is *pancyclic* if it contains a cycle of length i for each $3 \leq i \leq n$. Let L be the graph consisting of two vertex-disjoint copies of K_3 and an edge joining them.

Theorem 1.12 ([23]). *Let R and S be connected graphs. Then every 3-connected $\{R, S\}$-free graph is pancyclic if and only if R is an induced subgraph of $K_{1,3}$ and S is an induced subgraph of one of P_7, L, $N_{4,0,0}$, $N_{3,1,0}$, $N_{2,2,0}$ and $N_{2,1,1}$.*

A graph is said to be *Hamiltonian-connected* if there is a Hamiltonian path between any pair of distinct vertices.

Theorem 1.13 ([31]). *Every* 3*-connected $\{K_{1,3}, N\}$-free is Hamiltonian-connected.*

Theorem 1.14 ([4]). *Every* 3*-connected $\{K_{1,3}, L\}$-free is Hamiltonian-connected.*

For both Hamiltonian graphs and traceable graphs, no results for forbidden families of size bigger than 3 are known. For Hamiltonian-connected graphs, there is not even a characterization of the forbidden pairs. This suggest that in general finding a complete characterization (without restrictions on the size of the family of forbidden subgraphs) is a difficult problem. However, in this thesis we show complete characterization for several classes of graphs (see the next section for details).

1.3 Problems studied in this thesis

In this section we present and formally state the problems studied in this thesis, along with the main results we obtained.

Every time we say that "some large enough graphs satisfy some property", we mean that the number of such graphs that do not satisfy the property is finite.

Claw-free graphs

The graph $K_{1,3}$ is also called the *claw*. A graph is *claw-free* if it does not contain a $K_{1,3}$ as an induced subgraph. Claw-free graphs have been widely studied in the literature, as they are closely related to line graphs, and on the other side, there are many interesting results in connection with matching theory and Hamiltonian graphs theory. See [12] for a survey on claw-free graphs.

A 2-factor is a spanning subgraph such that every vertex has degree two. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degree of G, respectively.

Consider the following theorem about graphs having a 2-factor (see Section 3.2 for graph definitions).

Theorem 1.15 ([1]). *Let G be a connected graph with $\delta(G) \geq 2$ and $\Delta(G) \geq 3$.*

(i) If G is $\{Z_{1,3}, K_{1,3}\}$-free then G has a 2-factor.

(ii) If G is $\{Z_{1,3}, Y_3, W_2^3, K_{2,3}\}$-free and $|V(G)| \geq 9$ then G has a 2-factor.

Because $Z_{1,3}$ is an induced subgraph of itself, and all three graphs Y_3, W_2^3 and $K_{2,3}$ contain a $K_{1,3}$ as an induced subgraph, then every $\{Z_{1,3}, K_{1,3}\}$-free graph is also $\{Z_{1,3}, Y_3, W_2^3, K_{2,3}\}$-free. In this sense, we can say that (ii) is more general than (i) in Theorem 1.15. But on the other hand, we have the following result.

Theorem 1.16 ([1]). *Let G be a connected graph with $\delta(G) \geq 2$, $\Delta(G) \geq 3$ and $|V(G)| \geq 9$. If G is $\{Z_{1,3}, Y_3, W_2^3, K_{2,3}\}$-free, then G is also $K_{1,3}$-free.*

The interesting point about Theorem 1.16 is that even though no graph belonging to the family $\mathcal{H} = \{Z_{2,3}, Y_4, W_2^3, K_{2,3}\}$ is an induced subgraph of $K_{1,3}$, when considering the $\mathcal{H}$-free graphs under certain conditions, the graph $K_{1,3}$ is also forbidden. The authors of [1] were interested in finding a family of forbidden subgraphs implying a 2-factor that does not contain a star. But even though there is no star in $\{Y_4, Z_{2,3}, W_2^3, K_{2,3}\}$, by Theorem 1.16 it is somehow implicitly forbidden. That is why the authors of [1] called this phenomenon *implicit forbiddance*.

In this thesis, we further research the implicit forbiddance for $K_{1,3}$. Concretely, we look for other families of graphs that forbid $K_{1,3}$ implicitly. We do not consider the conditions on the minimum and maximum degree from Theorem 1.16 since those are necessary conditions related to the problem studied in [1]. We can state our problem as follows.

Problem 1.1. *Characterize all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free connected graph is $K_{1,3}$-free.*

In this thesis, we solve Problem 1.1. The solution is expressed in the following theorem (see Section 3.2 for graph definitions).

Theorem 1.17. *Let $\mathcal{F}$ be a family of connected graphs. Then the following are equivalent.*

- *Every large enough $\mathcal{F}$-free connected graph is $K_{1,3}$-free*
- *$\mathcal{F} \leq \mathcal{F}_m(l, q)$ for some $m \geq 1$, $l \geq 4$ and $q \geq 3$,*

where $\mathcal{F}_m(l, q) = \{K_{1,l}, W_q^3, T_q, D_q, Y_{m+2}, Z_{1,q}, \ldots, Z_{m,q}\}$.

Additionally, in Chapter 3 we show all the families of graphs that we get when restricting the size of the family of forbidden subgraphs (Theorem 3.4). Concretely, Theorem 3.4 solves the following problem.

Problem 1.2. *Given $k \geq 1$, characterize all the families of connected graphs $\mathcal{F}$ such that $|\mathcal{F}| \leq k$ and every large enough $\mathcal{F}$-free connected graph is $K_{1,3}$-free.*

In Chapter 3 we show the proofs for Theorems 1.17 and 3.4.

Star-free graphs

A star is a graph of the form $K_{1,t}$ with $t \geq 3$. In particular, $K_{1,3}$ is a star. We consider a natural extension of Problem 1.1 to star-free graphs.

Problem 1.3. *Given $t \geq 3$, characterize all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free connected graph is $K_{1,t}$-free.*

In this thesis, we solve Problem 1.3 for all $t \geq 3$. The solution is expressed in the following theorem (see Section 4.2 for graph definitions).

Theorem 1.18. *Let $t \geq 3$ and $\mathcal{F}$ be a family of connected graphs. Then the following are equivalent.*

- *Every large enough $\mathcal{F}$-free connected graph is $K_{1,t}$-free*
- *$\mathcal{F} \leq \mathcal{F}_m^t(l, q)$ for some $m \geq 1$, $l \geq 4$ and $q \geq 3$,*

where $\mathcal{F}_m^t(l, q) = \{K_{1,l}, W_q^t\} \cup \{Y_{m+2}^t, Z_{1,q}^t, \ldots, Z_{m,q}^t\} \cup \mathcal{T}^t(q) \cup \mathcal{D}^t(q) \cup \mathcal{YZ}^t(m, q)$.

Theorem 1.18 gives for each $t \geq 3$, a complete characterization for the families of forbidden subgraphs that imply the property of being $K_{1,t}$-free. In other words, it gives a characterization of the families of forbidden subgraphs that implicitly forbid $K_{1,t}$-free.

Theorem 1.18 is a generalization of Theorem 1.17. In particular, $\mathcal{F}_m^3(l, q) = \mathcal{F}_m(l, q)$. We show the proof of Theorem 1.18 in Chapter 4.

Graphs having a perfect matching

A *perfect matching* in a graph G is a set of disjoint edges covering all the vertices of G. It is clear that having even order is a necessary condition for having a perfect matching.

The following result was proved independently by Sumner [32] and Las Vergnas [24].

Theorem 1.19 ([32],[24]). *Every $K_{1,3}$-free connected graph of even order has a perfect matching.*

Plummer et al.[30] showed that $K_{1,3}$ is essentially the only graph with that property.

Theorem 1.20 ([30]). *Let H be a connected graph. If every large enough H-free connected graph of even order has a perfect matching then H is an induced subgraph of $K_{1,3}$.*

Fujita et al.[18] extended Theorem 1.20 by considering two forbidden subgraphs.

Theorem 1.21 ([18]). *Let H_1, H_2 be a pair of connected graphs. If every large enough $\{H_1, H_2\}$-free connected graph of even order has a perfect matching then one of H_1 and H_2 is an induced subgraph of $K_{1,3}$.*

Ota et al.[27] continued this line of research and characterized the families of forbidden subgraphs containing at most three graphs. (see to Section 5.2 for graph definitions).

Theorem 1.22 ([27]). *For every $l \geq 4$ and $r \geq 3$, there is an $n_0 = n_0(l, r)$ such that every $\{K_{1,l}, P_4, Z_{1,r}\}$-free connected graph of even order at least n_0 has a perfect matching.*

Theorem 1.23 ([27]). *For every $l \geq 4$, $m \geq 3$ and $r \geq 3$, there is an $n_0 = n_0(l, m, r)$ such that every $\{K_{1,l}, Y_m, Z^-_{1,r}\}$-free connected graph of even order at least n_0 has a perfect matching.*

Theorem 1.24 ([27]). *Let $\mathcal{F}$ be a family of connected graphs with $|\mathcal{F}| \leq 3$. If every large enough $\mathcal{F}$-free connected graph of even order has a perfect matching, then*

- *there is an $H \in \mathcal{H}$ such that H is an induced subgraph of $K_{1,3}$, or*
- *there exist $l \geq 4$ and $r \geq 3$ such that $\mathcal{F} \leq \{K_{1,l}, P_4, Z_{1,r}\}$, or*
- *there exist $l \geq 4$, $m \geq 3$ and $r \geq 3$ such that $\mathcal{F} \leq \{K_{1,l}, Y_m, Z^-_{1,r}\}$.*

In this thesis, we complete this line of research started in the 1970's by removing the restriction on the size of the family and so characterizing all the families of forbidden subgraphs implying a perfect matching in large enough graphs. Concretely, we solve the following problem.

Problem 1.4. *Characterize all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free connected graph of even order has a perfect matching.*

The solution is expressed in the following theorem.

Theorem 1.25. *Let $\mathcal{F}$ be a family of connected graphs. Then the following are equivalent.*

- *Every large enough $\mathcal{F}$-free connected graph of even order has a perfect matching.*
- *$\mathcal{F} \leq \mathcal{F}_m(l, q)$ for some $m \geq 1$, $l \geq 4$ and $q \geq 3$,*

where $\mathcal{F}_m(l, q) = \{K_{1,l}, Y_{m+2}, W_q, Z_{1,q}, \ldots, Z_{m,q}\}$.

Additionally, in Chapter 5 we show all the families of graphs that we get when restricting the size of the family of forbidden subgraphs (Theorem 5.8). Concretely, Theorem 5.8 solves the following problem.

Problem 1.5. *Given $k \geq 1$, characterize all the families of connected graphs $\mathcal{F}$ such that $|\mathcal{F}| \leq k$ and every large enough $\mathcal{F}$-free connected graph of even order has a perfect matching.*

In Chapter 5 we show the proofs for Theorems 1.25 and 5.8.

Graphs having a near perfect matching

A *near perfect matching* in a graph G is a set of disjoint edges covering all but one vertex of G. It is clear that having odd order is a necessary condition for having a near perfect matching.

Let T_n be the graph obtained by attaching 2 independent vertices to each end of a path on n vertices. The following result was proved in [18].

Theorem 1.26 ([18]). *Let $\mathcal{F}$ be a family of triangle-free connected graphs. Then the following are equivalent.*

- *Every large enough $\mathcal{F}$-free connected graph of odd order has a near perfect matching.*
- *$\mathcal{F} \leq \{T_n : n \geq 1\}$.*

Theorem 1.26 is a partial solution to the following problem.

Problem 1.6. *Characterize all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free connected graph of odd order has a near perfect matching.*

In this thesis we generalize Theorem 1.26 and give a complete answer to Problem 1.6. The solution is expressed in the following theorem (see Section 6.2 for graph definitions).

Theorem 1.27 ([18]). *Let $\mathcal{F}$ be a family of connected graphs. Then the following are equivalent.*

- *Every large enough $\mathcal{F}$-free connected graph of odd order has a near perfect matching.*
- *$\mathcal{F} \leq \mathcal{F}(l, n, m, q)$ for some $l \geq 5$, $n \geq 1$, $m \geq 1$ and $q \geq 3$,*

where $\mathcal{F}(l, n, m, q) = \{K_{1,l}\} \cup \mathcal{V}(q) \cup \mathcal{Z}(m, q) \cup \mathcal{T}(n) \cup \mathcal{Y}(n, q) \cup \mathcal{D}(n, m, q) \cup \mathcal{L}(n, m, q) \cup \mathcal{W}(n, q) \cup \mathcal{M}(n, q) \cup \mathcal{J}(n, q)$.

We show the proof of Theorem 1.27 in Chapter 6.

T-tough graphs

Let t be a positive real number. We say that a connected graph G is t-tough if for every cutset S of G, $t \cdot \omega(G - S) \leq |S|$, where $\omega(G - S)$ is the number of connected components of $G - S$. The *toughness* of G is the maximum t for which G is t-tough. See [2] for a survey on toughness.

Broersma[5] proposed to study the relation between forbidden subgraphs and the resulting toughness of G. Toughness also has some relation to the other classes of graphs studied in the present work as shown by the following theorems. Theorem 1.29 is an easy observation.

Theorem 1.28 ([9]). *Every 1-tough graph with even order has a perfect matching.*

Theorem 1.29. *Every $K_{1,3}$-free connected graph is $\frac{1}{2}$-tough. More generally, every $K_{1,l}$-free connected graph is $\frac{1}{l-1}$-tough.*

Following the same ideas as in the classes of graphs previously mentioned, we propose the following problem.

Problem 1.7. *Given a positive real number t, characterize all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free connected graph is t-tough.*

In this thesis, we solve Problem 1.7 for every positive real number t. The solution is expressed in the following two theorems (see Section 7.2 for graph definitions).

Theorem 1.30. *Let $0 < t \leq \frac{1}{2}$ and let $\mathcal{F}$ be a family of connected graphs. Then the following are equivalent.*

- *Every large enough $\mathcal{F}$-free connected graph is t-tough.*
- *$\mathcal{F} \leq \mathcal{F}_n^A(l, m, q)$ for some $l \geq n+2$, $m \geq 1$ and $q \geq 3$,*

where $n = \lfloor \frac{1}{t} \rfloor$ and $\mathcal{F}_n^A(l, m, r) = \{K_{1,l}, Y_{m+2}^n, Z_{1,q}^n, \ldots, Z_{m,q}^n\}$.

Theorem 1.31. *Let $t > \frac{1}{2}$ and let $\mathcal{F}$ be a family of connected graphs. Then the following are equivalent.*

- *Every large enough $\mathcal{F}$-free connected graph is t-tough.*
- *$\mathcal{F} \leq \mathcal{F}^B(l, m, q)$ for some $l \geq 3$, $m \geq 4$ and $q \geq 3$,*

where $\mathcal{F}^B(l, m, r) = \{K_{1,l}, P_m, Z_q\}$.

In Chapter 7 we show the proofs of Theorems 1.30 and 1.31.

Chapter 2

Definitions and Preliminaries

In this chapter we give some basic definitions of Graph Theory that are used throughout this thesis. For an introduction to Graph Theory, see [8].

2.1 Graphs and subgraphs

A *graph* is an ordered pair (V, E) where V is a non-empty finite set and E is a set of unordered pairs of elements of V. The elements of V are called *vertices* and elements of E, *edges*. If $G = (V, E)$ is a graph, define $V(G) = V$ and $E(G) = E$. The *order* of a graph G is the size of $V(G)$. The *size* of a graph G is the size of $E(G)$.

Let G_1 and G_2 be two graphs. A function $f : V(G_1) \to V(G_2)$ is an *isomorphism* if it is bijective and for every $v_1, v_2 \in V(G_1)$, $v_1v_2 \in E(G_1)$ if and only if $f(v_1)f(v_2) \in E(G_2)$. In such a case, we say that G_1 and G_2 are *isomorphic* and we write $G_1 \cong G_2$.

If $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are two graphs, we say that H is a *subgraph* of G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. If G is a graph and V is a non-empty subset of $V(G)$, the subgraph of G *induced* by V is (V, E), where E is the subset of $E(G)$ whose elements contain only elements of V. Such a subgraph is denoted by $G[V]$. If G and H are two graphs, we say that H is an *induced subgraph* of G if there is a non-empty set $V \subseteq V(G)$ such that $G[V] \cong H$. In such a case, we write $H \preceq G$. If $H \preceq G$, we do not distinguish between H and the vertex set of G defining H.

Let G be a graph. If H is a graph, we say that G is H*-free* if $H \npreceq G$. In such a case, we also say that H is *forbidden* in G. If $\mathcal{F}$ is a family of graphs, we say that G is $\mathcal{F}$-free if G is H-free for every $H \in \mathcal{F}$. In such a case, we also say that $\mathcal{F}$ is *forbidden* in G.

If G is a graph and $V \subset V(G)$, the graph $G - V$ is $G[V(G) - V]$. A subgraph H of a graph G is said to be *spanning* if $V(H) = V(G)$.

2.2 Paths, cycles and connectivity

Let G be a graph. A *track* in G is a sequence $v_1v_2\cdots v_k$ ($k \geq 1$) such that for all $1 \leq i \leq k-1$, $v_iv_{i+1} \in E(G)$. The *length* of a track $v_1v_2\cdots v_k$ is $k-1$. A *path* is a track whose vertices are pairwise distinct. The length of a path is the length of the underlying track. If $P = v_1v_2\cdots v_k$ is a path in G, the vertices v_1 and v_k are the *ends* of P. We also say that P is a path between v_1 and v_k. A *cycle* in G is a track $v_1v_2\cdots v_k$ of length at least three such that $v_1 = v_k$ and $v_2\cdots v_{k-1}$ are pairwise distinct. The length of a cycle is the length of the underlying track.

Let G be a graph. A path P in G is said to be an *induced path* if there is no edge of G between any two non-consecutive vertices of P. A *chord* in a cycle C of G is an edge of G between two non-consecutive vertices of C. A cycle is said to be *chordless* if it has no chords.

For $n \geq 1$, denote by P_n and C_n the graphs such that $V(P_n) = V(C_n) = \{v_1, \ldots, v_n\}$, $E(P_n) = \{v_iv_{i+1} : 1 \leq i \leq n-1\}$ and $E(C_n) = E(P_n) \cup \{v_nv_1\}$. The graphs P_n and C_n are called the path and the cycle of order n, respectively.

Let G be a graph, and $v, w \in V(G)$ be two vertices of G. If there is at least one path in G between v and w, define the *distance* between v and w as the length of a path between v and w of minimum length.

A graph is *connected* if there is a path between every pair of distinct vertices. If $k \geq 1$, a graph G is k-connected if $|V(G)| \geq k+1$ and for every $V \subseteq V(G)$ with $|V| = k-1$, $G - V$ is connected. The *connectivity* of a graph G is the maximum k for which G is k-connected, and we denote it by $\kappa(G)$. Notice that a graph of order at least two is connected if and only if it is 1-connected.

The *connected components* of a graph G are the maximal connected subgraph of G. Define $\omega(G)$ as the number of connected components of G. Notice that if G is connected, then $\omega(G) = 1$ and G itself is its unique connected component.

If G is a graph, a set $V \subset V(G)$ is a *cutset* of G if $G - V$ is not connected.

2.3 Neighborhood and vertex degree

Let G be a graph. Two vertices v_1, v_2 of G are *adjacent* if $v_1v_2 \in E(G)$. If $e = v_1v_2$ is an edge of G, we say that e is *incident* with v_1 and with v_2, and that v_1 and v_2 are the *ends* of e. Two edges of G are *disjoint* if they do not share an end.

Let G be a graph. If $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is the set of vertices adjacent with v. If $v \in V(G)$ and $i \geq 0$, define $N_G^i(v)$ as the set of vertices at distance i from v. Notice that $N_G^0(v) = \{v\}$ and $N_G^1(v) = N_G(v)$. If the graph G is clear from the context, we write $N(v)$ and $N^i(v)$ for $N_G(v)$ and $N_G^i(v)$, respectively.

Let G be a graph. The *degree* of a vertex $v \in V(G)$ is the size of $N_G(v)$. The minimum degree of G is the minimum degree over all the vertices of G, which is denoted by $\delta(G)$. Define similarly the maximum degree, which is denoted by $\Delta(G)$.

For $k \geq 0$, we say that a graph is *k-regular* if all its vertices have degree k. A graph is just *regular* if all its vertices have the same degree.

2.4 Complete graphs and Ramsey numbers

A graph G is *complete* if every two distinct vertices of G are adjacent. The complete graph of order n is denoted by K_n.

Let G be a graph. A *clique* of G is a non-empty set of pairwise adjacent vertices. An *independent set* of G is a non-empty set of pairwise non-adjacent vertices of G. Independent sets are also called *stable sets*.

For $l, r \geq 1$, the *Ramsey number* $R(l, r)$ is the minimum positive integer n such that any graph of order at least n contains either an independent set of size l or a clique of size r. The Ramsey number $R(l, r)$ exists for every pair of positive integers l and r (see for example Theorem 12.2 of [8]).

A graph G is said to be *bipartite* if there are two disjoint sets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$ and both A and B are independent sets of G. In other words, every edge of G has one end in A and the other one end in B. The sets A and B are called the *partite sets* of G.

A graph is *complete bipartite* if it is bipartite and every vertex in one partite set is adjacent to every vertex in the other partite set. For $n, m \geq 1$, denote by $K_{n,m}$ the complete bipartite graph with partite sets of sizes n and m.

2.5 Families of graphs

A family of graphs $\mathcal{F}$ is said to be *redundant* if there are two different graphs $H_1, H_2 \in \mathcal{F}$ such that $H_1 \preceq H_2$.

When we study the families of forbidden subgraphs implying some property in graphs, it is easy to see that we can restrict ourselves to considering only non-redundant families, as the following proposition shows.

Proposition 2.1. *Let $\mathcal{F}$ be a family of graphs. Let $\mathcal{F}_R = \{ H_1 \in \mathcal{F}\colon \exists H_2 \in \mathcal{F}$ such that $H_1 \neq H_2$ and $H_1 \preceq H_2 \}$ and $\mathcal{F}' = \mathcal{F} - \mathcal{F}_R$. Then $\mathcal{F}'$ is non-redundant and a graph G is $\mathcal{F}$-free if and only if G is $\mathcal{F}'$-free.*

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two families of graphs, we say that $\mathcal{F}_1 \leq \mathcal{F}_2$ if for each $H_2 \in \mathcal{F}_2$, there is an $H_1 \in \mathcal{F}_1$ such that H_1 is an induced subgraph of H_2. It is easy to see that the relation "$\leq$" defines a partial order in the set of non-redundant families of graphs. Furthermore, if $\mathcal{F}_1 \leq \mathcal{F}_2$ then any $\mathcal{F}_1$-free graph is also an $\mathcal{F}_2$-free graph (see for example Lemma 3 of [18]).

2.6 Particular classes of graphs

The *claw* is the graph $K_{1,3}$. A graph is *claw-free* if it is $K_{1,3}$-free. See [12] for a survey on claw-free graphs.

Let G be a graph. A *matching* in G is a non-empty set of disjoint edges of G. A *perfect matching* of G is a matching of G covering all its vertices. A *near perfect matching* of G is a matching of G covering all but one of its vertices. It is easy to see that if G has a perfect matching (near perfect matching) then G has even (odd) order. The *deficiency* of G is $|V(G)| - 2|M|$ where M is a maximum matching of G, which we denote by $\operatorname{def}(G)$. From the definition of deficiency, it is clear that G has a perfect matching (near perfect matching) if and only if $\operatorname{def}(G) = 0$ ($\operatorname{def}(G) = 1$).

If t is a real positive number, a connected graph G is said to be t-tough if for every cutset S of G, $t \cdot \omega(G - S) \leq |S|$. The *toughness* of G is the maximum t for which G is t-tough, which it is denoted by $\tau(G)$. See [2] for a survey on toughness.

A graph is said to be *Hamiltonian* if there is a cycle passing through all its vertices. Such a cycle is called a *Hamiltonian cycle*. A graph is said to be *traceable* if there is a path passing through all its vertices. Such path is called a *Hamiltonian path*. It is easy to see that every Hamiltonian graph is 2-connected and every traceable graph is connected.

There is a long standing conjecture relating toughness and Hamiltonicity.

Conjecture 2.1 ([9]). *There is a positive real number t_0 such that every t_0-tough graph is Hamiltonian.*

For $k \geq 1$, a k-factor of a graph G is a k-regular spanning subgraph of G. A 1-factor is the same as a perfect matching. A connected 2-factor is the same as a Hamiltonian cycle.

2.7 Preliminaries

Let G be a graph and let $S \subseteq V(G)$. For $S' \subseteq S$, define $B_S(S') = \{v \in V(G) : N(v) \cap S = S'\}$.

If we have two sets $N, S \subseteq V(G)$, the set $\{\ N \cap B_S(S') : \ S' \subseteq S\ \}$ gives a "partition" of the vertices of N according to how they are connected to S. If the size of each of those sets is bounded, we can bound the size of N as the following proposition shows.

Proposition 2.2. *Let G be a graph and $N, S \subseteq V(G)$. If there is a constant k such that for every $S' \subseteq S$, $|N \cap B_S(S')| \leq k$, then $|N| \leq 2^{|S|} \cdot k$.*

Proof. From the definition of $B_S(S')$, we have that $N = \bigcup_{S' \subseteq S} (N \cap B_S(S'))$, where the union is disjoint. Then $|N| = \sum_{S' \subseteq S} |N \cap B_S(S')| \leq |\{S' : S' \subseteq S\}| \cdot k = 2^{|S|} \cdot k$. □

We use the definition of $B_S(S')$ and Proposition 2.2 in several proofs throughout this thesis. Concretely, we use it in Chapters 3, 4, 5, 6 and 7.

Chapter 3

Claw-free graphs

The graph $K_{1,3}$ is also called the *claw*. A graph is *claw-free* if it does not contain a $K_{1,3}$ as an induced subgraph. Claw-free graphs have been widely studied in the literature. See [12] for a survey on claw-free graphs.

In this chapter, we study the relation between claw-free graphs and forbidden induced subgraphs. The main result in this chapter is Theorem 3.5, which shows a characterization of all families of forbidden subgraphs that imply the property of being claw-free in connected graphs of large enough order. All the new results we prove in this chapter can be found in [17].

3.1 Introduction

If we have several families of forbidden subgraphs implying some given property, it is important to compare them to understand which families lead to more general results. Concretely, if we have two families of graphs $\mathcal{F}_1$ and $\mathcal{F}_2$, and every $\mathcal{F}_1$-free graph is also $\mathcal{F}_2$-free, then we can say that $\mathcal{F}_2$ is more general, in the sense that a result that states that all $\mathcal{F}_2$-free graphs satisfy some property is more general than one that says that all $\mathcal{F}_1$-free graphs satisfy the same property.

To do such comparisons, one usually uses the relation "$\mathcal{F}_1 \leq \mathcal{F}_2$" that we defined in Section 1.2 (see also Section 2.5). But the authors of [1] showed that sometimes such a comparison might not be enough. Consider the following theorem about graphs having a 2-factor (see Section 3.2 for graph definitions). Remember that a 2-factor is a 2-regular spanning subgraph.

Theorem 3.1 ([1]). *Let G be a connected graph with $\delta(G) \geq 2$ and $\Delta(G) \geq 3$.*

(i) If G is $\{Z_{1,3}, K_{1,3}\}$-free then G has a 2-factor.

(ii) If G is $\{Z_{1,3}, Y_3, W_2^3, K_{2,3}\}$-free and $|V(G)| \geq 9$ then G has a 2-factor.

Because $Z_{1,3}$ is an induced subgraph of it self, and all three graphs Y_3, W_2^3 and $K_{2,3}$ contain a $K_{1,3}$ as an induced subgraph, we can say that (ii) is more general than (i). But on the other hand, we have the following result.

Theorem 3.2 ([1])**.** *Let G be a connected graph with $\delta(G) \geq 2$ and $\Delta(G) \geq 3$. If G is $\{Z_{1,3}, Y_3, W_2^3, K_{2,3}\}$-free and $|V(G)| \geq 9$, then G is $K_{1,3}$-free.*

Theorem 3.2 says that $\{Z_{1,3}, K_{1,3}\}$-free graphs and $\{Z_{1,3}, Y_3, W_2^3, K_{2,3}\}$-free are essentially (under some conditions) the same. This is not clear just by looking at the graphs in the families.

Another interesting point about Theorem 3.2 is that even though no graph of the family $\mathcal{H} = \{Z_{1,3}, Y_3, W_2^3, K_{2,3}\}$ is an induced subgraph of $K_{1,3}$, when considering the $\mathcal{H}$-free graphs under certain conditions, the graph $K_{1,3}$ is also forbidden. The authors of [1] were interested in finding a family of forbidden subgraphs implying a 2-factor that does not contain a star. But even though there is no star in $\{Z_{1,3}, Y_3, W_2^3, K_{2,3}\}$, by Theorem 3.2 it is somehow implicitly forbidden. That is why the authors of [1] called this phenomenon *implicit forbiddance*.

In the view of the previous results, in order to get more information about the implicit relation between families of forbidden subgraphs, it is important to research further this phenomenon. As a first step, we consider the case of $K_{1,3}$-free graphs, also in an effort to try to extend Theorem 3.2. We can state the problem in the following way.

Problem 3.1. *Characterize all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free connected graph is $K_{1,3}$-free.*

In this chapter, we give a full answer to Problem 3.1. In Chapter 4, we do a generalization to star-free graphs.

The rest of the chapter is organized as follows. In Section 3.2 we make all needed definitions and present our main results. In Sections 3.3 and 3.4 we give the proofs for those results. In Section 3.5, we show an application of our results. Finally, in Section 3.6 we make some discussion and propose some open problems.

3.2 Definitions and main results

Define $\mathbf{G}$ as the set of all non-redundant families of connected graphs. Define $\mathbf{H}$ as the set of families $\mathcal{H} \in \mathbf{G}$ such that there is a constant $n_0 = n_0(\mathcal{H})$ with the property that all $\mathcal{H}$-free connected graphs G with $|V(G)| \geq n_0$ are $K_{1,3}$-free. Then, our problem is reduced to finding all the elements in the set $\mathbf{H}$.

To state our results we define the following graphs (see Figure 3.1).

- Y_m is a path on m vertices with two extra vertices attached to the first vertex of the path. The last vertex of the path is called the tail of Y_m.
- W_q^h is the graph obtained by joining a K_q with h extra vertices.
- T_q is the graph obtained by identifying two degree one vertices of a claw with the two "extra" vertices of a $W_{2,q}$.
- T_q^- is T_q minus its only vertex of degree one.
- D_q is the graph obtained by attaching an extra vertex to a "non-added vertex" of a $W_{2,q}$.
- $Z_{m,r}^-$ is the graph obtained by identifying a vertex of a K_r with the end vertex of a path of order $m+1$.
- $Z_{m,r}$ is the graph obtained by identifying a vertex of a K_r with the tail of a Y_m.

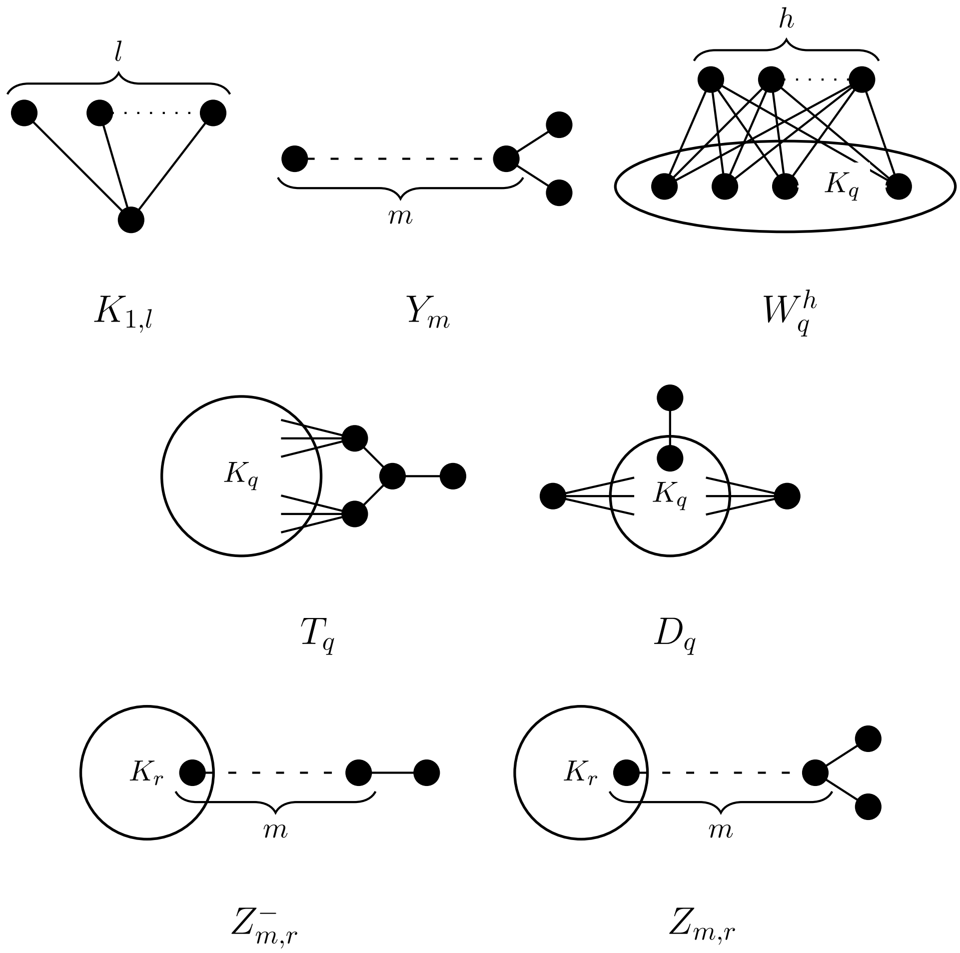


Figure 3.1: Some forbidden subgraphs

Define the following families of graphs.

- $\mathcal{H}_i^A(l,q,r) = \{K_{1,l}, Y_{i+2}, W_q^2, Z_{1,r}, \ldots, Z_{i,r}\}$ (for $i \geq 1$).
- $\mathcal{H}_i^B(l,m,q,r) = \{K_{1,l}, Y_m, W_q^2, Z_{1,r}, \ldots, Z_{i-1,r}, Z_{i,r}^-\}$ (for $i \geq 2$).
- $\mathcal{H}_i^C(l,q,r) = \{K_{1,l}, Y_{i+2}, W_q^3, D_q, T_q, Z_{1,r}, \ldots, Z_{i,r}\}$ (for $i \geq 1$).
- $\mathcal{H}_i^D(l,m,q,r) = \{K_{1,l}, Y_m, W_q^3, D_q, T_q, Z_{1,r}, \ldots, Z_{i-1,r}, Z_{i,r}^-\}$ (for $i \geq 3$).

Define the following subsets of $\mathbf{G}$.

