A Thesis for the Degree of Ph.D. in Science

Solvability of Initial-Boundary Value Problems for the Motion of a Vortex Filament

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主　論　文　要　旨

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主論文題目：

Solvability of Initial-Boundary Value Problems for the Motion of a Vortex Filament

（渦糸の運動に対する初期値－境界値問題の可解性）

（内容の要旨）

本論文は，３次元半空間における非圧縮非粘性流体中の渦糸の運動を記述する非線形偏微分方程式に対する初期値－境界値問題の可解性について論じたものである．本論文で扱う渦糸の運動のモデル方程式は局所誘導方程式(LIE)と軸方向の流れを考慮したLIEを一般化した方程式(一般化LIE)である．LIEは流体の渦度から速度を計算するBiot-Savartの法則に局所誘導近似を適用して得られる最も単純，かつ基本的な方程式である．一方，一般化LIEは接合漸近展開法を用いて渦糸の軸方向流の影響を取り込んだ方程式である．

第１章は序論で，本論文において扱うモデル方程式の導出，初期値問題に対する既存の結果，および本研究の目的を述べる．さらに関連論文の紹介をする．その後，今回新しく扱う初期値－境界値問題を設定し，本論文内で使用する関数空間や記号の説明をする．

第２章では，LIEに対する初期値－境界値問題の時間大域解の存在と一意性を証明する．初期値－境界値問題を扱う際には，解が存在するための必要条件として，初期値に両立条件を課す．この条件は構成する解の滑らかさに応じて複雑になり，一般的には両立条件は帰納的に定義される．LIEに対しては方程式の特殊な構造を利用することにより両立条件を明示的に表示することができた．この表示により，初期値－境界値問題を初期値問題へ帰着させ，初期値問題の解を用いて望みの初期値－境界値問題の解を構成することができた．

第３章では，一般化LIEに対する初期値－境界値問題に関連する線形問題を考察する．一般化LIEは非線形３階分散型偏微分方程式であり，その研究には線形化方程式の解析が重要である．この線形化方程式には，既存の線形偏微分方程式の理論が適用できず，初期値－境界値問題の解の存在や一意性は知られていなかった．そこで本章では，この線形化問題の解析の準備として，３階分散項を持つ２階放物型方程式系に対する初期値－境界値問題の解の存在と一意性を証明する．解を構成するために新しい放物型正則化を考案した．特に，問題の適切性を保存したままの正則化，すなわち境界条件の数を変えない形での正則化をすることにより，非線形問題へ応用できる形で解を構成することに成功した．

第４章では，第３章の結果を応用して，一般化LIEに対する初期値－境界値問題の時間局所解の存在と一意性を証明する．線形問題の解の存在定理と一般化LIEの持つ構造を利用することによって非線形問題の解をソボレフ空間において構成することができた．

SUMMARY OF Ph.D. DISSERTATION

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Title

Solvability of Initial-Boundary Value Problems for the Motion of a Vortex Filament

Abstract

This dissertation is concerned with the mathematical analysis on the motion of a vortex filament immersed in an incompressible and inviscid fluid in the three dimensional half space. A vortex filament is a space curve on which the vorticity of the fluid is concentrated. The existence and uniqueness of solutions to initial-boundary value problems describing the motion of a vortex filament in the three-dimensional half space is proved.

In Chapter 1, the background and the aim of the study are presented. Two model equations, the Localized Induction Equation (LIE) and the generalized LIE are introduced, and their related works are explained.

In Chapter 2, the initial-boundary value problem for the LIE is studied. The existence and uniqueness of the solution is proved. The proof is carried out first, by carefully analyzing the compatibility conditions for the initial-boundary value problem and second, by extending the initial datum to the whole space, and thus reducing the problem to an initial value problem. The solution to the initial value problem can then be used to construct the solution to the initial-boundary value problem.

In Chapter 3, we consider initial-boundary value problems for a second order parabolic system with a third order dispersive term. The system arises when we consider the linearized problem of the generalized LIE, and the existence and uniqueness of the solution for such linear system has not been studied. This motivated the author to consider a general linear parabolic-dispersive system and to prove the existence and uniqueness of the solution for the corresponding initial-boundary value problems. The crucial idea in the proof is to apply a new parabolic regularization, which made it possible to construct the solution in such a way that the existence theorem is applicable to the analysis of the generalized LIE.

In Chapter 4, we prove the solvability of initial-boundary value problems for the generalized LIE by utilizing the results of Chapter 3. Based on the existence theorems of the linear problems, we succeeded in constructing the solutions to the nonlinear problems in Sobolev spaces.

Contents

Chapter 1

Introduction

1.1 Background

The author is interested in the motion of a vortex filament. A vortex filament is a space curve in an incompressible and inviscid fluid where the vorticity is concentrated. The motion of a vortex filament has been studied for a long time, and the first model equation describing the motion was proposed by Da Rios [7] in 1906. The equation proposed by him is called the Localized Induction Equation or LIE for short. The LIE is the simplest model equation describing the motion of a vortex filament and is given by

$$\boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss}, \tag{1.1.1}$$

where $\boldsymbol{x}(s,t) = (x^1(s,t), x^2(s,t), x^3(s,t))$ is the position vector of the vortex filament parametrized by its arc length s at time t, the symbol $\times$ denotes the exterior product in the three dimensional Euclidean space, and subscripts denote differentiation with the respective variables. Later in this dissertation, we also use ∂_s and ∂_t to denote partial differentiation. The LIE is often said to be "rediscovered" by Arms and Hama [3]. This is because the original work by Da Rios was written in Italian and did not become well known. There is also a work by Murakami et al. [31] in 1937 where they derived the LIE independent of Da Rios' work, but again, the work was written in Japanese and is not well known. The work by Arms and Hama is the first work written in English that made the LIE known.

The LIE is derived by approximating the Biot–Savart law, which is an integral formula given by

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{\boldsymbol{\omega}(\boldsymbol{y}) \times (\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,\mathrm{d}\boldsymbol{y},$$

where $\boldsymbol{v}$ is the velocity and $\boldsymbol{\omega}$ is the vorticity of the fluid. This formula gives the velocity of incompressible fluid from the vorticity distribution. In Arms and Hama [3], they rewrote the Biot–Savart law in the case of a vortex filament as

$$\boldsymbol{v}(\boldsymbol{z}) = \frac{\Gamma}{4\pi}\int_{-\infty}^{\infty} \frac{\boldsymbol{x}_s(s) \times (\boldsymbol{z} - \boldsymbol{x}(s))}{|\boldsymbol{z} - \boldsymbol{x}(s)|^3}\,\mathrm{d}s,$$

for which the integral is well-defined for $\boldsymbol{z} \in \mathbf{R}^3$ except for points on the filament. Here, Γ is a real constant describing the intensity of the vorticity. They apply the Localized Induction approximation to derive the LIE, which is an approximation of the above integral when z is a point on the filament.

It is also well known that the LIE can be transformed into the nonlinear Schrödinger equation via the Hasimoto transformation. This is a transformation introduced by Hasimoto in [14] given by

$$\psi = \kappa \exp\left(\mathrm{i}\int_0^s \tau \mathrm{d}s\right), \tag{1.1.2}$$

where κ is the curvature and τ is the torsion of the filament. The above transformation of the unknown variable transforms (1.1.1) to

$$\psi_t = \mathrm{i}\psi_{ss} + \frac{\mathrm{i}}{2}\left|\psi\right|^2\psi.$$

The transformation (1.1.2) is valid as long as $\kappa \neq 0$. Since the transformation is defined in terms of the torsion, the above expression (1.1.2) is invalid if there is a point with zero curvature. There is no way to determine a priori which points of the filament have zero curvature. To overcome this, Koiso [24] constructed a transformation often called the generalized Hasimoto transformation, which is defined regardless of the value of the curvature. This rigorously justified the conversion between the LIE and the nonlinear Schrödinger equation in the sense that the transformation is well-defined and can also be reversed. This can be extended to a multi-dimensional analog, called the Schrödinger map, as in Chang, Shatah, and Uhlenbeck [6] and Nahmod, Shatah, Vega, and Zeng [32].

A generalized model which takes into account the effect of axial flow of the vortex filament was proposed in 1972 by Moore and Saffman [29] and later in 1991 by Fukumoto and Miyazaki [10]. The equation proposed in these two works are the same, but the method in which they derived it are different. The model equation is a generalization of the LIE given by

$$\boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + \alpha\left\{\boldsymbol{x}_{sss} + \frac{3}{2}\boldsymbol{x}_{ss} \times (\boldsymbol{x}_s \times \boldsymbol{x}_{ss})\right\}, \tag{1.1.3}$$

where $\alpha \in \mathbf{R}$ is a parameter describing the magnitude of the effect of axial flow. We refer to this equation as the generalized LIE. The "axial flow" of a vortex filament does not mean flow inside the filament. This model equation is derived by approximating the velocity field of a vortex tube. It is assumed that the velocity inside the tube is governed by the Navier–Stokes equations and the velocity outside the tube is governed by the Euler equations. Then, the velocity at the boundary of the tube is determined by using the so called matching method. Finally, the limit of the thickness of the tube tending to zero is taken to derive the model equation for the vortex filament. The details of the derivation of the model equation was given in Fukumoto and Miyazaki [10]. Note that even though (1.1.3) is a generalization of (1.1.1), the methods in which the two model equations were derived are completely different.

Again, by applying the aforementioned generalized Hasimoto transformation to (1.1.3), we obtain

$$\psi_t = \mathrm{i}\psi_{ss} + \frac{\mathrm{i}}{2}|\psi|^2\psi + \alpha\left\{\psi_{sss} + \frac{3}{2}|\psi|^2\psi_s\right\}, \tag{1.1.4}$$

which is called the Hirota equation.

Initial value problems for both (1.1.1) and (1.1.3) have been considered in many works. In Nishiyama and Tani [33], they proved the existence of a unique time-global solution for both (1.1.1) and (1.1.3) without applying the Hasimoto transformation. Their main method for proving the existence of the solution is a parabolic regularization, and they make use of conserved quantities to obtain a priori estimates. In 1997, Koiso [24] proved the existence and uniqueness of a time-global solution to a geometrically generalized version of (1.1.1) by applying the generalized Hasimoto transformation and using a known existence theorem for the nonlinear Schrödinger equation. Later in [25, 26], he also proved the unique solvability without using the generalized Hasimoto transformation and in a different geometrical setting. In 2008, Onodera [34, 35] considered a geometrically generalized version of (1.1.3) and proved the time-global unique solvability. He also considers the validity of the generalized Hasimoto transformation in his geometrical setting. In the geometrically generalized case, it is still unknown whether the transformation can be reversed.

Nishiyama and Tani [33] also considered an initial-boundary value problem for (1.1.1) in a finite interval. The boundary condition imposed there is the zero curvature condition at both ends of the filament, which is expressed at $s = 0$ by $\boldsymbol{x}_{ss}(0,t) = \mathbf{0}$. By direct calculation, we see that the solution $\boldsymbol{x}$ of the problem satisfies $\boldsymbol{x}_t(0,t) = \big(\boldsymbol{x}_s \times \boldsymbol{x}_{ss}\big)(0,t) = \mathbf{0}$, which means that the ends of the vortex filament are fixed at their initial positions. As will be explained in the next chapter, we consider an initial-boundary value problem for (1.1.1) with a different boundary condition, which allows the endpoint of the filament to move along the boundary.

The Hirota equation, and dispersive equations in general, have a vast history of studies, a part of which is mentioned below. Hirota [18] originally considered (1.1.4), with more general constant coefficients on each term, to obtain N-envelope-soliton solutions. By obtaining this type of solution, he discussed the relation between the N-envelope-solition and the classical solitons of the KdV equation and the Schrödinger equation. In 1997, Laurey [28] proved the unique solvability globally in time of the initial value problem for a class of third order dispersive equations which includes (1.1.4). In 2008, Segata [37] proved the time-global unique solvability of the initial value problem for (1.1.4) and also showed the asymptotic behavior of the solution as time goes to infinity. They both utilized the smoothing effect of dispersive equations in their analysis. Results related to this approach can also be found in Sjöberg [39], Kenig, Ponce, and Vega [19], and the references therein.

Besides the solvability of problems for the LIE, research have been done on many other aspects of vortex filaments. In Gutiérrez, Rivas, and Vega [12], they constructed a one-parameter family of self-similar solutions for the initial value problem that form a single sharp corner. They constructed the solutions by starting with a filament with a sharp corner and solving the problem reverse in time. Hasimoto [14] showed the existence of solitons that propagate along a linear filament, and numerically studied the shape and movement of the solitary wave. Fukumoto [9] gave an asymptotic formula for the velocity of the fluid induced by a closed vortex filament, called a vortex ring, and numerically studied the velocity distribution. Kida [21] constructed various exact solutions, which move steadily in time, of the LIE. The solutions studied there include the helicoidal filaments and solitary wave type filaments. Betchov [4] derived an equation, called the intrinsic equation, which is the LIE expressed in terms of the curvature and the torsion of the filament. He also studied solutions of the intrinsic equation in special settings. Klein and Majda [22] derived a different model equation for the motion of a vortex filament where the filament is assumed to be almost straight, but they take into account the effect of vortex stretching, and the model equation allows the filament to stretch. Many related results can also be found in [8, 11, 13, 17, 20, 27, 30], and the references therein.

1.2 Aim of the Present Study

The aim of this dissertation is to prove the unique solvability of initial-boundary value problems on the half-line for (1.1.1) and (1.1.3). The specific problems we consider are as follows.