- $\mathbf{F}_1 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,3}\}\ \}$.
- $\mathbf{F}_3 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, Y_m, K_r\}$ for some $l \geq 4$, $m \geq 3$, $r \geq 3\}$.
- $\mathbf{F}_4 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, Y_m, W_q^3, Z_{1,r}^-\}$ for some $l \geq 4$, $m \geq 3$, $q \geq 2$, $r \geq 3\}$.
- $\mathbf{F}_5 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, P_4, W_q^3, D_q, Z_{1,r}\}$ for some $l \geq 4$, $q \geq 2$, $r \geq 3\}$.
- $\mathbf{F}_6 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, Y_m, W_q^3, D_q, Z_{1,r}, Z_{2,r}^-\}$ for some $l \geq 4$, $m \geq 4$, $q \geq 2$, $r \geq 3\}$.
- $\mathbf{F}_i^A = \{\ \mathcal{H} \in \mathbf{G}$: $\mathcal{H} \leq \mathcal{H}_i^A(l,q,r)$ for some $l \geq 4$, $q \geq 2$, $r \geq 3\}$ $(i \geq 1)$.
- $\mathbf{F}_i^B = \{\ \mathcal{H} \in \mathbf{G}$: $\mathcal{H} \leq \mathcal{H}_i^B(l,m,q,r)$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$, $r \geq 3\}$ $(i \geq 2)$.
- $\mathbf{F}_i^C = \{\ \mathcal{H} \in \mathbf{G}$: $\mathcal{H} \leq \mathcal{H}_i^C(l,q,r)$ for some $l \geq 4$, $q \geq 2$, $r \geq 3\}$ $(i \geq 1)$.
- $\mathbf{F}_i^D = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \mathcal{H}_i^D(l,m,q,r)$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$, $r \geq 3\}$ $(i \geq 3)$.

First, we show that the families in the sets $\mathbf{F}_1$, $\mathbf{F}_3$, $\mathbf{F}_4$, $\mathbf{F}_5$, $\mathbf{F}_6$, $\mathbf{F}_i^A$, $\mathbf{F}_i^B$, $\mathbf{F}_i^C$, $\mathbf{F}_i^D$ actually families of forbidden subgraphs that implicitly forbid $K_{1,3}$.

Theorem 3.3. *The following statements hold.*

1. $\mathbf{F}_i \subseteq \mathbf{H}$ *for all* $i \in \{1,3,4,5,6\}$.
2. $\mathbf{F}_i^A \subseteq \mathbf{H}$ *for all* $i \geq 1$.
3. $\mathbf{F}_i^B \subseteq \mathbf{H}$ *for all* $i \geq 2$.
4. $\mathbf{F}_i^C \subseteq \mathbf{H}$ *for all* $i \geq 1$.
5. $\mathbf{F}_i^D \subseteq \mathbf{H}$ *for all* $i \geq 3$.

Then, we show that these families are exactly the maximal families of forbidden subgraph when the size of the family is limited to some positive integer k.

Theorem 3.4. *Let $k \geq 1$ and let $\mathcal{H} \in \mathbf{H}$ with $|\mathcal{H}| \leq k$. Then*

- $\mathcal{H} \in \mathbf{F}_i$ *for some* $i \in \{1, 3, 4, 5, 6\}$ *with* $i \leq k$ *or*
- $\mathcal{H} \in \mathbf{F}_i^A$ *for some* $1 \leq i \leq k-3$ *or*
- $\mathcal{H} \in \mathbf{F}_i^B$ *for some* $2 \leq i \leq k-3$ *or*
- $\mathcal{H} \in \mathbf{F}_i^C$ *for some* $1 \leq i \leq k-5$ *or*
- $\mathcal{H} \in \mathbf{F}_i^D$ *for some* $3 \leq i \leq k-5$.

Finally, we prove that the families in $\mathbf{F}_i^C$ give the characterization of families of forbidden subgraphs that implicitly forbid $K_{1,3}$. Theorem 3.5 is our main result in this chapter.

Theorem 3.5. *$\mathcal{H} \in \mathbf{H}$ if and only if $\mathcal{H} \in \mathbf{F}_i^C$ for some $i \geq 1$. That is,* $\mathbf{H} = \bigcup_{i \geq 1} \mathbf{F}_i^C$.

3.3 Proof of Theorems 3.3 and 3.5

First we show that it is enough to prove Theorem 3.3 only for $\mathbf{F}_i^C$ $(i \geq 1)$.

Lemma 3.6. *The following statements hold:*

(1) $\mathbf{F}_1 \subseteq \mathbf{F}_1^C$, $\mathbf{F}_3 \subseteq \mathbf{F}_4$, $\mathbf{F}_4 \subseteq \mathbf{F}_6$, $\mathbf{F}_5 \subseteq \mathbf{F}_1^C$ *and* $\mathbf{F}_6 \subseteq \mathbf{F}_3^D$.

(2) *Let $i \geq 1$, then $\mathbf{F}_i^A \subseteq \mathbf{F}_i^C$.*

(3) *Let $i \geq 2$, then $\mathbf{F}_i^B \subseteq \mathbf{F}_j^A$ for some $j \geq 1$.*

(4) *Let $i \geq 3$, then $\mathbf{F}_i^D \subseteq \mathbf{F}_j^C$ for some $j \geq 1$.*

Proof. Statements (1) and (2) are easy to verify.

Proof of (3): Let $i \geq 2$ and $\mathcal{H} \in \mathbf{F}_i^B$. Since $\mathcal{H} \leq \mathcal{H}_i^B(l, m, q, r)$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$ and $r \geq 3$, we have that $\mathcal{H} \leq \{Y_m\}$ for some $m \geq i+3$. Since $Z_{i,r}^- \preceq Z_{h,r}$ for all $h \geq i$ and all $r \geq 3$, then $\mathcal{H} \in \mathbf{F}_{m-2}^A$.

Proof of (4): Let $i \geq 3$ and $\mathcal{H} \in \mathbf{F}_i^D$. Since $\mathcal{H} \leq \mathcal{H}_i^D(l, m, q, r)$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$ and $r \geq 3$, we have that $\mathcal{H} \leq \{Y_m\}$ for some $m \geq i+3$. Since $Z_{i,r}^- \preceq Z_{h,r}$ for all $h \geq i$ and all $r \geq 3$, then $\mathcal{H} \in \mathbf{F}_{m-2}^C$. □

Below we prove several lemmas that are the main components of the proof of Theorem 3.3.

Lemma 3.7. *Let G be a connected graph with an induced $K_{1,3}$ of center x_0. If G is Y_m-free for some $m \geq 3$ then $N^{m+1}(x_0) = \emptyset$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = 3$ such that $\{x_0\} \cup Y$ is an induced $K_{1,3}$ in G. Suppose that $N^{m+1}(x_0) \neq \emptyset$. We will show that G contains a Y_m, which is a contradiction.

Let $P = x_0 x_1 \cdots x_{m+1}$ be an induced path of G with $x_i \in N^i(x_0)$ for all $0 \leq i \leq m+1$. Notice that $N^j(x_0) \cap N(Y) = \emptyset$ for all $3 \leq j \leq m+1$. Otherwise, an element $v \in N^j(x_0) \cap N(Y)$ would have a path of length 2 to x_0 (passing through some element of Y), contradicting that $v \in N^j(x_0)$. Then $N(Y) \cap P \subseteq \{x_0, x_1, x_2\}$.

Let $Y_1 = N(x_1) \cap Y$ and $Y_2 = N(x_2) \cap Y$. If $|Y_2| \geq 2$, then $Y_2 \cup \{x_2, \ldots, x_{m+1}\}$ contains a Y_m. If $|Y_2| = 1$, then $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_{m-1}\}$ is a Y_m.

Suppose now that $|Y_2| = 0$. If $|Y_1| \geq 2$, then $Y_1 \cup \{x_1, \ldots, x_m\}$ contains a Y_m. If $|Y_1| \leq 1$, then $(Y - Y_1) \cup \{x_0, \ldots, x_{m-1}\}$ contains a Y_m. □

Lemma 3.8. *Let G be a connected graph with an induced $K_{1,3}$ of center x_0. Suppose that G is $\{K_{1,l}, Z_{1,r}, D_q, W_q^3\}$-free for some $l \geq 4$, $r \geq 3$ and $q \geq 2$. Then $|N(x_0)| < 8 \cdot R(l, \max(r,q))$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = 3$, such that $\{x_0\} \cup Y$ is an induced $K_{1,3}$ in G. Let $Y' \subseteq Y$. We will show that $|N(x) \cap B_Y(Y')| < R(l, \max(r,q))$, and since $|Y| = 3$, by Proposition 2.2 we get that $|N(x_0)| \leq 2^3 \cdot R(l, \max(r,q))$.

If $|Y'| \leq 1$, then $|Y - Y'| \geq 2$. Then $|N(x_0) \cap B_Y(Y')| < R(l,r)$, since otherwise $(Y - Y') \cup \{x_0\} \cup (N(x_0) \cap B_Y(Y'))$ contains a $Z_{1,r}$ or a $K_{1,l}$.

If $|Y'| = 2$, then $|Y - Y'| = 1$. Then $|N(x_0) \cap B_Y(Y')| < R(l,q)$, since otherwise $Y' \cup (Y - Y') \cup \{x_0\} \cup (N(x_0) \cap B_Y(Y'))$ contains a D_q or a $K_{1,l}$.

If $|Y'| = 3$, then $|N(x_0) \cap B_Y(Y')| < R(l,q)$, since otherwise $Y' \cup (N(x_0) \cap B_Y(Y'))$ contains a W_q^3 or a $K_{1,l}$. □

Lemma 3.9. *Let G be a connected graph with an induced $K_{1,3}$ of center x_0. Suppose that G is $\{K_{1,l}, Z_{1,r}, Z_{2,r}, W_q^3, T_q\}$-free for some $l \geq 4$, $r \geq 3$ and $q \geq 2$. Then $|N^2(x_0)| < 8 \cdot R(l, \max(r,q)) \cdot |N(x_0)|$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = 3$ such that $\{x_0\} \cup Y$ is an induced $K_{1,3}$ in G. Let $x_1 \in N(x_0)$. Let $Y' \subseteq Y$. Call $N = N^2(x_0) \cap N(x_1)$. By Proposition 2.2, it suffices show that $|N \cap B_Y(Y')| < R(l, \max(r,q))$.

If $|Y'| = 1$, then $|Y - Y'| = 2$. Then $|N \cap B_Y(Y')| < R(l,r)$, since otherwise $(Y - Y') \cup \{x_0\} \cup Y' \cup (N \cap B_Y(Y'))$ contains a $Z_{2,r}$ or a $K_{1,l}$.

If $|Y'| = 2$, then $|Y - Y'| = 1$. Then $|N \cap B_Y(Y')| < R(l,q)$, since otherwise $(Y - Y') \cup \{x_0\} \cup Y' \cup (N \cap B_Y(Y'))$ contains a T_q or a $K_{1,l}$.

If $|Y'| = 3$, then $|N \cap B_Y(Y')| < R(l,q)$, since otherwise $Y' \cup (N \cap B_Y(Y'))$ contains a W_q^3 or a $K_{1,l}$.

Suppose now that $|Y'| = 0$, that is $N \cap B_Y(Y') \cap N(Y) = \emptyset$. Notice that if $x_1 \in Y$, then $N \cap B_Y(Y') = \emptyset$. Then we may suppose that $x_1 \notin Y$. Let $Y_1 = Y \cap N(x_1)$.

If $|Y_1| \geq 2$, then $|N \cap B_Y(Y')| < R(l,r)$, since otherwise $Y_1 \cup \{x_1\} \cup (N \cap B_Y(Y'))$ contains a $Z_{1,r}$ or a $K_{1,l}$.

If $|Y_1| \leq 1$, then $|Y - Y_1| \geq 2$. Then $|N \cap B_Y(Y')| < R(l,r)$, since otherwise $(Y - Y_1) \cup \{x_0, x_1\} \cup (N \cap B_Y(Y'))$ contains a $Z_{2,r}$ or a $K_{1,l}$. □

Lemma 3.10. *Let G be a connected graph with an induced $K_{1,3}$ of center x_0. Let $i \geq 2$ and suppose that G is $\{K_{1,l}, Z_{i-1,r}, Z_{i,r}, Z_{i+1,r}\}$-free for some $l \geq 4$ and $r \geq 3$. Then $|N^{i+1}(x_0)| < R(l,r) \cdot |N^i(x_0)|$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = 3$ such that $\{x_0\} \cup Y$ is an induced $K_{1,3}$ in G. Let $x_i \in N^i(x_0)$ and let $P = x_0 x_1 \cdots x_i$ be an induced path with $x_j \in N^j(x_0)$ for all $0 \leq j \leq i$. Let $N = N^{i+1}(x_0) \cap N(x_i)$. We will show that $|N| < R(l,r)$.

Let $Y_1 = Y \cap N(x_1)$ and $Y_2 = Y \cap N(x_2)$. As in the proof of Lemma 3.7, $N(Y) \cap P \subseteq \{x_0, x_1, x_2\}$.

If $|Y_2| \geq 2$, then $|N| < R(l,r)$, since otherwise $Y_2 \cup \{x_2, \ldots, x_i\} \cup N$ contains a $Z_{i-1,r}$ or a $K_{1,l}$. If $|Y_2| = 1$, then $|Y - Y_2| = 2$. Then $|N| < R(l,r)$, since otherwise $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_i\} \cup N$ contains a $Z_{i+1,r}$ or a $K_{1,l}$.

Suppose now that $|Y_2| = 0$, that is $N(x_2) \cap Y = \emptyset$.

If $|Y_1| \geq 2$, then $|N| < R(l,r)$, since otherwise $Y_1 \cup \{x_1, \ldots, x_i\} \cup N$ contains a $Z_{i,r}$ or a $K_{1,l}$. If $|Y_1| \leq 1$, then $|Y - Y_1| \geq 2$. Then $|N| < R(l,r)$, since otherwise $(Y - Y_1) \cup \{x_0, \ldots, x_i\} \cup N$ contains a $Z_{i+1,r}$ or a $K_{1,l}$. □

We show now the proof of Theorem 3.3.

Proof of Theorem 3.3. By Lemma 3.6, it is enough to show that $\mathbf{F}_i^C \subseteq \mathbf{H}$ for all $i \geq 1$. Let $m \geq 1$ and $\mathcal{H} \in \mathbf{F}_m^C$. Let $l \geq 4$, $q \geq 2$ and $r \geq 3$ such that $\mathcal{H} \leq \mathcal{H}_m^C(l,q,r)$.

Let G be a $\mathcal{H}$-free connected graph with an induced $K_{1,3}$ of center x. We will show that $|V(G)|$ is bounded by a function depending only on l, m, q and r.

Note that since G is Y_{m+2}-free, G is also $Z_{i,r}$-free for all $i \geq m+1$. Hence, G is $Z_{i,r}$-free for all $i \geq 1$. Thus, G satisfies all the conditions of Lemmas 3.7, 3.8, 3.9 and 3.10.

By Lemma 3.7, $N^{m+3}(x) = \emptyset$. Then we only need to show that $|N^i(x)|$ is bounded for all $1 \leq i \leq m+2$. By Lemmas 3.8 and 3.9, $|N(x)|$ and $|N^2(x)|$ are bounded.

By Lemma 3.10, $|N^{i+1}(x)| < R(l,r) \cdot |N^i(x)|$ for $2 \leq i \leq m+1$. Using an inductive argument, we get that for all $3 \leq i \leq m+2$, $|N^i(x)| < R(l,r)^{i-2} \cdot |N^2(x)|$. By Lemmas 3.8 and 3.9, we conclude that for all $3 \leq i \leq m+2$, $|N^i(x)| < R(l,r)^{i-2} \cdot 8^2 \cdot R(l, \max(r,q))^2$. □

We finish this section by showing the proof of Theorem 3.5, our main theorem in this chapter.

Proof of Theorem 3.5. By Theorem 3.3, we already know that for $i \geq 1$, every family of graphs in $\mathbf{F}_i^C$ is also in $\mathbf{H}$.

Let $\mathcal{H} \in \mathbf{H}$. Then there is a positive integer n_0 such that every $\mathcal{H}$-free connected graph of order at least n_0 is claw-free. Let n be an integer such that $n \geq \max(n_0, 4)$.

Consider the family $\mathcal{H}' = \mathcal{H}_n^C(n, n, n)$. All the graphs in $\mathcal{H}'$ are connected graphs of order at least n_0 containing an induced claw. Then it must be that no graph of $\mathcal{H}'$ is $\mathcal{H}$-free. In other words, for each $H' \in \mathcal{H}'$, there is an $H \in \mathcal{H}$ such that $H \preceq H'$. This is exactly the definition of $\mathcal{H} \leq \mathcal{H}'$. Since $\mathcal{H}' \in \mathbf{F}_n^C$, we conclude that $\mathcal{H}$ is also in $\mathbf{F}_n^C$. □

3.4 Proof of Theorem 3.4

First, we prove two lemmas that deal with the inductive part of the proof of Theorem 3.4.

Lemma 3.11. *Let $k \geq 4$ and let $\mathcal{H} \in \mathbf{H}$ with $|\mathcal{H}| \leq k$. Suppose that $\mathcal{H} \not\leq \{K_{1,3}\}$, $\mathcal{H} \notin \mathbf{F}_j^A$ for all $1 \leq j \leq k-3$ and $\mathcal{H} \notin \mathbf{F}_j^B$ for all $2 \leq j \leq k-3$. Suppose also that there are graphs $B_1, B_2, B_3, H_1 \in \mathcal{H}$ such that*

- *$B_1 = K_{1,l}$ for some $l \geq 4$.*
- *$B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq 3$.*
- *$B_3 = W_q^2$ for some $q \geq 2$.*
- *$H_1 = Z_{1,r_1}$ for some $r_1 \geq 3$.*

Then there are graphs $H_2, \ldots, H_{k-3}$ in $\mathcal{H}$ and integers $r_2, \ldots, r_{k-3}$ such that for all $2 \leq i \leq k-3$, $H_i = Z_{i,r_i}$ and $r_i \geq 3$. Additionally, $m \geq k$.

Proof. We prove by induction on i that there exists a graph $H_i \in \mathcal{H}$ with $H_i = Z_{i,r_i}$ for some $r_i \geq 3$.

Let $2 \leq i \leq k-3$ and suppose that there are graphs $H_1, \ldots, H_{i-1}$ in $\mathcal{H}$ such that $H_j = Z_{j,r_j}$ for some $r_j \geq 3$ and all $1 \leq j \leq i-1$. We will prove that there is a graph $H_i \in \mathcal{H}$ such that $H_i = Z_{i,r_i}$ for some $r_i \geq 3$.

Let $r' = \max(r_1, \ldots, r_{i-1})$. Since $\mathcal{H} \notin \mathbf{F}_{i-1}^A$ and $\mathcal{H} \leq \{K_{1,l}, W_q^2, Z_{1,r'}, \ldots, Z_{i-1,r'}\}$, then $\mathcal{H} \not\leq \{Y_{i+1}\}$. In particular, $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq i+2$.

Since $\mathcal{H} \notin \mathbf{F}_i^B$ and $\mathcal{H} \leq \{K_{1,l}, Y_m, W_q^2, Z_{1,r'}, \ldots, Z_{i-1,r'}\}$, then $\mathcal{H} \not\leq \{Z_{i,r}^-\}$ for all $r \geq 3$.

Since $\mathcal{H} \in \mathbf{H}$, there is a positive integer $n_0 = n_0(\mathcal{H})$ such that every $\mathcal{H}$-free connected graph of order at least n_0 is claw-free. Let $n = \max(n_0, 3)$.

Consider $G = Z_{i,n}$. Since G contains an induced claw, G must contain some graph in $\mathcal{H}$ as an induced subgraph. Since G contains neither $K_{1,4}$, P_{i+3} nor W_2^2 then $B_j \not\preceq G$ for all $j \in \{1, 2, 3\}$. Furthermore, since $Z_{j,3} \not\preceq G$ for all $1 \leq j \leq i-1$, then $H_j \not\preceq G$ for all $1 \leq j \leq i-1$. Then there must be some other graph $H_i \in \mathcal{H}$ such that $H_i \preceq G$.

Since $H_i \not\preceq K_{1,3}$, $H_i \not\preceq Y_{i+1}$ and that $H_i \not\preceq Z_{i,r}^-$ for all $r \geq 3$, then $H_i = Z_{i,r_i}$ for some $r_i \geq 3$. Notice that if $r_i = 2$, then it would contradict that $H_i \not\preceq Y_{i+1}$.

This concludes the inductive proof. We now prove that $m \geq k$. Let $i = k-3$. Let $r = \max(r_1, \ldots, r_i)$. If $\mathcal{H} \leq \{Y_{i+2}\}$, then $\mathcal{H} \leq \{K_{1,l}, Y_{i+2}, W_q^2, Z_{1,r}, \ldots, Z_{i,r}\}$, and hence $\mathcal{H} \leq \mathcal{H}_i^A(l, q, r)$ (with $i = k-3$), a contradiction. We conclude that $\mathcal{H} \not\leq \{Y_{i+2}\} = \{Y_{k-1}\}$ and so $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq k$. ☐

Lemma 3.12. *Let $k \geq 7$ and let $\mathcal{H} \in \mathbf{H}$ with $|\mathcal{H}| \leq k$. Suppose that $\mathcal{H} \not\leq \{K_{1,3}\}$, $\mathcal{H} \notin \mathbf{F}_j^C$ for all $1 \leq j \leq k-5$ and $\mathcal{H} \notin \mathbf{F}_j^D$ for all $3 \leq j \leq k-5$. Suppose also that there are graphs $B_1, \ldots, B_5, H_1, H_2 \in \mathcal{H}$ such that*

- *$B_1 = K_{1,l}$ for some $l \geq 4$.*
- *$B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq 3$.*
- *$B_3 = W_{q_1}^3$ for some $q_1 \geq 2$.*
- *$B_4 = D_{q_2}$ for some $q_2 \geq 2$.*
- *$B_5 = T_{q_3}^-$ or $B_5 = T_{q_3}$ for some $q_3 \geq 1$ and*
- *$H_1 = Z_{1,r_1}$ for some $r_1 \geq 3$.*
- *$H_2 = Z_{2,r_1}$ for some $r_2 \geq 3$.*

Then there are graphs $H_3, \ldots, H_{k-5}$ in $\mathcal{H}$ and integers $r_3, \ldots, r_{k-5}$ such that for all $3 \leq i \leq k-5$, $H_i = Z_{i,r_i}$ and $r_i \geq 3$. Additionally, $m \geq k-2$.

Proof. The proof of this lemma is essentially the same as the one of Lemma 3.11.

Let $3 \leq i \leq k-5$ and suppose that there are graphs $H_1, \ldots, H_{i-1}$ in $\mathcal{H}$ such that $H_j = Z_{j,r_j}$ for some $r_j \geq 3$ and all $1 \leq j \leq i-1$. We will prove that there is a graph $H_i \in \mathcal{H}$ such that $H_i = Z_{i,r_i}$ for some $r_i \geq 3$.

Let $r' = \max(r_1, \ldots, r_{i-1})$ and $q = \max(q_1, q_2, q_3)$. Since $\mathcal{H} \notin \mathbf{F}_{i-1}^C$ and $\mathcal{H} \leq \{K_{1,l}, W_q^3, D_q, T_q, Z_{1,r'}, \ldots, Z_{i-1,r'}\}$, then $\mathcal{H} \not\leq \{Y_{i+1}\}$. In particular, $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq i+2$.

Since $\mathcal{H} \notin \mathbf{F}_i^D$ and $\mathcal{H} \leq \{K_{1,l}, Y_m, W_q^3, D_q, T_q, Z_{1,r'}, \ldots, Z_{i-1,r'}\}$, then $\mathcal{H} \not\leq \{Z_{i,r}^-\}$ for all $r \geq 3$.

Let n_0 be as in Lemma 3.11. Let $n = \max(n_0, 3)$. Consider $G = Z_{i,n}$. Since G contains neither $K_{1,4}$, P_{i+3}, W_2^3, D_2, T_1^- then $B_j \not\preceq G$ for all $j \in \{1, 2, 3, 4, 5\}$. Furthermore, since $Z_{j,3} \not\preceq G$ for all $1 \leq j \leq i-1$, then $H_j \not\preceq G$ for all $1 \leq j \leq i-1$. Then there must be some other graph $H_i \in \mathcal{H}$ such that $H_i \preceq G$.

Since $H_i \not\preceq K_{1,3}$, $H_i \not\preceq Y_{i+1}$ and that $H_i \not\preceq Z_{i,r}^-$ for all $r \geq 3$, then $H_i = Z_{i,r_i}$ for some $r_i \geq 3$. Notice that if $r_i = 2$, then it would contradict that $H_i \not\preceq Y_{i+1}$.

This concludes the inductive proof.

We now prove that $m \geq k-2$. Let $i = k-5$. Let $r = \max(r_1, \ldots, r_i)$. If $\mathcal{H} \leq \{Y_{i+2}\}$, then $\mathcal{H} \leq \{K_{1,l}, Y_{i+2}, W_q^3, D_q, T_q, Z_{1,r}, \ldots, Z_{i,r}\}$, and hence $\mathcal{H} \leq \mathcal{H}_i^C(l, q, r)$ (with $i = k-5$), a contradiction. We conclude that $\mathcal{H} \not\leq \{Y_{i+2}\} = \{Y_{k-3}\}$ and so $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq k-2$. □

Proof of Theorem 3.4. Suppose that $\mathcal{H} \in \mathbf{H}$ and $|\mathcal{H}| \leq k$. Contrary to the theorem, suppose that

- $\mathcal{H} \notin \mathbf{F}_i$ for all $i \in \{1, 3, 4, 5, 6\}$ with $i \leq k$,
- $\mathcal{H} \notin \mathbf{F}_i^A$ for all $1 \leq i \leq k-3$,
- $\mathcal{H} \notin \mathbf{F}_i^B$ for all $2 \leq i \leq k-3$,
- $\mathcal{H} \notin \mathbf{F}_i^C$ for all $1 \leq i \leq k-5$ and
- $\mathcal{H} \notin \mathbf{F}_i^D$ for all $3 \leq i \leq k-5$.

Since $\mathcal{H} \in \mathbf{H}$, there is a positive integer $n_0 = n_0(\mathcal{H})$ such that every $\mathcal{H}$-free connected graph of order at least n_0 is claw-free. Let $n = \max(n_0, 3)$. We will consider different connected graphs G of order at least n containing an induced claw. Then for each of such graphs G there will be some $H \in \mathcal{H}$ such that $H \preceq G$.

Consider $G = K_{1,n}$. Then there is a graph $B_1 \in \mathcal{H}$ such that $B_1 \preceq G$. Since $\mathcal{H} \notin \mathbf{F}_1$, then $\mathcal{H} \not\leq \{K_{1,3}\}$, and so $B_1 \not\preceq K_{1,3}$. We conclude that

- $B_1 = K_{1,l}$ for some $l \geq 4$.

Consider $G = Y_n$. Since G contains no $K_{1,4}$, then $B_1 \not\preceq G$. Then $k \geq 2$ and there is a graph $B_2 \in \mathcal{H}$ such that $B_2 \preceq G$. Since $B_2 \not\preceq K_{1,3}$ then

- $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq 3$.

Consider $G = W_n^3$. Since G contains neither $K_{1,4}$ nor P_4, then $B_1 \not\preceq G$ and $B_2 \not\preceq G$. Then $k \geq 3$ and there is a graph $B_3 \in \mathcal{H}$ such that $B_3 \preceq G$. Since $\mathcal{H} \notin \mathbf{F}_3$, then $\mathcal{H} \not\leq \{K_r\}$ for all $r \geq 3$. Since $B_3 \not\preceq K_{1,3}$ and $B_3 \not\preceq K_r$ for all $r \geq 3$, then

- $B_3 = W_{q_1}^2$ or $B_3 = W_{q_1}^3$ for some $q_1 \geq 2$.

Consider $G = Z_{1,n}$. Since G contains neither $K_{1,4}$, P_4 nor W_2^2, then $B_i \not\preceq G$ for all $i \in \{1, 2, 3\}$. Then $k \geq 4$ and there is a graph $H_1 \in \mathcal{H}$ such that $H_1 \preceq G$ (the name H_1 will be better understand later in the proof). Since $\mathcal{H} \notin \mathbf{F}_4$, then $\mathcal{H} \not\leq \{Z_{1,r}^-\}$ for all $r \geq 3$. Since $H_1 \not\preceq K_{1,3}$ and $H_1 \not\preceq Z_{1,r}^-$ for all $r \geq 3$, then

- $H_1 = Z_{1,r_1}$ for some $r_1 \geq 3$.

Case 1: $\mathcal{H} \leq \{W_q^2\}$ for some $q \geq 2$.

Since $\mathcal{H} \leq \{W_q^2\}$ for some $q \geq 2$ then there is a graph B' in $\mathcal{H}$ such that $B' \preceq W_q^2$ for some $q \geq 2$. Notice it may be that $B' = B_3$ or not. Since $B' \not\preceq K_{1,3}$ and $B' \not\preceq K_r$ for all $r \geq 3$, then $B' = W_q^2$ for some $q \geq 2$.

By Lemma 3.11, there are graphs $H_2, \ldots, H_{k-3}$ in $\mathcal{H}$ such that $H_i = Z_{i,r_i}$ for some $r_i \geq 3$ and all $2 \leq i \leq k-3$. From the same lemma, we have that $m \geq k$ and so $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq k$. Notice that $\{B_1, B_2, B', H_1, \ldots, H_{k-3}\} \subseteq \mathcal{H}$. Since $|H| \leq k$, then $B' = B_3$ and $\mathcal{H}$ has no other graphs, namely, $\mathcal{H} = \{B_1, B_2, B_3, H_1, \ldots, H_{k-3}\}$.

Consider $G = Z_{k-2,n}$. Since G contains neither $K_{1,4}$, P_{k+1} nor W_2^2 then $B_i \not\preceq G$ for all $i \in \{1, 2, 3\}$. Furthermore, since $Z_{i,3} \not\preceq G$ for all $1 \leq i \leq k-3$, then $H_i \not\preceq G$ for all $1 \leq i \leq k-3$. Then G contains no graph of $\mathcal{H}$, which is a contradiction.

Case 2: $\mathcal{H} \not\leq \{W_{2,q}\}$ for all $q \geq 2$.

Since $\mathcal{H} \not\leq \{W_q^2\}$ for all $q \geq 2$, then

- $B_3 = W_{q_1}^3$ for some $q_1 \geq 2$.

Consider $G = D_n$. Since G contains neither $K_{1,4}$, P_4, W_2^3 nor $Z_{1,3}$, then $B_i \not\preceq G$ for all $i \in \{1, 2, 3\}$ and $H_1 \not\preceq G$. Then $k \geq 5$ and there is a graph $B_4 \in \mathcal{H}$ such that $B_4 \preceq G$. Since $B_4 \not\preceq K_{1,3}$, $B_4 \not\preceq K_r$ for all $r \geq 3$, $B_4 \not\preceq W_q^2$ for all $q \geq 2$ and $B_4 \not\preceq Z_{1,r}^-$ for all $r \geq 3$, then

- $B_4 = D_{q_2}$ for some $q_2 \geq 2$.

Since $\mathcal{H} \notin \mathbf{F}_5$, then $\mathcal{H} \not\leq \{P_4\}$. Then $B_2 = P_m$ for some $m \geq 5$, or $B_2 = Y_m$ for some $m \geq 3$.

Consider $G = T_n$. Since G contains neither $K_{1,4}$, P_5, Y_3, W_2^3, D_2 nor $Z_{1,3}$, then $B_i \not\preceq G$ for all $1 \leq i \leq 4$ and $H_1 \not\preceq G$. Then $k \geq 6$ and there is a graph $B_5 \in \mathcal{H}$ such that $B_5 \preceq G$. Since $\mathcal{H} \notin \mathbf{F}_6$, then $\mathcal{H} \not\leq \{Z_{2,r}^-\}$ for all $r \geq 3$. Since $B_5 \not\preceq K_{1,3}$, $B_5 \not\preceq W_q^2$ for all $q \geq 2$, and that $B_5 \not\preceq Z_{j,r}^-$ for $j \in \{1, 2\}$ and all $r \geq 3$, then

- $B_5 = T_{q_3}^-$ or $B_5 = T_{q_3}$ for some $q_3 \geq 1$.

Suppose that $\mathcal{H} \leq \{Y_3\}$. Since $\mathcal{H} \leq \{K_{1,l}, Y_3, W^3_{q_1}, D_{q_2}, T_{q_3}, Z_{1,r_1}\}$, then $\mathcal{H} \leq \mathcal{H}^C_1(l, \max(q_1, q_2, q_3), r_1)$, a contradiction (since $1 \leq k - 5$). Then we may suppose that $\mathcal{H} \not\leq \{Y_3\}$ and so $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq 4$.

Consider $G = Z_{2,n}$. Since G contains neither $K_{1,4}$, P_5, W^3_2, D_2, T^-_1 nor $Z_{1,3}$, then $B_i \not\preceq G$ for all $i \in \{1, 2, 3, 4, 5\}$ and $H_1 \not\preceq G$. Then $k \geq 7$ and there is a graph $H_2 \in \mathcal{H}$ such that $H_2 \preceq G$. Since $H_2 \not\preceq K_{1,3}$, $H_2 \not\preceq Y_3$ and $H_2 \not\preceq Z^-_{j,r}$ for $j \in \{1, 2\}$ and all $r \geq 3$, then

- $H_2 = Z_{2,r_2}$ for some $r_2 \geq 3$.

By Lemma 3.12, there are graphs $H_1 \ldots H_{k-5}$ in $\mathcal{H}$ such that $H_i = Z_{i,r_i}$ for some $r_i \geq 3$ and all $1 \leq i \leq k - 5$. From the same lemma, we have that $m \geq k - 2$ and so $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq k - 2$. Notice that $\{B_1, \ldots, B_5, H_1, \ldots, H_{k-5}\} \subseteq \mathcal{H}$. Since $|H| \leq k$, then $\mathcal{H}$ has no other graphs, namely, $\mathcal{H} = \{B_1, \ldots, B_5, H_1, \ldots, H_{k-5}\}$.

Consider $G = Z_{k-4,n}$. Since G contains neither $K_{1,4}$, P_{k-1}, W^3_2, D_2 nor T^-_1 then $B_i \not\preceq G$ for all $i \in \{1, 2, 3, 4, 5\}$. Furthermore, since $Z_{i,3} \not\preceq G$ for all $1 \leq i \leq k - 5$, then $H_i \not\preceq G$ for all $1 \leq i \leq k - 5$. Then G contains no graph of $\mathcal{H}$, which is a contradiction. □

3.5 Applications

In this section we show an application of Theorem 3.5. In particular we show a family of forbidden subgraphs implying traceability in large enough connected graphs.

Let N be the graph obtained by adding a pendant vertex to each vertex of a triangle (see Figure 3.2). The graph N is often called "net". Consider the following Theorem.

Theorem 3.13 ([13]). *Let R and S be connected graphs. Then every $\{R, S\}$-free connected graph has a Hamiltonian path if and only if $\{R, S\} \leq \{K_{1,3}, N\}$.*

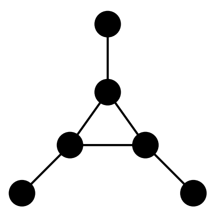


Figure 3.2: The graph N

We use now Theorem 3.5 to prove a variation of Theorem 3.13. We remove the limit on the number of forbidden subgraphs and replace it with the condition that the graph N is among the forbidden subgraphs.

Theorem 3.14. *Let $i \geq 1$ and $\mathcal{H} \in \mathbf{F}_i^C$ with $N \in \mathcal{H}$. Then there is an integer $n \geq 1$ such that every $\mathcal{H}$-free connected graph G with $|V(G)| \geq n$ has a Hamiltonian path.*

Proof. Let $i \geq 1$ and $\mathcal{H} \in \mathbf{F}_i^C$ with $N \in \mathcal{H}$. By Theorem 3.5, we know that every $\mathcal{H}$-free connected graph G with large enough order is $K_{1,3}$-free. Because $N \in \mathcal{H}$, then by Theorem 3.13, every $\mathcal{H}$-free connected graph G with large enough order has a Hamiltonian path. □

Theorem 3.14 is actually a case of implicit forbiddance, that we discussed in Section 3.1. Even though that, so far there was no result on forbidden subgraphs implying a Hamilton path with families of large or infinite size. So, we think the result is interesting by itself.

3.6 Conclusions and open problems

The characterization we were looking for is given by Theorem 3.5. We extend this characterization to star-free graphs in Chapter 4.

Regarding the characterization family $\mathcal{H}_i^C(l, q, r)$, notice that it is not necessary that the parameters "r" of all the $Z_{m,r}$ are the same. In other words, we could have r_1, r_2 and so on. But we think that doing so would make the proof of Theorem 3.3 more difficult to understand and it does not add real value to the result.

Theorem 3.4 characterizes all the families of forbidden subgraphs for claw-free graphs when restricting the size of the family. In the characterization there are four "irregular" families ($\mathbf{F}_3$, $\mathbf{F}_4$, $\mathbf{F}_5$ and $\mathbf{F}_6$) before the four infinite "regular" series ($\mathbf{F}_i^A$, $\mathbf{F}_i^B$, $\mathbf{F}_i^C$ and $\mathbf{F}_i^D$). We call them irregular because there is no easy way to see a pattern that describes them. They also include graphs that are claw-free, which are the result of the intersection of graphs that are not claw-free. These graphs become necessary because of the restriction in the size of the family. After $\mathbf{F}_6$, the families "stabilize" and appear the four infinite series.

When searching for forbidden subgraphs implying some property P on graphs, it makes sense to study only forbidden subgraphs that imply P on graphs that satisfy some condition, usually related to the necessary conditions for satisfying P. For example, in the case of graphs having a 2-factor, like in the Theorem 3.1, G should have minimum degree at least 2 and maximum degree at least 3. Minimum degree at least 2 is a necessary condition for having a 2-factor; maximum degree at least 3 is to avoid the trivial case of G being a cycle. Another example is the case of Hamiltonian graphs, which have a necessary condition of being 2-connected, as studied for example in [20, 7].

Usual necessary conditions in the literature (Hamilton cycle, Hamilton-connected[4], 2-factor) appear to be connectivity and minimum degree conditions. When studying properties with such necessary conditions, Theorem 3.5 might not be useful to understand if a star is being implicitly forbidden or not. To try to find generalizations of Theorem 3.5 that can also be used in these cases, we propose the following two problems.

Problem 3.2. *Let $k \geq 1$. Characterize all the families of connected graphs $\mathcal{H}$ satisfying the following property. Every large enough k-connected $\mathcal{H}$-free graph is claw-free.*

In this chapter we were able to resolve Problem 3.2 for the case $k = 1$.

Problem 3.3. *Let $d \geq 2$. Characterize all the families of connected graphs $\mathcal{H}$ satisfying the following property. Every large enough $\mathcal{H}$-free connected graph with minimum degree at least d is claw-free.*

Even a combination of Problems 3.2 and 3.3 is possible.

Chapter 4

Star-free graphs

In this chapter, we study the relation between star-free graphs and forbidden induced subgraphs. The main result in this chapter is Theorem 4.1, which shows for each $t \geq 3$, a characterization of all families of forbidden subgraphs that imply the property of being $K_{1,t}$-free in connected graphs of large enough order. This is a generalization to the some of the results found Chapter 3. All the new results we prove in this chapter can be found in [17].

4.1 Introduction

In this chapter we further research the phenomenon of *implicit forbiddance* that we described and studied in Chapter 3. Concretely, we extend our target class from claw-free graphs to star-free graphs, studying the implicit forbiddance for the latter. Our problem can be stated as follows.

Problem 4.1. *Given $t \geq 3$, characterize all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free connected graph is $K_{1,t}$-free.*

Problem 4.1 is an extension of Problem 3.1. In this chapter, we solve Problem 4.1 for every $t \geq 3$.

The rest of the chapter is organized as follows. In Section 4.2, we make all needed definitions and present our main results. In Section 4.3, we give the proofs for those results. Finally, in Section 4.4 we make some discussion, propose some open problems and comment on the cases $t = 1$ and $t = 2$ of Problem 4.1.

4.2 Definitions and main results

Define $\mathbf{G}$ as the set of all non-redundant families of connected graphs. Let $t \geq 3$ and define $\mathbf{H}(t)$ as the set of families $\mathcal{H} \in \mathbf{G}$ such that there is a constant $n_0 = n_0(t, \mathcal{H})$

with the property that all $\mathcal{H}$-free connected graphs G with $|V(G)| \geq n_0$ are $K_{1,t}$-free. Then, our problem is reduced to finding all the elements in the set $\mathbf{H}(t)$.

Let $t \geq 2$. To state our results we define the following graphs (see Figure 4.1).

- Y^t_m is a path on m vertices with $t-1$ extra vertices attached to the first vertex of the path. The last vertex of the path is called the tail of Y^t_m. $(m \geq 1)$
- $Y^t_{s,m}$ is the graph obtained by joining s degree one vertices of a $K_{1,t}$ with the first vertex of the path on m vertices. The last vertex of the path is called the tail of $Y^t_{s,m}$. $(m \geq 1, 1 \leq s \leq t)$
- $\widehat{Y}^t_{s,m}$ is the graph obtained by joining s degree one vertices of a $K_{1,t}$ with the first vertex of the path on m vertices and adding the edge between the center of the $K_{1,t}$ and the first vertex of the path. The last vertex of the path is called the tail of $\widehat{Y}^t_{s,m}$. $(m \geq 1, 1 \leq s \leq t)$
- W^t_q is the graph obtained by completely joining a K_q with t independent vertices. $(q \geq 1)$
- $T^t_{s,q}$ is the graph obtained by joining s degree one vertices of a $K_{1,t}$ with all the vertices of a K_q. $(q \geq 1, 1 \leq s \leq t)$
- $D^t_{s,q}$ is the graph obtained by joining s degree one vertices and the center of a $K_{1,t}$ with all the vertices of a K_q. $(q \geq 1, 0 \leq s \leq t)$
- $Z^t_{m,r}$, $Z^t_{s,m,r}$ and $\widehat{Z}^t_{s,m,r}$ are the graphs obtained by identifying a vertex of a K_r with the tail of a $Y^t_m, Y^t_{s,m}$ and $\widehat{Y}^t_{s,m}$, respectively. $(m \geq 1, r \geq 1, 1 \leq s \leq t)$

For $t \geq 3$, define the following families of graphs.

- $\mathcal{T}^t(q) = \{\ T^t_{s,q}\colon\ 2 \leq s \leq t-1\ \}$.
- $\mathcal{D}^t(q) = \{\ D^t_{s,q}\colon\ 2 \leq s \leq t-1\ \}$.
- $\mathcal{Y}^t(m) = \{\ Y^t_{s,m}\colon\ 2 \leq s \leq t-2\ \}$.
- $\mathcal{Z}^t(m,r) = \{\ Z^t_{s,m,r}\colon\ 2 \leq s \leq t-2\ \}$.
- $\widehat{\mathcal{Y}}^t(m) = \{\ \widehat{Y}^t_{s,m}\colon\ 2 \leq s \leq t-2\ \}$.
- $\widehat{\mathcal{Z}}^t(m,r) = \{\ \widehat{Z}^t_{s,m,r}\colon\ 2 \leq s \leq t-2\ \}$.
- $\mathcal{YZ}^t(m,r) = \mathcal{Y}^t(m+2) \cup \mathcal{Z}^t(1,r) \cup \ldots \cup \mathcal{Z}^t(m,r) \cup \widehat{\mathcal{Y}}^t(m+2) \cup \widehat{\mathcal{Z}}^t(1,r) \cup \ldots \cup \widehat{\mathcal{Z}}^t(m,r)$.

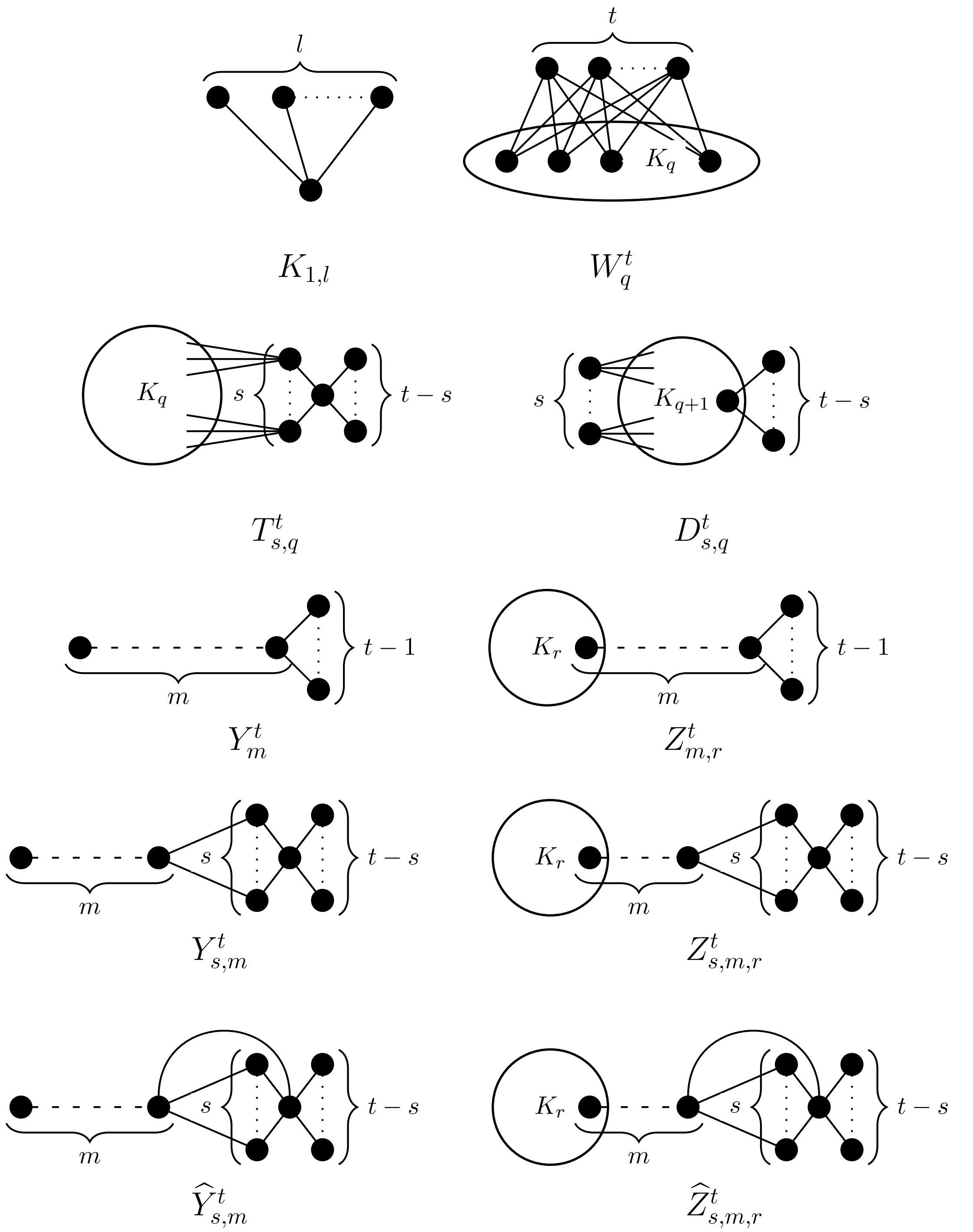


Figure 4.1: Some forbidden subgraphs

- $\mathcal{H}^t(m,l,q,r) = \{K_{1,l}, W_q^t\} \cup \{Y_{m+2}^t, Z_{1,r}^t, \ldots, Z_{m,r}^t\} \cup \mathcal{T}^t(q) \cup \mathcal{D}^t(q) \cup \mathcal{YZ}^t(m,r)$.

Notice that for the case $t = 3$, $\mathcal{Y}^t(m)$, $\mathcal{Z}^t(m,r)$, $\widehat{\mathcal{Y}}^t(m)$ and $\widehat{\mathcal{Z}}^t(m,r)$ are empty and both $\mathcal{T}^t(q)$ and $\mathcal{D}^t(q)$ have only one element.

For $t \geq 3$, define the following subset of $\mathbf{G}$.

- $\mathbf{F}(t) = \{\ \mathcal{H} \in \mathbf{G}$: $\mathcal{H} \leq \mathcal{H}^t(m,l,q,r)$ for some $m \geq 1$, $l \geq t+1$, $q \geq 2$, $r \geq 3\}$.

Our main result in this chapter is the following theorem. It gives the characterization of families of forbidden subgraphs for star-free graphs we described in Section 4.1.

Theorem 4.1. *Let $t \geq 3$, then $\mathbf{H}(t) = \mathbf{F}(t)$.*

Theorem 4.1 is a generalization of Theorem 3.5. Concretely, Theorem 3.5 is the case $t = 3$ of Theorem 4.1.

4.3 Proof of Theorem 4.1

First, we will prove the following theorem that shows that forbidding some family of $\mathbf{F}(t)$ is enough to imply that the graph is star-free provided it is large enough.

Theorem 4.2. *Let $t \geq 3$. Then $\mathbf{F}(t) \subseteq \mathbf{H}(t)$.*

Before giving the proof, we would like to comment on non-redundancy of the family $\mathcal{H}^t(m,l,q,r)$. It is not difficult to check that the family $\mathcal{H}^t(m,l,q,r)$ is non-redundant for the parameters used in the definition of $\mathbf{F}(t)$ ($m \geq 1$, $l \geq t+1$, $q \geq 2$, $r \geq 3$). These conditions were chosen so that $\mathcal{H}^t(m,l,q,r)$ is not redundant nor it contains any induced subgraph of $K_{1,t}$. Moreover, reducing by 1 any of the constants in the conditions would make $\mathcal{H}^t(m,l,q,r)$ either redundant or contain an induced subgraph of $K_{1,t}$. For example, if $q = 1$ then for all $m \geq 1$ and all $1 \leq s \leq t$ we have that $T^t_{s,q} \preceq Y^t_{s,m}$ and $T^t_{s,q} \preceq Z^t_{s,m}$; additionally W^t_q is a $K_{1,t}$.

We divide the proof of Theorem 4.2 into several lemmas that we state and prove bellow.

Lemma 4.3 is a generalization of Lemma 3.7.

Lemma 4.3. *Let $t \geq 3$ and let G be a graph with an induced $K_{1,t}$ of center x_0. If G is $(\{Y^t_m\} \cup \mathcal{Y}^t(m) \cup \widehat{\mathcal{Y}}^t(m))$-free for some $m \geq 3$, then $N^{m+1}(x_0) = \emptyset$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = t$ such that $\{x_0\} \cup Y$ is an induced $K_{1,t}$ in G. Suppose that $N^{m+1}(x_0) \neq \emptyset$. We will show that G contains a Y^t_m, some graph of $\mathcal{Y}^t(m)$ or some graph of $\widehat{\mathcal{Y}}^t(m)$, which is a contradiction.

Let $k = m+1$ and let $P = x_0x_1 \cdots x_k$ be an induced path of G with $x_i \in N^i(x_0)$ for all $0 \leq i \leq k$. Notice that $N^j(x_0) \cap N(Y) = \emptyset$ for all $3 \leq j \leq k$. Otherwise, an element $v \in N^j(x_0) \cap N(Y)$ would have a path of length 2 to x_0 (passing through some element of Y), contradicting that $v \in N^j(x_0)$. Then $N(Y) \cap P \subseteq \{x_0, x_1, x_2\}$.

Let $Y_1 = N(x_1) \cap Y$ and $Y_2 = N(x_2) \cap Y$. If $|Y_2| \geq t-1$, then $Y_2 \cup \{x_2, \ldots, x_{m+1}\}$ contains a Y^t_m. If $2 \leq |Y_2| \leq t-2$, then $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_{m+1}\}$ is a $Y^t_{s,m}$, where $s = |Y_2|$. If $|Y_2| = 1$, then $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_{m-1}\}$ is a Y^t_m.

Suppose now that $|Y_2| = 0$, that is $N(x_2) \cap Y = \emptyset$. If $|Y_1| \geq t-1$, then $Y_1 \cup \{x_1, \ldots, x_m\}$ contains a Y^t_m. If $2 \leq |Y_1| \leq t-2$, then $(Y - Y_1) \cup \{x_0\} \cup Y_1 \cup \{x_1, \ldots, x_m\}$ is a $\widehat{Y}^t_{s,m}$, where $s = |Y_1|$. If $|Y_1| \leq 1$, then $(Y - Y_1) \cup \{x_0, \ldots, x_{m-1}\}$ contains a Y^t_m. □

Lemma 4.4. *Let $t \geq 3$ and let G be a graph with an induced $K_{1,t}$ of center x_0. Suppose that G is $(\{K_{1,l}, Z^t_{1,r}, W^t_q\} \cup \mathcal{D}^t(q))$-free for some $l \geq t+1$, $r \geq 3$, $q \geq 2$. Then $|N(x_0)| < 2^t \cdot R(l, \max(r, q))$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = t$ such that $\{x_0\} \cup Y$ is an induced $K_{1,t}$ in G. Let $Y' \subseteq Y$. We will show that $|N(x_0) \cap B_Y(Y')| < R(l, \max(r, q))$, and since $|Y| = t$, by Proposition 2.2 we get that $|N(x_0)| < 2^t \cdot R(l, \max(r, q))$.

If $|Y'| \leq 1$, then $|Y - Y'| \geq t-1$ and so $|N(x_0) \cap B_Y(Y')| < R(l, r)$, since otherwise $(Y - Y') \cup \{x_0\} \cup (N(x_0) \cap B_Y(Y'))$ contains a $Z^t_{1,r}$ or a $K_{1,l}$.

If $2 \leq |Y'| \leq t-1$, then $|N(x_0) \cap B_Y(Y')| < R(l, q)$, since otherwise $Y' \cup (Y - Y') \cup \{x_0\} \cup (N(x_0) \cap B_Y(Y'))$ contains a $D^t_{s,q}$ or a $K_{1,l}$, where $s = |Y'|$.

If $|Y'| = t$, then $|N(x_0) \cap B_Y(Y')| < R(l, q)$, since otherwise $Y' \cup (N(x_0) \cap B_Y(Y'))$ contains a W^t_q or a $K_{1,l}$. □

Lemma 4.5. *Let $t \geq 3$ and let G be a graph with an induced $K_{1,t}$ of center x_0. Suppose that G is $(\{K_{1,l}, Z^t_{1,r}, Z^t_{2,r}, W^t_q\} \cup \widehat{\mathcal{Z}}^t(1, r) \cup \mathcal{T}^t(q))$-free for some $l \geq t+1$, $r \geq 3$, $q \geq 2$. Then $|N^2(x_0)| < 2^t \cdot R(l, \max(r, q)) \cdot |N(x_0)|$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = t$ such that $\{x_0\} \cup Y$ is an induced $K_{1,t}$ in G. Let $x_1 \in N(x_0)$. Let $Y' \subseteq Y$. Let $N = N^2(x_0) \cap N(x_1)$. By Proposition 2.2, it suffices to show that $|N \cap B_Y(Y')| < R(l, \max(r, q))$.

If $|Y'| = 1$, then $|Y - Y'| = t-1$ and so $|N \cap B_Y(Y')| < R(l, r)$, since otherwise $(Y - Y') \cup \{x_0\} \cup Y' \cup (N \cap B_Y(Y'))$ contains a $Z^t_{2,r}$ or a $K_{1,l}$.

If $2 \leq |Y'| \leq t-1$, then $|N \cap B_Y(Y')| < R(l, q)$, since otherwise $(Y - Y') \cup \{x_0\} \cup Y' \cup (N \cap B_Y(Y'))$ contains a $T^t_{s,q}$ or a $K_{1,l}$, where $s = |Y'|$.

If $|Y'| = t$, then $|N \cap B_Y(Y')| < R(l, q)$, since otherwise $Y' \cup (N \cap B_Y(Y'))$ contains a W^t_q or a $K_{1,l}$.

Suppose now that $|Y'| = 0$, that is $N \cap B_Y(Y') \cap N(Y) = \emptyset$. Notice that if $x_1 \in Y$, then $N \cap B_Y(Y') = \emptyset$. Then we may suppose that $x_1 \notin Y$. Let $Y_1 = Y \cap N(x_1)$.

If $|Y_1| \geq t-1$, then $|N \cap B_Y(Y')| < R(l, r)$, since otherwise $Y_1 \cup \{x_1\} \cup (N \cap B_Y(Y'))$ contains a $Z^t_{1,r}$ or a $K_{1,l}$.

If $2 \leq |Y_1| \leq t-2$, then $|N \cap B_Y(Y')| < R(l, r)$, since otherwise $(Y - Y_1) \cup \{x_0\} \cup Y_1 \cup \{x_1\} \cup (N \cap B_Y(Y'))$ contains a $\widehat{Z}^t_{s,1,r}$ or a $K_{1,l}$, where $s = |Y_1|$.

If $|Y_1| \leq 1$, then $|Y - Y_1| \geq t-1$ and so $|N \cap B_Y(Y')| < R(l, r)$, since otherwise $(Y - Y_1) \cup \{x_0, x_1\} \cup (N \cap B_Y(Y'))$ contains a $Z^t_{2,r}$ or a $K_{1,l}$. □

Lemma 4.6 is a generalization of Lemma 3.10.

Lemma 4.6. *Let $t \geq 3$ and let G be a graph with an induced $K_{1,t}$ of center x_0. Let $i \geq 2$ and suppose that G is $(\{K_{1,l}, Z^t_{i-1,r}, Z^t_{i,r}, Z^t_{i+1,r}\} \cup \mathcal{Z}^t(i-1,r) \cup \widehat{\mathcal{Z}}^t(i,r))$-free for some $l \geq t+1$ and $r \geq 3$. Then $|N^{i+1}(x_0)| < R(l,r) \cdot |N^i(x_0)|$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = t$ such that $\{x_0\} \cup Y$ is an induced $K_{1,t}$ in G. Let $x_i \in N^i(x)$ and let $P = x_0x_1 \cdots x_i$ be an induced path with $x_j \in N^j(x)$ for all $0 \leq j \leq i$. Let $N = N^{i+1}(x_0) \cap N(x_i)$. It suffices to show that $|N| < R(l,r)$.

Let $Y_1 = Y \cap N(x_1)$ and $Y_2 = Y \cap N(x_2)$. As in the proof of Lemma 4.3, $N(Y) \cap P \subseteq \{x_0, x_1, x_2\}$.

If $|Y_2| \geq t-1$, then $|N| < R(l,r)$, since otherwise $Y_2 \cup \{x_2, \ldots, x_i\} \cup N$ contains a $Z^t_{i-1,r}$ or a $K_{1,l}$.

If $2 \leq |Y_2| \leq t-2$, then $|N| < R(l,r)$, since otherwise $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_i\} \cup N$ contains a $Z^t_{s,i-1,r}$ or a $K_{1,l}$, where $s = |Y_2|$.

If $|Y_2| = 1$, then $|N| < R(l,r)$, since otherwise $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_i\} \cup N$ contains a $Z^t_{i+1,r}$ or a $K_{1,l}$.

Suppose now that $|Y_2| = 0$, that is $N(x_2) \cap Y = \emptyset$.

If $|Y_1| \geq t-1$, then $|N| < R(l,r)$, since otherwise $Y_1 \cup \{x_1, \ldots, x_i\} \cup N$ contains a $Z^t_{i,r}$ or a $K_{1,l}$.

If $2 \leq |Y_1| \leq t-2$, then $|N| < R(l,r)$, since otherwise $(Y - Y_1) \cup \{x_0\} \cup Y_1 \cup \{x_1, \ldots, x_i\} \cup N$ contains a $\widehat{Z}^t_{s,i,r}$ or a $K_{1,l}$, where $s = |Y_1|$.

If $|Y_1| \leq 1$, then $|Y - Y_1| \geq t-1$ and so $|N| < R(l,r)$, since otherwise $(Y - Y_1) \cup \{x_0, \ldots, x_i\} \cup N$ contains a $Z^t_{i+1,r}$ or a $K_{1,l}$. □

We use the above lemmas to prove Theorem 4.2.

Proof of Theorem 4.2. Let $\mathcal{H} \in \mathbf{F}(t)$. Let $m \geq 1$, $l \geq t+1$, $q \geq 2$ and $r \geq 3$ such that $\mathcal{H} \leq \mathcal{H}^t(m,l,q,r)$.

Let G be an $\mathcal{H}$-free connected graph. Suppose that there is an induced $K_{1,t}$ of center x. We will show that $|V(G)|$ is bounded by a function depending only on t, l, m, q and r.

Notice that since G is Y^t_{m+2}-free, then G is $Z^t_{i,r}$-free for all $i \geq m+1$. Since we also know that G is $Z^t_{i,r}$-free for all $1 \leq i \leq m$, we conclude that G is $Z^t_{i,r}$-free for all $i \geq 1$. Using a similar argument, we have that G is $\mathcal{Z}^t(i,r)$-free and $\widehat{\mathcal{Z}}^t(i,r)$-free for all $i \geq 1$. Thus, G satisfies all the conditions of Lemmas 4.3, 4.4, 4.5 and 4.6.

By Lemma 4.3, $N^{m+1}(x) = \emptyset$. Then we only need to show that $|N^i(x)|$ is bounded for all $1 \leq i \leq m$. By Lemmas 4.4 and 4.5, $|N(x)|$ and $|N^2(x)|$ are bounded. By Lemma 4.6, $|N^{i+1}(x)| < R(l,r) \cdot |N^i(x)|$ for all $2 \leq i \leq m-1$. Using an inductive argument we get that $|N^i(x)| < R(l,r)^{i-2} \cdot |N^2(x)|$ for all $3 \leq i \leq m$. We conclude that $|N^i(x)| < R(l,r)^{m-2} \cdot |N^2(x)|$ for all $3 \leq i \leq m$. □

Finally, we prove our main theorem.

Proof of Theorem 4.1. Let $t \geq 3$. By Theorem 4.2, we already know that every family of graphs in $\mathbf{F}(t)$ is also in $\mathbf{H}(t)$. It remains to show that every family of graphs in $\mathbf{H}(t)$ is also in $\mathbf{F}(t)$.

Let $\mathcal{H} \in \mathbf{H}(t)$. Then there is a positive integer n_0 such that every $\mathcal{H}$-free connected graph of order at least n_0 is $K_{1,t}$-free. Let n be an integer such that $n \geq \max(n_0, t+1)$.

Consider the family $\mathcal{H}' = \mathcal{H}^t(n,n,n,n)$. All the graphs in $\mathcal{H}'$ are connected graphs of order at least n_0 containing an induced $K_{1,t}$. Then it must be that no graph of $\mathcal{H}'$ is $\mathcal{H}$-free. In other words, for each $H' \in \mathcal{H}'$, there is an $H \in \mathcal{H}$ such that $H \preceq H'$. But this is exactly the definition of $\mathcal{H} \leq \mathcal{H}'$. Then since $\mathcal{H}'$ is in $\mathbf{F}(t)$, we conclude that $\mathcal{H}$ is also in $\mathbf{F}(t)$. □

4.4 Conclusions

The characterization we were looking for is given by Theorem 4.1.

We have solved the problem of characterizing $\mathbf{H}(t)$ for any $t \geq 3$, but it is also possible to consider $t = 1$ and $t = 2$. It is not difficult to see that $\mathbf{F}(t)$ is also the solution for $t = 1$ and $t = 2$. In these cases, the corresponding families $\mathcal{H}^1(m,l,q,r)$ and $\mathcal{H}^2(m,l,q,r)$ after removing redundant graphs are as follows.

- $\mathcal{H}^1(m,l,q,r) = \{K_{1,l}, K_q, P_m\}$
- $\mathcal{H}^2(m,l,q,r) = \{K_{1,l}, W_q^2, Y_m^2, Z_{1,r}^2, \ldots, Z_{m-2,r}^2\}$

The case $t = 1$ is an easy proposition that can also be found in [10, Proposition 9.4.1].

It is also possible to consider restricting the size of the families of forbidden subgraphs for $K_{1,t}$-free graphs for $t \geq 4$. We think that a complete characterization of such families may be difficult and very long. In particular, we think that there might be many "irregular" families and many "regular" infinite series of families.

In this thesis we considered connected graphs with no degree or connectivity conditions. For the same reasons as we explained in Section 3.6, we suggest to consider graphs with higher connectivity or with some minimum degree condition.

Problem 4.2. *Let $t \geq 3$ and $k \geq 1$. Characterize all the families of connected graphs $\mathcal{H}$ satisfying the following property. Every large enough k-connected $\mathcal{H}$-free graph is $K_{1,t}$-free.*

In this chapter we were able to resolve Problem 4.2 for the case $k = 1$.

Problem 4.3. *Let $t \geq 3$ and $d \geq 2$. Characterize all the families of connected graphs $\mathcal{H}$ satisfying the following property. Every large enough $\mathcal{H}$-free connected graph with minimum degree at least d is $K_{1,t}$-free.*

Even a combination of Problems 4.2 and 4.3 is possible.

Chapter 5

Perfect Matchings

In this chapter, we study the relation between perfect matchings and forbidden induced subgraphs. The main result in this chapter is Theorem 5.9, which shows a characterization of all families of forbidden subgraphs implying a perfect matching in graphs of large enough even order. All the new results we prove in this chapter can be found in [28].

5.1 Introduction

The following result was proved independently by Sumner [32] and Las Vergnas [24].

Theorem 5.1 ([32],[24]). *Every connected $K_{1,3}$-free graph of even order has a perfect matching.*

Plummer et al.[30] showed that $K_{1,3}$ is essentially the only graph with that property.

Theorem 5.2 ([30]). *Let H be a connected graph. If there exists a positive constant n_0 such that every connected H-free graph of even order at least n_0 has a perfect matching, then $H \preceq K_{1,3}$.*

Fujita et at.[18] extended Theorem 5.2 by considering two forbidden subgraphs.

Theorem 5.3 ([18]). *Let H_1, H_2 be a pair of connected graphs. If there exists a positive constant n_0 such that every connected $\{H_1, H_2\}$-free graph of even order at least n_0 has a perfect matching, then $H_1 \preceq K_{1,3}$ or $H_2 \preceq K_{1,3}$.*

In the view of these results, one can go further and consider the following more general problem. Characterize all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free connected graph of even order has a perfect matching.

By Theorem 5.3, we can observe that when considering a family $\mathcal{F}$ of two forbidden subgraphs, one of the graphs in $\mathcal{F}$ is an induced subgraph of $K_{1,3}$ and the other one is redundant, as it was pointed out in [27]. In this sense, even if we consider two forbidden subgraphs, $K_{1,3}$ is still essentially the only forbidden subgraph implying a perfect matching.

Ota et al.[27] continued this line of research and considered families of forbidden subgraphs of size 3. Their results show that there are families that do not contain an induced subgraph of $K_{1,3}$ when one considers at least three forbidden subgraphs (see Section 5.2 for graph definitions).

Theorem 5.4 ([27]). *For every $l \geq 4$ and $r \geq 3$, there is an $n_0 = n_0(l, r)$ such that every connected $\{K_{1,l}, P_4, Z_{1,r}\}$-free graph of even order at least n_0 has a perfect matching.*

Theorem 5.5 ([27]). *For every $l \geq 4$, $m \geq 3$ and $r \geq 3$, there is an $n_0 = n_0(l, m, r)$ such that every connected $\{K_{1,l}, Y_m, Z_{1,r}^-\}$-free graph of even order at least n_0 has a perfect matching.*

In [27], it was shown that these are essentially all the families of forbidden subgraphs of size at most 3.

Theorem 5.6 ([27]). *Let $\mathcal{H}$ be a family of connected graphs with $|\mathcal{F}| \leq 3$. If there exists a positive constant n_0 such that every connected $\mathcal{H}$-free graph of even order at least n_0 has a perfect matching, then*

- *there is an $H \in \mathcal{H}$ such that $H \preceq K_{1,3}$, or*
- *there exist $l \geq 4$ and $r \geq 3$ such that $\mathcal{F} \leq \{K_{1,l}, P_4, Z_{1,r}\}$, or*
- *there exist $l \geq 4$, $m \geq 3$ and $r \geq 3$ such that $\mathcal{F} \leq \{K_{1,l}, Y_m, Z_{1,r}^-\}$.*

In this chapter, we complete the characterization by finding all the families of forbidden subgraphs implying a perfect matching in graphs of large enough even order. The rest of the chapter is organized of follows. In Section 5.2 we make all needed definitions and present the main results. In Sections 5.3 and 5.4 we give the proofs for our results. Finally, in Section 5.5 we make some discussion and propose some open problems.

5.2 Definitions and main results

Define $\mathbf{G}$ as the set of all non-redundant families of connected graphs. Define $\mathbf{H}$ as the set of families $\mathcal{F} \in \mathbf{G}$ such that there is a constant $n_0 = n_0(\mathcal{F})$ with the property that all $\mathcal{F}$-free connected graphs of even order at least n_0 have a perfect matching. Then, our problem is reduced to finding all the elements in the set $\mathbf{H}$.

With the previous definitions, Theorems 5.2 and 5.3 say that If $\mathcal{H} \in \mathbf{H}$ and $|\mathcal{H}| \leq 2$ then $\mathcal{H} \leq \{K_{1,3}\}$.

To state our results we define the following graphs (see Figure 5.1)

- Y_m is a path on m vertices with two extra vertices attached to the first vertex of the path. The last vertex of the path is called the tail of Y_m. Notice that Y_1 is isomorphic to P_3 and the tail of Y_1 is the middle vertex of P_3.