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss}, & s > 0, t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_s(0,t) = \boldsymbol{e}_3, & t > 0 \end{cases}$$

for the LIE,

$$
(1.2.5) \qquad \begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + \alpha\Big\{\boldsymbol{x}_{sss} + \dfrac{3}{2}\boldsymbol{x}_{ss} \times (\boldsymbol{x}_s \times \boldsymbol{x}_{ss})\Big\}, & s > 0, t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_{ss}(0,t) = \boldsymbol{0}, & t > 0 \end{cases}
$$

for $\alpha < 0$, and

$$
(1.2.6) \qquad \begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + \alpha\Big\{\boldsymbol{x}_{sss} + \dfrac{3}{2}\boldsymbol{x}_{ss} \times (\boldsymbol{x}_s \times \boldsymbol{x}_{ss})\Big\}, & s > 0, t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_s(0,t) = \boldsymbol{e}_3, & t > 0, \\ \boldsymbol{x}_{ss}(0,t) = \boldsymbol{0}, & t > 0 \end{cases}
$$

for $\alpha > 0$. Here, $\boldsymbol{e}_3 = (0,0,1)$. Note that the number of boundary conditions changes depending on the sign of α, which will be addressed in more detail in Chapter 3 and 4.

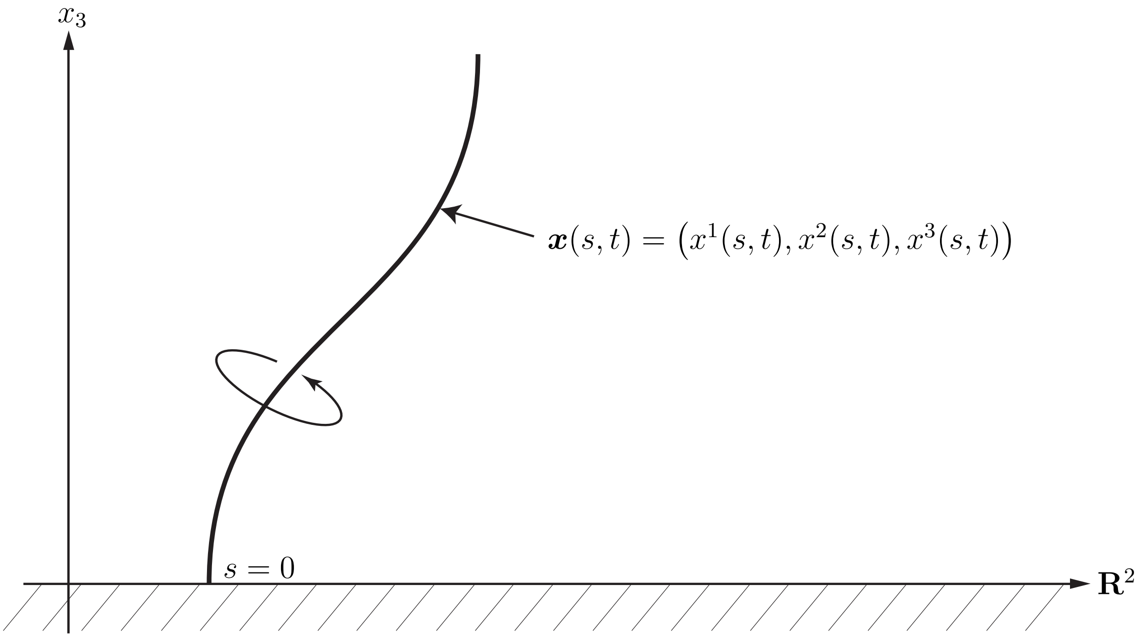

Figure 1.1: Vortex Filament in the Half Space

These initial-boundary value problems describe the motion of a vortex filament moving in the three dimensional half space, as shown in Figure 1.1.

For the initial-boundary value problem for the LIE, we prove the time-global solvability in Chapter 2 by reducing the problem to an initial value problem by extending the initial datum to the whole space. This is possible by carefully analyzing the compatibility conditions and giving an explicit expression for the n-th order compatibility condition for any natural number n. This allows us to prove that the extension of the initial datum by reflection with respect to the plain $\{\boldsymbol{x} \in \mathbf{R}^3; x_3 = 0\}$ is smooth. The LIE itself is also invariant under this reflection and thus, the solution to the initial value problem preserves the symmetry with respect to the plain. Under these circumstances, the restriction of the solution to the half space is the desired solution to the initial-boundary value problem.

The other two problems are approached in a more straight forward manner in which we consider the linearized problems. We prove the time-local solvability via an iteration argument based on the existence theorems for the linearized problems. Since the existence theorems for the linearized problems themselves are non-trivial, we devote Chapter 3 to this issue. Specifically, we consider general linear problems

$$\begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0 \end{cases} \tag{1.2.7}$$

for $\alpha < 0$, and

$$\begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}(0,t) = \boldsymbol{e}, & t > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0 \end{cases} \tag{1.2.8}$$

for $\alpha > 0$. Here, $\boldsymbol{u}(x,t) = (u^1(x,t), u^2(x,t), \ldots, u^m(x,t))$ is the unknown vector valued function, $\boldsymbol{u}_0(x)$, $\boldsymbol{w}(x,t) = (w^1(x,t), w^2(x,t), \ldots, w^k(x,t))$, and $\boldsymbol{f}(x,t) = (f^1(x,t), f^2(x,t), \ldots, f^m(x,t))$ are known vector valued functions, and $\boldsymbol{e}$ is an arbitrary constant vector. $\mathrm{A}(\boldsymbol{w}, \partial_x)$ is a second order differential operator of the form $\mathrm{A}(\boldsymbol{w}, \partial_x) = \mathrm{A}_0(\boldsymbol{w})\partial_x^2 + \mathrm{A}_1(\boldsymbol{w})\partial_x + \mathrm{A}_2(\boldsymbol{w})$. $\mathrm{A}_0,\ \mathrm{A}_1,\ \mathrm{A}_2$ are smooth matrices and $\mathrm{A}(\boldsymbol{w}, \partial_x)$ is strongly elliptic in the sense that for any bounded domain E in $\mathbf{R}^k$, there is a positive constant δ such that for any $\boldsymbol{w} \in E$

$$\mathrm{A}_0(\boldsymbol{w}) + \mathrm{A}_0(\boldsymbol{w})^* \geq \delta \mathrm{I},$$

where I is the unit matrix and $*$ denotes the adjoint of a matrix. These problems include a regularized form of the linearized problem for the generalized LIE. The equation linearized around $\boldsymbol{w}$ has the form

$$\boldsymbol{v}_t = \boldsymbol{w} \times \boldsymbol{v}_{ss} + \alpha\big\{\boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s)\big\} + \boldsymbol{f},$$

where $\boldsymbol{v}$ is the variation of the tangent vector of the filament. The solution to the initial-boundary value problem for the above system can be obtained by a parabolic regularization, which will be considered in detail in Chapter 3, of the form

$$\boldsymbol{v}_t = \alpha(-\varepsilon \boldsymbol{v}_t + \alpha \boldsymbol{v}_{ss})_s + \boldsymbol{w} \times \boldsymbol{v}_{ss} + 3\alpha \boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s) + \boldsymbol{f}$$

with $\varepsilon > 0$. It seems hard to obtain the estimate of the solution uniform in ε, which is needed to pass to the limit $\varepsilon \to +0$. To overcome this, we add the term $\delta \boldsymbol{v}_{ss}$ to the above system to obtain

$$\boldsymbol{v}_t = \alpha(-\varepsilon \boldsymbol{v}_t + \alpha \boldsymbol{v}_{ss})_s + \delta \boldsymbol{v}_{ss} + \boldsymbol{w} \times \boldsymbol{v}_{ss} + 3\alpha \boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s) + \boldsymbol{f}.$$

Then, by utilizing the dissipative property of the term $\delta \boldsymbol{v}_{ss}$, we are able to obtain the desired estimates uniform in ε. If we pass to the limit $\varepsilon \to +0$, we have a parabolic-dispersive system

$$\boldsymbol{v}_t = \alpha \boldsymbol{v}_{sss} + \big\{\delta \boldsymbol{v}_{ss} + \boldsymbol{w} \times \boldsymbol{v}_{ss} + 3\alpha \boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s)\big\} + \boldsymbol{f},$$

which satisfies the assumptions for (1.2.7) and (1.2.8). This is the motivation for considering problems (1.2.7) and (1.2.8).

As an application of the existence theorems obtained in Chapter 3, we prove the time-local solvability of

$$(1.2.9)\quad \begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\Big\{\boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s\big) - \dfrac{3}{2}|\boldsymbol{v}_s|^2 \boldsymbol{v}_s\Big\} \\ \qquad\qquad + \delta\big(\boldsymbol{v}_{ss} + |\boldsymbol{v}_s|^2 \boldsymbol{v}\big), & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t > 0 \end{cases}$$

for $\alpha < 0$, and

$$(1.2.10)\quad \begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\Big\{\boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s\big) - \dfrac{3}{2}|\boldsymbol{v}_s|^2 \boldsymbol{v}_s\Big\} \\ \qquad\qquad + \delta\big(\boldsymbol{v}_{ss} + |\boldsymbol{v}_s|^2 \boldsymbol{v}\big), & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t > 0, \\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t > 0 \end{cases}$$

for $\alpha > 0$ through a standard iteration scheme in Chapter 4. These two problems are regularized problems for (1.2.5) and (1.2.6), respectively, expressed in terms of the tangent vector $\boldsymbol{v}$. They are regularized as above so that we can apply the linear existence theorems obtained in Chapter 3. In fact, the second order terms correspond to the operator $\mathrm{A}(\boldsymbol{v}, \partial_s)$ and the lower order terms correspond to the forcing term $\boldsymbol{f}$ of the linear system. The extra regularizing term $\delta|\boldsymbol{v}_s|^2\boldsymbol{v}$ may seem unnecessary, but it actually plays an important role. If $|\boldsymbol{v}_0| \equiv 1$, a smooth solution $\boldsymbol{v}$ to (1.2.9) or (1.2.10) with $\delta = 0$ also satisfies $|\boldsymbol{v}| \equiv 1$. This property is one of the crucial components to derive energy estimates of the solution when $\delta = 0$, and by adding the regularizing term $\delta|\boldsymbol{v}_s|^2\boldsymbol{v}$, the same property holds for $\delta > 0$. Utilizing this property, we can obtain uniform estimates of the solution to (1.2.9) and (1.2.10) with respect to δ, allowing us to pass to the limit $\delta \to +0$. Finally, as in the case of the LIE, we can construct the desired solution $\boldsymbol{x}$ from $\boldsymbol{v}$ through the formula

$$\boldsymbol{x}(s,t) = \boldsymbol{x}_0(s) + \int_0^t \left\{ \boldsymbol{v} \times \boldsymbol{v}_s + \alpha \boldsymbol{v}_{ss} + + \frac{3}{2}\alpha \boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_s) \right\}(s,\tau)\mathrm{d}\tau.$$

1.3 Function Spaces

We define some function spaces that will be used throughout this dissertation and notations associated with the spaces.

For an open interval Ω, a non-negative integer m, and $1 \le p \le \infty$, $W^{m,p}(\Omega)$ is the Sobolev space containing all real-valued functions that have derivatives in the sense of distribution up to order m belonging to $L^p(\Omega)$ and $\dot{W}^{m,p}(\Omega)$ is the homogeneous Sobolev space. We set $H^m(\Omega) := W^{m,2}(\Omega)$ as the Sobolev space equipped with the usual inner product and $\dot{H}^m(\Omega) := \dot{W}^{m,2}(\Omega)$. We will particularly use the cases $\Omega = \mathbf{R}$ and $\Omega = \mathbf{R}_+$, where $\mathbf{R}_+ = \{x \in \mathbf{R}; x > 0\}$. When $\Omega = \mathbf{R}_+$, the norm in $H^m(\Omega)$ is denoted by $\|\cdot\|_m$ and we simply write $\|\cdot\|$ for $\|\cdot\|_0$. Otherwise, for a Banach space X, the norm in X is written as $\|\cdot\|_X$. The inner product in $L^2(\mathbf{R}_+)$ is denoted by $(\cdot,\cdot)$ and the inner product in $L^2(\mathbf{R})$ is denoted by $\langle \cdot,\cdot \rangle$.

For $0 < T < \infty$ and a Banach space X, $C^m([0,T];X)$ denotes the space of functions that are m times continuously differentiable in t with respect to the norm of X.

We define the Sobolev–Slobodetskiĭ space. For $0 < T \le \infty$, we denote $Q_T := \mathbf{R}_+ \times (0,T)$, and for $h > 0$ and a positive integer l, we define the space $H_h^{l,l/2}(Q_T)$ as the space of functions defined on Q_T with finite norm

$$|||u|||^2_{H_h^{l,l/2}(Q_T)} := |||u|||^2_{H_h^{l,0}(Q_T)} + |||u|||^2_{H_h^{0,l/2}(Q_T)},$$

where

$$\begin{aligned} |||u|||^2_{H_h^{l,0}(Q_T)} &:= \int_0^T \mathrm{e}^{-2ht}\|u(\cdot,t)\|^2_{\dot{H}^l}\mathrm{d}t, \\ |||u|||^2_{H_h^{0,l/2}(Q_T)} &:= h^l \int_0^T \mathrm{e}^{-2ht}\|u(\cdot,t)\|^2\mathrm{d}t \\ &\quad + \int_0^T \mathrm{e}^{-2ht}\int_0^\infty \left\| \frac{\partial^{[l/2]}u_0(\cdot,t-r)}{\partial t^{[l/2]}} - \frac{\partial^{[l/2]}u_0(\cdot,t)}{\partial t^{[l/2]}} \right\|^2 r^{-1-l+2[\frac{l}{2}]}\mathrm{d}r\mathrm{d}t, \end{aligned}$$

$[\frac{l}{2}]$ is the integer part of $\frac{l}{2}$ and u_0 is the extension of u by zero into $t < 0$ if $\frac{l}{2}$ is not an integer. When $\frac{l}{2}$ is an integer,

$$|||u|||^2_{H_h^{0,l/2}(Q_T)} := \int_0^T \mathrm{e}^{-2ht}\left(h^l\|u(\cdot,t)\|^2 + \left\| \frac{\partial^{l/2}u}{\partial t^{l/2}}(\cdot,t) \right\|^2 \right)\mathrm{d}t$$

and we also impose that $\frac{\partial^j u}{\partial t^j}(x,0)=0$ for $j=0,1,\ldots,\frac{l}{2}-1$. When $T=\infty$, the following equivalent norm for the space $H_h^{l,l/2}(Q_\infty)$ will be used.

$$\|u\|^2_{H_h^{l,l/2}(Q_\infty)} := \sum_{j=0}^{l} \int_{-\infty}^{\infty} \left\| \frac{\partial^j \tilde{u}}{\partial x^j}(\cdot,\tau) \right\|^2 |\tau|^{l-j} \mathrm{d}\eta,$$

where a tilde denotes the Laplace transform with respect to t defined by

$$\tilde{u}(x,\tau) = \int_0^\infty \mathrm{e}^{-\tau t} u(x,t) \mathrm{d}t,$$

where $\tau = h + \mathrm{i}\eta$ with $h>0$. The equivalence was shown in Solonnikov [38].