- Let $\{v_1, \ldots, v_q\}$ be the vertices of a K_q. $W_{q,t}$ is the graph obtained by adding 2 extra vertices which are adjacent to every vertex $\{v_1, \ldots, v_q\}$ and then attaching one pendant vertex to each vertex of $\{v_1, \ldots, v_t\}$ $(q \geq t)$. Define also $W_q = W_{q,q}$.

- $W^-_{q,t}$ is the graph obtained by attaching one pendant vertex to t different vertices of a K_q $(q \geq t)$.

- $Z^-_{m,r}$ is the graph obtained by identifying a vertex of a K_r with the end vertex of a P_n.

- $Z_{m,r}$ is the graph obtained by identifying a vertex of a K_r with the tail of a Y_m.

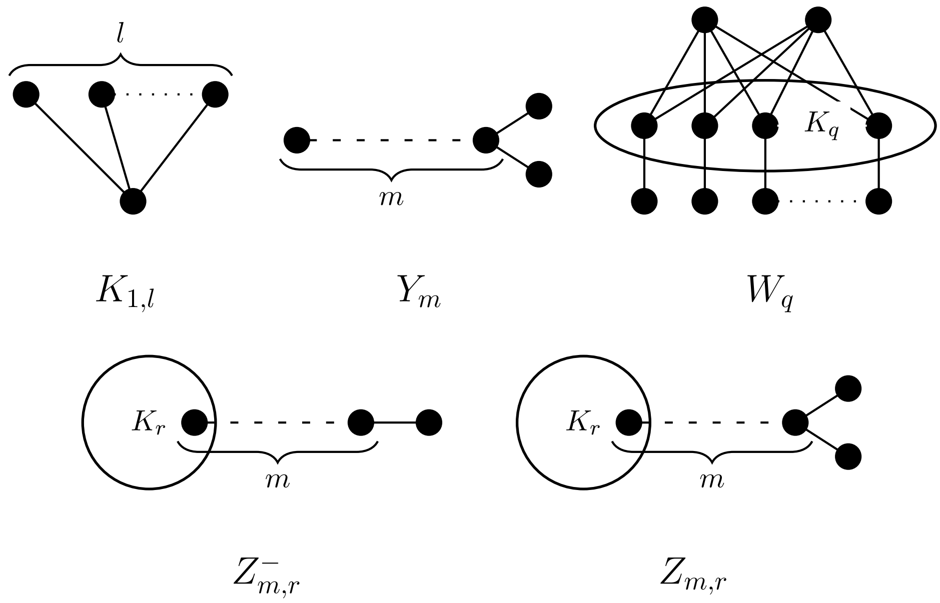


Figure 5.1: Some forbidden subgraphs

Define the following families of graphs.

- $\mathcal{H}_i^A(l,q,r) = \{K_{1,l}, Y_{i+2}, W_q, Z_{1,r}, \ldots, Z_{i,r}\}$ (for $i \geq 1$).
- $\mathcal{H}_i^B(l,m,q,r) = \{K_{1,l}, Y_m, W_q, Z_{1,r}, \ldots, Z_{i-1,r}, Z_{i,r}^-\}$ (for $i \geq 2$).

Notice that the size of both $\mathcal{H}_i^A(l,q,r)$ and $\mathcal{H}_i^B(l,m,q,r)$ is $i+3$.

Define the following subsets of $\mathbf{G}$.

- $\mathbf{H}_1 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,3}\}\ \}$.
- $\mathbf{H}_2 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, P_4, Z_{1,r}\}$ for some $l \geq 4$, $r \geq 3\}$.
- $\mathbf{H}_3 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, Y_m, Z_{1,r}^-\}$ for some $l \geq 4$, $m \geq 3$ and $r \geq 3\}$.
- $\mathbf{H}_i^A = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \mathcal{H}_i^A(l,q,r)$ for some $l \geq 4$, $q \geq 2$ and $r \geq 3\}$ $(i \geq 1)$.
- $\mathbf{H}_i^B = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \mathcal{H}_i^B(l,m,q,r)$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$ and $r \geq 3\}$ $(i \geq 2)$.

With these notation and definitions, Theorem 5.4 says that $\mathbf{H}_2 \subseteq \mathbf{H}$ and Theorem 5.5 says that $\mathbf{H}_3 \subseteq \mathbf{H}$. Also, Theorem 5.6 says that if $\mathcal{H} \in \mathbf{H}$ and $|\mathcal{H}| \leq 3$ then $\mathcal{H} \in \mathbf{H}_1 \cup \mathbf{H}_2 \cup \mathbf{H}_3$.

Main Results

Our main results in this chapter are the following three theorems. First we show that the new families from the sets $\mathbf{H}_i^A$ and $\mathbf{H}_i^B$ imply a perfect matching.

Theorem 5.7. *For every $i \geq 1$, $\mathbf{H}_i^A \subseteq \mathbf{H}$ and for every $i \geq 2$, $\mathbf{H}_i^B \subseteq \mathbf{H}$.*

Next, we show that these two families are exactly the maximal families when the size of the family of forbidden subgraphs is limited to some positive integer k.

Theorem 5.8. *Let $k \geq 1$ be an integer and $\mathcal{H} \in \mathbf{H}$ with $|\mathcal{H}| \leq k$. Then*

- *$\mathcal{H} \in \mathbf{H}_i$ for some $i \in \{1,2,3\}$ and $1 \leq k \leq 3$ or*
- *$\mathcal{H} \in \mathbf{H}_i^A$ for some $1 \leq i \leq k-3$ or*
- *$\mathcal{H} \in \mathbf{H}_i^B$ for some $2 \leq i \leq k-3$.*

Finally and to complete the characterization, we need to show that there are no infinite family in $\mathbf{H}$ that is not already considered by the families in $\mathbf{H}_i^A$ and in $\mathbf{H}_i^B$. Furthermore, we show that it is enough to consider only the families in $\mathbf{H}_i^A$. We prove this fact in the following theorem.

Theorem 5.9. *$\mathcal{H} \in \mathbf{H}$ if and only if $\mathcal{H} \in \mathbf{H}_i^A$ for some $i \geq 1$.*

Definitions

For our proofs we need the following definitions and theorems.

For a graph G, let $c_o(G)$ be the number of odd components of G. We use the following Tutte's Theorem [33] as our main tool in the proofs (see also Theorem 9.5 of [8]).

Theorem 5.10 ([33]). *A graph G has a perfect matching if and only if for every $S \subseteq V(G)$, $c_o(G-S) \leq |S|$.*

A set $S \subseteq V(G)$ not satisfying the inequality in Theorem 5.10 is called a *Tutte set*. A Tutte set S is said to be minimal if it does not contain any proper Tutte set.

Let S be a Tutte set, $x \in S$ and $X \subseteq S$. Define

- $\mathcal{C}_S(x) = \{C\colon C$ is a component of $G-S$ such that $N_G(x) \cap V(C) \neq \emptyset \}$
- $\mathcal{C}_S(X) = \bigcup_{x \in X} \mathcal{C}_S(x)$

Because S is usually clear from the context, we write $\mathcal{C}(x)$ and $\mathcal{C}(S)$ instead of $\mathcal{C}_S(x)$ and $\mathcal{C}_S(X)$, respectively.

5.3 Proof of Theorem 5.7

First we prove a few lemmas. Lemmas 5.11 and 5.12 were proven in [27] as part of the proof of Theorem 1. We prove them again here for completeness.

Lemma 5.11. *Let G be a connected graph of even order and $S \subseteq V(G)$ a minimal Tutte set. Then for every non-empty subset $X \subseteq S$, $|\mathcal{C}(X)| \geq |X| + 2$.*

Proof. By the definition of Tutte set, $c_o(G-S) \geq |S|+1$. But since $|V(G)|$ is even, then $c_o(G-S) \geq |S|+2$.

Let $S' = S - X$. By minimality of S, $c_o(G-S') \leq |S'|$. Since each component of $G-S$ not in $\mathcal{C}(X)$ is a component of $G-S'$, then $\mathcal{C}(S) - \mathcal{C}(X) \subseteq \mathcal{C}(S')$ and so $c_o(G-S) - |\mathcal{C}(X)| \leq c_o(G-S')$. Then we have that

$$|S| + 2 - |\mathcal{C}(X)| \leq c_o(G-S) - |\mathcal{C}(X)| \leq c_o(G-S') \leq |S'| = |S| - |X|$$

We conclude that $|\mathcal{C}(X)| \geq |X| + 2$. □

Lemma 5.12. *Let $l \geq 4$ and let G be a $K_{1,l}$-free connected graph of even order and $S \subseteq V(G)$ a minimal Tutte set. Then for every non-empty set $A \subseteq S$, there is a component $C_0 \in \mathcal{C}(A)$ such that $|N_G(C_0) \cap A| \leq l-2$.*

Proof. Let k be the number of adjacencies (x, C) with $x \in A$ and $C \in \mathcal{C}(A)$. Clearly,

$$k = \sum_{x \in A} |\mathcal{C}(x)| \quad \text{and} \quad k = \sum_{C \in \mathcal{C}(A)} |N_G(C) \cap A|.$$

Suppose that for all components $C \in \mathcal{C}(A)$, $|N_G(C) \cap A| \geq l-1$. By Lemma 5.11, $|\mathcal{C}(A)| \geq |A|+2$. Then $\sum_{C \in \mathcal{C}(A)} |N_G(C) \cap A| \geq (l-1) \cdot (|A|+2)$, and so $k \geq (l-1) \cdot (|A|+2)$.

On the other hand, since G is $K_{1,l}$-free then $|\mathcal{C}(x)| \leq l-1$ for all $x \in A$. But then $\sum_{x \in A} |\mathcal{C}(x)| \leq (l-1) \cdot |A|$, and so $k \leq (l-1) \cdot |A|$, a contradiction. □

Lemma 5.13. *Let G be a connected graph of even order and $S \subseteq V(G)$ a minimal Tutte set. Suppose that G is $\{K_{1,l}, Z_{1,r}\}$-free for some $l \geq 4$ and $r \geq 3$. Let $q \geq 2$ and let $A \subseteq S$ be a clique with $|A| \geq (l-2) \cdot q + r$. Suppose that there are two vertices $y_1 \in D_1$ and $y_2 \in D_2$ for two different components $D_1, D_2 \in \mathcal{C}(A)$ such that $N(y_1) \cap A = N(y_2) \cap A = A$. Then G contains an induced W_q.*

Proof. We will prove by induction that there are q vertices $v_1, \ldots, v_q$ in q different components $C_1, \ldots, C_q$ of $\mathcal{C}(A) - \{D_1, D_2\}$ and q vertices $a_1, \ldots, a_q$ of A such that $v_i a_j \in E(G)$ if and only if $i = j$. Then the set $\{y_1, y_2, a_1, \ldots, a_q, v_1, \ldots, v_q\}$ will induce a W_q in G.

To so do, we find components $C_1, \ldots, C_q$ in such a way that they satisfy the following condition. If for $1 \leq i \leq q+1$ we define $A_i = A - \bigcup_{j=1}^{i-1} (A \cap N(C_j))$, then for all $1 \leq i \leq q$, $C_i \in \mathcal{C}(A_i)$ and $|A_i \cap N(C_i)| \leq l-2$.

Let $1 \leq i \leq q$ and suppose that we have found $C_1, \ldots, C_{i-1}$. We find C_i in the following way. Since

$$|A_i| = |A| - \left| \bigcup_{j=1}^{i-1} (A \cap N(C_j)) \right| \geq (l-2) \cdot q + r - (l-2) \cdot (i-1) \geq l-2+r$$

then A_i is not empty. Then by Lemma 5.12, there is a component $C_i \in \mathcal{C}(A_i)$ such that $|A_i \cap N(C_i)| \leq l-2$. By the way A_i is defined, $N(C_j) \cap A_i = \emptyset$ for all $1 \leq j \leq i-1$ and so $C_i \notin \{C_1, \ldots, C_{i-1}\}$. If $C_i = D_1$ then $y_1 \in C_i$ and so $|A_i| = |N(y_1) \cap A_i| \leq |N(C_i) \cap A_i| \leq l-2$, contradicting that $|A_i| \geq l-2+r$. Then $C_i \neq D_1$ and similarly we get that $C_i \neq D_2$. Thus, we have found the required C_i.

For $1 \leq i \leq q$, choose $a_i \in A_i \cap N(C_i)$ and $v_i \in C_i \cap N(a_i)$.

Let $1 \leq i \leq q$, we will prove that $a_i \notin N(v_j)$ for all $1 \leq j \leq q$ with $j \neq i$. Since for all $1 \leq j \leq i-1$, $A_i \cap N(C_j) = \emptyset$ then $a_i \notin N(v_j)$ for all $1 \leq j \leq i-1$. Suppose that $a_i \in N(v_j)$ for some $i+1 \leq j \leq q$. Since

$$|A_{q+1}| = |A| - \left| \bigcup_{j=1}^{q} (A \cap N(C_j)) \right| \geq (l-2) \cdot q + r - (l-2) \cdot q = r$$

and $A_{q+1} \cap N(v_j) = A_{q+1} \cap N(v_i) = \emptyset$ then $A_{q+1} \cup \{a_j\} \cup \{v_i, v_j\}$ contains a $Z_{1,r}$ which is a contradiction.

Thus, we have proved that the set $\{y_1, y_2, a_1, \dots, a_q, v_1, \dots, v_q\}$ induces a W_q. $\square$

Now we prove the main theorem of this section.

Proof of Theorem 5.7. We first show that for all $i \geq 2$ and $\mathcal{H} \in \mathbf{H}_i^B$, there is some $m \geq 1$ such that $\mathcal{H} \in \mathbf{H}_m^A$. This implies that it is enough to show that $\mathbf{H}_i^A \subseteq \mathbf{H}$ for all $i \geq 1$.

Let $i \geq 2$ and $\mathcal{H} \in \mathbf{H}_i^B$. Then $\mathcal{H} \leq \{K_{1,l}, Y_m, W_q, Z_{2,r}, \dots, Z_{i-1,r}, Z_{i,r}^-\}$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$ and $r \geq 3$. Since $Z_{i,r} \preceq Z_{h,r}^+$ for all $h \geq i$ and $r \geq 3$, then $\mathcal{H} \in \mathbf{H}_{m-2}^A$.

Let $i \geq 1$ and $\mathcal{H} \in \mathbf{H}_i^A$. Let $l \geq 4$, $q \geq 2$, and $r \geq 3$ such that $\mathcal{H} \leq \mathcal{H}_i^A(l,q,r)$. Let G be a connected $\mathcal{H}$-free graph of even order with no perfect matching. We will show that $|V(G)|$ is bounded by a function depending only on i, l, q, r.

By Theorem 5.10, G has a Tutte set $S \subseteq V(G)$. We may assume that S is chosen to be minimal. Since G is connected, then $S \neq \emptyset$. Let $x \in S$. By Lemma 5.11, there are three different components $C_1, C_2, C_3 \in \mathcal{C}(x)$. Let $y_i \in (C_i \cap N_G(x))$ for $i \in \{1,2,3\}$. Let $Y = \{y_1, y_2, y_3\}$. Notice that $\{x\} \cup Y$ is a claw in G.

Claim 5.7.1. $|N(x)| \leq 8 \cdot R(l, (l-2) \cdot q + r)$

Proof. Let $Y' \subseteq Y$. Since $\{x\} \cup (N(x) \cap B_Y(Y'))$ contains no $K_{1,l}$, then $N(x) \cap B_Y(Y')$ contains no independent set of size at least l. Let N be the largest clique in $N(x) \cap B_Y(Y')$. If we show that $|N| < (l-2) \cdot q + r$, then $|N(x) \cap B_Y(Y')| < R(l, (l-2) \cdot q + r)$, and since $|Y| = 3$, by Proposition 2.2 we get that $|N(x)| \leq 2^3 \cdot R(l, (l-2) \cdot q + r)$.

If $|Y'| \leq 1$, then $|Y - Y'| \geq 2$. Then $|N| < r$, since otherwise $(Y - Y') \cup \{x\} \cup N$ contains a $Z_{1,r}$.

If $|Y'| \geq 2$, then at least two vertices of Y are adjacent to all the vertices of N. Since the elements of Y are in different components and $|Y'| \geq 2$, then it must be that $N \subseteq S$. Then by Lemma 5.13, $|N| < (l-2) \cdot q + r$. $\square$

Notice that since G is $Z_{m,r}$-free for all $1 \leq m \leq i$ and G is Y_{i+2}-free, then G is $Z_{m,r}$-free for all $m \geq 1$.

Claim 5.7.2. $|N^2(x)| \leq |N(x)| \cdot 8 \cdot R(l, (l-2) \cdot q + r)$

Proof. Let $x_1 \in N(x)$. We will show that $|N^2(x) \cap N(x_1)| \leq 8 \cdot R(l, (l-2) \cdot q + r)$.

Let $Y_1 = N(x_1) \cap Y$ and $Y' \subseteq Y$. Let $N' = N^2(x) \cap N(x_1) \cap B_Y(Y')$. Since $\{x_1\} \cup N'$ contains no $K_{1,l}$, then N' contains no independent set of size at least l. Let N be the largest clique in N'. If we show that $|N| < (l-2) \cdot q + r$, then

$|N'| < R(l,(l-2)\cdot q+r)$, and since $|Y| = 3$, by Proposition 2.2 we get that $|N^2(x) \cap N(x_1)| \leq 2^3 \cdot R(l,(l-2)\cdot q+r)$.

If $|Y'| = 1$, then $|Y - Y'| = 2$. Then $|N| < r$, since otherwise $(Y - Y') \cup \{x\} \cup Y' \cup N$ contains a $Z_{2,r}$. If $|Y'| \geq 2$, then we get that $|N| < (l-2)\cdot q + r$ in the same way as Claim 5.7.1.

Suppose now that $|Y'| = 0$, that is $N \cap N(Y) = \emptyset$.

If $|Y_1| \geq 2$, then $|N| < r$ since otherwise $Y_1 \cup \{x_1\} \cup N$ contains a $Z_{1,r}$. If $|Y_1| \leq 1$, then $|Y - Y_1| \geq 2$. Then $|N| < r$ since otherwise $(Y - Y_1) \cup \{x\} \cup \{x_1\} \cup N$ contains a $Z_{2,r}$. □

Let $m \geq 2$. Since G is $\{Z_{m,r}, Z_{m+1,r}, Z_{m+2,r}\}$-free, then by Lemma 3.10 we have that $|N^{m+1}(x)| < R(l,r) \cdot |N^m(x)|$. By an inductive argument we get that for all $m \geq 3$, $|N^m(x)| < R(l,r)^{m-2} \cdot |N^2(x)|$. Then by Claims 5.7.1 and 5.7.2, we have that for all $m \geq 3$, $|N^m(x)| < R(l,r)^{m-2} \cdot 8 \cdot R(l,(l-2)\cdot q+r) \cdot 8 \cdot R(l,(l-2)\cdot q+r)$ $= 64 \cdot R(l,r)^{m-2} \cdot R(l,(l-2)\cdot q+r)^2$.

Additionally, since G is Y_{i+2}-free then by Lemma 3.7, $N^{i+3}(x) = \emptyset$. It follows that $|V(G)| < (i+2) \cdot 64 \cdot R(l,r)^i \cdot R(l,(l-2)\cdot q+r)^2 + 1$. □

5.4 Proof of Theorems 5.8 and 5.9

In this section we prove the other two main results of this chapter.

Proof of Theorem 5.8. Suppose that $\mathcal{H} \in \mathbf{H}$ and $|\mathcal{H}| \leq k$. Since $\mathcal{H} \in \mathbf{H}$, there is a positive integer n_0 such that every $\mathcal{H}$-free connected graph of even order at least n_0 has a perfect matching. Let n be an odd integer such that $n \geq \max(n_0, 4)$.

Contrary to the theorem, suppose that $\mathcal{H} \notin \mathbf{H}_i$ for all $1 \leq i \leq 3$, $\mathcal{H} \notin \mathbf{H}_i^A$ for all $1 \leq i \leq k-3$ and $\mathcal{H} \notin \mathbf{H}_i^B$ for all $2 \leq i \leq k-3$. Since $\mathcal{H} \notin \mathbf{H}_i$ for all $1 \leq i \leq 3$, then by Theorem 5.6, $k \geq 4$.

We will consider several connected graphs G of even order at least n containing no perfect matching. Then for each of such graphs G there will be some $H \in \mathcal{H}$ such that $H \preceq G$.

Consider $G = K_{1,n}$. Then there is a graph $H_1 \in \mathcal{H}$ such that $H_1 \preceq G$. All connected subgraphs of G are stars. Since $\mathcal{H} \notin \mathbf{H}_1$, then $\mathcal{H} \not\preceq \{K_{1,3}\}$, and so $H_1 \not\preceq K_{1,3}$. We conclude that

- $H_1 = K_{1,l}$ for some $l \geq 4$.

Consider $G = Y_n$. Since G contains no $K_{1,4}$ then $H_1 \not\preceq G$. Then there is a graph $H_2 \in \mathcal{H}$ such that $H_2 \preceq G$. Since $H_2 \not\preceq K_{1,3}$ then

- $H_2 = P_m$ for some $m \geq 4$ or $H_2 = Y_m$ for some $m \geq 4$.

Since $\mathcal{H} \notin \mathbf{H}_3$, then $\mathcal{H} \not\leq \{Z_{1,r}^-\}$ for all $r \geq 3$.

Consider $G = Z_{1,n+1}$. Since G contains neither $K_{1,4}$ nor P_4 then $H_1 \not\preceq G$ and $H_2 \not\preceq G$. Then there is a graph $H_4 \in \mathcal{H}$ such that $H_4 \preceq G$ (H_3 will be used later). Since $H_4 \not\preceq K_{1,3}$ and $H_4 \not\preceq Z_{1,r}^-$ for all $r \geq 3$, then

- $H_4 = Z_{1,r_1}$ for some $r_1 \geq 3$ (Notice that $Z_{1,2} = K_{1,3}$).

Since $\mathcal{H} \notin \mathbf{H}_2$, then $\mathcal{H} \not\leq \{P_4\}$. Then $H_2 = P_m$ for some $m \geq 5$, or $H_2 = Y_m$ for some $m \geq 4$.

Consider $G = W_n$. Since G contains neither $K_{1,4}, P_5, Y_4$ nor $Z_{1,3}$, then $H_1 \not\preceq G$, $H_2 \not\preceq G$ and $H_4 \not\preceq G$. Then there is a graph $H_3 \in \mathcal{H}$ such that $H_3 \preceq G$. Since $H_3 \not\preceq K_{1,3}$, $H_3 \not\preceq P_4$ and $H_3 \not\preceq Z_{1,r}^-$ for all $r \geq 3$, then

- $H_3 = W_{q,t}^-$ for some $q \geq 3$ and $t \geq 2$ with $q \geq t$, or $H_3 = W_{q,t}$ for some $q \geq 2$ and $t \geq 0$ with $q \geq t$.

Claim 5.8.1. *There are graphs $H_4, \ldots, H_k$ in $\mathcal{H}$ and integers $r_1, \ldots, r_{k-3}$ such that for all $4 \leq i \leq k$, $H_i = Z_{i-3,r_{i-3}}$ and $r_{i-3} \geq 3$. In particular $|\mathcal{H}| = k$*

Proof. The proof is by induction on i. We have already showed the base case $i = 4$.

We prove now the inductive case. If $k = 4$ there is nothing to prove. Suppose that $k \geq 5$ and let $5 \leq i \leq k$. Suppose that we have already proved that there are graphs $H_4, \ldots, H_{i-1}$ in $\mathcal{H}$ such that for all $4 \leq j \leq i-1$, $H_j \preceq Z_{j-3,r_{j-3}}^+$ for some $r_{j-3} \geq 3$. We will prove that there is a graph $H_i \in \mathcal{H}$ such that $H_i = Z_{i-3,r_{i-3}}$ for some $r_{i-3} \geq 3$.

Let $r' = \max(r_1, \ldots, r_{i-4})$. Since $1 \leq i-4 \leq k-4$, then $\mathcal{H} \notin \mathbf{H}_{i-4}^A$. Then since $\mathcal{H} \leq \{K_{1,l}, W_q, Z_{1,r'}, \ldots, Z_{i-4,r'}\}$, $\mathcal{H} \not\leq \{Y_{i-2}\}$. In particular, $H_2 = P_m$ for some $m \geq i-1$ or $H_2 = Y_m$ for some $m \geq i-1$. Since $2 \leq i-3 \leq k-3$, then $\mathcal{H} \notin \mathbf{H}_{i-3}^B$. Then since $\mathcal{H} \leq \{K_{1,l}, Y_m, W_q, Z_{1,r'}, \ldots, Z_{i-4,r'}\}$, $\mathcal{H} \not\leq \{Z_{i-3,r}\}$ for all $r \geq 3$.

If $i-3$ is even, consider $G = Z_{i-3,n}$ and if $i-3$ is odd, consider $G = Z_{i-3,n+1}$. Since G contains neither $K_{1,4}, P_i, W_{3,2}^-$ nor $W_{2,0}$ and that $Z_{j-3,3} \not\preceq G$ for all $4 \leq j \leq i-1$, then G contains no H_j for all $1 \leq j \leq i-1$. Then there is a graph $H_i \in \mathcal{H}$ such that $H_i \preceq G$. Since $H_i \not\preceq Y_{i-2}$ and $H_i \not\preceq Z_{i-3,r}$ for all $r \geq 3$, then $H_i = Z_{i-3,r_{i-3}}$ for some $r_{i-3} \geq 3$ (Notice that $Z_{i-3,2}$ contains a Y_{i-2}). □

Let $r = \max(r_1, \ldots, r_{k-3})$.

Suppose first that $\mathcal{H} \leq \{Y_{k-1}\}$. Since $|\mathcal{H}| = k$, then there must be some $1 \leq i \leq k$ such that $H_i \preceq Y_k$. But the only graph for which that is possible is H_2. Then $H_2 = P_m$ with $m \leq k-1$ or $H_2 = Y_m$ with $m \leq k-1$ and so $H_2 \preceq Y_{k-1}$. We conclude that $\mathcal{H} \leq \mathcal{H}_{k-3}^A(l,q,r)$ and so $\mathcal{H} \in \mathbf{H}_{k-3}^A$.

Suppose now that $\mathcal{H} \not\leq \{Y_{k-1}\}$. Then $H_2 = P_m$ for some $m \geq k$ or $H_2 = Y_m$ for some $m \geq k$.

If $k-2$ is even, consider $G = Z_{k-2,n}$ and if $k-2$ is odd, consider $G = Z_{k-2,n+1}$. Since G contains neither $K_{1,4}$, $P_{k+1}, W_{3,2}^{-}$ nor $W_{2,0}$ and that $Z_{j-3,3} \not\preceq G$ for all $4 \leq j \leq k$, then G contains no H_j for all $1 \leq j \leq k$. We conclude that G is $\mathcal{H}$-free, a contradiction. □

Proof of Theorem 5.9. By Theorem 5.7, we already know that for $i \geq 1$, every family of graphs $\mathcal{H}$ in $\mathbf{H}_i^A$ is also in $\mathbf{H}$.

Let $\mathcal{H} \in \mathbf{H}$. Then there is a positive integer n_0 such that every $\mathcal{H}$-free connected graph of even order at least n_0 has a perfect matching. Let n be an odd integer such that $n \geq \max(n_0, 4)$.

Consider the following family of graphs. $\mathcal{H}' = \{ K_{1,n}, Y_{n+2}, W_n \} \cup \{ Z_{m,n}\colon m$ is even and $1 \leq m \leq n\} \cup \{ Z_{m,n+1}\colon m$ is odd and $1 \leq m \leq n \}$. Notice that $\mathcal{H}' \leq \mathcal{H}_n^A(n, n, n+1)$ and so $\mathcal{H}' \in \mathbf{H}_n^A$.

Since n is odd, all the graph in $\mathcal{H}'$ are connected graph of even order at least n_0. Furthermore, none of them have a perfect matching. Then it must be that no graph of $\mathcal{H}'$ is $\mathcal{H}$-free. In other words, for each $H_2 \in \mathcal{H}'$, there is an $H_1 \in \mathcal{H}$ such that $H_1 \preceq H_2$. We conclude that $\mathcal{H} \leq \mathcal{H}'$. But since $\mathcal{H}'$ is in $\mathbf{H}_n^A$ then $\mathcal{H}$ is also in $\mathbf{H}_n^A$. □

5.5 Conclusions and open problems

The characterization we were looking for is given by Theorem 5.9. It is interesting to notice that by this result, every infinite family in $\mathbf{H}$ has a finite subfamily that is enough to imply a perfect matching (under the assumptions of connectedness and large enough even order). This fact might be surprising but it is easily explained as follows.

The graph that is needed to bound the diameter of the graph G ($N^{i+3}(x) = \emptyset$ near the end of Theorem 5.7), in this case Y_i, has the property that $Y_i \preceq Y_{i+1}$. This implies that it is not necessary to forbid Y_m for all m greater than some integer m_0, but it is enough to forbid Y_m just for one value of m. This phenomenon does not happens for example when considering the problem for near perfect matchings, as we showed in Chapter 6. In that case, the family that characterize the corresponding set $\mathbf{H}$ has an infinite number of graphs and any proper subfamily is not enough to imply that the graph has a near perfect matching .

Regarding the characterization family $\mathcal{H}_i^A(l, q, r)$, notice that it is not necessary that the parameters "r" of all the $Z_{m,r}$ are the same. In other words, we could have r_1, r_2 and so on. But we think that doing so would make the proof of Theorem 5.7 more difficult to understand and it does not add real value to the result.

About the characterization itself, may be it is interesting to notice that there is only one set of families of size 4, namely $\mathbf{H}_2^A$. For size 3, there are two sets: $\mathbf{H}_2$ and $\mathbf{H}_3$. And for all other sizes $i \geq 5$, there are also two sets: $\mathbf{H}_{i-2}^A$ and $\mathbf{H}_{i-2}^B$.

In this chapter we considered connected graphs with no degree conditions. But it is also possible to consider graphs with higher connectivity or with some minimum degree condition. In this line of research, we propose the following problems.

Problem 5.1. *Let $k \geq 1$. Characterize all the families of connected graphs $\mathcal{H}$ satisfying the following property. Every $\mathcal{H}$-free k-connected graph of large enough even order has a perfect matching.*

In this chapter we were able to resolve Problem 5.1 for the case $k = 1$.

Problem 5.2. *Let $d \geq 3$. Characterize all the families of connected graphs $\mathcal{H}$ satisfying the following property. Every $\mathcal{H}$-free connected graph of large enough even order and minimum degree at least d has a perfect matching.*

Even a combination of Problems 5.1 and 5.2 is possible.

Chapter 6

Near Perfect Matchings

In this chapter, we study the relation between near perfect matchings and forbidden induced subgraphs. The main result in this chapter is Theorem 6.5, which shows a characterization of all families of forbidden subgraphs implying a near perfect matching in graphs of large enough odd order. All the new results we prove in this chapter can be found in [26].

6.1 Introduction

Fujita et al.[18] studied the relation between forbidden induced subgraphs and the resulting deficiency.

Let T_n be the graph obtained by attaching 2 independent vertices to each end of a path on n vertices. For $d \geq 0$, let $\mathcal{T}_d = \{K_{1,3+d}\} \cup \{\ T_n \colon n \geq 2\ \}$.

Theorem 6.1 ([18])**.** *Let $d \geq 0$. Then every $\mathcal{T}_d$-free connected graph G with $|V(G)| \equiv d \pmod 2$ satisfies $\operatorname{def}(G) \leq d$.*

In the same paper, the authors also proved that for the near perfect matching case ($d = 1$), the family T_d is essentially the only family with that property when forbidding only triangle-free graphs. In other words, they proved the following theorem.

Theorem 6.2 ([18])**.** *Let $\mathcal{F}$ be a family of triangle-free connected graphs. If there exists a positive constant n_0 such that every $\mathcal{F}$-free graph of odd order at least n_0 has a near perfect matching, then $\mathcal{F} \leq \mathcal{T}_1$.*

In this chapter, we complete this line of research by characterizing all families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free graph of odd order has a near perfect matching (without any restriction on the graphs of $\mathcal{F}$). Our main result of this chapter is Theorem 6.5 that we state in Section 6.3.

The rest of the chapter is organized as follows. In Section 6.2, we give some definitions and cite some useful known results. In Section 6.3, we define several graphs and families of graphs and state our results. In Section 6.4, we give the proofs for our results. Finally, in Section 6.5, we make some final remarks.

6.2 Definitions and useful results

Define $\mathbf{G}$ as the set of all non-redundant families of connected graphs. Define $\mathbf{H}$ as the set of families $\mathcal{F} \in \mathbf{G}$ such that there is a constant $n_0 = n_0(\mathcal{F})$ with the property that all $\mathcal{F}$-free connected graphs G of odd order with $|V(G)| \geq n_0$ have a near perfect matching. Clearly, our problem is reduced to finding all the elements in the set $\mathbf{H}$.

With the previous definitions, Theorem 6.2 says that If $\mathcal{F} \in \mathbf{H}$ and every graph in $\mathcal{F}$ is triangle-tree then $\mathcal{F} \leq \mathcal{T}_1$.

Let G be a graph. An *odd component* of G is a connected component of G of odd order. Let $c_o(G)$ be the number of odd components of G. We use the following theorem from Berge (see Theorem 3.1.14 of [25]) as our main tool in the proofs.

Theorem 6.3 ([25])**.** *Let G be a graph. Then* $\operatorname{def}(G) = \max_{S \subseteq V(G)} (c_o(G-S) - |S|)$.

The formula in Theorem 6.3 is called the Tutte-Berge Formula. Theorem 6.3 can be rewritten for the case of near perfect matching as follows.

Theorem 6.4. *A graph G of odd order has a near perfect matching if and only if for every set $S \subseteq V(G)$, $c_o(G-S) \leq |S| + 1$.*

A set $S \subseteq V(G)$ not satisfying the inequality in Theorem 6.4 is a *Tutte set.* A Tutte set S is said to be *minimal* if for every $S' \subset S$, S' is not a Tutte set. Given a Tutte set S, for each $x \in S$ define

$$\mathcal{C}_S(x) = \{C\colon C \text{ is a component of } G-S \text{ such that } N_G(x) \cap V(C) \neq \emptyset\ \}.$$

Because S is usually clear from the context, we write $\mathcal{C}(x)$ instead of $\mathcal{C}_S(x)$. For a subset $X \subseteq S$, define

- $\mathcal{C}(X) = \bigcup_{x \in X} \mathcal{C}(x)$
- $\mathcal{C}_o(X) =$ the set of odd components in $\mathcal{C}(X)$

Let G be a connected graph. If $v, w \in V(G)$ and $H \subseteq V(G)$, let $\operatorname{dist}(v, w)$ be the distance from v to w, and $\operatorname{dist}(H, v) = \min_{w \in H} \operatorname{dist}(v, w)$.

6.3 Graphs and Families

In this section, we define the graphs and families of graphs necessary to state our results. At the end of this section we state Theorem 6.5 that is our main result of this chapter.

In the drawings of the graphs we define in this section, the white vertices are a minimal Tutte set for the corresponding graph. We give more detail about this in Section 6.4.

First, we define some auxiliary graphs that we use later to define the graphs and families we need.

- $A_{s,q}$ is the graph obtained by joining a set $\{w_1, \ldots, w_s\}$ of s independent vertices with a K_q, and then attaching one pendant vertex to each vertex of the K_q. The vertices $w_1, \ldots, w_s$ are called the heads of $A_{s,q}$.

- $B_{m,q}$ is the graph obtained by identifying the last vertex of a P_m with a vertex of a K_q. The first vertex of the P_m is called the head of $B_{m,q}$.

- C^A_n is an induced path $p_1 \ldots p_n$.

- $C^B_{n_1,n_2}$ is an induced path $p_1 \ldots p_{n_1+n_2}$ and a vertex z adjacent only to p_{n_1} and p_{n_1+1}.

- $C^C_{n_1,n_2}$ is an induced path $p_1 \ldots p_{n_1+n_2+1}$ and a vertex z adjacent only to p_{n_1}, p_{n_1+1} and p_{n_1+2}.

- $C^D_{n_1,n_2}$ is an induced path $p_1 \ldots p_{n_1+n_2-1}$ and a vertex z adjacent only to p_{n_1}.

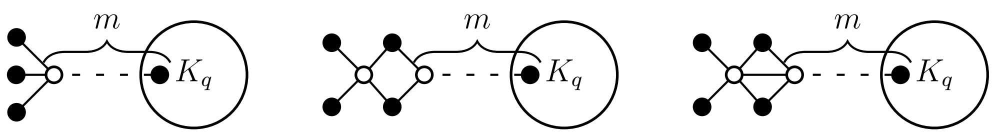
- $C^E_{n_1,n_2}$ is an induced path $p_1 \ldots p_{n_1+n_2+1}$ and a vertex z adjacent only to p_{n_1} and p_{n_1+2}.

If $C \in \{C^A_n, C^B_{n_1,n_2}, C^C_{n_1,n_2}, C^D_{n_1,n_2}, C^E_{n_1,n_2}\}$ is a subgraph of a graph G in which two vertices x_1 and x_2 of G are the ends of the path in C, then we call C a *connector* between x_1 and x_2. If C is a connector other than C^A_n, then z is called the *balancer* of C. A connector is said to be odd if it has an odd number of vertices. We say that C^A_n, $C^B_{n_1,n_2}$, $C^C_{n_1,n_2}$, $C^D_{n_1,n_2}$, $C^E_{n_1,n_2}$ are connectors of type A, B, C, D and E, respectively.

We define now the graphs and families we need to state our results. The families with a "hat" on top are defined only to be used later in our proofs. We divide the graphs in "types", which is indicated by the letter we use to define them. In the definitions, $\bar{n}$, $\bar{m}$ and $\bar{q}$ are positive integers.

Type "Z" graphs:

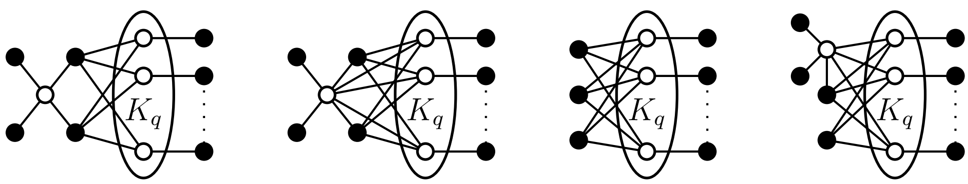
- $Z^1_{m,q}$ is the graph obtained by adding three pendant vertices to the head of a $B_{m,q}$.
- $Z^2_{m,q}$ is the graph obtained by putting together a $K_{1,4}$ and a $B_{m,q}$ of head x, and adding an edge between x and two degree one vertices of the $K_{1,4}$.
- $Z^3_{m,q}$ is the graph obtained by adding the edge between x and the center of the $K_{1,4}$ in a $Z^2_{m,q}$.
- $\mathcal{Z}(\bar{m},\bar{q}) = \bigcup_{i=1}^{3}(\{Z^i_{1,\bar{q}}, Z^i_{2,\bar{q}}, \ldots, Z^i_{\bar{m},\bar{q}}, Z^i_{\bar{m}+2,1}\})$.
- $\widehat{\mathcal{Z}}(\bar{q}) = \bigcup_{i=1}^{3}(\{Z^i_{m,\bar{q}} : m \geq 1\})$.



$Z^1_{m,q}$ $Z^2_{m,q}$ $Z^3_{m,q}$

Type "V" graphs:

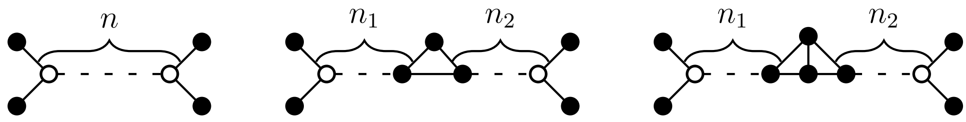
- V^1_q is the graph obtained by identifying the heads w_1 and w_2 of a $A_{2,q}$ with two degree one vertices of a $K_{1,4}$.
- V^2_q is the graph obtained by identifying the heads w_1, w_2 and w_3 of a $A_{3,q}$ with two degree one vertices and the center of a $K_{1,4}$.
- V^3_q is $A_{3,q}$.
- V^4_q is the graph obtained by putting together a $A_{3,q}$ of heads w_1, w_2, w_3 and two extra vertices z_1 and z_2 and adding the edges w_1w_2, w_1z_1 and w_1z_2.
- $\mathcal{V}(q) = \{V^1_q, V^2_q, V^3_q, V^4_q\}$



V^1_q V^2_q V^3_q V^4_q

Type "T" graphs:

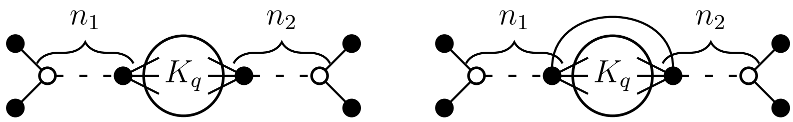
- Let T_n^A, T_{n_1,n_2}^B and T_{n_1,n_2}^C be the graphs obtained by adding two pendant vertices y_1^a, y_1^b to one end of a C_n^A, a C_{n_1,n_2}^B or a C_{n_1,n_2}^C, respectively, and two pendant vertices y_2^a, y_2^b to the other end. Notice that $T_n^A = T_n$.
- $\mathcal{T}^A(\bar{n}) = \{ \ T_n^A\colon\ n \text{ is odd and } n \geq \bar{n}+1 \ \}$
- $\mathcal{T}^B(\bar{n}) = \{ \ T_{n_1,n_2}^B\colon\ n_1+n_2 \text{ is even, } n_1,n_2 \geq 1 \text{ and } n_1+n_2 \geq \bar{n}+1 \ \}$
- $\mathcal{T}^C(\bar{n}) = \{ \ T_{n_1,n_2}^C\colon\ n_1+n_2 \text{ is odd, } n_1,n_2 \geq 1 \text{ and } n_1+n_2 \geq \bar{n}+1 \ \}$
- $\mathcal{T}(\bar{n}) = \mathcal{T}^A(\bar{n}) \cup \mathcal{T}^B(\bar{n}) \cup \mathcal{T}^C(\bar{n})$



T_n^A T_{n_1,n_2}^B T_{n_1,n_2}^C

Type "Y" graphs:

- $Y_{n_1,n_2,q}^1$ is the graph obtained by joining all the vertices of a K_q with the last vertex of a P_{n_1} and with the last vertex of a P_{n_2}, and adding two pendant vertices to the first vertex of the P_{n_1} and two pendant vertices to the first vertex of the P_{n_2}.
- $Y_{n_1,n_2,q}^2$ is the graph obtained by adding the edge between the last vertex of the P_{n_1} and the last vertex of the P_{n_2} in a $Y_{n_1,n_2,q}^1$.
- $\mathcal{Y}^1(\bar{n},\bar{q}) = \{Y_{n_1,n_2,\bar{q}}^1\colon\ n_1,n_2 \geq 1 \text{ and } n_1+n_2+1 \leq \bar{n} \ \}$.
- $\mathcal{Y}^2(\bar{n},\bar{q}) = \{Y_{n_1,n_2,\bar{q}}^2\colon\ n_1,n_2 \geq 1 \text{ and } n_1+n_2 \leq \bar{n} \ \}$.
- $\mathcal{Y}(\bar{n},\bar{q}) = \mathcal{Y}^1(\bar{n},\bar{q}) \cup \mathcal{Y}^2(\bar{n},\bar{q})$
- $\widehat{\mathcal{Y}}(\bar{q}) = \{ \ Y_{n_1,n_2,\bar{q}}^1\colon\ n_1,n_2 \geq 1 \ \} \cup \{ \ Y_{n_1,n_2,\bar{q}}^2\colon\ n_1,n_2 \geq 1 \ \}$.



$Y_{n_1,n_2,q}^1$ $Y_{n_1,n_2,q}^2$

Type "D" graphs:

- $D^1_{n_1,n_2,m,q}$ is the graph obtained by identifying the p_{n_1} of the path of the connector in a $T^A_{n_1+n_2-1}$ and the head of a $B_{m,q}$.

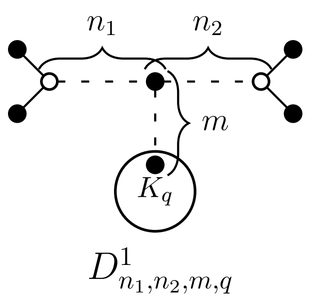
- $D^2_{n_1,n_2,m,q}$ is the graph obtained by identifying the balancer z of the connector in a $T^B_{n_1,n_2}$ and the head of a $B_{m,q}$.

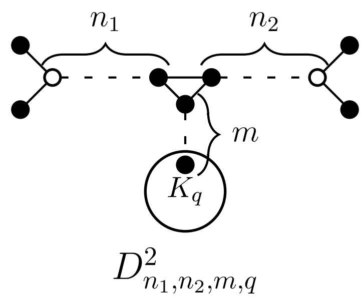
- $\mathcal{D}^1(\bar{n},\bar{m},\bar{q}) = \{\ D^1_{n_1,n_2,1,\bar{q}},\ D^1_{n_1,n_2,2,\bar{q}},\ \ldots,\ D^1_{n_1,n_2,\bar{m},\bar{q}},\ D^1_{n_1,n_2,\bar{m}+2,1}$: n_1, n_2 are even, $n_1, n_2 \geq 2$, $n_1 \leq n_2$ and $n_1 + n_2 - 1 \leq \bar{n}\ \}$.

- $\mathcal{D}^2(\bar{n},\bar{m},\bar{q}) = \{\ D^2_{n_1,n_2,1,\bar{q}},\ D^2_{n_1,n_2,2,\bar{q}},\ \ldots,\ D^2_{n_1,n_2,\bar{m},\bar{q}},\ D^2_{n_1,n_2,\bar{m}+2,1} : n_1 + n_2 \leq \bar{n}$ and $n_1, n_2 \geq 1\}$.

- $\mathcal{D}(\bar{n},\bar{m},\bar{q}) = \mathcal{D}^1(\bar{n},\bar{m},\bar{q}) \cup \mathcal{D}^2(\bar{n},\bar{m},\bar{q})$

- $\widehat{\mathcal{D}}(\bar{q}) = \{\ D^1_{n_1,n_2,m,\bar{q}}, D^2_{n_1,n_2,m,\bar{q}} : n_1, n_2 \geq 1$ and $m \geq 1\ \}$.





Type "L" graphs:

- $L^{1A}_{n,m,q}$, $L^{1B}_{n_1,n_2,m,q}$, $L^{1C}_{n_1,n_2,m,q}$, $L^{1D}_{n_1,n_2,m,q}$ and $L^{1E}_{n_1,n_2,m,q}$ are the graphs obtained by identifying the head of a $B_{m,q}$ and the p_n of a C^A_n, a $C^B_{n_1,n_2}$, a $C^C_{n_1,n_2}$, a $C^D_{n_1,n_2}$ and a $C^E_{n_1,n_2}$ (respectively) and adding a pendant vertex to p_n and two pendant vertices to p_1.

- $L^{2A}_{n,m,q}$, $L^{2B}_{n_1,n_2,m,q}$, $L^{2C}_{n_1,n_2,m,q}$, $L^{2D}_{n_1,n_2,m,q}$ and $L^{2E}_{n_1,n_2,m,q}$ are the graphs obtained by putting together a $B_{m,q}$ of head x and a C^A_n, a $C^B_{n_1,n_2}$, a $C^C_{n_1,n_2}$, a $C^D_{n_1,n_2}$ and a $C^E_{n_1,n_2}$ (respectively) of ends p_1 and p_n, adding two vertices z_1 and z_2, adding the edges xz_1, xz_2, p_nz_1, p_nz_2 and adding two pendant vertices to p_1.

- $L^{3A}_{n,m,q}$, $L^{3B}_{n_1,n_2,m,q}$, $L^{3C}_{n_1,n_2,m,q}$, $L^{3D}_{n_1,n_2,m,q}$ and $L^{3E}_{n_1,n_2,m,q}$ are the graphs obtained by adding the edge xp_n in a $L^{2A}_{n,m,q}$, in a $L^{2B}_{n_1,n_2,m,q}$, in a $L^{2C}_{n_1,n_2,m,q}$, in a $L^{2D}_{n_1,n_2,m,q}$ and in a $L^{2E}_{n_1,n_2,m,q}$ (respectively).

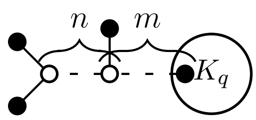
- $\mathcal{L}^{iA}(\bar{n},\bar{m},\bar{q}) = \{\ L^{iA}_{n,1,\bar{q}},\ L^{iA}_{n,2,\bar{q}},\ \ldots,\ L^{iA}_{n,\bar{m},\bar{q}},\ L^{iA}_{n,\bar{m}+2,1}$: n is odd, $3 \leq n \leq \bar{n}\ \}$.

- $\mathcal{L}^{iB}(\bar{n},\bar{m},\bar{q}) = \{\, L^{iB}_{n_1,n_2,1,\bar{q}},\, L^{iB}_{n_1,n_2,2,\bar{q}},\, \ldots,\, L^{iB}_{n_1,n_2,\bar{m},\bar{q}},\, L^{iB}_{n_1,n_2,\bar{m}+2,1}$: n_1+n_2 is even, $n_1+n_2 \le \bar{n}$ and $n_1, n_2 \ge 1\}$.

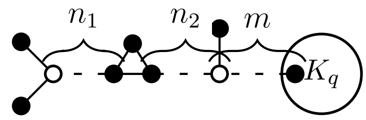
- $\mathcal{L}^{iC}(\bar{n},\bar{m},\bar{q}) = \{\, L^{iC}_{n_1,n_2,1,\bar{q}},\, L^{iC}_{n_1,n_2,2,\bar{q}},\, \ldots,\, L^{iC}_{n_1,n_2,\bar{m},\bar{q}},\, L^{iC}_{n_1,n_2,\bar{m}+2,1}$: n_1+n_2 is odd, $n_1+n_2 \le \bar{n}$ and $n_1, n_2 \ge 1\}$.

- $\mathcal{L}(\bar{n},\bar{m},\bar{q}) = \bigcup_{i=1}^{3} (\mathcal{L}^{iA}(\bar{n},\bar{m},\bar{q}) \cup \mathcal{L}^{iB}(\bar{n},\bar{m},\bar{q}) \cup \mathcal{L}^{iC}(\bar{n},\bar{m},\bar{q}))$.

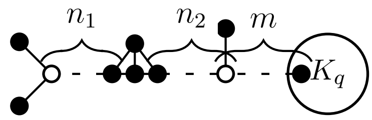
- $\widehat{\mathcal{L}}(\bar{q}) = \bigcup_{i=1}^{3} (\{L^{iA}_{n,m,\bar{q}}$: n is odd, $n, m \ge 1\} \cup \{L^{iB}_{n_1,n_2,m,\bar{q}}$: n_1+n_2 is even and $n_1, n_2, m \ge 1\} \cup \{L^{iC}_{n_1,n_2,m,\bar{q}},\, L^{iD}_{n_1,n_2,m,\bar{q}},\, L^{iE}_{n_1,n_2,m,\bar{q}}$: n_1+n_2 is odd and $n_1, n_2, m \ge 1\,\})$.



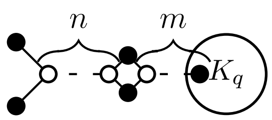
$L^{1A}_{n,m,q}$



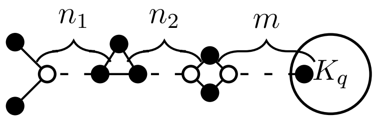
$L^{1B}_{n_1,n_2,m,q}$



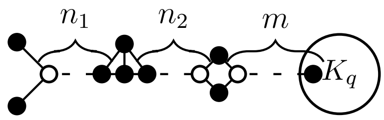
$L^{1C}_{n_1,n_2,m,q}$



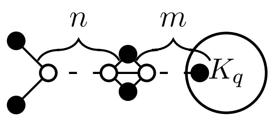
$L^{2A}_{n,m,q}$



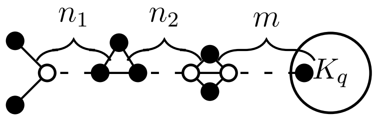
$L^{2B}_{n_1,n_2,m,q}$



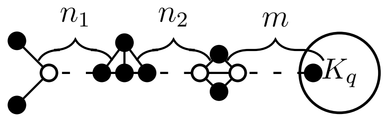
$L^{2C}_{n_1,n_2,m,q}$



$L^{3A}_{n,m,q}$



$L^{3B}_{n_1,n_2,m,q}$



$L^{3C}_{n_1,n_2,m,q}$

Type "W" graphs:

- $W^{1A}_{n,q}$, $W^{1B}_{n_1,n_2,q}$, $W^{1C}_{n_1,n_2,q}$, $W^{1D}_{n_1,n_2,q}$ and $W^{1E}_{n_1,n_2,q}$ are the graphs obtained by putting together a $A_{2,q}$ of heads w_1, w_2 and a C^A_n, a $C^B_{n_1,n_2}$, a $C^C_{n_1,n_2}$, a $C^D_{n_1,n_2}$ and a $C^E_{n_1,n_2}$ (respectively) of ends p_1, p_n, adding the edges w_1p_n and w_2p_n and adding two pendant vertices to p_1.

- $W^{2A}_{n,q}$, $W^{2B}_{n_1,n_2,q}$, $W^{2C}_{n_1,n_2,q}$, $W^{2D}_{n_1,n_2,q}$ and $W^{2E}_{n_1,n_2,q}$ are the graphs obtained by identifying the w_2 of a $A_{3,q}$ of heads w_1, w_2, w_3 and the p_n of a C^A_n, a $C^B_{n_1,n_2}$, a $C^C_{n_1,n_2}$, a $C^D_{n_1,n_2}$ and a $C^E_{n_1,n_2}$ (respectively) of ends p_1, p_n, adding the edges w_1p_n and w_3p_n and adding two pendant vertices to p_1.

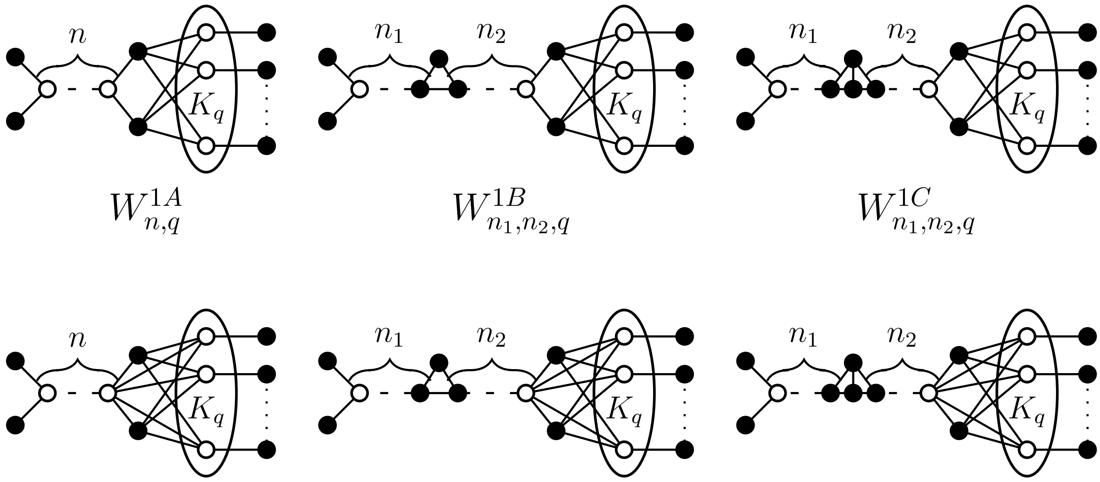
- $\mathcal{W}^{iA}(\bar{n},\bar{q}) = \{\ W^{iA}_{n,\bar{q}}$: n is odd, $3 \le n \le \bar{n}\ \}$.

- $\mathcal{W}^{iB}(\bar{n},\bar{q}) = \{\ W^{iB}_{n_1,n_2,\bar{q}}$: $n_1 + n_2$ is even, $n_1 + n_2 \le \bar{n}$ and $n_1, n_2 \ge 1\ \}$.

- $\mathcal{W}^{iC}(\bar{n},\bar{q}) = \{\ W^{iC}_{n_1,n_2,\bar{q}}$: $n_1 + n_2$ is odd, $n_1 + n_2 \le \bar{n}$ and $n_1, n_2 \ge 1\ \}$.

- $\mathcal{W}(\bar{n},\bar{q}) = \bigcup_{i=1}^{2}(\mathcal{W}^{iA}(\bar{n},\bar{q}) \cup \mathcal{W}^{iB}(\bar{n},\bar{q}) \cup \mathcal{W}^{iC}(\bar{n},\bar{q}))$.

- $\widehat{\mathcal{W}}(\bar{q}) = \bigcup_{i=1}^{2}$ ($\{W^{iA}_{n,\bar{q}}$: n is odd and $n \ge 1\} \cup \{W^{iB}_{n_1,n_2,\bar{q}}$: $n_1 + n_2$ is even and $n_1, n_2 \ge 1\} \cup \{W^{iC}_{n_1,n_2,\bar{q}}$, $W^{iD}_{n_1,n_2,\bar{q}}$, $W^{iE}_{n_1,n_2,\bar{q}}$: $n_1 + n_2$ is odd and $n_1, n_2 \ge 1\ \}$).



$W^{1A}_{n,q}$ $W^{1B}_{n_1,n_2,q}$ $W^{1C}_{n_1,n_2,q}$

$W^{2A}_{n,q}$ $W^{2B}_{n_1,n_2,q}$ $W^{2C}_{n_1,n_2,q}$

Type "M" graphs:

- $M^{A}_{n,q}$, $M^{B}_{n_1,n_2,q}$, $M^{C}_{n_1,n_2,q}$, $M^{D}_{n_1,n_2,q}$ and $M^{E}_{n_1,n_2,q}$ are the graphs obtained by identifying the w_2 of a $A_{2,q}$ of heads w_1, w_2 and the p_n of a C^{A}_{n}, a $C^{B}_{n_1,n_2}$, a $C^{C}_{n_1,n_2}$, a $C^{D}_{n_1,n_2}$ and a $C^{E}_{n_1,n_2}$ (respectively) of ends p_1, p_n and adding two pendant vertices to p_1.

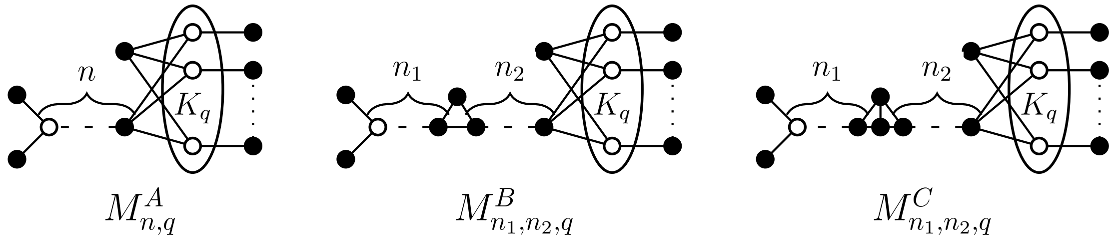
- $\mathcal{M}^{A}(\bar{n},\bar{q}) = \{\ M^{A}_{n,\bar{q}}$: n is even, $3 \le n+1 \le \bar{n}\ \}$.

- $\mathcal{M}^{B}(\bar{n},\bar{q}) = \{\ M^{B}_{n_1,n_2,\bar{q}}$: $n_1 + n_2$ is odd, $n_1 + n_2 + 1 \le \bar{n}$ and $n_1, n_2 \ge 1\ \}$.

- $\mathcal{M}^{C}(\bar{n},\bar{q}) = \{\ M^{C}_{n_1,n_2,\bar{q}}$: $n_1 + n_2$ is even, $n_1 + n_2 + 1 \le \bar{n}$ and $n_1, n_2 \ge 1\ \}$.

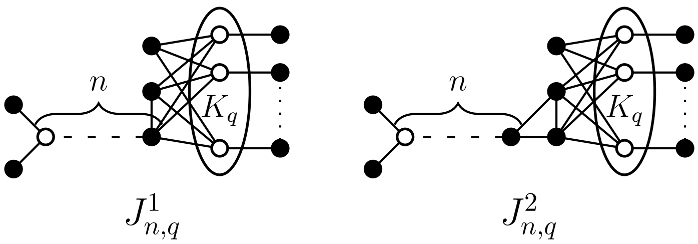
- $\mathcal{M}(\bar{n},\bar{q}) = \mathcal{M}^{A}(\bar{n},\bar{q}) \cup \mathcal{M}^{B}(\bar{n},\bar{q}) \cup \mathcal{M}^{C}(\bar{n},\bar{q})$.

- $\widehat{\mathcal{M}}(\bar{q}) = \{M^{A}_{n,\bar{q}}$: n is even and $n \ge 2\} \cup \{M^{B}_{n_1,n_2,\bar{q}}$: $n_1{+}n_2$ is odd and $n_1, n_2 \ge 1\}$ $\cup$ $\{M^{C}_{n_1,n_2,\bar{q}}$, $M^{iD}_{n_1,n_2,\bar{q}}$, $M^{iE}_{n_1,n_2,\bar{q}}$: $n_1 + n_2$ is even and $n_1, n_2 \ge 1\ \}$.



Type "J" graphs:

- $J^1_{n,q}$ is the graph obtained by identifying the w_3 of a $A_{3,q}$ of heads w_1, w_2, w_3 and the p_n of a C^A_n of ends p_1, p_n, adding the edge w_2w_3 and adding two pendant vertices to p_1.
- $J^2_{n,q}$ is the graph obtained by putting together a $A_{3,q}$ of heads w_1, w_2, w_3 and a C^A_n of ends p_1, p_n, adding the edges w_2w_3, w_2p_n and w_3p_n and adding two pendant vertices to p_1.
- $\mathcal{J}^1(\bar{n}, \bar{q}) = \{\ J^1_{n,\bar{q}}$: n is odd, $4 \leq n+1 \leq \bar{n}\ \}$.
- $\mathcal{J}^2(\bar{n}, \bar{q}) = \{\ J^2_{n,\bar{q}}$: n is even, $3 \leq n+1 \leq \bar{n}\ \}$.
- $\mathcal{J}(\bar{n}, \bar{q}) = \mathcal{J}^1(\bar{n}, \bar{q}) \cup \mathcal{J}^2(\bar{n}, \bar{q})$.
- $\widehat{\mathcal{J}}(\bar{q}) = \{\ J^1_{n,\bar{q}}$: n is odd, $n \geq 1\ \} \cup \{\ J^2_{n,\bar{q}}$: n is even, $n \geq 2\ \}$.



Now, we are ready to define the family of graph characterizing the set $\mathbf{H}$ and state the main theorem of this chapter. Let l,n,m and q be positive integers.

- $\mathcal{F}(l, n, m, q) = \{K_{1,l}\} \cup \mathcal{V}(q) \cup \mathcal{Z}(m, q) \cup \mathcal{T}(n) \cup \mathcal{Y}(n, q) \cup \mathcal{D}(n, m, q) \cup \mathcal{L}(n, m, q) \cup \mathcal{W}(n, q) \cup \mathcal{M}(n, q) \cup \mathcal{J}(n, q)$.

Theorem 6.5. *Let $\mathcal{F} \in \mathbf{G}$ be a non-redundant family of connected graphs. Then $\mathcal{F} \in \mathbf{H}$ if and only if $\mathcal{F} \leq \mathcal{F}(l, n, m, q)$ for some $l \geq 5$, $n \geq 1$, $m \geq 1$ and $q \geq 3$.*

Theorem 6.5 is our main result of this chapter. We prove it in the next section.

We define two additional families that will be used during the proof of Theorem 6.5. Let l,m and q be positive integers.

- $\mathcal{F}_A(l,m,q) = \{K_{1,l}\} \cup \{V_q^1, V_q^2, V_q^3\} \cup \mathcal{Z}(m,q)$.
- $\widehat{\mathcal{F}}(q) = \widehat{\mathcal{Z}}(q) \cup \widehat{\mathcal{Y}}(q) \cup \widehat{\mathcal{D}}(q) \cup \widehat{\mathcal{L}}(q) \cup \widehat{\mathcal{W}}(q) \cup \widehat{\mathcal{M}}(q) \cup \widehat{\mathcal{J}}(q)$.

6.4 Proof of Theorem 6.5

In this section we prove Theorem 6.5. We divide the proof of Theorem 6.5 in the following two theorems.

Theorem 6.6. *Let G be a connected graph of odd order. Suppose that there is a minimal Tutte set $S \subseteq V(G)$, such that there is a vertex $x \in S$ with $|\mathcal{C}_o(x)| \geq 4$. Suppose that G is $\mathcal{F}_A(l,m,q)$-free for some $l \geq 5, m \geq 1$ and $q \geq 3$. Then $|V(G)|$ is bounded by a function depending only on l, m and q.*

Theorem 6.7. *Let G be a connected graph of odd order. Suppose that there is a minimal Tutte set $S \subseteq V(G)$ such that for every $x \in S$, $|\mathcal{C}_o(x)| \leq 3$. Suppose that G is $\mathcal{F}(l,n,m,q)$-free for some $l \geq 5, n \geq 1, m \geq 1$ and $q \geq 3$. Then $|V(G)|$ is bounded by a function depending only on l, n, m and q.*

Theorems 6.6 and 6.7 are proved later in this section. We present now the proof of Theorem 6.5 assuming Theorems 6.6 and 6.7.

Proof of Theorem 6.5. Let $\mathcal{F} \in \mathbf{G}$ be a non-redundant family of connected graphs.

Suppose that $\mathcal{F} \leq \mathcal{F}(l,n,m,q)$ for some $l \geq 5, n \geq 1, m \geq 1$ and $q \geq 3$. Let G be a $\mathcal{F}$-free connected graph of odd order. Suppose that G does not have a near perfect matching. By Theorem 6.4, G has a Tutte set S. Take S to be minimal. Then by Theorems 6.6 and 6.7, $|V(G)|$ is bounded by a function depending only on l, n, m and q. We conclude that $\mathcal{F} \in \mathbf{H}$.

Suppose now that $\mathcal{F} \in \mathbf{H}$. Then there is a positive integer n_0 such that every $\mathcal{F}$-free connected graph of odd order at least n_0 has a near perfect matching. Let n be an integer such that $n \geq \max(n_0, 5)$.

Consider the family $\mathcal{F}_1 = \mathcal{F}(n,n,n,n)$. We construct another family of graphs $\mathcal{F}_2$ as follows. First, add to $\mathcal{F}_2$ all the graphs of $\mathcal{F}_1$ that have an odd number of vertices. It is easy to check that the graphs of type "V", "T", "W", "M" and "J" that are in $\mathcal{F}_1$ have an odd number of vertices, and so are also in $\mathcal{F}_2$. Modify graphs of type "Z", "Y", "D" and "L" that are in $\mathcal{F}_1$ and do not have an odd number of vertices by adding 1 to the parameter q in those graphs that the parameter q is at least 3, and adding 1 to the parameter m in those graphs that the parameter q is 1, and add all of them to $\mathcal{F}_2$. Notice that the family $\mathcal{F}_2$ satisfies $\mathcal{F}_2 \leq \mathcal{F}(n,n,n+1,n+1)$.

By the way $\mathcal{F}_2$ was constructed, all the graphs in $\mathcal{F}_2$ are connected graph of odd order at least n_0. In the drawings of Section 6.3, we showed a minimal Tutte set for each graph. It is not difficult to check that all the components that remain after removing the Tutte set S have odd order and satisfy $c_o(G-S) \geq |S|+3$. We can conclude then that none of the graphs in $\mathcal{F}_2$ has a near perfect matching.

Then it must be that no graph of $\mathcal{F}_2$ is $\mathcal{F}$-free. In other words, for each $H_2 \in \mathcal{F}_2$, there is an $H \in \mathcal{F}$ such that $H \preceq H_2$. We conclude that $\mathcal{F} \leq \mathcal{F}_2$. But since $\mathcal{F}_2 \leq \mathcal{F}(n,n,n+1,n+1)$ then $\mathcal{F} \leq \mathcal{F}(n,n,n+1,n+1)$. This completes the "only if" part of the proof. □

Auxiliary Lemmas

In this section, we prove some auxiliary lemmas that we use later in the proofs of Theorems 6.6 and 6.7.

Lemma 6.8. *Let G be a connected graph of odd order. Suppose that there is a minimal Tutte set $S \subseteq V(G)$. Then for every nonempty subset $X \subseteq S$, $|\mathcal{C}_o(X)| \geq |X|+2$.*

Proof. By definition of Tutte set, $c_o(G-S) \geq |S|+2$. But since $|V(G)|$ is odd, then $c_o(G-S) \geq |S|+3$.