Finally, we define some auxiliary function spaces, which will be used in Chapters 3 and 4. Let l be a non-negative integer and define the following.

$$X_T^l := \bigcap_{j=0}^{l} \Big(C^j\big([0,T]; H^{2+3(l-j)}(\mathbf{R}_+)\big) \cap H^j\big(0,T; H^{3+3(l-j)}(\mathbf{R}_+)\big) \Big),$$

$$Y_T^l := \left\{ f;\ f \in \bigcap_{j=0}^{l-1} C^j\big([0,T]; H^{2+3(l-1-j)}(\mathbf{R}_+)\big),\ \frac{\partial^l f}{\partial t^l} \in L^2\big(0,T; H^1(\mathbf{R}_+)\big) \right\},$$

$$Z_T^l := \left\{ w;\ w \in \bigcap_{j=0}^{l-1} C^j\big([0,T]; H^{2+3(l-1-j)}(\mathbf{R}_+)\big),\ \frac{\partial^l w}{\partial t^l} \in L^\infty\big(0,T; H^1(\mathbf{R}_+)\big) \right\}.$$

For any function space described above, we say that a vector valued function belongs to the function space if each of its components does.

The contents of this dissertation are as follows. In Chapter 2, we prove the time-global solvability of the initial-boundary value problem for the LIE. In Chapter 3, we consider initial-boundary value problems for a second order parabolic system with a third order dispersive term and prove the solvability. From here on, we refer to this system as a parabolic-dispersive system. This parabolic-dispersive system is considered to analyze the generalized LIE, for which we prove the time-local solvability of initial-boundary value problems in Chapter 4. Finally, in Appendix A, we address the initial-boundary value problem for the LIE to demonstrate the generalized Hasimoto transformation. We do this to communicate the idea of the generalized Hasimoto transformation while refraining from using technical terms of differential geometry as much as possible.

Chapter 2

Localized Induction Equation in the Half Space

2.1 Problem Setting

We consider the initial-boundary value problem for the motion of a vortex filament in the half space in which the filament is allowed to move on the boundary:

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss}, & s > 0,\ t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_s(0,t) = \boldsymbol{e}_3, & t > 0, \end{cases} \tag{2.1.1}$$

where $\boldsymbol{e}_3 = (0,0,1)$. We assume that

$$|\boldsymbol{x}_{0s}(s)| = 1 \quad \text{for} \quad s \geq 0, \qquad x_0^3(0) = 0, \tag{2.1.2}$$

for the initial datum. The first condition states that the initial vortex filament is parametrized by its arc length and the second condition states that the curve is parameterized starting from the boundary. Here we observe that by taking the inner product of $\boldsymbol{e}_3$ with the equation, taking the trace at $s = 0$, and noting the boundary condition we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{e}_3 \cdot \boldsymbol{x})\,|_{s=0} &= \boldsymbol{e}_3 \cdot (\boldsymbol{x}_s \times \boldsymbol{x}_{ss})|_{s=0} \\ &= \boldsymbol{x}_s \cdot (\boldsymbol{x}_s \times \boldsymbol{x}_{ss})|_{s=0} \\ &= 0, \end{aligned}$$

where " $\cdot$ " denotes the inner product and $|_{s=0}$ denotes the trace at $s = 0$. This implies that if the end of the vortex filament is on the boundary initially, then it will stay on the boundary, but is not necessarily fixed. This is the reason for the expression "allowed to move on the boundary".

By introducing new variables $\boldsymbol{v}(s,t) := \boldsymbol{x}_s(s,t)$ and $\boldsymbol{v}_0(s) := \boldsymbol{x}_{0s}(s)$, (2.1.1) and (2.1.2) respectively become

$$\begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss}, & s > 0,\ t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t > 0, \end{cases} \tag{2.1.3}$$

and

$$|\boldsymbol{v}_0(s)| = 1, \quad s \geq 0. \tag{2.1.4}$$

Once we solve (2.1.3), the solution $\boldsymbol{x}$ of (2.1.1) and (2.1.2) can be constructed by

$$\boldsymbol{x}(s,t) = \boldsymbol{x}_0(s) + \int_0^t \boldsymbol{v}(s,\tau) \times \boldsymbol{v}_s(s,\tau)\,\mathrm{d}\tau.$$

Thus from now on, we concentrate on the initial-boundary value problem (2.1.3) under the condition (2.1.4). Note that if the initial datum satisfies (2.1.4), then any smooth solution $\boldsymbol{v}$ of (2.1.3) satisfies

$$|\boldsymbol{v}(s,t)| = 1, \quad s \geq 0,\ t \geq 0. \tag{2.1.5}$$

This can be confirmed by taking the inner product of the equation in (2.1.3) with $\boldsymbol{v}$.

2.2 Compatibility Conditions

We derive necessary conditions for a smooth solution to exist for (2.1.3) with (2.1.4).

Suppose that $\boldsymbol{v}(s,t)$ is a smooth solution of (2.1.3) with (2.1.4) defined in $\mathbf{R}_+ \times [0,T]$ for some positive T. We have already seen that for all $(s,t) \in \mathbf{R}_+ \times [0,T]$

$$|\boldsymbol{v}(s,t)|^2 = 1. \tag{2.2.1}$$

By differentiating the boundary condition with respect to t, we see that

$$\partial_t^n \boldsymbol{v}\big|_{s=0} = \mathbf{0} \quad \text{for} \quad n \in \mathbf{N},\ t > 0. \tag*{$(B)_n$}$$

We next show

Lemma 2.2.1 *For a smooth solution $\boldsymbol{v}(s,t)$ under consideration,*

$$\boldsymbol{v} \times \partial_s^{2n} \boldsymbol{v}\big|_{s=0} = \mathbf{0}, \tag*{$(C)_n$}$$

$$\partial_s^j \boldsymbol{v} \cdot \partial_s^l \boldsymbol{v}\big|_{s=0} = 0 \quad \text{for} \quad j + l = 2n+1 \tag*{$(D)_n$}$$

hold.

Proof. We prove them by induction. From $(B)_1$ and by taking the trace of the equation we see that

$$\mathbf{0} = \boldsymbol{v}_t \left.\right|_{s=0} = \boldsymbol{v} \times \boldsymbol{v}_{ss} \left.\right|_{s=0},$$

and thus, $(C)_1$ holds. By taking the exterior product of $\boldsymbol{v}_s$ and $(C)_1$ we have

$$\left\{ (\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss})\, \boldsymbol{v} - (\boldsymbol{v}_s \cdot \boldsymbol{v})\, \boldsymbol{v}_{ss} \right\} \left.\right|_{s=0} = \mathbf{0}.$$

On the other hand, by differentiating (2.2.1) with respect to s, we have $\boldsymbol{v} \cdot \boldsymbol{v}_s \equiv 0$. Combining these two and the fact that $\boldsymbol{v}$ is a non-zero vector, we arrive at

$$\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss} \left.\right|_{s=0} = 0.$$

Finally, by differentiating (2.2.1) with respect to s three times and setting $s = 0$, we have

$$0 = 2\left(\boldsymbol{v} \cdot \boldsymbol{v}_{sss} + 3\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss} \right) \left.\right|_{s=0} = 2\boldsymbol{v} \cdot \boldsymbol{v}_{sss} \left.\right|_{s=0},$$

which implies that $(D)_1$ holds. Suppose that the statements hold up to $n-1$ for some $n \geq 2$. By differentiating $(C)_{n-1}$ with respect to t we have

$$\boldsymbol{v} \times \left(\partial_s^{2(n-1)} \boldsymbol{v}_t \right)\Big|_{s=0} = \mathbf{0},$$

where we have used $(B)_1$. We see that

$$\partial_s^{2(n-1)} \boldsymbol{v}_t = \partial_s^{2(n-1)} \left(\boldsymbol{v} \times \boldsymbol{v}_{ss} \right) = \sum_{k=0}^{2(n-1)} \begin{pmatrix} 2(n-1) \\ k \end{pmatrix} \left(\partial_s^k \boldsymbol{v} \times \partial_s^{2(n-1)-k+2} \boldsymbol{v} \right),$$

where $\begin{pmatrix} 2(n-1) \\ k \end{pmatrix}$ is the binomial coefficient. We have

$$\sum_{k=0}^{2(n-1)} \begin{pmatrix} 2(n-1) \\ k \end{pmatrix} \left\{ \boldsymbol{v} \times \left(\partial_s^k \boldsymbol{v} \times \partial_s^{2(n-1)-k+2} \boldsymbol{v} \right) \right\} \Big|_{s=0} = \mathbf{0}. \tag{2.2.2}$$

We examine each term in the summation. When $2 \leq k \leq 2(n-1)$ is even, we see from the assumptions of induction $(C)_{k/2}$ and $(C)_{(2(n-1)-k+2)/2}$ that both $\partial_s^k \boldsymbol{v}$ and $\partial_s^{2(n-1)-k+2} \boldsymbol{v}$ are parallel to $\boldsymbol{v}$, so that

$$\partial_s^k \boldsymbol{v} \times \partial_s^{2(n-1)-k+2} \boldsymbol{v} \Big|_{s=0} = \mathbf{0}.$$

When $1 \leq k \leq 2(n-1)$ is odd, we rewrite the exterior product in (2.2.2) as

$$\boldsymbol{v} \times \left(\partial_s^k \boldsymbol{v} \times \partial_s^{2(n-1)-k+2} \boldsymbol{v} \right) = \left(\boldsymbol{v} \cdot \partial_s^{2(n-1)-k+2} \boldsymbol{v} \right) \partial_s^k \boldsymbol{v} - \left(\boldsymbol{v} \cdot \partial_s^k \boldsymbol{v} \right) \partial_s^{2(n-1)-k+2} \boldsymbol{v}.$$

Since $2(n-1) - k + 2$ is also odd, by $(D)_{(k-1)/2}$ and $(D)_{(2(n-1)-k+1)/2}$ we have

$$\boldsymbol{v} \cdot \partial_s^k \boldsymbol{v} \Big|_{s=0} = \boldsymbol{v} \cdot \partial_s^{2(n-1)-k+2} \boldsymbol{v} \Big|_{s=0} = 0.$$

Thus, only the term with $k = 0$ remains and we get

$$\boldsymbol{v} \times \big(\boldsymbol{v} \times \partial_s^{2n} \boldsymbol{v}\big)\big|_{s=0} = \boldsymbol{0}.$$

Here, we note that

$$\boldsymbol{v} \times (\boldsymbol{v} \times \partial_s^{2n} \boldsymbol{v}) = (\boldsymbol{v} \cdot \partial_s^{2n} \boldsymbol{v})\boldsymbol{v} - \partial_s^{2n} \boldsymbol{v},$$

where we used (2.2.1). Taking the exterior product of this with $\boldsymbol{v}$ we see that $(C)_n$ holds. Taking the exterior product of $\partial_s^{2n+1-2k} \boldsymbol{v}$ with $(C)_k$ and using $(D)_{n-k}$ for $1 \leq k \leq n$ yields

$$\big(\partial_s^{2k} \boldsymbol{v} \cdot \partial_s^{2n+1-2k} \boldsymbol{v}\big)\boldsymbol{v}\big|_{s=0} = \boldsymbol{0}.$$

Since $\boldsymbol{v}$ is a non-zero vector, we have for $1 \leq k \leq n$

$$\partial_s^{2k} \boldsymbol{v} \cdot \partial_s^{2n+1-2k} \boldsymbol{v}\big|_{s=0} = 0. \tag{2.2.3}$$

Finally, by differentiating (2.2.1) with respect to s $(2n+1)$ times, we have

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} \big(\partial_s^{j} \boldsymbol{v} \cdot \partial_s^{2n+1-j} \boldsymbol{v}\big)\Big|_{s=0} = 0.$$

Since every term except when $j = 0,\ 2n+1$ is of the form (2.2.3), we see that

$$\boldsymbol{v} \cdot \partial_s^{2n+1} \boldsymbol{v}\big|_{s=0} = 0,$$

which, together with (2.2.3), finishes the proof of $(D)_n$. □

Worth noting are the following two properties which will be used in later parts of this chapter. For a natural number n,

$$\boldsymbol{e}_3 \times \partial_s^{2n} \boldsymbol{v}\big|_{s=0} = \boldsymbol{0}, \quad \boldsymbol{e}_3 \cdot \partial_s^{2n+1} \boldsymbol{v}\big|_{s=0} = 0.$$

These are special cases of $(C)_n$ and $(D)_n$ with the boundary condition substituted in.