Let $S' = S - X$. By minimality of S, $c_o(G-S') \leq |S'|+1$. Since each component of $G-S$ not in $\mathcal{C}(X)$ is a component of $G-S'$, then $c_o(G-S)-|\mathcal{C}_o(X)| \leq c_o(G-S')$. Then we have that

$$|S|+3-|\mathcal{C}_o(X)| \leq c_o(G-S)-|\mathcal{C}_o(X)| \leq c_o(G-S') \leq |S'|+1 = |S|-|X|+1.$$

We conclude that $|\mathcal{C}_o(X)| \geq |X|+2$. □

Lemma 6.9. *Let G be a connected graph of odd order. Suppose that there is a minimal Tutte set $S \subseteq V(G)$. If G is $K_{1,l}$-free for some $l \geq 4$ then for every nonempty set $A \subseteq S$, there is a component $C \in \mathcal{C}(A)$ such that $|N(C) \cap A| \leq l-2$.*

Proof. Let k be the number of pairs (x, C) with $x \in A$, $C \in \mathcal{C}(A)$ and $C \in \mathcal{C}(x)$. Clearly,

$$k = \sum_{x \in A} |\mathcal{C}(x)| \quad \text{and} \quad k = \sum_{C \in \mathcal{C}(A)} |N(C) \cap A|.$$

Suppose that for all components $C \in \mathcal{C}(A)$, $|N(C) \cap A| \geq l-1$. By Lemma 6.8, $|\mathcal{C}(A)| \geq |A|+2$. Then $\sum_{C \in \mathcal{C}(A)} |N(C) \cap A| \geq (l-1) \cdot (|A|+2)$, and so $k \geq (l-1) \cdot (|A|+2)$.

On the other hand, since G is $K_{1,l}$-free then $|\mathcal{C}(x)| \leq l-1$ for all $x \in A$. But then we have that $\sum_{x \in A} |\mathcal{C}(x)| \leq (l-1) \cdot |A|$, and so $k \leq (l-1) \cdot |A|$, a contradiction. □

Lemma 6.10. *Let G be a connected graph of odd order. Suppose that there is a minimal Tutte set $S \subseteq V(G)$. Suppose that G is $\{K_{1,l}, Z^1_{1,r}, Z^1_{2,r}\}$-free for some $l \geq 5$ and $r \geq 3$. Let $A \subseteq S$ be a clique, and $Z \subseteq G - S$ and $X \subseteq S - A$ be two sets of vertices. Suppose that one of the following is true:*

(i) $X = \emptyset$.

(ii) $X = \{x\}$ *and there are three pairwise non-adjacent vertices* $z_1, z_2, z_3 \in Z \cap N(x)$. *Also* $N(x) \cap A = N(z_1) \cap A = N(z_2) \cap A = \emptyset$ *and* $A \subseteq N(z_3)$.

(iii) $X = \{x\}$ *and there are two non-adjacent vertices* $z_1, z_2 \in Z \cap N(x)$. *Also* $N(z_1) \cap A = N(z_2) \cap A = \emptyset$ *and* $A \subseteq N(x)$.

(iv) For every $x \in X$, $|\mathcal{C}(x)| = 3$ *and x is adjacent to three vertices of Z that are in three different components of* $G - S$.

Let $q \geq 3$ and suppose that $|A| \geq (2q + |Z|) \cdot (l - 2) + r$.

Then there are q vertices $y_1, \ldots, y_q$ in q different components of $\mathcal{C}(A)$ and q different vertices $a_1, \ldots, a_q$ of A such that

- $y_i a_j \in E(G)$ *if and only if* $i = j$ *and*
- $y_i \notin N(Z) \cup N(X)$ *for all* $1 \leq i \leq q$.

Proof. Let m be the number of components of $G - S$ that meet the vertices of Z. Clearly $m \leq |Z|$.

First, we construct a sequence $A_1 \supseteq \ldots \supseteq A_{2q+m+1}$ of subsets of A such that there are distinct components $D_1, \ldots, D_{2q+m}$ of $G - S$ with the property that for all $1 \leq i \leq 2q + m$, $D_i \in \mathcal{C}(A_i)$ and $N(D_i) \cap A_{i+1} = \emptyset$.

The construction is by induction. Let $A_1 = A$. For $1 \leq i \leq 2q + m$, suppose that we have constructed $A_1, \ldots, A_i$ and $D_1, \ldots, D_{i-1}$. By Lemma 6.9 there is a component $C \in \mathcal{C}(A_i)$ such that $|N(C) \cap A_i| \leq l - 2$. Take $D_i = C$ and $A_{i+1} = A_i - N(D_i)$. Since $|A_{i+1}| = |A_i| - |N(D_i) \cap A_i| \geq |A_i| - (l - 2)$, we have that

$$|A_{2q+1}| \geq |A_1| - (2q + m) \cdot (l - 2) = (2q + |Z|) \cdot (l - 2) + r - (2q + m) \cdot (l - 2) \geq r.$$

Among the components $D_1, \ldots, D_{2q+m}$, at most m of them meet Z. Choose $D_{i_1}, \ldots, D_{i_{2q}}$ so that $i_1 \leq i_2 \leq \ldots \leq i_{2q}$ and they do not meet Z.

For $1 \leq j \leq 2q$, choose $y_j \in D_{i_j}$ and $a_j \in N(y_j) \cap A_{i_j}$. Since $D_{i_1}, \ldots, D_{i_{2q}}$ do not meet Z, then $y_j \notin N(Z)$ for all $1 \leq j \leq 2q$.

Suppose there is a $1 \leq j \leq 2q$ such that $y_j \in N_G(x)$ for some $x \in X$. If condition (i) of the lemma holds, no such x exists. If condition (ii) holds, then $\{z_1, z_2, y_j, x, z_3\} \cup A_{2q+1}$ contains a $Z^1_{2,r}$ which is a contradiction. If condition (iii) holds, then $\{z_1, z_2, y_j, x\} \cup A_{2q+1}$ contains a $Z^1_{1,r}$ which is a contradiction. If condition (iv) holds, since $D_{i_j} \in \mathcal{C}(x)$, then $|\mathcal{C}(x)| \geq 4$ which is a contradiction. Then $y_j \notin N(X)$ for all $1 \leq j \leq 2q$.

For $1 \leq j \leq 2q$, define $Y_j = N(a_j) \cap \{y_1, \ldots, y_{2q}\}$. Since $y_j \in Y_j$, then $|Y_j| \geq 1$ for all $1 \leq j \leq 2q$. Suppose there is a $1 \leq j \leq 2q$ such that $|Y_j| \geq 3$. Then $Y_j \cup \{a_j\} \cup A_{2q+1}$ contains a $Z^1_{1,r}$ which is a contradiction. We conclude that for all $1 \leq j \leq 2q$, $1 \leq |Y_j| \leq 2$. Then there is a way to choose q different pairs (a_j, y_j) from $\{(a_1, y_1) \ldots (a_{2q}, y_{2q})\}$ satisfying the conditions required by the lemma. □

Lemma 6.11. *Let G be a connected graph. If G is $\mathcal{F}(\bar{l}, \bar{n}, \bar{m}, \bar{q})$-free for some $\bar{l} \geq 5$, $\bar{n} \geq 1$, $\bar{m} \geq 1$ and $\bar{q} \geq 3$, then G is also $\widehat{\mathcal{F}}(\bar{q})$-free.*

Proof. Let G be an $\mathcal{F}(\bar{l}, \bar{n}, \bar{m}, \bar{q})$-free connected graph for given integers $\bar{l} \geq 5$, $\bar{n} \geq 1$, $\bar{m} \geq 1$ and $\bar{q} \geq 3$. We will show that G is $\widehat{\mathcal{F}}(\bar{q})$-free.

Let $i \in \{1, 2, 3\}$ and $m \geq 1$. Consider $Z = Z^i_{m,\bar{q}}$. If $1 \leq m \leq \bar{m}$, then $Z \in \mathcal{Z}(\bar{m}, \bar{q})$. If $m \geq \bar{m} + 1$, since $\bar{q} \geq 2$ then Z contains a $Z^i_{\bar{m}+2,1} \in \mathcal{Z}(\bar{m}, \bar{q})$. We conclude that G is $\widehat{\mathcal{Z}}(\bar{q})$-free.

Let $n_1, n_2 \geq 1$. Consider $Y = Y^1_{n_1,n_2,\bar{q}}$. If $n_1 + n_2 + 1 \leq \bar{n}$, then $Y \in \mathcal{Y}(\bar{n}, \bar{q})$. Suppose that $n_1 + n_2 + 1 \geq \bar{n} + 1$. If $n_1 + n_2 + 1$ is odd, since $\bar{q} \geq 1$ then Y contains a $T^A_{n_1+n_2+1} \in \mathcal{T}(\bar{n})$. If $n_1 + n_2 + 1$ is even, since $\bar{q} \geq 2$ then Y contains a $T^C_{n_1,n_2} \in \mathcal{T}(\bar{n})$.

Let $n_1, n_2 \geq 1$. Consider $Y = Y^2_{n_1,n_2,\bar{q}}$. If $n_1 + n_2 \leq \bar{n}$, then $Y \in \mathcal{Y}(\bar{n}, \bar{q})$. Suppose that $n_1 + n_2 \geq \bar{n} + 1$. If $n_1 + n_2$ is odd, then Y contains a $T^A_{n_1+n_2} \in \mathcal{T}(\bar{n})$. If $n_1 + n_2$ is even, since $\bar{q} \geq 1$ then Y contains a $T^B_{n_1+n_2} \in \mathcal{T}(\bar{n})$. We conclude that G is $\widehat{\mathcal{Y}}(\bar{q})$-free.

Let $n_1, n_2 \geq 1$ and $m \geq 1$. Consider $D^1 = D^1_{n_1,n_2,m,\bar{q}}$. Suppose first that at least one of n_1, n_2 is odd. By symmetry, we may suppose that n_1 is odd. If $n_1 \geq \bar{n} + 1$, then D^1 contains a $T^A_{n_1} \in \mathcal{T}(\bar{q})$. If $n_1 = 1$, then D^1 contains a $Z^1_{m,\bar{q}} \in \widehat{\mathcal{Z}}(\bar{q})$. Since n_1 is odd, then $n_1 \neq 2$. Suppose that $3 \leq n_1 \leq \bar{n}$. If $1 \leq m \leq \bar{m}$, then D^1 contains a $L^{1A}_{n_1,m,\bar{q}} \in \mathcal{L}(\bar{n}, \bar{m}, \bar{q})$. If $m \geq \bar{m} + 1$, then D^1 contains a $L^{1A}_{n_1,\bar{m}+2,1} \in \mathcal{L}(\bar{n}, \bar{m}, \bar{q})$. Suppose now that both n_1, n_2 are even. Then $n_1, n_2 \geq 2$. If $n_1 + n_2 - 1 \geq \bar{n} + 1$, since $n_1 + n_2 - 1$ is odd then D^1 contains a $T^A_{n_1+n_2-1} \in \mathcal{T}(\bar{n})$. Suppose that $n_1 + n_2 - 1 \leq \bar{n}$. If $1 \leq m \leq \bar{m}$, then $D^1 \in \mathcal{D}^1(\bar{n}, \bar{m}, \bar{q})$. If $m \geq \bar{m} + 1$, then D^1 contains a $D^1_{n_1,n_2,\bar{m}+2,1} \in \mathcal{D}^1(\bar{n}, \bar{m}, \bar{q})$.

Let $n_1, n_2 \geq 1$ and $m \geq 1$. Consider $D^2 = D^2_{n_1,n_2,m,\bar{q}}$. Suppose first that $n_1 + n_2 \geq \bar{n} + 1$. If $n_1 + n_2$ is odd, D^2 contains a $T^A_{n_1+n_2} \in \mathcal{T}(\bar{n})$. If $n_1 + n_2$ is even, D^2 contains a $T^B_{n_1+n_2} \in \mathcal{T}(\bar{n})$. Suppose now that $n_1 + n_2 \leq \bar{n}$. If $1 \leq m \leq \bar{m}$, then $D^2 \in \mathcal{D}^2(\bar{n}, \bar{m}, \bar{q})$. If $m \geq \bar{m} + 1$, then D^2 contains a $D^2_{n_1,n_2,\bar{m}+2,1} \in \mathcal{D}^2(\bar{n}, \bar{m}, \bar{q})$. We conclude that G is $\widehat{\mathcal{D}}(\bar{q})$-free.

Let $i \in \{1, 2, 3\}$, $n \geq 1$ and $m \geq 1$, with n odd. Consider $L^A = L^{iA}_{n,m,\bar{q}}$. If $n \geq \bar{n} + 1$, then L^A contains a $T^A_n \in \mathcal{T}(\bar{n})$. If $n \leq \bar{n}$, then $L^A \in \mathcal{L}^{iA}(\bar{n}, \bar{m}, \bar{q})$ or L^A

contains a $L^{iA}_{n,\bar{m}+2,1} \in \mathcal{L}^{iA}(\bar{n}, \bar{m}, \bar{q})$.

Let $i \in \{1, 2, 3\}$, $n_1, n_2 \geq 1$ and $m \geq 1$, with $n_1 + n_2$ even. Consider $L^B = L^{iB}_{n_1,n_2,m,\bar{q}}$. If $n_1 + n_2 \geq \bar{n} + 1$, then L^B contains a $T^B_{n_1,n_2} \in \mathcal{T}(\bar{n})$. If $n_1 + n_2 \leq \bar{n}$, then $L^B \in \mathcal{L}^{iB}(\bar{n}, \bar{m}, \bar{q})$ or L^B contains a $L^{iB}_{n,\bar{m}+2,1} \in \mathcal{L}^{iB}(\bar{n}, \bar{m}, \bar{q})$.

Let $i \in \{1, 2, 3\}$, $n_1, n_2 \geq 1$ and $m \geq 1$, with $n_1 + n_2$ odd. Consider $L^C = L^{iC}_{n_1,n_2,m,\bar{q}}$. If $n_1 + n_2 \geq \bar{n} + 1$, then L^C contains a $T^C_{n_1,n_2} \in \mathcal{T}(\bar{n})$. If $n_1 + n_2 \leq \bar{n}$, then $L^C \in \mathcal{L}^{iC}(\bar{n}, \bar{m}, \bar{q})$ or L^C contains a $L^{iC}_{n,\bar{m}+2,1} \in \mathcal{L}^{iC}(\bar{n}, \bar{m}, \bar{q})$.

Let $i \in \{1, 2, 3\}$, $n_1, n_2 \geq 1$ and $m \geq 1$, with $n_1 + n_2$ odd. Consider $L^{iD} = L^{iD}_{n_1,n_2,m,\bar{q}}$ and $L^{iE} = L^{iE}_{n_1,n_2,m,\bar{q}}$. Since $n_1 + n_2$ is odd then one of n_1, n_2 is odd. If $n_2 = 1$, then both L^{iD} and L^{iE} contain a $Z^i_{m,\bar{q}} \in \widehat{\mathcal{Z}}(\bar{q})$. If n_2 is odd and $n_2 \geq 3$, then both L^{iD} and L^{iE} contain a $L^{iA}_{n_2,m,\bar{q}}$, which we have considered earlier. If $n_1 = 1$, then L^{iD} contains a graph of type "Z^1" and L^{iE} contains a graph of type "Z^2", which we considered earlier. If n_1 is odd and $n_1 \geq 3$, then L^{iD} contains a graph of type "L^{1A}" and L^{iE} contains a graph of type "L^{2A}", which we have considered earlier. We conclude that G is $\widehat{\mathcal{L}}(\bar{q})$-free.

Let $i \in \{1, 2\}$ and $n \geq 1$, with n odd. Consider $W^A = W^{iA}_{n,\bar{q}}$. If $n \geq \bar{n} + 1$, then W^A contains a $T^A_n \in \mathcal{T}(\bar{n})$. If $n \leq \bar{n}$, then $W^A \in \mathcal{W}^{iA}(\bar{n}, \bar{q})$.

Let $i \in \{1, 2\}$ and $n_1, n_2 \geq 1$, with $n_1 + n_2$ even. Consider $W^B = W^{iB}_{n_1,n_2,\bar{q}}$. If $n_1 + n_2 \geq \bar{n} + 1$, then W^B contains a $T^B_{n_1,n_2} \in \mathcal{T}(\bar{n})$. If $n_1 + n_2 \leq \bar{n}$, then $W^B \in \mathcal{W}^{iB}(\bar{n}, \bar{q})$.

Let $i \in \{1, 2\}$ and $n_1, n_2 \geq 1$, with $n_1 + n_2$ odd. Consider $W^C = W^{iC}_{n_1,n_2,\bar{q}}$. If $n_1 + n_2 \geq \bar{n} + 1$, then W^C contains a $T^C_{n_1,n_2} \in \mathcal{T}(\bar{n})$. If $n_1 + n_2 \leq \bar{n}$, then $W^C \in \mathcal{W}^{iC}(\bar{n}, \bar{q})$.

Let $i \in \{1, 2\}$ and $n_1, n_2 \geq 1$, with $n_1 + n_2$ odd. Consider $W^{iD} = W^{iD}_{n_1,n_2,\bar{q}}$ and $W^{iE} = W^{iE}_{n_1,n_2,\bar{q}}$. Since $n_1 + n_2$ is odd then one of n_1, n_2 is odd. If $n_2 = 1$, then both W^{iD} and W^{iE} contain a $V^i_{\bar{q}} \in \mathcal{V}(\bar{q})$. If n_2 is odd and $n_2 \geq 3$, then both W^{iD} and W^{iE} contain a $W^{iA}_{n_2,\bar{q}}$, which we have considered earlier. If $n_1 = 1$, then W^{iD} contains a graph of type "Z^1" and W^{iE} contains a graph of type "Z^2", which we considered earlier. If n_1 is odd and $n_1 \geq 3$, then W^{iD} contains a graph of type "L^{1A}" and W^{iE} contains a graph of type "L^{2A}", which we considered earlier. We conclude that G is $\widehat{\mathcal{W}}(\bar{q})$-free.

Let $n \geq 2$, with n even. Consider $M^A = M^A_{n,\bar{q}}$. If $n + 1 \geq \bar{n} + 1$, then M^A contains a $T^A_{n+1} \in \mathcal{T}(\bar{n})$. If $3 \leq n + 1 \leq \bar{n}$, then $M^A \in \mathcal{M}^A(\bar{n}, \bar{q})$.

Let $n_1, n_2 \geq 1$, with $n_1 + n_2$ odd. Consider $M^B = M^B_{n_1,n_2,\bar{q}}$. If $n_1 + n_2 + 1 \geq \bar{n} + 1$, then M^B contains a $T^B_{n_1,n_2+1} \in \mathcal{T}(\bar{n})$. If $n_1 + n_2 + 1 \leq \bar{n}$, then $M^B \in \mathcal{M}^B(\bar{n}, \bar{q})$.

Let $n_1, n_2 \geq 1$, with $n_1 + n_2$ even. Consider $M^C = M^C_{n_1,n_2,\bar{q}}$. If $n_1 + n_2 + 1 \geq \bar{n} + 1$, then M^C contains a $T^C_{n_1,n_2+1} \in \mathcal{T}(\bar{n})$. If $n_1 + n_2 \leq \bar{n}$, then $M^C \in \mathcal{M}^C(\bar{n}, \bar{q})$.

Let $n_1, n_2 \geq 1$, with $n_1 + n_2$ even. Consider $M^D = M^D_{n_1,n_2,\bar{q}}$ and $M^E = M^E_{n_1,n_2,\bar{q}}$.

If n_2 is even (and so $n_2 \geq 2$), then both M^D and M^E contain a $M^A_{n_2,\bar{q}}$. Suppose that n_2 is odd. Since $n_1 + n_2$ is even, then n_1 is odd. If $n_1 = 1$, then M^D contains a $Z^1_{n_2,\bar{q}} \in \widehat{\mathcal{Z}}(\bar{q})$ and M^E contains a $Z^2_{n_2,\bar{q}} \in \widehat{\mathcal{Z}}(\bar{q})$. If $n_1 \geq 3$, then M^D contains a $L^{1A}_{n_1,n_2,\bar{q}} \in \widehat{\mathcal{L}}(\bar{q})$ and M^E contains a $L^{2A}_{n_1,n_2,\bar{q}} \in \widehat{\mathcal{L}}(\bar{q})$. We conclude that G is $\widehat{\mathcal{M}}(\bar{q})$-free.

Let $n \geq 1$, with n odd. Consider $J^1 = J^1_{n,\bar{q}}$. If $n+1 \geq \bar{n}+1$, then J^1 contains a $T^B_{n,1} \in \mathcal{T}(\bar{n})$. If $4 \leq n+1 \leq \bar{n}$, then $J^1 \in \mathcal{J}^1(\bar{n}, \bar{q})$. If $n = 1$, then J^1 contains a $V^4_{\bar{q}} \in \mathcal{V}(\bar{q})$.

Let $n \geq 2$, with n even. Consider $J^2 = J^2_{n,\bar{q}}$. If $n+1 \geq \bar{n}+1$, then J^2 contains a $T^C_{n,1} \in \mathcal{T}(\bar{n})$. If $3 \leq n+1 \leq \bar{n}$, then $J^2 \in \mathcal{J}^2(\bar{n}, \bar{q})$. □

Proof of Theorem 6.6

In this section we present the proof of Theorem 6.6.

Proof of Theorem 6.6. Let G be a connected graph of odd order and $S \subseteq V(G)$ a minimal Tutte set. Suppose that G is $\mathcal{F}_A(l, m, q)$-free for some $l \geq 5, m \geq 1$ and $q \geq 3$.

Let $x_0 \in S$ with $|\mathcal{C}_o(x_0)| \geq 4$. Let $C_1, \ldots, C_4$ be four different components in $\mathcal{C}_o(x_0)$. For $1 \leq i \leq 4$, let $y_i \in C_i \cap N_G(x)$. Let $Y = \{y_1, \ldots, y_4\}$.

Define the function $f(z) = (2q + z) \cdot (l - 2) + q$. Let $k = f(4)$.

Claim 6.6.1. $|N(x_0)| < 2^4 \cdot R(l, k)$.

Proof. Let $Y' \subseteq Y$ and let $N = N(x_0) \cap B_Y(Y')$. By Proposition 2.2 it is enough to show that $|N| < R(l, k)$. Since $\{x_0\} \cup N$ contains no $K_{1,l}$ then N contains no independent set of size l. We will show that N contains no clique of size k. Let A be a clique of N.

Since the elements of Y are in different components of $G - S$, if $|Y'| \geq 2$ then $A \subseteq S$.

If $|Y'| \geq 3$ then $|A| < f(3) < k$, since otherwise $Y' \cup A$ could be extended to a V^3_q by Lemma 6.10 (condition (i)). If $|Y'| = 2$ then $|A| < f(4) = k$, since otherwise $Y \cup \{x_0\} \cup A$ could be extended to a V^2_q by Lemma 6.10 (condition (iii)). If $|Y'| \leq 1$, since $(Y - Y') \cup \{x_0\} \cup A$ contains no $Z^1_{1,r}$, then $|A| < q < k$. □

Claim 6.6.2. $|N^2(x_0)| < 2^4 \cdot R(l, k) \cdot |N(x_0)|$

Proof. Let $x_1 \in N(x_0)$. We will show that $|N^2(x_0) \cap N(x_1)| < 2^4 \cdot R(l, k)$.

Let $Y' \subseteq Y$. Let $N = N^2(x_0) \cap N(x_1) \cap B_Y(Y')$. By Proposition 2.2 it is enough to show that $|N| < R(l, k)$. Since $\{x_1\} \cup N$ contains no $K_{1,l}$ then N contains no independent set of size l. We will show that N contains no clique of size k. Let A be a clique of N.

As in Claim 6.6.1, if $|Y'| \geq 2$ then $A \subseteq S$.

If $|Y'| \geq 3$ then $|A| < f(3) < k$, since otherwise $Y' \cup A$ could be extended to a V_q^3 by Lemma 6.10 (condition (i)). If $|Y'| = 2$ then $|A| < f(4) = k$, since otherwise $Y \cup \{x_0\} \cup A$ could be extended to a V_q^1 by Lemma 6.10 (condition (ii)). If $|Y'| = 1$, since $(Y - \{y_1\}) \cup \{x_0, y_1\} \cup A$ (with $y_1 \in Y'$) contains no $Z_{2,q}^1$, then $|A| < q < k$.

Suppose now that $|Y'| = 0$. Let $Y_1 = Y \cap N(x_1)$. If $|Y_1| \geq 3$, since $Y_1 \cup \{x_1\} \cup A$ contains no $Z_{1,q}^1$, then $|A| < q < k$. If $|Y_1| = 2$, since $Y \cup \{x_0, x_1\} \cup A$ contains no $Z_{1,q}^3$, then $|A| < q < k$. If $|Y_1| \leq 1$, since $(Y - Y_1) \cup \{x_0, x_1\} \cup A$ contains no $Z_{2,q}^1$, then $|A| < q < k$. □

Claim 6.6.3. *Let $i \geq 3$. Then $|N^i(x_0)| < R(l, q) \cdot |N^{i-1}(x_0)|$.*

Proof. Let $P = x_0 \ldots x_{i-1}$ be an induced path such that $x_j \in N^j(x)$ for all $0 \leq j \leq i - 1$. Let $N = N(x_{i-1}) \cap N^i(x_0)$. We will show that $|N| < R(l, q)$.

Since $\{x_{i-1}\} \cup N$ contains no $K_{1,l}$ then N contains no independent set of size l. We will show that N contains no clique of size q. Let A be a clique of N and suppose that then $|A| \geq q$.

As in Lemma 6.11, since G is $\mathcal{Z}(m, q)$-free, then G is also $\widehat{\mathcal{Z}}(q)$-free.

Since P is an induced path, $N(x_j) \cap Y = \emptyset$ for all $3 \leq j \leq i - 1$. Let $Y_1 = N(x_1) \cap Y$ and $Y_2 = N(x_2) \cap Y$. Also, $N(A) \cap Y = \emptyset$.

If $|Y_2| \geq 3$ then $Y_2 \cup \{x_2, \ldots, x_{i-1}\} \cup A$ contains a $Z_{i-2,q}^1$. If $|Y_2| = 2$ then $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_{i-1}\} \cup A$ contains a $Z_{i-2,q}^2$. If $|Y_2| = 1$, then $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_{i-1}\} \cup A$ contains a $Z_{i,q}^1$.

Suppose now that $|Y_2| = 0$. If $|Y_1| \geq 3$, then $Y_1 \cup \{x_1, \ldots, x_{i-1}\} \cup A$ contains a $Z_{i-1,q}^1$. If $|Y_1| = 2$, then $(Y - Y_1) \cup \{x_0\} \cup Y_1 \cup \{x_1, \ldots, x_{i-1}\} \cup A$ contains a $Z_{i-1,q}^3$. If $|Y_1| \leq 1$, then $(Y - Y_1) \cup \{x_0, x_1, \ldots, x_{i-1}\} \cup A$ contains a $Z_{i,q}^1$. □

Claim 6.6.4. $N^{m+3}(x_0) = \emptyset$.

Proof. Suppose that $N^{m+3}(x_0) \neq \emptyset$. Let $P = x_0, \ldots x_{m+3}$ an induced path such that $x_j \in N^j(x)$ for all $0 \leq j \leq m + 3$. Since P is an induced path, $N(x_j) \cap Y = \emptyset$ for all $3 \leq j \leq m + 3$. Let $Y_1 = N(x_1) \cap Y$ and $Y_2 = N(x_2) \cap Y$.

If $|Y_2| \geq 3$, then $Y_2 \cup \{x_2, \ldots, x_{m+3}\}$ contains a $Z_{m+2,1}^1$. If $|Y_2| = 2$, then $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_{m+3}\}$ is a $Z_{m+2,1}^2$. If $|Y_2| = 1$, then $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_{m+1}\}$ is a $Z_{m+2,1}^1$.

Suppose now that $|Y_2| = 0$. If $|Y_1| \geq 3$, then $Y_1 \cup \{x_1, \ldots, x_{m+2}\}$ contains a $Z_{m+2,1}^1$. If $|Y_1| = 2$, then $(Y - Y_1) \cup \{x_0\} \cup Y_1 \cup \{x_1, \ldots, x_{m+2}\}$ is a $Z_{m+2,1}^3$. If $|Y_1| \leq 1$, then $(Y - Y_1) \cup \{x_0, x_1, \ldots, x_{m+1}\}$ contains a $Z_{m+2,1}^1$. □

Using the above claims we will show that $|N^i(x_0)|$ is bounded for all $i \geq 1$ and that $N^i(x_0) = \emptyset$ for some $i \geq 1$.

By Claims 6.6.1 and 6.6.2, we have that $|N^1(x_0)|$ and $|N^2(x_0)|$ are bounded. By Claim 6.6.4, $N^{m+3}(x_0) = \emptyset$. It remains to show that $|N^i(x_0)|$ is bounded for all $i \geq 3$.

By Claim 6.6.3, $|N^i(x_0)| < R(l, q) \cdot |N^{i-1}(x_0)|$ for all $i \geq 3$. Using an inductive argument, we can show that $|N^i(x_0)| < R(l, q)^{i-2} \cdot |N^2(x_0)|$ for all $i \geq 3$. By Claims 6.6.1 and 6.6.2, we get that $|N^i(x_0)| < R(l, q)^{i-2} \cdot R(l, k)^2$ for all $i \geq 3$. By Claim 6.6.4, we get that $|N^i(x_0)| < R(l, q)^{m+1} \cdot R(l, k)^2$ for all $i \geq 3$. □

Proof of Theorem 6.7

In this section we present the proof of Theorem 6.7. First, we prove a lemma to find a starting structure. In proof of the Theorem 6.7 itself, we use this structure to divide the vertices of the graph into sets according to the distance to the structure.

Lemma 6.12. *Let G be a connected graph of odd order and $S \subseteq V(G)$ a minimal Tutte set. Suppose that for every $x \in S$, $|\mathcal{C}_o(x)| \leq 3$. Suppose that G is $\mathcal{F}(\bar{l}, \bar{n}, \bar{m}, \bar{q})$-free for some $\bar{l} \geq 5, \bar{n} \geq 1, \bar{m} \geq 1$ and $\bar{q} \geq 3$. Then there are two different vertices $x_1, x_2 \in S$ such that $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)| = 1$. Also, there is an odd connector C between x_1 and x_2 such that $C - \{x_1, x_2\} \subseteq D$, with $D \in \mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)$, satisfying:*

- *if C is a C^A_n then n is odd and $3 \leq n \leq \bar{n}$,*
- *if C is a $C^B_{n_1,n_2}$ then $n_1 + n_2$ is even and $n_1 + n_2 \leq \bar{n}$,*
- *if C is a $C^C_{n_1,n_2}$ then $n_1 + n_2$ is odd and $n_1 + n_2 \leq \bar{n}$ and*
- *If C is a $C^D_{n_1,n_2}$ or a $C^E_{n_1,n_2}$, then $n_1 + n_2$ is odd. Also, if n_1 is odd and n_2 is even then $1 \leq n_1 \leq \bar{n}$ and $2 \leq n_2 \leq \bar{m}$, and if n_2 is odd and n_1 is even then $1 \leq n_2 \leq \bar{n}$ and $2 \leq n_1 \leq \bar{m}$.*

Proof. As in Lemma 6.8, we have that $|\mathcal{C}_o(S)| \geq |S| + 3$. Since $|\mathcal{C}_o(x)| \leq 3$ for all $x \in S$, by Lemma 6.8 we get that $|\mathcal{C}_o(x)| = 3$ for all $x \in S$.

Claim 6.12.1. *There are two different vertices $x_1, x_2 \in S$ such that $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)| = 1$.*

Proof. Suppose that for all $x_1, x_2 \in S$, $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)| \neq 1$. Let $x_1 \in S$ and let $S' = S - \{x_1\}$. Suppose that for any $x \in S'$, $|\mathcal{C}_o(x) - \mathcal{C}_o(x_1)| \leq 1$. Then,

$$|\mathcal{C}_o(S)| = |\mathcal{C}_o(x_1)| + \sum_{x \in S'} |\mathcal{C}_o(x) - \mathcal{C}_o(x_1)| \leq |\mathcal{C}_o(x_1)| + |S'| = 3 + |S| - 1 = |S| + 2.$$

And so $|\mathcal{C}_o(S)| \leq |S| + 2$, which is a contradiction. We conclude that there are two different vertices $x_1, x_2 \in S$ such that $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)| \leq 1$.

Suppose that $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)| = 0$. Let G' be the bipartite graph with vertex set $V(G') = S \cup \mathcal{C}(S)$ and edge set $E(G') = \{(x, C)\colon x \in S \text{ and } C \in \mathcal{C}(x)\ \}$.

Suppose that G' is not connected. Since G is connected, then S can be partitioned in two nonempty sets S_1, S_2 such that $\mathcal{C}_o(S_1) \cap \mathcal{C}_o(S_2) = \emptyset$. By the minimality of S, we have that $|\mathcal{C}_o(S_1)| \leq |S_1| + 1$ and $|\mathcal{C}_o(S_2)| \leq |S_2| + 1$. But then, $|\mathcal{C}_o(S)| = |\mathcal{C}_o(S_1)| + |\mathcal{C}_o(S_2)| \leq |S_1| + 1 + |S_2| + 1 = |S| + 2$, which is a contradiction. We conclude that G' is connected.

Let $P = z_1 \cdots z_k$ be a shortest path from x_1 to x_2 in $G'(x_1 = z_1,\ x_2 = z_k)$. Since G' is bipartite, the vertices of P alternate between S and $\mathcal{C}(S)$. Let $C_1, C_2, C_3 \in \mathcal{C}_o(x_1)$ three different components. We may suppose that $z_2 = C_1$. Since $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)| = 0$, then $z_3 \neq x_2$. Notice that $z_3 \in S$ and $C_1 \in \mathcal{C}_o(z_3)$. We may suppose that $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(z_3)| \geq 2$. Hence we may suppose that $C_2 \in \mathcal{C}_o(z_3)$.

Consider z_4 and z_5. Notice that $z_5 \in S$ and z_5 might be x_2. Since $z_4 \in \mathcal{C}_o(z_3) \cap \mathcal{C}_o(z_5)$ then we may suppose that $|\mathcal{C}_o(z_3) \cap \mathcal{C}_o(z_5)| \geq 2$. Hence at least one of C_1 or C_2 is in $\mathcal{C}_o(z_5)$. This contradicts the minimality of P. We conclude that $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)| = 1$. □

Let $x_1, x_2 \in S$ be two different vertices with $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)| = 1$. Let $D \in \mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)$ be a component of $G - S$.

Claim 6.12.2. *There is an odd connector C between x_1 and x_2 such that $C - \{x_1, x_2\} \subseteq D$.*

Proof. Suppose first that x_1 and x_2 are adjacent. Let $z \in D \cap N(x_1)$. If $zx_2 \in E(G)$ then $\{x_1, x_2, z\}$ is a $C^B_{1,1}$. If $zx_2 \notin E(G)$ then $\{x_1, x_2, z\}$ is a $C^D_{1,2}$.

Suppose now that x_1 and x_2 are not adjacent. Let $P = p_1 \cdots p_n$ be a shortest path between x_1 and x_2 in $D \cup \{x_1, x_2\}$ with $p_1 = x_1$ and $p_2 = x_2$. If n is odd then P is C^A_n. If n is even, since $|D|$ is odd, then $D - P \neq \emptyset$. Then there is some vertex $z \in D - P$ such that $N(z) \cap (P - \{p_1, p_k\}) \neq \emptyset$.