By passing to the limit $t \to 0$ in $(C)_n$, we derive a necessary condition for the initial datum.

Definition 2.2.2 *For $n \in \mathbf{N} \cup \{0\}$, we say that the initial datum $\boldsymbol{v}_0$ satisfies the compatibility condition $(A)_n$ if the following conditions are satisfied for $0 \leq k \leq n$*

$$\begin{cases} \boldsymbol{v}_0|_{s=0} = \boldsymbol{e}_3, & k = 0, \\ \big(\boldsymbol{v}_0 \times \partial_s^{2k} \boldsymbol{v}_0\big)\big|_{s=0} = \boldsymbol{0}, & k \neq 0. \end{cases}$$

From the proof of Lemma 2.2.1, we see that if $\boldsymbol{v}_0$ satisfies (2.1.4) and the compatibility condition $(A)_n$, then $\boldsymbol{v}_0$ also satisfies $(D)_k$ for $0 \leq k \leq n$ with $\boldsymbol{v}$ replaced by $\boldsymbol{v}_0$ as long as the trace exists.

2.3 Extension of the Initial Data

For the initial datum $\boldsymbol{v}_0$ defined on the half-line, we extend it to the whole line by

$$\widetilde{\boldsymbol{v}}_0(s) = \begin{cases} \boldsymbol{v}_0(s), & s \geq 0, \\ -\overline{\boldsymbol{v}}_0(-s), & s < 0, \end{cases} \tag{2.3.1}$$

where $\overline{\boldsymbol{v}} = (v^1, v^2, -v^3)$ for $\boldsymbol{v} = (v^1, v^2, v^3) \in \mathbf{R}^3$.

Proposition 2.3.1 *For any integer* $m \geq 2$, *if* $\boldsymbol{v}_{0s} \in H^m(\mathbf{R}_+)$ *satisfies* (2.1.4) *and the compatibility condition* $(A)_{[\frac{m}{2}]}$, *then* $\widetilde{\boldsymbol{v}}_{0s} \in H^m(\mathbf{R})$. *Here,* $[\frac{m}{2}]$ *indicates the largest integer not exceeding* $\frac{m}{2}$.

Proof. Fix an arbitrary integer $m \geq 2$. We will prove by induction on k that $\partial_s^{k+1}\widetilde{\boldsymbol{v}}_0 \in L^2(\mathbf{R})$ for any $0 \leq k \leq m$. Specifically we show that the derivatives of $\widetilde{\boldsymbol{v}}_0$ in the distribution sense on the whole line $\mathbf{R}$ up to order $m+1$ have the form

$$\big(\partial_s^{k+1}\widetilde{\boldsymbol{v}}_0\big)(s) = \begin{cases} \left(\partial_s^{k+1}\boldsymbol{v}_0\right)(s), & s > 0, \\ -(-1)^{k+1}\big(\overline{\partial_s^{k+1}\boldsymbol{v}_0}\big)(-s), & s < 0, \end{cases} \tag{2.3.2}$$

for $0 \leq k \leq m$.

Since $\boldsymbol{v}_0 \in L^\infty(\mathbf{R}_+)$ and $\boldsymbol{v}_{0s} \in H^2(\mathbf{R}_+)$, Sobolev's embedding theorem states $\boldsymbol{v}_{0s} \in L^\infty(\mathbf{R}_+)$ and thus $\boldsymbol{v}_0 \in W^{1,\infty}(\mathbf{R}_+)$, so that the trace $\boldsymbol{v}_0(0)$ exists. By definition (2.3.1) we have

$$\widetilde{\boldsymbol{v}}_0(-0) = \big(-v_0^1(0), -v_0^2(0), v^3(0)\big),$$

but from $(A)_0$, $v_0^1(0) = v_0^2(0) = 0$. These imply that $\widetilde{\boldsymbol{v}}_0(+0) = \widetilde{\boldsymbol{v}}_0(-0)$, so that we obtain

$$\partial_s\widetilde{\boldsymbol{v}}_0(s) = \begin{cases} \left(\partial_s\boldsymbol{v}_0\right)(s), & s > 0, \\ -(-1)\big(\overline{\partial_s\boldsymbol{v}_0}\big)(-s), & s < 0, \end{cases}$$

and the case $k = 0$ is proved.

Suppose that (2.3.2) with $k+1$ replaced by k holds for some $k \in \{1, 2, \ldots, m\}$. We check that the derivative $\partial_s^k\widetilde{\boldsymbol{v}}_0$ does not have a jump discontinuity at $s = 0$. When k is even, from the definition of $\overline{\partial_s^k\boldsymbol{v}_0}$,

$$\big(\partial_s^k\widetilde{\boldsymbol{v}}_0\big)(-0) = \big(-\partial_s^k v_0^1(0), -\partial_s^k v_0^2(0), \partial_s^k v_0^3(0)\big),$$

but from $(A)_{\frac{k}{2}}$ we have

$$\mathbf{0} = \boldsymbol{v}_0 \times \partial_s^k\boldsymbol{v}_0\big|_{s=0} = \boldsymbol{e}_3 \times \partial_s^k\boldsymbol{v}_0(0),$$

which implies that $\partial_s^k \boldsymbol{v}_0(0)$ is parallel to $\boldsymbol{e}_3$ and that the first and second components are zero. When k is odd,

$$\left(\partial_s^k \widetilde{\boldsymbol{v}}_0\right)(-0) = \left(\partial_s^k v_0^1(0), \partial_s^k v_0^2(0), -\partial_s^k v_0^3(0)\right),$$

but $(A)_{[\frac{k}{2}]}$ implies $(D)_{[\frac{k}{2}]}$ and particularly

$$0 = \boldsymbol{v}_0 \cdot \partial_s^k \boldsymbol{v}_0\big|_{s=0} = \boldsymbol{e}_3 \cdot \partial_s^k \boldsymbol{v}_0(0) = (\partial_s^k v_0^3)(0),$$

so the third component is zero. In both cases, we have $\left(\partial_s^k \widetilde{\boldsymbol{v}}_0\right)(+0) = \left(\partial_s^k \widetilde{\boldsymbol{v}}_0\right)(-0)$, so that we can verify (2.3.2). This finishes the proof of the proposition. $\square$

2.4 Existence and Uniqueness of Solution

Using $\widetilde{\boldsymbol{v}}_0$, we consider the following initial value problem:

$$\boldsymbol{u}_t = \boldsymbol{u} \times \boldsymbol{u}_{ss}, \qquad s \in \mathbf{R},\ t > 0, \tag{2.4.1}$$

$$\boldsymbol{u}(s,0) = \widetilde{\boldsymbol{v}}_0(s), \qquad s \in \mathbf{R}. \tag{2.4.2}$$

By Proposition 2.3.1, the existence and uniqueness theorem (in Nishiyama and Tani [33]) of a strong solution $\boldsymbol{u}$ is applicable. Specifically we use the following theorem.

Theorem 2.4.1 (T. Nishiyama and A. Tani [33]) *For a non-negative integer m, if $\widetilde{\boldsymbol{v}}_{0s} \in H^{2+m}(\mathbf{R})$ and $|\widetilde{\boldsymbol{v}}_0| \equiv 1$, then the initial value problem* (2.4.1) *and* (2.4.2) *has a unique solution $\boldsymbol{u}$ such that*

$$\boldsymbol{u} - \widetilde{\boldsymbol{v}}_0 \in C\left([0,\infty); H^{3+m}(\mathbf{R})\right) \cap C^1\left([0,\infty); H^{1+m}(\mathbf{R})\right)$$

and $|\boldsymbol{u}| \equiv 1$.

From Proposition 2.3.1, the assumptions of this theorem are satisfied if $\boldsymbol{v}_{0s} \in H^{2+m}(\mathbf{R}_+)$ satisfies the compatibility condition $(A)_{[\frac{2+m}{2}]}$ and (2.1.4).

Now we define the operator T by

$$(\mathrm{T}\boldsymbol{w})(s) = -\overline{\boldsymbol{w}}(-s),$$

for $\mathbf{R}^3$-valued function $\boldsymbol{w}$ defined on $s \in \mathbf{R}$. It is easy to verify that $\mathrm{T}\widetilde{\boldsymbol{v}}_0 = \widetilde{\boldsymbol{v}}_0$ and that $\mathrm{T}\left(\boldsymbol{u} \times \boldsymbol{u}_{ss}\right) = (\mathrm{T}\boldsymbol{u}) \times (\mathrm{T}\boldsymbol{u})_{ss}$. Taking these into account and applying the operator T to (2.4.1) and (2.4.2), we have

$$\begin{cases} (\mathrm{T}\boldsymbol{u})_t = (\mathrm{T}\boldsymbol{u}) \times (\mathrm{T}\boldsymbol{u})_{ss}, & s \in \mathbf{R},\ t > 0, \\ (\mathrm{T}\boldsymbol{u})(s,0) = (\mathrm{T}\widetilde{\boldsymbol{v}}_0)(s) = \widetilde{\boldsymbol{v}}_0(s), & s \in \mathbf{R}, \end{cases}$$

in other words, $\mathrm{T}\boldsymbol{u}$ is also a solution of (2.4.1) and (2.4.2). Thus we have $\mathrm{T}\boldsymbol{u} = \boldsymbol{u}$ by the uniqueness of the solution. Therefore, for any $t \in [0, T]$

$$\boldsymbol{u}(0,t) = (\mathrm{T}\boldsymbol{u})(0,t) = -\overline{\boldsymbol{u}}(0,t),$$

which is equivalent to $u^1(0,t) = u^2(0,t) = 0$. Therefore, it holds that $u^3(0,t) = -1$ or 1 because $|\boldsymbol{u}| \equiv 1$, but in view of $\widetilde{\boldsymbol{v}}_0(0) = \boldsymbol{v}_0(0) = \boldsymbol{e}_3$, we obtain $\boldsymbol{u}(0,t) = \boldsymbol{e}_3$ by the continuity in t.

This shows that the restriction of $\boldsymbol{u}$ to $\mathbf{R}_+$ is a solution of our initial-boundary value problem. Using this function $\boldsymbol{v} := \boldsymbol{u}|_{\mathbf{R}_+}$, we can construct the solution $\boldsymbol{x}$ to the original equation as we stated in Section 2.2. Thus we have

Theorem 2.4.2 (M. Aiki and T. Iguchi [1]) *For a non-negative integer m, if $\boldsymbol{x}_{0ss} \in H^{2+m}(\mathbf{R}_+)$ and $\boldsymbol{x}_{0s}$ satisfies the compatibility condition $(A)_{[\frac{2+m}{2}]}$ and* (2.1.2)*, then there exists a unique solution $\boldsymbol{x}$ of* (2.1.1) *such that*

$$\boldsymbol{x} - \boldsymbol{x}_0 \in C\big([0,\infty); H^{4+m}(\mathbf{R}_+)\big) \cap C^1\big([0,\infty); H^{2+m}(\mathbf{R}_+)\big),$$

and $|\boldsymbol{x}_s| \equiv 1$.

Proof. The uniqueness is left to be proved. Suppose that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are solutions as in the theorem. Then, by extending $\boldsymbol{x}_i$ $(i = 1, 2)$ by

$$\widetilde{\boldsymbol{x}}_i(s,t) = \begin{cases} \boldsymbol{x}_i(s,t) & s \ge 0, t > 0, \\ \overline{\boldsymbol{x}}_i(-s,t) & s < 0, t > 0, \end{cases}$$

we see that $\widetilde{\boldsymbol{x}}_i$ are solutions of the LIE in the whole space. Thus $\boldsymbol{x}_1 = \boldsymbol{x}_2$ follows from the uniqueness of the solution to the initial value problem. $\square$

Chapter 3

Initial-Boundary Value Problems for a Parabolic-Dispersive System

3.1 Problem Setting

In this chapter, we prove the unique solvability of the following initial-boundary value problems: for $\alpha < 0$,

$$(3.1.1) \qquad \begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}_x(0, t) = \boldsymbol{0}, & t > 0; \end{cases}$$

for $\alpha > 0$,

$$(3.1.2) \qquad \begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}(0, t) = \boldsymbol{e}, & t > 0, \\ \boldsymbol{u}_x(0, t) = \boldsymbol{0}, & t > 0. \end{cases}$$

Here, $\boldsymbol{u}(x,t) = (u^1(x,t), u^2(x,t), \dots, u^m(x,t))$ is the unknown vector valued function, $\boldsymbol{u}_0(x)$, $\boldsymbol{w}(x,t) = (w^1(x,t), w^2(x,t), \dots, w^k(x,t))$, and $\boldsymbol{f}(x,t) = (f^1(x,t), f^2(x,t), \dots, f^m(x,t))$ are known vector valued functions, $\boldsymbol{e}$ is an arbitrary constant vector, subscripts denote derivatives with the respective variables, $\mathrm{A}(\boldsymbol{w}, \partial_x)$ is a second order differential operator of the form $\mathrm{A}(\boldsymbol{w}, \partial_x) = \mathrm{A}_0(\boldsymbol{w})\partial_x^2 + \mathrm{A}_1(\boldsymbol{w})\partial_x + \mathrm{A}_2(\boldsymbol{w})$ with smooth matrices A_0, A_1, and A_2. Furthermore, $\mathrm{A}(\boldsymbol{w}, \partial_x)$ is assumed to be strongly elliptic in the sense that for any bounded domain E in $\mathbf{R}^k$, there is a positive constant δ such that for any $\boldsymbol{w} \in E$

$$\mathrm{A}_0(\boldsymbol{w}) + \mathrm{A}_0(\boldsymbol{w})^* \geq \delta \mathrm{I},$$

where I is the unit matrix and $*$ denotes the adjoint of a matrix. Note here that the number of boundary conditions imposed changes depending on the sign of α, much like the KdV equation. This is because the number of fundamental solutions of $\boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx}$ that are bounded in x changes depending on the sign of α.