Let $d = \min\{i : p_i \in N(z)\}$ and $d' = \max\{i : p_i \in N(z)\}$. Since P is the shortest path between x_1 and x_2 in $D \cup \{x_1, x_2\}$, then $d \leq d' \leq d + 2$. Let $n_1 = d$ and $n_2 = n - d' + 1$. If $d' = d$ then $P \cup \{z\}$ is $C^D_{n_1,n_2}$. If $d' = d+1$ then $P \cup \{z\}$ is $C^B_{n_1,n_2}$. If $d' = d + 2$ and $p_{d+1} \in N(z)$ then $P \cup \{z\}$ is $C^C_{n_1,n_2}$. If $d' = d + 2$ and $p_{d+1} \notin N(z)$ then $P \cup \{z\}$ is $C^E_{n_1,n_2}$. □

Let C be an odd connector between x_1 and x_2 with $C - \{x_1, x_2\} \subseteq D$. Let $p_1 \ldots p_n$ be named as in Claim 6.12.2. If C is not of type A, let z be the balancer of C.

Let $C_1^a, C_1^b \in \mathcal{C}_o(x_1) - \mathcal{C}_o(x_2)$ and $C_2^a, C_2^b \in \mathcal{C}_o(x_2) - \mathcal{C}_o(x_1)$ four different components of $G - S$. Let $y_1^a \in V(C_1^a) \cap N(x_1)$, $y_1^b \in V(C_1^b) \cap N(x_1)$, $y_2^a \in V(C_2^a) \cap N(x_2)$ and $y_2^b \in V(C_2^b) \cap N(x_2)$. Let $H = \{y_1^a, y_1^b\} \cup C \cup \{y_2^a, y_2^b\}$.

If C is a C_n^A, since C is an odd connector then n is odd, since $x_1 \neq x_2$ then $n \geq 3$, and since $H \notin \mathcal{T}^A(\bar{n})$ then $n \leq \bar{n}$. If C is a C_{n_1,n_2}^B, since C is an odd connector then $n_1 + n_2$ is even, and since $H \notin \mathcal{T}^B(\bar{n})$ then $n_1 + n_2 \leq \bar{n}$. If C is a C_{n_1,n_2}^C, since C is an odd connector then $n_1 + n_2$ is odd, and since $H \notin \mathcal{T}^C(\bar{n})$ then $n_1 + n_2 \leq \bar{n}$.

Suppose that C is a C_{n_1,n_2}^D or a C_{n_1,n_2}^E. Since C is an odd connector then $n_1 + n_2$ is odd. Suppose that n_1 is odd and n_2 is even. Since $\{y_1^a, y_1^b\} \cup \{p_1, \ldots, p_{n_1}\} \cup \{z, p_{n_1+1}\} \notin \mathcal{T}^A(\bar{n})$ then $n_1 \leq \bar{n}$. If C is a C_{n_1,n_2}^D, since $H - \{y_b^2\}$ is not a $L_{n_1,\bar{m}+2,1}^{1A}$ then $n_2 + 1 < \bar{m} + 2$ and so $n_2 \leq \bar{m}$. If C is a C_{n_1,n_2}^E, since $H - \{y_b^2\}$ is not a $L_{n_1,\bar{m}+2,1}^{2A}$ then $n_2 + 1 < \bar{m} + 2$ and so $n_2 \leq \bar{m}$. We conclude that $n_1 \leq \bar{n}$ and $n_2 \leq \bar{m}$. If n_2 is odd and n_1 is even, in the same way we get that $n_2 \leq \bar{n}$ and $n_1 \leq \bar{m}$. □

We present now the proof of Theorem 6.7.

Proof of Theorem 6.7. Let G be a connected graph of odd order and $S \subseteq V(G)$ a minimal Tutte set. Suppose that for every $x \in S$, $|\mathcal{C}_o(x)| \leq 3$. Suppose that G is $\mathcal{F}(\bar{l}, \bar{n}, \bar{m}, \bar{q})$-free. By Lemma 6.11, G is also $\widehat{\mathcal{F}}(\bar{q})$-free. We will prove that $|V(G)|$ is bounded by a function depending only on $\bar{l}, \bar{n}, \bar{m}$ and $\bar{q}$.

Let the following elements be as in the statement and proof of Lemma 6.12.

- $x_1, x_2 \in S$ with $|\mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)| = 1$
- $D \in \mathcal{C}_o(x_1) \cap \mathcal{C}_o(x_2)$
- C be an odd connector between x_1 and x_2 $C - \{x_1, x_2\} \subseteq D$
- $x_1 = p_1 \ldots p_n = x_2$ the induced path in C
- If C is not of type A, let z be the balancer of C.
- $C_1^a, C_1^b \in \mathcal{C}_o(x_1) - \mathcal{C}_o(x_2)$ and $C_2^a, C_2^b \in \mathcal{C}_o(x_2) - \mathcal{C}_o(x_1)$
- $y_1^a \in V(C_1^a) \cap N(x_1)$, $y_1^b \in V(C_1^b) \cap N(x_1)$, $y_2^a \in V(C_2^a) \cap N(x_2)$ and $y_2^b \in V(C_2^b) \cap N(x_2)$.

Notice that C is of type A if and only if n is odd.

Additionally, let $y_1^c \in V(D) \cap N(x_1)$ and $y_2^c \in V(D) \cap N(x_2)$. Notice that y_1^c and y_2^c might be the same vertex, and that one or both of them might be part of the connector C.

Let $H = \{y_1^a, y_1^b, y_1^c\} \cup C \cup \{y_2^a, y_2^b, y_2^c\}$. Define $f(z) = (2\bar{q} + z) \cdot (l - 2) + \bar{q}$. We will prove that:

(i) $|N^0(H)| \leq \bar{n} + \bar{m} + 8$

(ii) $|N^1(H)| < 2^{|H|} \cdot R(\bar{l}, k)$, where $k = f(n+3)$

(iii) $|N^i(H)| < R(\bar{l}, \bar{q}) \cdot |N^{i-1}(H)|$ for all $2 \leq i \leq \bar{m} + 1$

(iv) $N^{\bar{m}+2}(H) = \emptyset$

By proving (i), (ii), (iii) and (iv) and using a similar argument to the one used at the end of the proof of Theorem 6.6, we can show that $|N^i(H)|$ is bounded for every $i \geq 0$ and that $N^{\bar{m}+2}(H) = \emptyset$, which is enough to prove the theorem. Notice that $k = f(n+3) \leq f(|H|) = f(|N^0(H)|)$.

By Lemma 6.12, $|N^0(H)| = |H| \leq \bar{n} + \bar{m} + 8$ (the worst case is when C is of type E). Hence, (i) is true. For the rest of the proof, fix $i \geq 1$.

If $i = 1$, let $H' \subseteq H$ with $H' \neq \emptyset$ and let $N_1 = B_H(H') \cap N^1(H)$. We will prove that $|N_1| < R(\bar{l}, k)$, which implies (ii) by Proposition 2.2.

If $i \geq 2$, let $z_1, \ldots, z_{i-1}$ an induced path, with $z_j \in N^j(H)$ for all $1 \leq j \leq i-1$ and let $N_i = N(z_{i-1}) \cap N^i(H)$. We will prove that if $2 \leq i \leq \bar{m}+1$ then $|N_i| < R(\bar{l}, q)$, and that if $i \geq \bar{m} + 2$ then $N_i = \emptyset$, which implies (iii) and (iv), respectively.

Let $y \in H'$ (remember that $H' \neq \emptyset$). Since $N_1 \cup \{y\}$ does not contain a $K_{1,\bar{l}}$, then N_1 contains no independent set of size $\bar{l}$. If $2 \leq i \leq \bar{m}+1$, since $N_i \cup \{z_{i-1}\}$ does not contain a $K_{1,\bar{l}}$, then N_i contains no independent set of size $\bar{l}$. Let K_i be a clique in N_i. We will show that if $i = 1$ then $|K_1| < k$ and that if $2 \leq i \leq \bar{m}+1$ then $|K_i| < \bar{q}$.

In the rest of the proof, we will show that

(v) if $i = 1$ then $|K_1| < k$, if $2 \leq i \leq \bar{m}+1$ then $|K_i| < \bar{q}$ and if $i \geq \bar{m}+2$ then $N_i = \emptyset$.

To do so, for $i = 1$ we will divide into cases according to what vertices of H are in H' and so how K_1 is connected to H; for $i \geq 2$ we will divide into cases according to what vertices of H are in $N(z_1)$ and so how z_1 is connected to H. Because the division into cases is exactly the same for all three $i = 1$, $2 \leq i \leq \bar{m}+1$ and $i \geq \bar{m}+2$, we will consider them simultaneously.

In order to do it, we define two auxiliary sets R and A_i as follows. If $i = 1$, let $R = H'$ and $A_i = K_i$. If $2 \leq i \leq m+1$, let $R = N(z_1) \cap H$ and $A_i = \{z_1, \ldots, z_{i-1}\} \cup K_i$. If $i \geq \bar{m}+2$, let $R = N(z_1) \cap H$ and $A_i = \{z_1, \ldots z_{i-1}\} \cup N_i$. According to what vertices of H are in R we can know how K_1 is connected to H (if $i = 1$) and how z_1 is connected to H (if $i \geq 2$). For each case, we will use some vertices of H and put them together with A_i to find some graph from $\mathcal{F}(\bar{l}, \bar{n}, \bar{m}, \bar{q})$ or from $\widehat{\mathcal{F}}(\bar{q})$.

When we apply Lemma 6.10, we will say that some subgraph of G can be extended to some graph from $\mathcal{F}(\bar{l}, \bar{n}, \bar{m}, \bar{q})$ or from $\widehat{\mathcal{F}}(\bar{q})$, without explicitly writing that we are using Lemma 6.10. All the applications use condition (iv) of Lemma 6.10, except when we extend to $V_{\bar{q}}^3$, that we use condition (i). Lemma 6.10 is only applied when $i = 1$. Before applying it, we will always show that $A_1 \subseteq S$ and use A_1 as the set A of Lemma 6.10. The value $k = f(n+3)$ corresponds to the applications of Lemma 6.10 to $\{y_1^a, y_1^b\} \cup C \cup \{y_2^a, y_2^b\} \cup A_1$, that contains at most $n+3$ elements of $G - S$, and is the induced subgraph of G with the largest amount of elements of $G - S$ to which we will apply the Lemma.

Let $Y = \{y_1^a, y_1^b, y_2^a, y_2^b\}$ and $P = \{p_1, \ldots, p_n\}$. We divide the proof into four claims, according to the size of $R \cap Y$.

Claim 6.7.1. *If there are three pairwise non-adjacent vertices y_1, y_2, y_3 of H that are in at least two different components of $G - S$ and such that $y_i \in R$ for $i \in \{1, 2, 3\}$, then (v) holds. In particular, if $|R \cap Y| \geq 3$ then (v) holds.*

Proof. If $i = 1$, since y_1, y_2, y_3 are in at least two different components of $G - S$, A_i is connected to more than one component of $G - S$ and so $A_i \subseteq S$, allowing us to apply Lemma 6.10.

Consider $V_i = A_i \cup \{y_1, y_2, y_3\}$. If $i = 1$ and $|K_1| > f(3)$, V_i can be extended to a $V_{\bar{q}}^3$. If $2 \leq i \leq m+1$ and $|K_i| \geq \bar{q}$, then V_i contains a $Z_{i-1,\bar{q}}^1$. If $i \geq \bar{m}+2$, then V_i contains a $Z_{\bar{m}+2,1}^1$. □

Claim 6.7.2. *If $|R \cap Y| = 2$ then (v) holds.*

Proof. If $i = 1$, since $|R \cap Y| = 2$ then A_i is connected to more than one component of $G - S$ and so $A_i \subseteq S$, allowing us to apply Lemma 6.10.

Suppose first that $|R \cap \{y_1^a, y_1^b\}| = |R \cap \{y_2^a, y_2^b\}| = 1$. By symmetry, we may suppose that $R \cap Y = \{y_1^a, y_2^a\}$. By Claim 6.7.1, we may assume that $y_1^c \notin R$.

Suppose that (v) does not hold. That is, if $i = 1$ then $|K_1| \geq k = f(n+4)$, if $2 \leq i \leq \bar{m}+1$ then $|K_i| \geq q$ and if $i \geq \bar{m}+2$ then $N_i \neq \emptyset$.

Consider $V_i = \{y_1^a, y_1^b, y_1^c, x_1, y_2^a\} \cup A_i$. If $x_1 \in R$, then V_i can be extended to a $V_{\bar{q}}^4$, or V_i contains a $L_{1,1,i-1,\bar{q}}^{1B}$, or V_i contains a $L_{1,1,\bar{m}+2,1}^{1B}$ (depending on i). If $x_1 \notin R$, then V_i can be extended to a $M_{2,\bar{q}}^A$, or V_i contains a $L_{3,i-1,\bar{q}}^{1A}$, or V_i contains a $L_{3,\bar{m}+2,1}^{1A}$ (depending on i).

Suppose now that $|R \cap \{y_1^a, y_1^b\}| = 2$ or $|R \cap \{y_2^a, y_2^b\}| = 2$. By symmetry, we may suppose that $R \cap Y = \{y_2^a, y_2^b\}$. By Claim 6.7.1, we may assume that $((P - \{x_1, x_2\}) \cup \{y_1^c\}) \cap R = \emptyset$.

If $x_1 \in R$, then $\{y_1^a, y_1^b, y_1^c, x_1\} \cup A_i$ contains a $Z_{i,\bar{q}}^1$ or a $Z_{\bar{m}+2,\bar{q}}^1$ (depending on i). Suppose that $x_1 \notin R$.

Consider $V_i = \{y_1^a, y_1^b\} \cup C \cup \{y_2^a, y_2^b\} \cup A_i$. Suppose that $x_2 \in R$. If $i = 1$, depending on the type of C, V_i can be extended to a $W^{2A}_{n,\bar{q}}$, $W^{2B}_{n_1,n_2,\bar{q}}$, $W^{2C}_{n_1,n_2,\bar{q}}$, $W^{2D}_{n_1,n_2,\bar{q}}$ or $W^{2E}_{n_1,n_2,\bar{q}}$. If $i \geq 2$, depending on the type of C, V_i contains an $L^{3A}_{n,i-1,\bar{q}}$, $L^{3B}_{n_1,n_2,i-1,\bar{q}}$, $L^{3C}_{n_1,n_2,i-1,\bar{q}}$, $L^{3D}_{n_1,n_2,i-1,\bar{q}}$ or $L^{3E}_{n_1,n_2,i-1,\bar{q}}$ or the corresponding graphs with $m = \bar{m} + 2$ and $q = 1$ if $i \geq \bar{m} + 2$.

Suppose that $x_2 \notin R$. If $i = 1$, depending on the type of C, V_i can be extended to a $W^{1A}_{n,\bar{q}}$, $W^{1B}_{n_1,n_2,\bar{q}}$, $W^{1C}_{n_1,n_2,\bar{q}}$, $W^{1D}_{n_1,n_2,\bar{q}}$ or $W^{1E}_{n_1,n_2,\bar{q}}$. If $i \geq 2$, depending on the type of C, V_i contains a $L^{2A}_{n,i-1,\bar{q}}$, $L^{2B}_{n_1,n_2,i-1,\bar{q}}$, $L^{2C}_{n_1,n_2,i-1,\bar{q}}$, $L^{2D}_{n_1,n_2,i-1,\bar{q}}$ or $L^{2E}_{n_1,n_2,i-1,\bar{q}}$ or the corresponding graphs with $m = \bar{m} + 2$ and $q = 1$ if $i \geq \bar{m} + 2$. □

Claim 6.7.3. *Suppose $|R \cap Y| = 0$. If $1 \leq i \leq \bar{m} + 1$ then $|K_i| < \bar{q}$. If $i \geq \bar{m} + 1$ then $N_i = \emptyset$. In particular, (v) holds.*

Proof. Suppose that $N_i \neq \emptyset$ and that $|K_i| \geq \bar{q}$.

Suppose that $|R \cap P| = 1$ and let $1 \leq d \leq n$ such that $p_d \in R$. Then $\{y_1^a, y_1^b, p_1, \ldots, p_n, y_2^a, y_2^b\} \cup A_i$ contains a $D^1_{d,n-d+1,i,\bar{q}}$ or a $D^1_{d,n-d+1,\bar{m}+2,1}$ (depending on i).

Suppose that $|R \cap P| \geq 2$. Let $d = min\{i : p_i \in R\}$ and $d' = max\{i : p_i \in R\}$. Clearly, $d' \geq d+1$. If $d' = d+1$, consider $V_i = \{y_1^a, y_1^b, p_1, \ldots, p_n, y_2^a, y_2^b\} \cup A_i$. If $i = 1$, V_i contains a $Y^2_{d,n-d,\bar{q}}$. If $2 \leq i \leq m+1$, V_i contains a $D^2_{d,n-d,i-1,\bar{q}}$. If $i \geq m+2$, V_i contains a $D^2_{d,n-d,\bar{m}+2,1}$. If $d' \geq d+2$, consider $V_i = \{y_1^a, y_1^b, p_1, \ldots, p_d, p_{d'}, \ldots, p_n, y_2^a, y_2^b\} \cup A_i$. If $i = 1$, V_i contains a $Y^1_{d,n-d'+1,\bar{q}}$. If $2 \leq i \leq m + 1$, V_i contains a $D^1_{d,n-d'+1,i-1,\bar{q}}$. If $i \geq m + 2$, V_i contains a $D^1_{d,n-d'+1,\bar{m}+2,1}$.

We may suppose that $|R \cap P| = 0$. Suppose that C is not of type A and that $z \in R$. If C is a $C^B_{n_1,n_2}$, then $\{y_1^a, y_1^b, p_1, \ldots, p_n, y_2^a, y_2^b\} \cup \{z\} \cup A_i$ contains a $D^2_{n_1,n_2,i,\bar{q}}$ or a $D^2_{n_1,n_2,\bar{m}+2,1}$. If C is a $C^C_{n_1,n_2}$ or a $C^E_{n_1,n_2}$, then $\{y_1^a, y_1^b, p_1, \ldots, p_{n_1}, p_{n_1+2}, \ldots, p_n, y_2^a, y_2^b\}$ $\cup$ $\{z\} \cup A_i$ contains a $D^1_{n_1+1,n_2+1,i,\bar{q}}$ or a $D^1_{n_1+1,n_2+1,\bar{m}+2,1}$. If C is a $C^D_{n_1,n_2}$ then $\{y_1^a, y_1^b, p_1, \ldots, p_n, y_2^a, y_2^b\} \cup \{z\} \cup A_i$ contains a $D^1_{n_1,n_2,i+1,\bar{q}}$ or a $D^1_{n_1,n_2,\bar{m}+2,\bar{q}}$.

We may suppose that C is of type A or that $z \notin R$. Since $R \neq \emptyset$ then $R \cap \{y_1^c, y_2^c\} \neq \emptyset$. Suppose that $y_1^c \in R$. If $y_1^c \in C$ then we have already consider all the cases where $y_1^c \in R$. If $y_1^c \notin C$, then $p_2 \neq y_1^c$ and $\{y_1^a, y_1^b, p_2, p_1, y_1^c\} \cup A_i$ contains a $Z^1_{i+1,\bar{q}}$ or a $Z^1_{\bar{m}+2,1}$. Similarly, if $y_2^c \in R$. □

Claim 6.7.4. *If $|R \cap Y| = 1$ then (v) holds.*

Proof. By symmetry, we may suppose that $R \cap Y = \{y_2^a\}$. If $i = 1$ and $R \cap (H - \{x_1, x_2, y_2^a\}) \neq \emptyset$ then A_i is connected to more than one component of $G - S$ and so $A_i \subseteq S$, allowing as to apply Lemma 6.10.

Suppose that (v) does not hold. That is, if $i = 1$ then $|K_1| \geq k = f(n + 3)$, if $2 \leq i \leq \bar{m} + 1$ then $|K_i| \geq q$ and if $i \geq \bar{m} + 2$ then $N_i \neq \emptyset$.

Suppose that $x_1 \in R$. If $y_1^c \in R$, consider $V_i = \{y_1^a, y_1^b, y_1^c, x_1, y_2^a\} \cup A_i$. If $i = 1$, V_i can be extended to a $V_{\bar{q}}^4$. If $2 \le i \le \bar{m}+1$, V_i contains a $L_{1,1,i-1,\bar{q}}^{1B}$. If $i \ge \bar{m}+2$, V_i contains a $L_{1,1,\bar{m}+2,1}^{1B}$. If $y_1^c \notin R$, then $\{y_1^a, y_1^b, y_1^c, x_1\} \cup A_i$ contains a contains a $Z_{i,\bar{q}}^1$ or a $Z_{\bar{m}+2,1}^1$. We may suppose that $x_1 \notin R$.

Start supposing that $R \cap (C - \{x_1, x_2\}) \neq \emptyset$. By Claim 6.7.1, we may suppose that R does not contain two non-adjacent vertices of $C - \{x_1, x_2\}$.

Suppose first that either C is not of type A and $z \notin R$ or C is of type A. Suppose that $|R \cap (P - \{x_1, x_2\})| = 1$ and let $2 \le d \le n-1$ such that $p_d \in R \cap (P - \{x_1, x_2\})$. If d is even, then $\{y_1^a, y_1^b, p_1, \ldots, p_d, y_2^a\} \cup A_i$ can be extended to a $M_{d,\bar{q}}^A$ or contains a $L_{d+1,i-1,\bar{q}}^{1A}$ or a $L_{d+1,\bar{m}+2,1}^{1A}$ (depending on i). Suppose then that d is odd. If $d \le n-2$, then $p_{d+1} \neq x_2$ and so $\{y_1^a, y_1^b, p_1, \ldots, p_d, p_{d+1}\} \cup A_i$ contains a $L_{d,i,\bar{q}}^{1A}$ or a $L_{d,\bar{m}+2,1}^{1A}$. Suppose then that $d = n-1$. Since d is odd, then n is even and C is not of type A. If C is not $C_{n,1}^D$, then $N(z) \cap (P - x_2) \neq \emptyset$. Consider $V_i = \{y_1^a, y_1^b, p_1, \ldots, p_{n-1}, z, y_2^a\} \cup A_i$. If $i = 1$, then V_i can be extended to some graph of type M^B, M^C, M^D or M^E (depending on how z is connected to $P - x_2$). If $i \ge 2$, then V_i contains some graph of type L^{1B}, L^{1C}, L^{1D} or L^{1E} (depending on how z is connected to $P - x_2$). Suppose then that C is $C_{n,1}^D$ and consider $V_i = \{y_1^a, y_1^b, p_1, \ldots, p_n, y_2^b, z\} \cup A_i$. If $x_2 \in R$, then V_i contains a $Y_{n-1,1,\bar{q}}^2$ or a $D_{n-1,1,i-1,\bar{q}}^2$ or a $D_{n-1,1,\bar{m}+2,1}^2$. If $x_2 \notin R$, then V_i contains a $D_{n-1,2,i,\bar{q}}^1$ or a $D_{n-1,2,\bar{m}+2,1}^1$.

Suppose that $|R \cap (P - \{x_1, x_2\})| \ge 2$. Since R does not contains two non-adjacent vertices of $C - \{x_1, x_2\}$, then we may suppose that $|R \cap (P - \{x_1, x_2\})| = 2$ and that there is a $2 \le d \le n-2$ such that $p_d, p_{d+1} \in R$. If d is even, then $\{y_1^a, y_1^b, p_1, \ldots, p_d, y_2^a\} \cup A_i$ can be extended to a $M_{d,\bar{q}}^A$ or contains a $L_{d+1,i-1,\bar{q}}^{1A}$ or a $L_{d+1,\bar{m}+2,1}^{1A}$. If d is odd, then $\{y_1^a, y_1^b, p_1, \ldots, p_d, p_{d+1}, y_2^a\} \cup A_i$ can be extended to a $J_{d-1,\bar{q}}^1$ or contains a $L_{d,1,i-1,\bar{q}}^{1B}$ or a $L_{d,1,\bar{m}+2,1}^{1B}$.

Suppose now that C is not of type A and that $z \in R$. Since R does not contains two non-adjacent vertices of $C - \{x_1, x_2\}$, then $(R \cap (P - \{x_1, x_2\})) \subseteq N(z)$. Since C is not of type A, then n is even. Let $1 \le d \le n$ be the smallest integer such that $z \in N(p_d)$. Suppose that $d = n$ and consider $V_i = \{y_1^a, y_1^b, p_1, \ldots, p_n, y_2^a, y_2^b, z\} \cup A_i$. If $x_2 \in R$, then V_i can be extended to a $W_{n,1,\bar{q}}^{2D}$ or contains a $L_{n,1,i-1,\bar{q}}^{3D}$ or a $L_{n,1,\bar{m}+2,1}^{3D}$. If $x_2 \notin R$, then V_i can be extended to a $W_{n,1,\bar{q}}^{1D}$ or contains a $L_{n,1,i-1,\bar{q}}^{2D}$ or a $L_{n,1,\bar{m}+2,1}^{2D}$. We may suppose then that $1 \le d \le n-1$.

Suppose that d is odd and consider $V_i = \{y_1^a, y_1^b, p_1, \ldots, p_d, z, y_2^a\} \cup A_i$. If $p_d \in R$, then V_i can be extended to a $J_{d,\bar{q}}^1$ or contains a $L_{d,1,i-1,\bar{q}}^{1B}$ or a $L_{d,1,\bar{m}+2,1}^{1B}$. If $p_d \notin R$, then V_i can be extended to a $M_{d+1,\bar{q}}^A$ or contains a $L_{d+2,i-1,\bar{q}}^{1A}$ or a $L_{d+2,\bar{m}+2,1}^{1A}$. We may suppose then that d is even. If $d = n-1$ then n is odd and so C is of type A, a contradiction. Then $d \le n-2$ and so $p_{d+1} \neq x_2$. If $p_d \in R$, then $\{y_1^a, y_1^b, p_1, \ldots, p_d, y_2^a\} \cup A_i$ can be extended to a $M_{d,\bar{q}}^A$ or contains a $L_{d+1,i-1,\bar{q}}^{1A}$ or a $L_{d+1,\bar{m}+2,1}^{1A}$. Suppose then that

$p_d \notin R$. If $p_{d+1}z \notin E(G)$ then $p_{d+1} \notin R$, and so $\{y_1^a, y_1^b, p_1, \ldots p_d, p_{d+1}, z, y_2^a\} \cup A_i$ can be extended to a $M^D_{d,2,\bar{q}}$ or contains a $L^{1D}_{d,3,i-1,\bar{q}}$ or a $L^{1D}_{d,3,\bar{m}+2,1}$. We may suppose then that $p_{d+1}z \in E(G)$. Consider $V_i = \{y_1^a, y_1^b, p_1, \ldots, p_d, p_{d+1}, z, y_2^a\} \cup A_i$. If $p_{d+1} \in R$, then V_i can be extended to a $J^2_{d,\bar{q}}$ or contains a $L^{1C}_{d,1,i-1,\bar{q}}$ or a $L^{1C}_{d,1,\bar{m}+2,1}$. If $p_{d+1} \notin R$, then V_i can be extended to a $M^B_{d,1,\bar{q}}$ or contains a $L^{1B}_{d,2,i-1,\bar{q}}$ or a $L^{1B}_{d,2,\bar{m}+2,1}$.

We may suppose now that $R \cap (C - \{x_1, x_2\}) = \emptyset$. If $x_2 \notin R$, depending on the type of C, $\{y_1^a, y_1^b, p_1, \ldots, p_n, y_2^a, y_2^b\} \cup A_i$ contains a $L^{1A}_{n,i+1,\bar{q}}$, $L^{1B}_{n_1,n_2,i+1,\bar{q}}$, $L^{1C}_{n_1,n_2,i+1,\bar{q}}$, $L^{1D}_{n_1,n_2,i+1,\bar{q}}$ or $L^{1E}_{n_1,n_2,i+1,\bar{q}}$ or the corresponding graphs with $m = \bar{m} + 2$ and $q = 1$ (if $i \geq \bar{m} + 2$). If $x_2 \in R$, depending on the type of C, $\{y_1^a, y_1^b, p_1, \ldots, p_n, y_2^b\} \cup A_i$ contains a $L^{1A}_{n,i,\bar{q}}$, $L^{1B}_{n_1,n_2,i,\bar{q}}$, $L^{1C}_{n_1,n_2,i,\bar{q}}$, $L^{1D}_{n_1,n_2,i,\bar{q}}$ or $L^{1E}_{n_1,n_2,i,\bar{q}}$ or the corresponding graphs with $m = \bar{m} + 2$ and $q = 1$ if $i \geq \bar{m} + 2$. □

By the discussion done before, proving Claims 6.7.1, 6.7.2, 6.7.3 and 6.7.4 completes the proof of the theorem. □

6.5 Final Remarks

In this chapter we considered the problem of characterizing the families of forbidden induced subgraphs that imply a near perfect matching in large enough connected graphs. We were able to characterize all such families and the characterization is given by Theorem 6.5 with the family $\mathcal{F}(l, n, m, q)$. In particular, Theorem 6.5 extends the previous result given in [18], that is Theorem 6.1.

Even though we did not mention it before, it is important to notice that the family $\mathcal{F}(l, n, m, q)$ itself is part of $\mathbf{H}$. To see this, one needs to check that the family $\mathcal{F}(l, n, m, q)$ is non-redundant. A complete and detailed proof of this fact would be tedious, would take several pages, and it is not essential. However, it is easy to check by looking at the "subfamilies" composing $\mathcal{F}(l, n, m, q)$ and the conditions given to the parameters in the definition of those subfamilies. Part of the proof of the non-redundancy of $\mathcal{F}(l, n, m, q)$ is actually included in the proof of Lemma 6.11.

Nevertheless, we would like to remark that this is an important fact in our result, since the $\mathcal{F}(l, n, m, q)$ could be replaced in Theorem 6.5 by some other redundant family that includes $\mathcal{F}(l, n, m, q)$ without changing the truth of Theorem 6.5. For example, $\mathcal{F}(n) = \{$ G: G is a connected graph of odd order with no near perfect matching and $|V(G)| \geq n$ $\}$ is such a family. Proving Theorem 6.5 with $\mathcal{F}(n)$ is of course no meaningful result.

In Chapter 5, we did a similar characterization for graphs with perfect matchings. There are two big differences between the results. The first is that the families for near perfect matchings are much more complicated. Actually, the characterization for perfect matchings uses only 2 types of graphs ($Z^1_{m,q}$ and V^3_q, see Section 6.3 for definitions), compared to 34 types we use in our characterization. The second is that for perfect matchings it is enough to consider finite families, but for near perfect matching it is necessary to consider also infinite families (the family $\mathcal{F}(l, n, m, q)$ defined in section Section 6.3 contains infinite many graphs).

Also, as we considered perfect matchings and near perfect matchings, it seems natural to consider the same problem for graphs of higher deficiency.

Problem 6.1. *Giver $d \geq 0$, characterize all the families of connected graphs $\mathcal{F}$ such that every large enough $\mathcal{F}$-free connected graph G with $|V(G)| \equiv d \pmod 2$ has deficiency d.*

In this thesis we solved Problem 6.1 for $d = 0, 1$. We think that the higher the deficiency, the more complicated the families of forbidden subgraphs will be. We leave the characterization of forbidden families for graphs of deficiency higher that 1 as an open problem.

Chapter 7

Toughness

In this chapter, we study the relation between toughness and forbidden induced subgraphs. The main result in this chapter is Theorem 7.2, which shows for each $t > 0$, a characterization of all families of forbidden subgraphs that imply the property of being t-tough in connected graphs of large enough order. All the new results we prove in this chapter can be found in [29].

7.1 Introduction

Let G be a connected graph. Broersma[5] proposed to study the relation between forbidden subgraphs in G and the resulting toughness of G. Consider the following problem.

Problem 7.1. *Let t be a positive real number. Characterize the families of connected graphs $\mathcal{F}$ such that every connected $\mathcal{F}$-free graph is t-tough.*

If the problem is stated this way, it is easy to see that the family $\mathcal{F}(n) = \{K_{1,n+1}\}$ (with $n = \lfloor \frac{1}{t} \rfloor$) is essentially the only answer, as the following proposition shows.

Proposition 7.1. *Let $t > 0$ and let $\mathcal{F}$ be a family of connected graphs. Then every connected $\mathcal{F}$-free is t-tough if and only if $\mathcal{F} \leq \{K_{1,n+1}\}$, where $n = \lfloor \frac{1}{t} \rfloor$.*

Proof. The only if part is a consequence of the fact that $K_{1,n+1}$ itself is not t-tough. For the if part, we need to show that every connected $K_{1,n+1}$-free graph is t-tough. For $1 \leq t \leq \frac{1}{2}$, it is a consequence of Lemma 7.7. For $t > \frac{1}{2}$, we have $n = 1$ and so $K_{1,n+1} = P_3$, which implies that the $K_{1,n+1}$-free graphs are complete graphs. □

We consider then allowing a finite number of exceptions. In other words, we study the following problem.

Problem 7.2. *Let t be a positive real number. Characterize the families of connected graphs $\mathcal{F}$ such that every **large enough** connected $\mathcal{F}$-free graph is t-tough.*

In this chapter, we solve Problem 7.2. The answer is expressed in Theorem 7.2, that we state in the following section.

The rest of this chapter is organized as follows. In Section 7.2, we make all needed definitions and present our main results. In Section 7.3, we give the proofs for the case $t > \frac{1}{2}$. In Section 7.4, we give the proofs for the case $0 \leq t \leq \frac{1}{2}$. Finally, in Section 7.5, we make some analysis of our results and propose some open problems.

7.2 Definitions and main results

Define $\mathbf{G}$ as the set of all non-redundant families of connected graphs. Let $t > 0$. Define $\mathbf{H}(t)$ as the set of families $\mathcal{F} \in \mathbf{G}$ such that there is a constant $n_0 = n_0(t, \mathcal{F})$ with the property that all $\mathcal{F}$-free connected graphs G with $|V(G)| \geq n_0$ are t-tough. The answer to Problem 7.2 is reduced to finding all the elements in the set $\mathbf{H}(t)$.

Define the following graphs (See Figure 7.1).

- Y_m^n is the graph obtained from identifying the center of a $K_{1,n}$ with the first vertex of a path on m vertices. The last vertex of the path is called the tail of the Y_m^n.

- $Z_{m,r}^n$ is the graph obtained by identifying one vertex of a K_r with the tail of a Y_m^n.

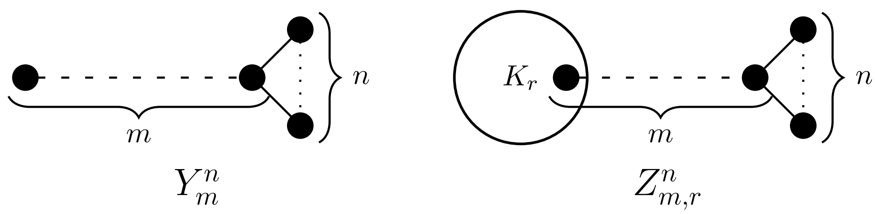


Figure 7.1: Some forbidden subgraphs

Define the following families of graphs:

- Let $\mathcal{F}^A(m, l, r) = \{K_{1,l}, P_m, Z_{1,r}^1\}$.