Problems (3.1.1) and (3.1.2) are considered to prove the unique solvability of the following nonlinear problems: for $\alpha < 0$,

$$
\begin{cases}
\boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha \Big\{ \boldsymbol{v}_{sss} + \dfrac{3}{2} \boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_s) \\
\qquad\qquad\qquad\qquad + \dfrac{3}{2} \boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{ss}) \Big\}, & s > 0, t > 0, \\
\boldsymbol{v}(s, 0) = \boldsymbol{v}_0(s), & s > 0, \\
\boldsymbol{v}_s(0, t) = \boldsymbol{0}, & t > 0;
\end{cases}
\tag{3.1.3}
$$

for $\alpha > 0$,

$$
\begin{cases}
\boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha \Big\{ \boldsymbol{v}_{sss} + \dfrac{3}{2} \boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_s) \\
\qquad\qquad\qquad\qquad + \dfrac{3}{2} \boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{ss}) \Big\}, & s > 0, t > 0, \\
\boldsymbol{v}(s, 0) = \boldsymbol{v}_0(s), & s > 0, \\
\boldsymbol{v}(0, t) = \boldsymbol{e}_3, & t > 0, \\
\boldsymbol{v}_s(0, t) = \boldsymbol{0}, & t > 0,
\end{cases}
\tag{3.1.4}
$$

where $\boldsymbol{v} = (v^1(s,t), v^2(s,t), v^3(s,t))$ is the tangent vector of the vortex filament parameterized by its arc length s at time t, $\boldsymbol{e}_3 = (0, 0, 1)$, the symbol $\times$ is the exterior product in the three dimensional Euclidean space, and α is a real constant describing the magnitude of the effect of axial flow. These two problems are the problems for the generalized LIE written in term of its tangent vector. We refer to the equation written in terms of $\boldsymbol{v}$ as the vortex filament equation to differentiate it from the generalized LIE.

As far as the author knows, there are no results on initial-boundary value problems for the above equation. As mentioned in Chapter 1, Segata [37] proved the unique solvability and showed the asymptotic behavior in time of the solution to the Hirota equation, given by

$$
\mathrm{i} q_t = q_{xx} + \frac{1}{2} |q|^2 q + \mathrm{i} \alpha \Big\{ q_{xxx} + \frac{3}{2} |q|^2 q_x \Big\},
\tag{3.1.5}
$$

which can be obtained by applying the Hasimoto transformation to the vortex filament equation. Since there are many results regarding the initial value problem for the Hirota equation and other Schrödinger type equations, it may be more natural to see if the available theories from these results can be utilized to solve the initial-boundary value problem for (3.1.5), instead of considering (3.1.3) and (3.1.4) directly. Admittedly, problem (3.1.3) and (3.1.4) can be transformed into an initial-boundary value problem for the Hirota equation. But, in light of the possibility that a new boundary condition may be considered for the vortex filament equation in the future, it would be beneficial to develop the analysis of the vortex filament equation itself because the Hasimoto transformation may not be applicable under the new boundary condition. For example, (3.1.3) and (3.1.4) model the motion of a vortex filament moving in the three dimensional half space, but if we consider a boundary that is not flat, it is non-trivial as to if we can apply the Hasimoto transformation or not, thus we consider the vortex filament equation directly.

We begin with the following linearized system with given $\boldsymbol{w}$ and $\boldsymbol{f}$.

$$\boldsymbol{v}_t = \boldsymbol{w} \times \boldsymbol{v}_{ss} + \alpha\bigl\{\boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s)\bigr\} + \boldsymbol{f}.$$

The solution to the initial-boundary value problem for the above system can be obtained by a parabolic regularization, which will be considered in detail later in this chapter, of the form

$$\boldsymbol{v}_t = \alpha(-\varepsilon\boldsymbol{v}_t + \alpha\boldsymbol{v}_{ss})_s + \boldsymbol{w} \times \boldsymbol{v}_{ss} + 3\alpha\boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s) + \boldsymbol{f}$$

with $\varepsilon > 0$. It seems hard to obtain the estimate of the solution uniform in ε, which is needed to pass to the limit $\varepsilon \to +0$. To overcome this, we added the term $\delta\boldsymbol{v}_{ss}$ to the above system to obtain

$$\boldsymbol{v}_t = \alpha(-\varepsilon\boldsymbol{v}_t + \alpha\boldsymbol{v}_{ss})_s + \delta\boldsymbol{v}_{ss} + \boldsymbol{w} \times \boldsymbol{v}_{ss} + 3\alpha\boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s) + \boldsymbol{f}.$$

Then, by utilizing the dissipative property of the term $\delta\boldsymbol{v}_{ss}$, we are able to obtain the desired estimates uniform in ε. If we pass to the limit $\varepsilon \to +0$, we have a parabolic-dispersive system

$$\boldsymbol{v}_t = \alpha\boldsymbol{v}_{sss} + \bigl\{\delta\boldsymbol{v}_{ss} + \boldsymbol{w} \times \boldsymbol{v}_{ss} + 3\alpha\boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s)\bigr\} + \boldsymbol{f}, \tag{3.1.6}$$

which satisfies the assumptions for problems (3.1.1) and (3.1.2). The dissipative property of the second order part allows us to construct the solution to the initial-boundary value problems for

$$\boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\left\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_s) + \frac{3}{2}\boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{ss})\right\} + \delta\bigl(\boldsymbol{v}_{ss} + |\boldsymbol{v}_s|\boldsymbol{v}\bigr).$$

Then, we can pass to the limit $\delta \to +0$ by using the uniform estimates derived from the property $|\boldsymbol{v}| = 1$.

This is the motivation for considering (3.1.1) and (3.1.2). Note that the limit $\delta \to +0$ cannot be considered in general for (3.1.1) and (3.1.2).

At first glance, one may think that (3.1.6) can be treated by using the known results of KdV and KdV–Burgers equations such as Hayashi and Kaikina [15], Hayashi, Kaikina, and Ruiz Paredes [16], or Bona and Zhang [5]. However, this seems hard to do because the vortex filament equation in (3.1.3) and (3.1.4) has second order derivatives in the nonlinear term and the linear estimates obtained in the KdV and KdV–Burgers theory is insufficient to treat the nonlinear terms as a regular perturbation. Thus, a new technique is needed.

Our key method to prove the solvability of (3.1.1) and (3.1.2) is a new parabolic regularization. For (3.1.2), we can regularize the system with a fourth order dissipation term, transforming it into a standard parabolic system. We cannot do this for (3.1.1) because a fourth order parabolic system requires two boundary conditions to solve, but the original problem imposes only one boundary condition. Thus, a standard regularization cannot be applied to (3.1.1). To prove the unique solvability of (3.1.1), we introduce a new type of regularization

$$\begin{cases} \boldsymbol{u}_t = \alpha\left(\boldsymbol{u}_{xx} - \varepsilon\boldsymbol{u}_t\right)_x + \mathrm{A}(\boldsymbol{w},\partial_x)\boldsymbol{u} + \boldsymbol{g}, & x>0, t>0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x>0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t>0 \end{cases} \tag{3.1.7}$$

with $\varepsilon > 0$. Here, $\boldsymbol{g}$ is a given data which we determine later so that the compatibility conditions are satisfied. To construct the solution of the above system, we first consider the following problem.

$$\begin{cases} \boldsymbol{u}_t = \alpha\left(\boldsymbol{u}_{xx} - \varepsilon\boldsymbol{u}_t\right)_x + \boldsymbol{g}, & x>0, t>0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x>0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t>0. \end{cases} \tag{3.1.8}$$

Problem (3.1.7) is a parabolic regularization of (3.1.1) whose principal term is the first term in the right-hand side of the equation in (3.1.8). In fact, for a vector $\boldsymbol{C}$, if we substitute $\boldsymbol{u}(x,t) = \mathrm{e}^{\tau t + \mathrm{i}\xi x}\boldsymbol{C}$ into $\boldsymbol{u}_t = \alpha\left(\boldsymbol{u}_{xx} - \varepsilon\boldsymbol{u}_t\right)_x$, we obtain the dispersion relation $\tau = -\alpha(\xi^2 + \varepsilon\tau)\mathrm{i}\xi$, so that for a non-trivial solution to exist, we need

$$\Re\tau = -\frac{\alpha^2\varepsilon\xi^4}{1+\alpha^2\varepsilon^2\xi^2},$$

which indicates that the equation is parabolic in nature.

Since the proof for the case $\alpha > 0$ is fairly standard, we concentrate on the case $\alpha < 0$, and give a remark on the case $\alpha > 0$ at the end of this chapter.

Now, we state the main theorems. The compatibility conditions mentioned in the theorems are defined in the next section.

Theorem 3.1.1 (M. Aiki and T. Iguchi [2]) *For any $T > 0$ and an arbitrary non-negative integer l, if $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y_T^l$, and $\boldsymbol{w} \in Z_T^l$ satisfy the compatibility conditions* *up to order l, a unique solution $\boldsymbol{u}$ of* (3.1.1) *exists such that $\boldsymbol{u} \in X_T^l$. Furthermore, the solution satisfies*

$$\|\boldsymbol{u}\|_{X_T^l} \leq C\big(\|\boldsymbol{u}_0\|_{2+3l} + \|\boldsymbol{f}\|_{Y_T^l}\big),$$

where the constant C depends on T, $\|\boldsymbol{w}\|_{Z_T^l}$, and δ.

Theorem 3.1.2 (M. Aiki and T. Iguchi [2]) *For any $T > 0$ and an arbitrary non-negative integer l, if $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y_T^l$, and $\boldsymbol{w} \in Z_T^l$ satisfy the compatibility conditions up to order l, a unique solution $\boldsymbol{u}$ of* (3.1.2) *exists such that $\boldsymbol{u} \in X_T^l$. Furthermore, the solution satisfies*

$$\|\boldsymbol{u}\|_{X_T^l} \leq C\big(\|\boldsymbol{u}_0\|_{2+3l} + \|\boldsymbol{f}\|_{Y_T^l}\big),$$

where the constant C depends on T, $\|\boldsymbol{w}\|_{Z_T^l}$, and δ.

The function spaces X_T^l, Y_T^l, and Z_T^l are defined in Chapter 1.

The contents of this chapter are as follows. In Section 3.2, we consider the compatibility conditions and the necessary corrections to the given data required for the regularized problem. In Section 3.3, we construct and estimate the solution to the regularized problem. Then in Section 3.4, we construct and estimate the solution of the parabolic-dispersive system in appropriate function spaces and prove Theorem 3.1.1. Finally in Section 3.5, we give a remark on the proof of Theorem 3.1.2.

3.2 Compatibility Conditions

We will construct the solution of (3.1.1) by taking the limit $\varepsilon \to +0$ in the following regularized problem.

$$\text{(3.2.1)} \qquad \begin{cases} \boldsymbol{u}_t = -\alpha\varepsilon\boldsymbol{u}_{tx} + \alpha\boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{g}, & x > 0, t > 0, \\ \boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}_x(0, t) = \boldsymbol{0}, & t > 0. \end{cases}$$

Since the derivation of the compatibility conditions for the regularized problem is complicated and the required corrections for the given data are not standard, we devote this section to clarify these matters.

3.2.1 Compatibility Conditions for (3.1.1)

We first derive the compatibility conditions for the original problem (3.1.1). We denote the right-hand side of the equation in (3.1.1) as

$$\text{(3.2.2)} \qquad \boldsymbol{Q}_1(\boldsymbol{u}, \boldsymbol{f}, \boldsymbol{w}) = \alpha\boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f},$$

and we also use the notation $\boldsymbol{Q}_1(x,t) := \boldsymbol{Q}_1(\boldsymbol{u},\boldsymbol{f},\boldsymbol{w})$ and sometimes omit the (x,t) for simplicity. We successively define

$$\boldsymbol{Q}_n := \alpha\partial_x^3\boldsymbol{Q}_{n-1} + \sum_{j=0}^{n-1}\left(\begin{array}{c} n-1 \\ j \end{array}\right)\mathrm{B}_j\boldsymbol{Q}_{n-1-j} + \partial_t^{n-1}\boldsymbol{f}, \tag{3.2.3}$$

where $\mathrm{B}_j = \big(\partial_t^j\mathrm{A}_0(\boldsymbol{w})\big)\partial_x^2 + \big(\partial_t^j\mathrm{A}_1(\boldsymbol{w})\big)\partial_x + \partial_t^j\mathrm{A}_2(\boldsymbol{w})$. The above definition (3.2.3) gives the formula for the expression of $\partial_t^n\boldsymbol{u}$ which only contains x derivatives of $\boldsymbol{u}$ and mixed derivatives of $\boldsymbol{w}$ and $\boldsymbol{f}$. From the boundary condition in (3.1.1), we arrive at the following definition for the compatibility conditions.

Definition 3.2.1 (Compatibility conditions for (3.1.1)). *For $n \in \mathbf{N}\cup\{0\}$, we say that $\boldsymbol{u}_0$, $\boldsymbol{f}$, and $\boldsymbol{w}$ satisfy the n-th order compatibility condition for* (3.1.1) *if*

$$\boldsymbol{u}_{0x}(0,0) = \mathbf{0}$$

when $n = 0$, and

$$\big(\partial_x\boldsymbol{Q}_n\big)(0,0) = \mathbf{0}$$

when $n \geq 1$. We also say that the data satisfy the compatibility conditions for (3.1.1) *up to order n if they satisfy the k-th order compatibility condition for all k with $0 \leq k \leq n$.*

Now that we have defined the compatibility conditions, we discuss an approximation of the data via smooth functions which keep the compatibility conditions. Recall that X_T^l, Y_T^l, and Z_T^l are function spaces defined in Chapter 1 to which the solution and given data belong. Data in these function spaces with index l are smooth enough for the l-th order compatibility condition to have meaning in a point-wise sense, but the $(l+1)$-th order compatibility condition does not. Utilizing the method due to Rauch and Massey [36], we can get the following.

Lemma 3.2.2 *Fix non-negative integers l and N with $N > l$. For any $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y_T^l$, and $\boldsymbol{w} \in Z_T^l$ satisfying the compatibility conditions for* (3.1.1) *up to order l, there exist sequences $\{\boldsymbol{u}_{0n}\}_{n\geq 1}$ in $H^{2+3N}(\mathbf{R}_+)$, $\{\boldsymbol{f}_n\}_{n\geq 1}$ in Y_T^N, and $\{\boldsymbol{w}_n\}_{n\geq 1}$ in Z_T^N such that for any $n \geq 1$, $\boldsymbol{u}_{0n}$, $\boldsymbol{f}_n$, and $\boldsymbol{w}_n$ satisfy the compatibility conditions for* (3.1.1) *up to order N and*

$$\boldsymbol{u}_{0n} \to \boldsymbol{u}_0 \text{ in } H^{2+3l}(\mathbf{R}_+), \quad \boldsymbol{f}_n \to \boldsymbol{f} \text{ in } Y_T^l, \text{ and } \boldsymbol{w}_n \to \boldsymbol{w} \text{ in } Z_T^l \text{ as } n \to \infty.$$

From Lemma 3.2.2, we can assume that the given data are arbitrarily smooth and satisfy the necessary compatibility conditions in the proceeding arguments.

3.2.2 Compatibility Conditions for (3.2.1)

In this subsection, we define the compatibility conditions for (3.2.1). We set

$$\boldsymbol{P}_1(\boldsymbol{u},\boldsymbol{g},\boldsymbol{w}) = \alpha\boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w},\partial_x)\boldsymbol{u} + \boldsymbol{g}. \tag{3.2.4}$$

We also write $\boldsymbol{P}_1(x,t)$ and $\boldsymbol{P}_1$ as we did with $\boldsymbol{Q}_1$ in the previous subsection. Setting $\boldsymbol{\phi}_1(x) := \boldsymbol{u}_t(x,0)$ and taking the trace $t=0$ of the equation, we have

$$\alpha\varepsilon\boldsymbol{\phi}_1' + \boldsymbol{\phi}_1 = \boldsymbol{P}_1(\cdot,0). \tag{3.2.5}$$

A prime denotes the derivative with respect to x. Note that $\boldsymbol{P}_1(x,0)$ is expressed by given data only. Solving the above ordinary differential equation for $\boldsymbol{\phi}_1$, we have

$$\boldsymbol{\phi}_1(x) = \mathrm{e}^{-\frac{x}{\alpha\varepsilon}}\left\{\boldsymbol{\phi}_1(0) + \frac{1}{\alpha\varepsilon}\int_0^x \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_1(y,0)\mathrm{d}y\right\}.$$

Since we are looking for square integrable solutions, we impose that $\lim_{x\to\infty}\boldsymbol{\phi}_1(x) = \boldsymbol{0}$. Thus we have

$$\boldsymbol{\phi}_1(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_1(y,0)\mathrm{d}y.$$

By direct calculation, we see that

$$\boldsymbol{\phi}_1'(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_1'(y,0)\mathrm{d}y,$$

where we have used integration by parts. We also note here that $\boldsymbol{\phi}_1$ is expressed with given data only. From the boundary condition in (3.2.1), we see that the first order compatibility condition is given by

$$\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_1'(y,0)\mathrm{d}y = \boldsymbol{0}.$$

In the same manner, we will derive the n-th order compatibility condition for $n \geq 2$. Taking the t derivative of the equation (3.2.4) $(n-1)$ times, taking the trace $t=0$, and setting $\boldsymbol{\phi}_n(x) := \partial_t^n\boldsymbol{u}(x,0)$, we have

$$\alpha\varepsilon\boldsymbol{\phi}_n' + \boldsymbol{\phi}_n = (\partial_t^{n-1}\boldsymbol{P}_1)(\cdot,0).$$

We denote

$$\boldsymbol{P}_n := \partial_t^{n-1}\boldsymbol{P}_1.$$

We will prove by induction that $\boldsymbol{\phi}_n$ and $\boldsymbol{P}_n(x,0)$ are expressed by using given data only. Since $\boldsymbol{P}_n = \partial_t^{n-1}\boldsymbol{P}_{n-1} = \partial_t^{n-1}(\alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w})\boldsymbol{u} + \boldsymbol{g})$, it holds that

$$\boldsymbol{P}_n(\cdot,0) = \alpha\boldsymbol{\phi}_{n-1}''' + \sum_{j=0}^{n-1}\begin{pmatrix} n-1 \\ j \end{pmatrix}\mathrm{B}_j\boldsymbol{\phi}_{n-1-j} + \partial_t^{n-1}\boldsymbol{g}(\cdot,0). \tag{3.2.6}$$

For $n \geq 2$, assume that $\boldsymbol{\phi}_k$ and $\boldsymbol{P}_k(\cdot,0)$ are expressed with given data for $1 \leq k \leq n-1$. Formula (3.2.6) implies that $\boldsymbol{P}_n(\cdot,0)$ is expressed with given data. Solving for $\boldsymbol{\phi}_n$ yields

$$\boldsymbol{\phi}_n(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_n(y,0)\mathrm{d}y.$$

This proves that $\boldsymbol{\phi}_n$ is also expressed by using given data only. Again by direct calculation, we have

$$\boldsymbol{\phi}_n'(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_n'(y,0)\mathrm{d}y,$$

and arrive at the n-th order compatibility condition

$$\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_n'(y,0)\mathrm{d}y = \mathbf{0}.$$

Now we can define the following.

Definition 3.2.3 (Compatibility conditions for (3.2.1)). *For $n \in \mathbf{N}\cup\{0\}$, we say that $\boldsymbol{u}_0$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfy the n-th order compatibility condition for* (3.2.1) *if*

$$\boldsymbol{u}_{0x}(0) = \mathbf{0}$$

when $n = 0$, and

$$\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_n'(y,0)\mathrm{d}y = \mathbf{0}$$

when $n \geq 1$. We also say that the data satisfy the compatibility conditions for (3.2.1) *up to order n if the data satisfy the k-th order compatibility condition for all k with $0 \leq k \leq n$. For the definition of $\boldsymbol{P}_n$, see* (3.2.4) *and* (3.2.6).

We note that for $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y_T^l$, and $\boldsymbol{w} \in Z_T^l$, the compatibility conditions up to order l have pointwise meaning, but the $(l+1)$-th order compatibility condition does not.

3.2.3 Corrections to the Data

Since we regularized the problem, we must make corrections to the data to assure that the compatibility conditions hold. Fix a large positive integer N and suppose that $\boldsymbol{u}_0 \in H^{2+3N}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y_T^N$, and $\boldsymbol{w} \in Z_T^N$ satisfy the compatibility conditions for (3.1.1) up to order N. We will make corrections to the forcing term so that the data satisfy the compatibility conditions for (3.2.1) up to order N. More specifically, we prove the following.

Proposition 3.2.4 *Fix a positive integer N. For $\boldsymbol{u}_0 \in H^{2+3N}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y_T^N$, and $\boldsymbol{w} \in Z_T^N$ satisfying the compatibility conditions for* (3.1.1) *up to order N, we can define $\boldsymbol{g} \in Y_T^N$ in the form $\boldsymbol{g} = \boldsymbol{f} + \boldsymbol{h}_\varepsilon$ such that $\boldsymbol{u}_0$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfy the compatibility conditions for* (3.2.1) *up to order N and $\boldsymbol{h}_\varepsilon \to \mathbf{0}$ in Y_T^N as $\varepsilon \to +0$.*

Proof. We write the equation in (3.2.1) as

$$\boldsymbol{u}_t = -\alpha\varepsilon \boldsymbol{u}_{tx} + \mathrm{P}(x,t,\partial_x)\boldsymbol{u} + \boldsymbol{g},$$

i.e. $\mathrm{P}(x,t,\partial_x)\boldsymbol{u} = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w},\partial_x)\boldsymbol{u}$. Setting $\boldsymbol{\phi}_1(x) := \boldsymbol{u}_t(x,0)$ and taking the trace $t=0$ of the equation, we have

$$\alpha\varepsilon\boldsymbol{\phi}_1' + \boldsymbol{\phi}_1 = \mathrm{P}(\cdot,0,\partial_x)\boldsymbol{u}_0 + \boldsymbol{f}(\cdot,0) + \boldsymbol{h}_\varepsilon(\cdot,0) = \boldsymbol{Q}_1(\cdot,0) + \boldsymbol{h}_\varepsilon(\cdot,0) \tag{3.2.7}$$

by using the notation in (3.2.2). As before, solving the above ordinary differential equation for $\boldsymbol{\phi}_1$ under the constraint $\lim_{x\to\infty} \boldsymbol{\phi}_1(x) = \mathbf{0}$ we have

$$\boldsymbol{\phi}_1(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\big\{\boldsymbol{Q}_1(y,0) + \boldsymbol{h}_\varepsilon(y,0)\big\}\mathrm{d}y.$$

We give an ansatz for the form of $\boldsymbol{h}_\varepsilon$, namely

$$\boldsymbol{h}_\varepsilon(x,t) = \left(\sum_{j=0}^N \boldsymbol{C}_{j,\varepsilon}\frac{t^j}{j!}\right)\mathrm{e}^{-x},$$

where $\boldsymbol{C}_{j,\varepsilon}$, $j = 0,1,...,N$, are constant vectors depending on ε to be determined later. From Definition 3.2.3, the first order compatibility condition is

$$\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\big\{\boldsymbol{Q}_1'(y,0) + \boldsymbol{h}_\varepsilon'(y,0)\big\}\mathrm{d}y = \mathbf{0}.$$

Substituting the ansatz for $\boldsymbol{h}_\varepsilon(x,t)$, we have

$$\boldsymbol{C}_{0,\varepsilon}\left(1 - \frac{1}{\alpha\varepsilon}\right)^{-1} = \int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{Q}_1'(y,0)\mathrm{d}y.$$

Since $\boldsymbol{Q}_1'(0,0) = \mathbf{0}$ from the compatibility condition for (3.1.1), we have by integration by parts

$$\boldsymbol{C}_{0,\varepsilon} = (\alpha\varepsilon - 1)\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{Q}_1''(y,0)\mathrm{d}y.$$

If we limit ourselves to $0 < \varepsilon < \min\{1, 1/|\alpha|\}$, then from

$$\mathrm{e}^{\frac{y}{\alpha\varepsilon}}|\boldsymbol{Q}_1''(y,0)| \le \mathrm{e}^{-y}|\boldsymbol{Q}_1''(y,0)|,$$

and for $y > 0$

$$\mathrm{e}^{\frac{y}{\alpha\varepsilon}}|\boldsymbol{Q}_1''(y,0)| \to 0 \text{ as } \varepsilon \to +0,$$

we see that $\boldsymbol{C}_{0,\varepsilon} \to \mathbf{0}$ as $\varepsilon \to +0$. We will show by induction that $\boldsymbol{C}_{j,\varepsilon}$ can be chosen so that $\boldsymbol{C}_{j,\varepsilon} \to \mathbf{0}$ for $1 \le j \le N$ and $\boldsymbol{g} = \boldsymbol{f} + \boldsymbol{h}_\varepsilon$, $\boldsymbol{u}_0$, and $\boldsymbol{w}$ satisfy the compatibility conditions for (3.2.1) up to order N. Suppose that the above statement holds for $0 \le j \le n-2$ for some n with $2 \le n \le N$.