- Let $\mathcal{F}_n^B(m, l, r) = \{K_{1,l}, Y_{m+2}^n, Z_{1,r}^n, \ldots, Z_{m,r}^n\}$.

Define the following subsets of $\mathbf{G}$:

- Let $\mathbf{F}^A = \{\ \mathcal{F} \in \mathbf{G}$: $\mathcal{F} \leq \mathcal{F}^A(m, l, r)$ for some $m \geq 4$, $l \geq 3$ and $r \geq 3\ \}$.

- Let $\mathbf{F}_n^B = \{\ \mathcal{F} \in \mathbf{G}$: $\mathcal{F} \leq \mathcal{F}_n^B(m, l, r)$ for some $m \geq 1$, $l \geq n + 2$ and $r \geq 3\ \}$.

Our main result in this chapter and the answer to Problem 7.2 is the following theorem.

Theorem 7.2. *Let t be a positive real number. Then,*

- *If $t > \frac{1}{2}$, $\mathbf{H}(t) = \mathbf{F}^A$.*
- *If $0 < t \leq \frac{1}{2}$, $\mathbf{H}(t) = \mathbf{F}^B_n$, where $n = \lfloor \frac{1}{t} \rfloor$.*

In the rest of this section we give some definitions we need for our proofs.

Let G be a connected graph. If $v, w \in V(G)$, we write $v \sim w$ when $vw \in E(G)$. If $S \subseteq V(G)$ is a cutset of G and $x \in S$, define

$$\mathcal{C}_S(x) = \{ \ C : C \text{ is a component of } G - S \text{ such that } N(x) \cap V(C) \neq \emptyset \ \}.$$

For $X \subseteq S$, define $\mathcal{C}_S(X) = \bigcup_{x \in X} \mathcal{C}_S(x)$. If there is no ambiguity about the set S, we write $\mathcal{C}(x)$ instead of $\mathcal{C}_S(x)$.

Let G be a connected graph. If $0 < t \leq 1$, it is easy to see that the cutset condition is not needed in the definition of t-tough. That is, G is t-tough if for every non-empty set $S \subseteq V(G)$, it holds that $t \cdot w(G - S) \leq |S|$. In accordance to this, we make the following non-standard but useful definition. A non-empty set $S \subseteq V(G)$ is a *t-tough cut* if $w(G - S) > \frac{1}{t}|S|$. A t-tough cut $S \subseteq V(G)$ is a *minimal t-tough cut* if for every $S' \subset S$, S' is not a t-tough cut.

Let $S \subseteq V(G)$ be a t-tough cut, $x \in S$ and let $\mathcal{D} \subseteq \mathcal{C}_S(x)$ be a set of components. A set $A \subseteq V(G)$ is a *selection* for x from $\mathcal{D}$ if $A \subseteq N(x)$ and for every $C \in \mathcal{D}$, $|A \cap V(C)| = 1$. In other words, A is made by taking exactly one vertex of $N(x)$ from each $C \in \mathcal{D}$. A set $A \subseteq V(G)$ is a *selection* for x if A is selection for x from $\mathcal{C}_S(x)$.

The following is a direct corollary of Hall's marriage Theorem (see Theorem 9.4 of [8] or Corollary 1.1.4 of [25]). We use it later in Section 7.4.

Theorem 7.3. *Let G be a bipartite graph with partite sets X and Y with $X = \{x_1, \ldots x_k\}$. Suppose that for all $X' \subseteq X$, $|N(X')| \geq n|X'|$. Then there are pairwise disjoint subsets $Y_1, \ldots, Y_k$ of Y such that for all $1 \leq i \leq k$, $Y_i \subseteq N(x_i)$ and $|Y_i| = n$.*

7.3 Case $t > \frac{1}{2}$

Theorem 7.4. *Let $t > \frac{1}{2}$. Then $\mathbf{F}^A \subseteq \mathbf{H}(t)$.*

Proof. Let $\mathcal{F} \in \mathbf{F}^A$. Let $m \geq 4$, $l \geq 3$ and $r \geq 3$ such that $\mathcal{F} \leq \mathcal{F}^A(m, l, r)$. Let G be a connected $\mathcal{F}$-free graph. Suppose that G is not t-tough. We will show that $|V(G)|$ is bounded by a function depending only on t, m, l and r.

Since G is not t-tough, there is a cutset $S \subseteq V(G)$ such that $|S| < t \cdot w(G-S)$. We may suppose that S is minimal under inclusion.

Claim 7.4.1. *There is a vertex $y \in N(S) - S$ such that $|N(y) \cap S| < l \cdot t$.*

Proof. Suppose on the contrary that for all $y \in N(S) - S$, $|N(y) \cap S| \geq l \cdot t$.

Let k be the number of pairs (x, C) with $x \in S$ and $C \in \mathcal{C}(x)$. Clearly,

$$k = \sum_{x \in S} |\mathcal{C}(x)| \quad \text{and} \quad k = \sum_{C \in \mathcal{C}(S)} |N(C) \cap S|.$$

Since G is $K_{1,l}$-free then $|\mathcal{C}(x)| < l$ for all $x \in S$. Then, we have that

$$k = \sum_{x \in S} |\mathcal{C}(x)| < l \cdot |S| < l \cdot t \cdot w(G-S).$$

Let $C \in \mathcal{C}(S)$ and let $y \in V(C) \cap N(S)$. Then $|N(C) \cap S| \geq |N(y) \cap S| \geq l \cdot t$. We conclude that $|N(C) \cap S| \geq l \cdot t$ for all $C \in \mathcal{C}(S)$. Then, we have that

$$k = \sum_{C \in \mathcal{C}(S)} |N(C) \cap S| \geq l \cdot t \cdot |\mathcal{C}(S)| = l \cdot t \cdot w(G-S),$$

a contradiction, completing the proof. □

Let y_1 be a vertex in $N(S) - S$ as in Claim 7.4.1 and let $x_0 \in S \cap N(y_1)$. Let $C_1 \in \mathcal{C}(x_0)$ such that $y_1 \in C_1$.

We show now that $|\mathcal{C}(x_0)| \geq 2$. Since S is a cutset, then $|\mathcal{C}(S)| = w(G-S) \geq 2$. If $|S| = 1$, then $|\mathcal{C}(x_0)| = |\mathcal{C}(S)| \geq 2$. Suppose then that $|S| \geq 2$. If $|\mathcal{C}(x_0)| \leq 1$, then since G is connected $S' = S - \{x_0\}$ is still a cutset with $w(G-S') \geq w(G-S)$. Then $t \cdot w(G-S') \geq t \cdot w(G-S) > |S| > |S'|$. This contradicts the minimality of S. We conclude that $|\mathcal{C}(x_0)| \geq 2$.

Thus, there is a component $C_2 \in \mathcal{C}(x_0)$ with $C_2 \neq C_1$. Let $y_2 \in N(x_0) \cap V(C_2)$.

Since G is P_m-free then $N^{m-1}(x_0) = \emptyset$. Then it suffices to show that $N^i(x_0)$ is bounded for all $1 \leq i \leq m-2$.

First, we show that $|N(x_0)| < 2 \cdot R(l, r) + t \cdot l$. Since $\{x_0\} \cup N(x_0)$ has no $K_{1,l}$ then $N(x_0)$ has no independent set of size l. Since $\{y_1, x_0\} \cup (N(x_0) - N(y_1))$ does not contain a $Z^1_{1,r}$ then $N(x_0) - N(y_1)$ does not contain a clique of size r. In the same way, $|N(x_0) - N(y_2)|$ does not contain a clique of size r. Then we have that $|(N(x_0) - N(y_1)) \cup (N(x_0) - N(y_2))| < 2 \cdot R(l, r)$

Let $X = N(x_0) \cap N(y_1) \cap N(y_2)$. Since y_1 and y_2 are in different components of $G - S$, X has neighbors in more than one component of $G - S$ and so $X \subseteq S$. By the way y_1 was chosen, $|X| < l \cdot t$. We conclude that $|N(x_0)| < 2 \cdot R(l, r) + t \cdot l$.

Let $i \geq 1$. We show that $|N^{i+1}(x_0)| < R(l, r) \cdot |N^i(x_0)|$. Let $x_i \in N^i(x_0)$. It is enough to show that $|N(x_i) \cap N^{i+1}(x_0)| < R(l, r)$. Since $\{x_i\} \cup (N(x_i) \cap N^{i+1}(x_0))$ has no $K_{1,l}$ then $N(x_i) \cap N^{i+1}(x_0)$ has no independent set of size l. Let $x_{i-1} \in N^{i-1}(x_0)$.

Notice that if $i = 1$ then $x_{i-1} = x_0$. Since $\{x_{i-1}, x_i\} \cup (N(x_i) \cap N^{i+1}(x_0))$ does not contain a $Z^1_{1,r}$ then $N(x_i) \cap N^{i+1}(x_0)$ does not contain a clique of size r. We conclude that $|N(x_i) \cap N^{i+1}(x_0)| < R(l, r)$.

Using an inductive argument, we get that for all $i \geq 0$

$$|N^i(x_0)| < R(l,r)^{i-1} \cdot |N(x_0)| < R(l,r)^{i-1} \cdot (2 \cdot R(l,r) + t \cdot l).$$

Since $N^{m-1}(x_0) = \emptyset$, then $|N^i(x_0)| < R^{m-2}(l,r) \cdot (2 \cdot R(l,r) + t \cdot l)$ for all $1 \leq i \leq m-2$. □

Theorem 7.5. *Let $t > \frac{1}{2}$. Then $\mathbf{H}(t) \subseteq \mathbf{F}^A$.*

Proof. Let $\mathcal{F} \in \mathbf{H}(t)$. Then there is a positive integer n_0 such that every $\mathcal{F}$-free connected graph of order at least n_0 is t-tough. Let n_1 be an integer such that $n_1 \geq \max(n_0, 3)$.

Consider the family $\mathcal{F}' = \mathcal{F}^A(n_1, n_1, n_1)$. K_{1,n_1} has toughness $\frac{1}{n_1} < \frac{1}{2}$. P_{n_1} has toughness $\frac{1}{2}$. Z^1_{1,n_1} has toughness $\frac{1}{2}$. Thus, all the graphs in $\mathcal{F}'$ have toughness at most $\frac{1}{2}$ and so none of them is t-tough.

Since $n_1 \geq n_0$ then all the graphs in $\mathcal{F}'$ are connected graphs of order at least n_0. Then it must be that no graph of $\mathcal{F}'$ is $\mathcal{F}$-free. In other words, for each graph $H' \in \mathcal{F}'$, there is a graph $H \in \mathcal{F}$ such that $H \preceq H'$. This is exactly the definition of $\mathcal{F} \leq \mathcal{F}'$. Then since $\mathcal{F}'$ is in $\mathbf{F}^A$, we conclude that $\mathcal{F}$ is also in $\mathbf{F}^A$. □

7.4 Case $0 < t \leq \frac{1}{2}$

Theorem 7.6. *Let $0 < t \leq \frac{1}{2}$. Then $\mathbf{F}^B_n \subseteq \mathbf{H}(t)$, where $n = \lfloor \frac{1}{t} \rfloor$.*

We divide the proof of this theorem in several lemmas that we state and prove bellow.

Lemma 7.7. *Let G be a connected graph. Let $0 < t \leq 1$, and S be a minimal t-tough cut. Then $|\mathcal{C}_S(X)| > \frac{1}{t}|X|$ for all nonempty $X \subseteq S$. In particular, $|\mathcal{C}_S(x)| > \frac{1}{t}$ for all $x \in S$.*

Proof. By the definition of t-tough cut, $w(G - S) > \frac{1}{t}|S|$. Let $S' = S - X$. By the minimality of S, $w(G - S') \leq \frac{1}{t}|S'|$.

Since each component of $G - S$ not in $\mathcal{C}_S(X)$ is a component of $G - S'$, then $\mathcal{C}_S(S) - \mathcal{C}_S(X) \subseteq \mathcal{C}_{S'}(S')$ and so $w(G - S) - |\mathcal{C}_S(X)| \leq w(G - S')$. Then we have that

$$\frac{1}{t}|S| - |\mathcal{C}_S(X)| < w(G - S) - |\mathcal{C}_S(X)| \leq w(G - S') \leq \frac{1}{t}|S'| = \frac{1}{t}(|S| - |X|)$$

Then, we conclude that $|\mathcal{C}_S(X)| > \frac{1}{t}|X|$. □

Lemma 7.8. *Let G be a connected graph. Let $n \geq 2$, $0 < t \leq \frac{1}{n}$, S be a minimal t-tough cut and $x_0 \in S$. If G is Y_m^n-free for some $m \geq 1$, then $N^{m'}(x_0) = \emptyset$, where $m' = 2 \cdot \max(n, m+1) + m$.*

Proof. Suppose that $N^{m'}(x) \neq \emptyset$. Let $P = x_0 \cdots x_{m'}$ be a path such that $x_i \in N^i(x_0)$. Notice that P is an induced path. If $v \in P$ with $v = x_i$, we use the notation $v^{+j} = x_{i+j}$ and $v^{-j} = x_{i-j}$.

Let $q = \max(n, m+1)$. We construct a subsequence $v_1, \ldots, v_q$ of $x_0, \ldots, x_{m'}$ and sets $A_1, \ldots, A_q$ satisfying the following properties:

(i) $v_i \in S$ for all $1 \leq i \leq q$,

(ii) v_{i+1} is either v_i^{+1} or v_i^{+2} for all $1 \leq i \leq q-1$,

(iii) A_i is a selection for v_i for all $1 \leq i \leq q$ and

(iv) $|A_i - A_{i+1}| \leq n-1$ for all $1 \leq i \leq q-1$.

We do the construction by induction. Choose $v_1 = x_0$ and let A_1 be any selection for x_0. Let $1 \leq i < q$ and suppose we have chosen $v_1, \ldots, v_i$ and $A_1, \ldots, A_i$. We choose v_{i+1} and A_{i+1} in the following way.

Note that by condition (ii), if $v_i = x_h$ then $h \leq 2i - 2 \leq 2q - 4$. Thus, we have that $m' = 2q + m > h + m$, and hence v_i^{+j} exists for all $1 \leq j \leq m$.

For all $j \geq 3$, since the distance between v_i and v_i^{+j} is j then $N(v_i) \cap N(v_i^{+j}) = \emptyset$ and so $A_i \cap N(v_i^{+j}) = \emptyset$. Let $Y_1 = A_i \cap N(v_i^{+1})$ and $Y_2 = A_i \cap N(v_i^{+2})$.

Suppose $|Y_2| = 1$ and let $y \in Y_2$. We have that $y \sim v_i$, $y \sim v_i^{+2}$ and $y \nsim v_i^{+j}$ for all $3 \leq j \leq m-1$. Since y and the vertices of A_i are in different components of $G - S$, then $N(y) \cap A_i = \emptyset$. By Lemma 7.7, $|A_i| > \frac{1}{t} \geq n$ and so $|A_i - \{y\}| \geq n$. Since the vertices of A_i are in different components, $A_i - \{y\}$ is an independent set. But then, $(A_i - \{y\}) \cup \{v_i, y, v_i^{+2}, v_i^{+3}, \ldots, v_i^{+m-1}\}$ contains a Y_m^n which is a contradiction.

Suppose now that $|Y_2| = 0$ and $|Y_1| \leq 1$. We have that $(A_i - Y_1) \cap N(v_i^{+1}) = \emptyset$. Since $|A_i| \geq n+1$ then $|A_i - Y_1| \geq n$. But then, $(A_i - Y_1) \cup \{v_i, v_i^{+1}, v_i^{+2}, \ldots, v_i^{+m-1}\}$ contains a Y_m^n which is a contradiction. Then, we have that either $|Y_2| \geq 2$, or $|Y_2| = 0$ and $|Y_1| \geq 2$.

If $|Y_2| \geq 2$, then v_i^{+2} has neighbors in more than one component of $G - S$ and so $v_i^{+2} \in S$. Choose $v_{i+1} = v_i^{+2}$ and let A_{i+1} be any selection for v_i^{+2} such that $Y_2 \subseteq A_{i+1}$. Let $y \in Y_2$. In a similar way to the case $|Y_2| = 1$, since $(A_i - A_{i+1}) \cup \{v_i, y, v_i^{+2}, v_i^{+3}, \ldots, v_i^{+m-1}\}$ does not contain a Y_m^n then $|A_i - A_{i+1}| \leq n-1$.

If $|Y_2| = 0$ and $|Y_1| \geq 2$, then $v_i^{+1} \in S$. Choose $v_{i+1} = v_i^{+1}$ and let A_{i+1} be any selection for v_i^{+1} such that $Y_1 \subseteq A_{i+1}$. Since $(A_i - A_{i+1}) \cup \{v_i, v_i^{+1}, v_i^{+2}, \ldots, v_i^{+m-1}\}$ does not contain a Y_m^n then $|A_i - A_{i+1}| \leq n-1$.

Claim 7.8.1. $|A_q| \leq 2(n-1)$.

Proof. As before, for $j \geq 3$ we have that $A_q \cap N(v_q^{-j}) = \emptyset$.

Suppose first that $A_q \cap N(v_q^{-2}) \neq \emptyset$ and let $y \in A_q \cap N(v_q^{-2})$.
Since $(A_q - N(v_q^{-2})) \cup \{v_q, y, v_q^{-2}, \dots v_q^{-(m-1)}\}$ does not contain a Y_n^m, then $|A_q - N(v_q^{-2})| \leq n-1$. Since $(A_q \cap N(v_q^{-2})) \cup \{v_q^{-2}, \dots, v_q^{-(m+1)}\}$ does not contain a Y_n^m, then $|A_q \cap N(v_q^{-2})| \leq n-1$. Then $|A_q| = |A_q - N(v_q^{-2})| + |A_q \cap N(v_q^{-2})| \leq (n-1)+(n-1) = 2(n-1)$.

Suppose now that $A_q \cap N(v_q^{-2}) = \emptyset$.
Since $(A_q - N(v_q^{-1})) \cup \{v_q, v_q^{-1}, v_q^{-2}, \dots, v_q^{-(m-1)}\}$ does not contain a Y_n^m, then $|A_q - N(v_q^{-1})| \leq n-1$. Since $(A_q \cap N(v_q^{-1})) \cup \{v_q^{-1}, v_q^{-2}, \dots, v_q^{-m}\}$ does not contain a Y_n^m, then $|A_q \cap N(v_q^{-1})| \leq n-1$. Then $|A_q| = |A_q - N(v_q^{-1})| + |A_q \cap N(v_q^{-1})| \leq (n-1)+(n-1) = 2(n-1)$. □

By Lemma 7.7, we have that

$$|A_1 \cup \dots \cup A_q| \geq |C_S(v_1) \cup \dots \cup C_S(v_q)| = |C_S(v_1, \dots, v_q)| > n \cdot q.$$

On the other hand, we have that

$$|A_1 \cup \dots \cup A_q| = |A_1 - \bigcup_{i=2}^{q} A_i| + |A_2 - \bigcup_{i=3}^{q} A_i| + \dots + |A_{q-1} - A_q| + |A_q| \leq$$

$$|A_1 - A_2| + |A_2 - A_3| + \dots + |A_{q-1} - A_q| + |A_q| \leq (n-1)(q-1) + 2(n-1) = (n-1)(q+1).$$

Then we have that $(n-1)(q+1) > n \cdot q$. Then, we get that $q < n-1$, which contradicts the way q was taken ($q = \max(n, m+1)$). □

Lemma 7.9. *Let G be a connected graph. Let $n \geq 2$, $0 < t \leq \frac{1}{n}$, S be a minimal t-tough cut. Let $X \subseteq S$ be a clique. If G is $\{K_{1,l}, Z_{1,r}^n\}$-free for some $r \geq 3$ and some $l \geq n+2$, then $|X| < l(r-1)$.*

Proof. Let $\mathcal{Y} = \mathcal{C}(X)$. For each $x \in X$ let $\mathcal{Y}_x = \mathcal{C}(x)$.

Claim 7.9.1. *For each $x \in X$ there is a set $Y_x \subseteq V(G)$ that is a selection for x from some set $\mathcal{Y}'_x \subseteq \mathcal{Y}_x$ such that $|Y_x| = n$, and so that for all $x_1, x_2 \in X$ ($x_1 \neq x_2$), $\mathcal{Y}'_{x_1} \cap \mathcal{Y}'_{x_2} = \emptyset$.*

Proof. Let G' be the bipartite graph with vertex set $V(G') = X \cup \mathcal{Y}$ and edge set $E(G') = \{(x, C) : x \in X, C \in \mathcal{Y}_x\}$.

Since $X \subseteq S$, by Lemma 7.7, for all $X' \subseteq X$, $|N_{G'}(X')| = |\mathcal{C}(X')| > n|X'|$. Then, by applying Theorem 7.3 to G', for each $x \in X$ there is a set $\mathcal{Y}'_x \subseteq \mathcal{Y}_x$ such that $|\mathcal{Y}'_x| = n$ and so that for all $x_1, x_2 \in X$ ($x_1 \neq x_2$), $\mathcal{Y}'_{x_1} \cap \mathcal{Y}'_{x_2} = \emptyset$.

For each $x \in X$, let $Y_x \subseteq V(G)$ be a selection for x from $\mathcal{Y}'_x$. Then, the claim follows. □

Let $x \in X$. If $|X - N(Y_x)| \geq r-1$ then $Y_x \cup \{x\} \cup (X - N(Y_x))$ contains a $Z^n_{1,r}$ which is a contradiction. Then for all $x \in X$, $|X - N(Y_x)| < r-1$.

We may suppose that $|X| \geq l$. Let $x_1, \ldots, x_l \in X$. If there is a vertex $x \in X - \bigcup_{i=1}^{l}(X - N(Y_{x_i}))$ then for all $1 \leq i \leq l$, $N(x) \cap Y_{x_i} \neq \emptyset$. Since the Y_{x_i}'s are selections from pairwise disjoint $\mathcal{Y}'_{x_i}$'s, then $N(x) \cup \bigcup_{i=1}^{l} Y_{x_i}$ contains a $K_{1,l}$, which is a contradiction. Then $X = \bigcup_{i=1}^{l}(X - N(Y_{x_i}))$. But then $|X| = |\bigcup_{i=1}^{l}(X - N(Y_{x_i}))| < l(r-1)$. $\square$

Lemma 7.10. *Let G be a connected graph. Let $n \geq 2$, $0 < t \leq \frac{1}{n}$, S be a minimal t-tough cut and $x_0 \in S$. Let $X \subseteq N(x_0)$ be a clique. If G is $Z^n_{1,r}$-free for some $r \geq 3$, then $|X| < q$ where $q = r(l+1)$.*

Proof. Let $X_1 = X - S$ and $X_2 = X \cap S$. By Lemma 7.9, $|X_2| < l(r-1)$. We will show that $|X_1| < r$ and so $|X| = |X_1| + |X_2| < r + l(r-1) < r(l+1) = q$.

Let Y_0 be a selection for x_0. By Lemma 7.7, $|Y_0| \geq n+1$. Let Y be any subset of Y_0 with $|Y| = n+1$.

Since $X_1 \cap S = \emptyset$, then there is a component C of $G - S$ such that $X_1 \subseteq V(C)$. Let $Y' = Y \cap V(C)$. Then $|Y'| \leq 1$ and so $|Y - Y'| \geq n$. Since $X_1 \subseteq V(C)$ then there are no edges between $Y - Y'$ and X_1. Then since $(Y - Y') \cup \{x_0\} \cup X_1$ does not contain a $Z^n_{1,r}$ it must be that $X_1 < r$. $\square$

Lemma 7.11. *Let G be a connected graph. Let $n \geq 2$, $0 < t \leq \frac{1}{n}$, S be a minimal t-tough cut, $x_0 \in S$. Let $x_1 \in N(x_0)$ and let $X \subseteq N(x_1) \cap N^2(x_0)$ be a clique. If G is $\{Z^n_{1,r}, Z^n_{2,r}\}$-free for some $r \geq 3$, then $|X| < q$ where $q = r(l+1)$.*

Proof. If $x_1 \in S$, then by Lemma 7.10, $|X| < r(l+1)$. Then we may suppose that $x_1 \notin S$.

Let $X_1 = X - S$ and $X_2 = N \cap S$. By Lemma 7.9, $|X_2| < l(r-1)$. We will show that $|X_1| < r$ and so $|X| = |X_1| + |X_2| < r + l(r-1) < r(l+1) = q$.

Let Y_0 be a selection for x_0. By lemma 7.7, $|Y_0| \geq n+1$. Let Y be any subset of Y_0 with $|Y| = n+1$.

Since $X_1 \cap S = \emptyset$, then there is a component C of $G - S$ such that $X_1 \subseteq V(C)$. We may suppose that $x_1 \in V(C)$. Let $Y' = Y \cap V(C)$. Then $|Y'| \leq 1$ and so $|Y - Y'| \geq n$. Furthermore, since $x_1 \in V(C)$ there are no edges between x_1 and $Y - Y'$ and since $X_1 \subseteq V(C)$, no edges between X_1 and $Y - Y'$. But then, since $(Y - Y') \cup \{x_0, x_1\} \cup X_1$ does not contain a $Z^n_{2,r}$, $|X_1| < r$. $\square$

Lemma 7.12. *Let G be a connected graph. Let $n \geq 2$, $0 < t \leq \frac{1}{n}$, S be a minimal t-tough cut, $x_0 \in S$ and $i \geq 0$. If G is $\{K_{1,l}, Z^n_{1,r}, \ldots, Z^n_{i+1,r}\}$-free for some $r \geq 3$ and some $l \geq n+2$, then $|N^{i+1}(x_0)| < |N^i(x_0)| \cdot R(l,q)$ where $q = r(l+1)$.*

Proof. Let $x_i \in N^i(x_0)$. Notice that if $i = 0$ then $x_i = x_0$. We will show that $|N(x_i) \cap N^{i+1}(x_0)| < R(l, q)$.

Since $\{x_i\} \cup (N(x_i) \cap N^{i+1}(x_0))$ does not contain a $K_{1,l}$, then $N(x_i) \cap N^{i+1}(x_0)$ does not contain an independent set of size at least l. Let $X \subseteq N(x_i) \cap N^{i+1}(x_0)$ be a clique. We will show that $|X| < q$ and so we can conclude that $|N(x_i) \cap N^{i+1}(x_0)| < R(l, q)$.

Let $P = x_0 \cdots x_i$ be a path from x_0 to x_i such that for all $0 \le j \le i$, $x_j \in N^j(x_0)$. Notice that P is an induced path.

Let $k = \max\{\ j\colon\ 0 \le j \le i \text{ and } x_j \in S\}$. Since $x_0 \in S$, such an index k exists. If $k = i$ or $k = i - 1$ then the result follows from Lemma 7.10 and Lemma 7.11, respectively (take x_k as the x_0 in the corresponding lemma).

Suppose that $k \le i - 2$. Let Y be a selection for x_k. By Lemma 7.7, $|Y| \ge n + 1$. Let P' be the subpath of P going from x_k to x_i. Then P' is a shortest path from x_k to x_i, and hence $|N(Y) \cap P| \subseteq \{x_k, x_{k+1}, x_{k+2}\}$. Furthermore, $|N(Y) \cap X| = \emptyset$.

Let $Y_1 = Y \cap N(x_{k+1})$ and $Y_2 = Y \cap N(x_{k+2})$. By the way k was chosen, none of x_{k+1} and x_{k+2} is in S and so $|Y_1| \le 1$ and $|Y_2| \le 1$.

Suppose that $|Y_2| = 1$ and let $y \in Y_2$. Since $(Y - \{y\}) \cup \{x_k, y, x_{k+2}, \dots, x_i\} \cup X$ does not contain a $Z^n_{i-k+1,r}$ then $|X| < r < r(l+1) = q$. Then, we may suppose that $|Y_2| = 0$.

Since $|Y_1| \le 1$, then $|Y - Y_1| \ge n$. Then, since $(Y - Y_1) \cup \{x_k, x_{k+1}, x_{k+2}, \dots, x_i\} \cup X$ does not contain a $Z^n_{i-k+1,r}$ we have that $|X| < r < q$. □

We use the above lemmas to prove Theorem 7.6.

Proof of Theorem 7.6. Let $\mathcal{F} \in \mathbf{F}^B_n$. Let $m \ge 1$, $l \ge n + 2$, and $r \ge 3$ such that $\mathcal{F} \le \mathcal{F}^B_n(m, l, r)$.

Let G be an $\mathcal{F}$-free connected graph. Suppose that G is not t-tough. We will show that $|V(G)|$ is bounded by a function depending only on t, l, m and r.

Since G is not t-tough, G has a t-tough cut. We may suppose that S is a minimal t-tough cut. Let $x_0 \in S$.

Notice that since G is Y^n_{m+2}-free, then G is $Z^n_{i,r}$-free for all $i \ge m + 1$. Since we also know that G is $Z^n_{i,r}$-free for all $1 \le i \le m$, we conclude that G is $Z^n_{i,r}$-free for all $i \ge 1$. Notice also that since $n = \lfloor \frac{1}{t} \rfloor$, then $t \le \frac{1}{n}$. Then G satisfies all the conditions of Lemmas 7.8 and 7.12.

Let $m' = 2 \cdot \max(n, m + 1) + m$. By Lemma 7.8, $N^{m'}(x_0) = \emptyset$. Then we only need to show that $N^i(x_0)$ is bounded for all $1 \le i \le m' - 1$.

Let $q = r(l + 1)$. By Lemma 7.12, $|N^{i+1}(x_0)| < R(l, q) \cdot |N^i(x_0)|$ for all $i \ge 0$. Using an inductive argument we get that $|N^i(x_0)| < R(l, q)^{i-1}$ for all $i \ge 1$. Since $N^{m'}(x_0) = \emptyset$, we conclude that $|N^i(x_0)| < R(l, q)^{m'-2}$ for all $1 \le i \le m' - 1$. □

Theorem 7.13. *Let $0 < t \leq \frac{1}{2}$. Then $\mathbf{H}(t) \subseteq \mathbf{F}_n^B$, where $n = \lfloor \frac{1}{t} \rfloor$.*

Proof. Let $\mathcal{F} \in \mathbf{H}(t)$. Then there is a positive integer n_0 such that every $\mathcal{F}$-free connected graph of order at least n_0 is t-tough. Let n_1 be an integer such that $n_1 \geq \max(n_0, n+2)$.

Consider the family $\mathcal{F}' = \mathcal{F}_n(n_1, n_1, n_1)$. Notice that $\mathcal{F}' \in \mathbf{F}_n^B$. K_{1,n_1} has toughness $\frac{1}{n_1} < \frac{1}{n+1}$. $Y_{n_1+2}^n$ has toughness $\frac{1}{n+1}$. Z_{m,n_1}^n has toughness $\frac{1}{n+1}$ for all $1 \leq m \leq n_1$. Thus, all the graphs in $\mathcal{F}'$ have toughness at most $\frac{1}{n+1}$. Since $n = \lfloor \frac{1}{t} \rfloor$, then $t > \frac{1}{n+1}$ and so no graph of $\mathcal{F}'$ is t-tough.

In the same way as in Theorem 7.5, we get that $\mathcal{F}$ is in $\mathbf{F}_n^B$. □

7.5 Discussion

The characterization of forbidden induced subgraphs for toughness is given by Theorem 7.2, which is a direct consequence of Theorems 7.4, 7.5, 7.6 and 7.13.

It is not difficult to check that for every $n \geq 2$, the family $\mathcal{F}_n^B(m, l, r)$ is non-redundant for the constants used in the condition of the definition of $\mathbf{F}_n^B$ ($m \geq 1$, $l \geq n+2$, $r \geq 3$). Moreover, reducing by 1 any of these constants would make $\mathcal{F}_n^B(m, l, r)$ redundant. In the same way, $\mathcal{F}^A(m, l, r)$ is non-redundant for the constants used in the definition of $\mathbf{F}^A$ ($m \geq 4$, $l \geq 3$, $r \geq 3$) and reducing by 1 any of the constants would make it redundant. On the other hand, increasing the constants in the definitions of either $\mathbf{F}^A$ or $\mathbf{F}_n^B$ would not change the truth of Theorem 7.2. In this sense, we can say that the constants in the definition of both $\mathbf{F}^A$ and $\mathbf{F}_n^B$ are optimal for Theorem 7.2.

It is easy to see that the proof for Theorem 7.6 can be extended without any change from $0 < t \leq \frac{1}{2}$ to $0 < t \leq 1$, since for $\frac{1}{2} < t \leq 1$, the condition of cutset for the set S is still not necessary in the definition of t-tough. Also, $\mathcal{F}^A(m, l, r) \leq \mathcal{F}_1^B(m, l, r)$ for all $m \geq 1$, $l \geq n+2$ and $r \geq 3$. Using these two observations, the case $\frac{1}{2} < t \leq 1$ of Theorem 7.4 can be proved by using Theorem 7.6. Thus, we have shown two different proofs for $\frac{1}{2} < t \leq 1$. Nevertheless, notice that $\mathcal{F}_n^B(m, l, r)$ is redundant for $n = 1$.

Even though for the case $0 < t \leq \frac{1}{2}$ the resulting family of forbidden subgraphs depends on t, for the case $t > \frac{1}{2}$, it does not. We comment now on a possible explanation for this. It is well known that $\lceil 2t \rceil$-connectivity is a necessary condition for t-toughness. Since in Problem 7.2 we are imposing a condition of only 1-connectivity on G, the lack of connectivity must be compensated by forbidding very small graphs in G. Actually, even the forbidden subgraphs for t-toughness with t close to $\frac{1}{2}$ happen to be so small that they are enough for all $t > \frac{1}{2}$, and they don't need to be changed.

Moreover, the proof of Theorem 7.4, can be easily adapted to show that the connectivity of a $\mathcal{F}^A(m, l, r)$-free graph goes to infinity when the size of the graph goes to infinity. The adaptation consists mainly in taking the set S to be a minimal cutset of G and showing that the size of S is constant with respect to the size of G.

Consequently, to find out more about the case $t > \frac{1}{2}$, we propose the following problem.

Problem 7.3. *Let t be a positive real number. Characterize the connected graph families $\mathcal{F}$ such that every large enough $\lceil 2t \rceil$-connected $\mathcal{F}$-free graph is t-tough.*

Since for all $0 < t \leq \frac{1}{2}$, $\lceil 2t \rceil = 1$, we have solved Problem 7.3 for all $0 < t \leq \frac{1}{2}$. Even though it might be an interesting problem, we think that for any $t > \frac{1}{2}$, the families for Problem 7.3 are quite complicated and giving a full characterization might be very difficult.

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