We define $\boldsymbol{P}_n(x,0)$ and $\boldsymbol{\phi}_n(x)$ as in Subsection 3.3.2 and we have

$$\boldsymbol{\phi}_n(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_n(y,0)\mathrm{d}y, \tag{3.2.8}$$

and the n-th order compatibility condition for (3.2.1) is

$$\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_n'(y,0)\mathrm{d}y = \mathbf{0}.$$

We rewrite this condition as

$$-\boldsymbol{P}_n'(0,0) + \int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_n''(y,0)\mathrm{d}y = \mathbf{0} \tag{3.2.9}$$

by integration by parts. We recall that $\boldsymbol{P}_n(x,0)$ was successively defined by

$$\boldsymbol{P}_n(\cdot,0) = \alpha\boldsymbol{\phi}_{n-1}''' + \sum_{j=0}^{n-1}\begin{pmatrix} n-1 \\ j\end{pmatrix}\mathrm{B}_j\boldsymbol{\phi}_{n-1-j} + \partial_t^{n-1}\boldsymbol{g}(\cdot,0)$$

with $\boldsymbol{P}_1(x,0) = \alpha\boldsymbol{u}_{0xxx}(x) + \mathrm{A}(\boldsymbol{w}(x,0),\partial_x)\boldsymbol{u}_0(x) + \boldsymbol{g}(x,0)$. Substituting (3.2.8) with n replaced by j for $\boldsymbol{\phi}_j$ and using integration by parts, we have

$$\begin{aligned}\boldsymbol{P}_n(\cdot,0) = \alpha\boldsymbol{P}_{n-1}'''(\cdot,0) + \sum_{j=0}^{n-1}\begin{pmatrix} n-1 \\ j\end{pmatrix}\mathrm{B}_j\boldsymbol{P}_{n-1-j}(\cdot,0) + \partial_t^{n-1}\boldsymbol{g}(\cdot,0)& \\ - \alpha\varepsilon\Big\{\alpha\boldsymbol{\phi}_{n-1}'''' + \sum_{j=0}^{n-1}\begin{pmatrix} n-1 \\ j\end{pmatrix}\mathrm{B}_j\boldsymbol{\phi}_{n-1-j}'\Big\}(\cdot,0).&\end{aligned}$$

Also recall that

$$\boldsymbol{Q}_n = \alpha\partial_x^3\boldsymbol{Q}_{n-1} + \sum_{j=0}^{n-1}\begin{pmatrix} n-1 \\ j \end{pmatrix}\mathrm{B}_j\boldsymbol{Q}_{n-1-j} + \partial_t^{n-1}\boldsymbol{f},$$

with $\boldsymbol{Q}_1(x,0) = \alpha\boldsymbol{u}_{0xxx}(x) + \mathrm{A}(\boldsymbol{w}(x,0),\partial_x)\boldsymbol{u}_0(x) + \boldsymbol{f}(x,0)$. Thus, setting $\boldsymbol{R}_n := \boldsymbol{P}_n - \boldsymbol{Q}_n$, we have

$$\begin{aligned}\boldsymbol{R}_n(\cdot,0) = \alpha\boldsymbol{R}_{n-1}'''(\cdot,0) + \sum_{j=0}^{n-1}\begin{pmatrix} n-1 \\ j \end{pmatrix}\mathrm{B}_j\boldsymbol{R}_{n-1-j}(\cdot,0) + \partial_t^{n-1}\boldsymbol{h}_\varepsilon(\cdot,0)& \\ - \alpha\varepsilon\biggl\{\alpha\boldsymbol{\phi}_{n-1}'''' + \sum_{j=0}^{n-1}\begin{pmatrix} n-1 \\ j \end{pmatrix}\mathrm{B}_j\boldsymbol{\phi}_{n-1-j}'\biggr\},&\end{aligned}$$

with $\boldsymbol{R}_1(x,0) = \boldsymbol{h}_\varepsilon(x,0)$. We prove by induction that $\boldsymbol{R}_n(x,0) = \partial_t^{n-1}\boldsymbol{h}_\varepsilon(x,0) + o(1) \quad (\varepsilon \to +0)$. The case $n=1$ is obvious from the definition of $\boldsymbol{R}_1(x,0)$. Suppose that it holds for $\boldsymbol{R}_k(x,0)$ for $1 \le k \le n-1$. From the above expression for $\boldsymbol{R}_n(x,0)$, the assumption of induction on $\boldsymbol{R}_n$, and the assumption of induction that $\boldsymbol{C}_{j,\varepsilon} \to \mathbf{0}$ for $0 \le j \le n-2$, we see that

$$\boldsymbol{R}_n(\cdot,0) = \partial_t^{n-1}\boldsymbol{h}_\varepsilon + o(1) - \alpha\varepsilon\biggl\{\alpha\boldsymbol{\phi}_{n-1}'''' + \sum_{j=0}^{n-1}\begin{pmatrix} n-1 \\ j \end{pmatrix}\mathrm{B}_j\boldsymbol{\phi}_{n-1-j}'\biggr\}.$$

Again, from (3.2.8) and Lebesgue's dominated convergence theorem, we see that the last two terms are $o(1)$, which proves $\boldsymbol{R}_n(x,0) = \boldsymbol{P}_n(x,0) - \boldsymbol{Q}_n(x,0) = \partial_t^{n-1}\boldsymbol{h}_\varepsilon(x,0) + o(1)$ $(\varepsilon \to +0)$. Here, we have used the fact that $\boldsymbol{P}_k(x,0)$ for $1 \le k \le n-1$ are uniformly bounded with respect to ε. We note that from the expressions of $\boldsymbol{R}_n(x,0)$ and $\boldsymbol{h}_\varepsilon$, the terms in $o(1)$ are composed of terms such that their x derivatives are also $o(1)$. Substituting for $\boldsymbol{P}_n(x,0)$ and the ansatz for $\boldsymbol{h}_\varepsilon$ in (3.2.9) yield,

$$\begin{aligned}\boldsymbol{C}_{n-1,\varepsilon} &= \boldsymbol{Q}_n'(0,0) + \int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{Q}_n''(y,0)\mathrm{d}y + o(1) \\ &= \int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{Q}_n''(y,0)\mathrm{d}y + o(1) \quad (\varepsilon \to +0),\end{aligned}$$

where we have used the assumption of induction that $\boldsymbol{u}_0$, $\boldsymbol{f}$, and $\boldsymbol{w}$ satisfy the n-th order compatibility condition for (3.1.1), i.e. $\boldsymbol{Q}_n'(0,0) = \mathbf{0}$. By using the above expression to define $\boldsymbol{C}_{n-1,\varepsilon}$, we see that $\boldsymbol{C}_{n-1,\varepsilon} \to \mathbf{0}$ as $\varepsilon \to +0$ and $\boldsymbol{u}_0$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfy the compatibility conditions for (3.2.1) up to order n. Furthermore, from the explicit form we see that $\boldsymbol{h}_\varepsilon \to \mathbf{0}$ in Y_T^N. This finishes the proof of the proposition. $\square$

The corrections to the data associated with (3.1.8) can be treated in the same way.

3.3 Construction and Estimate of Solution for the Regularized System

We construct the solution $\boldsymbol{u}$ to problem (3.1.8) as $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, where $\boldsymbol{u}_1$ is defined as the solution to the initial value problem

$$\begin{cases} \boldsymbol{u}_{1t} = \alpha\left(\boldsymbol{u}_{1xx} - \varepsilon\boldsymbol{u}_{1t}\right)_x + \boldsymbol{G}, & x \in \mathbf{R}, t > 0, \\ \boldsymbol{u}_1(x,0) = \boldsymbol{U}_0, & x \in \mathbf{R}, \end{cases} \tag{3.3.1}$$

and $\boldsymbol{u}_2$ is defined as the solution to the initial-boundary value problem

$$\begin{cases} \boldsymbol{u}_{2t} = \alpha\left(\boldsymbol{u}_{2xx} - \varepsilon\boldsymbol{u}_{2t}\right)_x, & x > 0, t > 0, \\ \boldsymbol{u}_2(x,0) = \mathbf{0}, & x > 0, \\ \boldsymbol{u}_{2x}(0,t) = -\boldsymbol{u}_{1x}(0,t) =: \boldsymbol{\Phi}(t), & t > 0. \end{cases} \tag{3.3.2}$$

Here, $\boldsymbol{G}$ and $\boldsymbol{U}_0$ are extensions of $\boldsymbol{g}$ and $\boldsymbol{u}_0$ to $x < 0$, respectively.

3.3.1 Construction and Estimate of $\boldsymbol{u}_1$

First we solve (3.3.1). By applying the Fourier transform with respect to x, we obtain the ordinary differential equation

$$\begin{cases} \hat{\boldsymbol{u}}_{1t} = \dfrac{1}{1 + \mathrm{i}\alpha\varepsilon\xi}\big(- \mathrm{i}\alpha\xi^3\hat{\boldsymbol{u}}_1 + \hat{\boldsymbol{G}}\big), \\ \hat{\boldsymbol{u}}_1(\xi,0) = \hat{\boldsymbol{U}}_0, \end{cases} \tag{3.3.3}$$

where $\hat{\boldsymbol{u}}_1$ is the Fourier transform defined by

$$\hat{\boldsymbol{u}}_1(\xi,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi}\boldsymbol{u}_1(x,t)\mathrm{d}x.$$

Problem (3.3.3) can be explicitly solved as

$$\hat{\boldsymbol{u}}_1(\xi,t) = \mathrm{e}^{c(\xi)t}\hat{\boldsymbol{U}}_0 + \int_0^t \mathrm{e}^{c(\xi)(t-\tau)}\frac{1}{1+\mathrm{i}\alpha\varepsilon\xi}\hat{\boldsymbol{G}}(\xi,\tau)\mathrm{d}\tau,$$

where $c(\xi)$ is given by

$$c(\xi) = \frac{-\alpha^2\varepsilon\xi^4 - \mathrm{i}\alpha\xi^3}{1 + \alpha^2\varepsilon^2\xi^2}.$$

Now we derive an estimate for $\boldsymbol{u}_1$. The estimate derived here will be of parabolic nature, and will not be uniform in ε.

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{u}_1\|^2_{L^2(\mathbf{R})} + \alpha^2\varepsilon^2\|\boldsymbol{u}_{1x}\|^2_{L^2(\mathbf{R})}\right\} &= \langle \boldsymbol{u}_1, \boldsymbol{u}_{1t}\rangle + \alpha^2\varepsilon^2\langle \boldsymbol{u}_{1x}, \boldsymbol{u}_{1xt}\rangle \\ &\le \frac{1}{2}\left(\|\boldsymbol{u}_1\|^2_{L^2(\mathbf{R})} + \alpha^2\varepsilon^2\|\boldsymbol{u}_{1x}\|^2_{L^2(\mathbf{R})}\right) - \alpha^2\varepsilon\|\boldsymbol{u}_{1xx}\|^2_{L^2(\mathbf{R})} + \|\boldsymbol{G}\|^2_{L^2(\mathbf{R})}. \end{aligned}$$

Similarly, for an integer l we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\partial_x^l \boldsymbol{u}_1\|^2_{L^2(\mathbf{R})}+\alpha^2\varepsilon^2\|\partial_x^{l+1}\boldsymbol{u}_1\|^2_{L^2(\mathbf{R})}\Big\}$$
$$\leq \frac{1}{2}\Big(\|\partial_x^l \boldsymbol{u}_1\|^2_{L^2(\mathbf{R})}+\alpha^2\varepsilon^2\|\partial_x^{l+1}\boldsymbol{u}_1\|^2_{L^2(\mathbf{R})}\Big)-\alpha^2\varepsilon\|\partial_x^{l+2}\boldsymbol{u}_1\|^2_{L^2(\mathbf{R})}+\|\partial_x^l \boldsymbol{G}\|^2_{L^2(\mathbf{R})}.$$

We also obtain estimates for the mixed x and t derivatives of $\boldsymbol{u}_1$, which will come in use later, in the same way as above

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\partial_x^l\partial_t^m \boldsymbol{u}_1\|^2_{L^2(\mathbf{R})}+\alpha^2\varepsilon^2\|\partial_x^{l+1}\partial_t^m\boldsymbol{u}_1\|^2_{L^2(\mathbf{R})}\Big\}$$
$$\leq \frac{1}{2}\Big(\|\partial_x^l\partial_t^m \boldsymbol{u}_1\|^2_{L^2(\mathbf{R})}+\alpha^2\varepsilon^2\|\partial_x^{l+1}\partial_t^m\boldsymbol{u}_1\|^2_{L^2(\mathbf{R})}\Big)-\alpha^2\varepsilon\|\partial_x^{l+2}\partial_t^m\boldsymbol{u}\|^2_{L^2(\mathbf{R})}+\|\partial_x^l\partial_t^m \boldsymbol{G}\|^2_{L^2(\mathbf{R})}.$$

To finish the estimate, we must estimate the t derivatives of $\boldsymbol{u}_1$ at $t=0$ in terms of $\boldsymbol{U}_0$ and $\boldsymbol{G}$. Set $\boldsymbol{\phi}_{1n}(x):=\partial_t^n\boldsymbol{u}_1(x,0)$. As before, by taking the trace $t=0$ of the equation and solving for $\boldsymbol{\phi}_{11}$ under the constraint $\lim\limits_{x\to\infty}\boldsymbol{\phi}_{11}(x)=\mathbf{0}$ yield

$$\boldsymbol{\phi}_{11}(x)=-\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\big\{\alpha\boldsymbol{U}_0'''(y)+\boldsymbol{G}(y,0)\big\}\mathrm{d}y.$$

Through direct calculation, we see that

$$\partial_x^k\boldsymbol{\phi}_{11}(x)=-\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\big\{\alpha\partial_y^{k+3}\boldsymbol{U}_0(y)+\partial_y^k\boldsymbol{G}(y,0)\big\}\mathrm{d}y.$$

Also from direct calculation, we obtain

$$\|\partial_x^k\boldsymbol{\phi}_{11}\|_{L^2(\mathbf{R})}\leq C\|\partial_x^{k+2}\boldsymbol{U}_0\|_{L^2(\mathbf{R})}+\|\partial_x^k\boldsymbol{G}(\cdot,0)\|_{L^2(\mathbf{R})}. \tag{3.3.4}$$

Here, we have used

$$\begin{aligned}
-\frac{1}{\varepsilon}&\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\partial_y^{k+3}\boldsymbol{U}_0(y)\mathrm{d}y\\
&=-\frac{1}{\varepsilon}\Big[\mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\partial_y^{k+2}\boldsymbol{U}_0(y)\Big]_{y=x}^\infty+\frac{1}{\alpha\varepsilon^2}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\partial_y^{k+2}\boldsymbol{U}_0(y)\mathrm{d}y\\
&=\frac{1}{\varepsilon}\partial_x^{k+2}\boldsymbol{U}_0(x)+\frac{1}{\alpha\varepsilon^2}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\partial_y^{k+2}\boldsymbol{U}_0(y)\mathrm{d}y.
\end{aligned}$$

As shown from the above calculation, the constant C in (3.3.4) is not uniform in ε.

Taking the t derivative of the equation $n-1$ times, we obtain

$$\boldsymbol{\phi}_{1n}'=\frac{1}{\alpha\varepsilon}\big\{-\boldsymbol{\phi}_{1n}+\alpha\boldsymbol{\phi}_{1(n-1)}'''+\partial_t^{n-1}\boldsymbol{G}(\cdot,0)\big\}.$$

As before, we obtain an expression

$$\partial_x^k\boldsymbol{\phi}_{1n}(x)=-\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\big\{\alpha\partial_y^{k+3}\boldsymbol{\phi}_{1(n-1)}(y)+\partial_t^{n-1}\partial_y^k\boldsymbol{G}(y,0)\big\}\mathrm{d}y,$$

and an estimate

$$\|\partial_x^k \boldsymbol{\phi}_{1n}\|_{L^2(\mathbf{R})} \leq C\biggl(\|\boldsymbol{U}_0\|_{H^{k+2n}(\mathbf{R})} + \sum_{j=0}^{n-1} \|\partial_t^{n-1-j}\boldsymbol{G}(\cdot,0)\|_{H^{k+2j}(\mathbf{R})}\biggr),$$

where C is a positive constant depending on ε. Combining these estimates yields

$$\begin{aligned}
\sup_{0\leq t\leq T}&\Bigl\{\|\partial_t^m\partial_x^l \boldsymbol{u}_1(\cdot,t)\|^2_{L^2(\mathbf{R})} + \|\partial_t^m\partial_x^{l+1} \boldsymbol{u}_1(\cdot,t)\|^2_{L^2(\mathbf{R})}\Bigr\} + \int_0^T \|\partial_t^m\partial_x^{l+2} \boldsymbol{u}_1(\cdot,t)\|^2_{L^2(\mathbf{R})}\mathrm{d}t \\
&\leq C\mathrm{e}^T\biggl(\|\boldsymbol{U}_0\|^2_{H^{l+2m}(\mathbf{R})} + \sum_{j=0}^{n-1}\|\partial_t^{n-1-j}\boldsymbol{G}(\cdot,0)\|^2_{H^{l+2j}(\mathbf{R})}\biggr) \\
&\qquad\qquad\qquad\qquad + C\int_0^T \mathrm{e}^{T-t}\|\partial_t^m\partial_x^l\boldsymbol{G}(\cdot,t)\|^2_{L^2(\mathbf{R})}\mathrm{d}t.
\end{aligned}$$

From the boundedness of the extension, we have the following estimate on the half-line.

$$\begin{aligned}
\sup_{0\leq t\leq T}&\Bigl\{\|\partial_t^m\partial_x^l \boldsymbol{u}_1(\cdot,t)\|^2 + \|\partial_t^m\partial_x^{l+1} \boldsymbol{u}_1(\cdot,t)\|^2\Bigr\} + \int_0^T \|\partial_t^m\partial_x^{l+2} \boldsymbol{u}_1(\cdot,t)\|^2\mathrm{d}t \\
&\leq C\mathrm{e}^T\biggl(\|\boldsymbol{u}_0\|^2_{l+2m} + \sum_{j=0}^{n-1}\|\partial_t^{n-1-j}\boldsymbol{g}(\cdot,0)\|^2_{l+2j}\biggr) + C\int_0^T \mathrm{e}^{T-t}\|\partial_t^m\partial_x^l\boldsymbol{g}(\cdot,t)\|^2\mathrm{d}t.
\end{aligned}$$

3.3.2 Construction and Estimate of $\boldsymbol{u}_2$

In this subsection, we solve (3.3.2). First we derive the compatibility conditions for (3.3.2) and check that they are satisfied. Suppose that the initial datum and the forcing term satisfy the compatibility conditions for (3.1.8) up to some finite order. The 0-th order compatibility condition for (3.3.2) is $\boldsymbol{u}_{1x}(0,0) = \mathbf{0}$. From the definition of $\boldsymbol{u}_1$ and the compatibility condition for (3.1.8), we have

$$-\boldsymbol{u}_{1x}(0,0) = -\boldsymbol{u}_{0x}(0,0) = \mathbf{0},$$

so that the 0-th order compatibility condition for (3.3.2) is satisfied. Now we check the first order compatibility condition. Taking the t derivative of the boundary condition, we have $\boldsymbol{u}_{2tx}(0,0) = -\boldsymbol{u}_{1tx}(0,0)$. Taking the trace $t=0$ of the equation in (3.3.2) and setting $\boldsymbol{\phi}_{21}(x) := \boldsymbol{u}_{2t}(x,0)$ yield

$$\boldsymbol{\phi}_{21}' = -\frac{1}{\alpha\varepsilon}\boldsymbol{\phi}_{21}.$$

Solving for $\boldsymbol{\phi}_{21}$, we have

$$\boldsymbol{\phi}_{21}(x) = \boldsymbol{\phi}_{21}(0)\mathrm{e}^{-\frac{1}{\alpha\varepsilon}x}.$$

For $\boldsymbol{\phi}_{21}$ to be integrable, $\boldsymbol{\phi}_{21}(0)$ must be a zero vector. Thus, $\boldsymbol{\phi}_{21}(x) = \mathbf{0}$ for any $x > 0$, from which we can deduce that the first order compatibility condition for (3.3.2) is $\boldsymbol{u}_{1tx}(0,0) = \mathbf{0}$. Taking the trace $t = 0$ of the equation in (3.3.1) and setting $\boldsymbol{\phi}_{11}(x) := \boldsymbol{u}_{1t}(x,0)$, we have

$$\boldsymbol{\phi}_{11}' = -\frac{1}{\alpha\varepsilon}\boldsymbol{\phi}_{11} + \frac{1}{\alpha\varepsilon}\big\{\alpha \boldsymbol{U}_0''' + \boldsymbol{G}(\cdot,0)\big\}.$$

As before, solving for $\boldsymbol{\phi}_{11}$ and using the integrability of $\boldsymbol{\phi}_{11}$ gives

$$\boldsymbol{\phi}_{11}(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\big\{\alpha \boldsymbol{U}_0'''(y) + \boldsymbol{G}(y,0)\big\}\mathrm{d}y.$$

If x is restricted to $x > 0$, $\boldsymbol{U}_0$ and $\boldsymbol{G}$ can be replaced with $\boldsymbol{u}_0$ and $\boldsymbol{g}$, respectively, because they are extensions of the respective functions. Taking the trace $t = 0$ of the equation in (3.1.8), setting $\boldsymbol{\phi}_1(x) := \boldsymbol{u}_t(x,0)$, and solving for $\boldsymbol{\phi}_1$ we have

$$\boldsymbol{\phi}_1(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\big\{\alpha \boldsymbol{u}_0'''(y) + \boldsymbol{g}(y,0)\big\}\mathrm{d}y = \boldsymbol{\phi}_{11}(x). \tag{3.3.5}$$

Taking the t derivative of the boundary condition in (3.1.8) and taking the trace $x = 0$ and $t = 0$, we see that $\boldsymbol{\phi}_1'(0) = \boldsymbol{u}_{tx}(0,0) = \mathbf{0}$, which gives

$$\boldsymbol{u}_{1tx}(0,0) = \boldsymbol{\phi}_{11}'(0) = \boldsymbol{\phi}_1'(0) = \mathbf{0},$$

where we have used (3.3.5). This shows that the first order compatibility condition for (3.3.2) is satisfied.

In the same manner, we set $\boldsymbol{\phi}_{1n}(x) := \partial_t^n \boldsymbol{u}_1(x,0)$, $\boldsymbol{\phi}_{2n}(x) := \partial_t^n \boldsymbol{u}_2(x,0)$, and $\boldsymbol{\phi}_n(x) := \partial_t^n \boldsymbol{u}(x,0)$, where $\boldsymbol{\phi}_{2n}$ and $\boldsymbol{\phi}_n$ can be expressed by using given data only as in Subsection 3.2.2. We will show that the n-th order compatibility condition for (3.3.2) is satisfied by proving that $\boldsymbol{\phi}_{1n} = \boldsymbol{\phi}_n$ and $\boldsymbol{\phi}_{2n} = \mathbf{0}$. We prove this by induction. Suppose that $\boldsymbol{\phi}_{1k} = \boldsymbol{\phi}_k$ and $\boldsymbol{\phi}_{2k} = \mathbf{0}$ for $k = 1, 2, \ldots, n-1$. We note that from the compatibility conditions for (3.1.8), $\boldsymbol{\phi}_k'(0) = \mathbf{0}$ for $0 \le k \le n$. By taking the derivative of the respective equations $(n-1)$ times with respect to t and taking the trace $t = 0$, these functions satisfy

$$\begin{aligned}
\boldsymbol{\phi}_{1n} &= \alpha\boldsymbol{\phi}_{1(n-1)}''' - \alpha\varepsilon\boldsymbol{\phi}_{1n}' + \partial_t^{n-1}\boldsymbol{G}(\cdot,0),\\
\boldsymbol{\phi}_{2n} &= \alpha\boldsymbol{\phi}_{2(n-1)}''' - \alpha\varepsilon\boldsymbol{\phi}_{2n}',\\
\boldsymbol{\phi}_{n} &= \alpha\boldsymbol{\phi}_{n-1}''' - \alpha\varepsilon\boldsymbol{\phi}_{n}' + \partial_t^{n-1}\boldsymbol{g}(\cdot,0).
\end{aligned}$$

First, we see from $\boldsymbol{\phi}_{2(n-1)} = \mathbf{0}$ that

$$\boldsymbol{\phi}_{2n}' = -\frac{1}{\alpha\varepsilon}\boldsymbol{\phi}_{2n}.$$

As before, from this equation and the necessity of $\boldsymbol{\phi}_{2n}$ to be integrable, we see that $\boldsymbol{\phi}_{2n} = \mathbf{0}$. This implies that, through the boundary condition, the n-th order compatibility condition for (3.3.2) is $\partial_t^n \boldsymbol{u}_{1x}(0,0) = \mathbf{0}$. Solving the above equations for $\boldsymbol{\phi}_{1n}$ and $\boldsymbol{\phi}_n$, we have

$$\boldsymbol{\phi}_{1n}(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\big\{\alpha\boldsymbol{\phi}'''_{1(n-1)}(y) + \partial_t^{n-1}\boldsymbol{G}(y,0)\big\}\mathrm{d}y,$$
$$\boldsymbol{\phi}_n(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\big\{\alpha\boldsymbol{\phi}'''_{n-1}(y) + \partial_t^{n-1}\boldsymbol{g}(y,0)\big\}\mathrm{d}y.$$

Again, from the assumption of induction and the fact that $\boldsymbol{U}_0$ and $\boldsymbol{G}$ are extensions of $\boldsymbol{u}_0$ and $\boldsymbol{g}$, respectively, we see that $\boldsymbol{\phi}_{1n}(x) = \boldsymbol{\phi}_n(x)$. We have

$$\partial_t^n \boldsymbol{u}_{1x}(0,0) = \boldsymbol{\phi}'_{1n}(0) = \boldsymbol{\phi}'_n(0) = \mathbf{0},$$

which shows that the n-th order compatibility condition for (3.3.2) is satisfied.

Now we construct $\boldsymbol{u}_2$. We saw that $\frac{\mathrm{d}^k\boldsymbol{\Phi}}{\mathrm{d}t^k}(0) = \partial_t^k \boldsymbol{u}_{1x}(0,0) = \mathbf{0}$ for $0 \le k \le n$, thus, we construct and estimate $\boldsymbol{u}_2$ in Sobolev–Slobodetskiĭ spaces. Taking the Laplace transform with respect to t of the equation yields

$$\begin{cases} \tau\tilde{\boldsymbol{u}}_2 = \alpha\tilde{\boldsymbol{u}}_{2xxx} - \alpha\varepsilon\tau\tilde{\boldsymbol{u}}_{2x}, & x > 0, \\ \tilde{\boldsymbol{u}}_{2x}(0,\tau) = -\tilde{\boldsymbol{u}}_{1x}(0,\tau) = \tilde{\boldsymbol{\Phi}}(\tau), \end{cases}$$

where $\tau = h + \mathrm{i}\eta$ with $h > 0$ and $\eta \in \mathbf{R}$. We show the following properties about the characteristic roots of the above ordinary differential equation.

Lemma 3.3.1 *For $h > 0$ and $\varepsilon > 0$, the characteristic equation, $\lambda^3 - \varepsilon\tau\lambda - \frac{\tau}{\alpha} = 0$, has exactly one root λ satisfying $\Re\lambda < 0$. We denote this root as μ. Furthermore, there are positive constants η_0 and C such that for $|\eta| \ge \eta_0$ the following holds.*

$$\left|\mu + \sqrt{\frac{\varepsilon}{2}}\,(1+\mathrm{i})\,|\eta|^{1/2}\right| \le C.$$

Proof. First, we look at the asymptotic behavior of the roots as $\eta \to +\infty$. Dividing the characteristic equation by $\eta^{3/2}$ and setting $\tilde{\lambda} := \frac{\lambda}{\eta^{1/2}}$, we have

$$\tilde{\lambda}^3 - \frac{\varepsilon h}{\eta}\tilde{\lambda} - \mathrm{i}\varepsilon\tilde{\lambda} - \frac{h}{\alpha\eta^{3/2}} - \mathrm{i}\frac{1}{\alpha\eta^{1/2}} = 0. \tag{3.3.6}$$

Passing to the limit $\eta \to +\infty$, we have

$$\tilde{\lambda}^3 - \mathrm{i}\varepsilon\tilde{\lambda} = 0.$$

The roots are $\tilde{\lambda} = 0,\ \pm\sqrt{\frac{\varepsilon}{2}}(1+\mathrm{i})$. The root $-\sqrt{\frac{\varepsilon}{2}}(1+\mathrm{i})$ corresponds to the desired root of the original characteristic equation. We must consider the root 0 in more detail. By setting $\tilde{\lambda} = 0 + c_1\eta^{-1/2} + \mathrm{O}(\eta^{-1})$ and substituting it into (3.3.6), the coefficients of the terms $\mathrm{O}(\eta^{-1/2})$ yield,

$$-\mathrm{i}\varepsilon c_1 - \mathrm{i}\frac{1}{\alpha} = 0.$$

This gives $c_1 = -\frac{1}{\alpha\varepsilon} > 0$, and hence only one root with a negative real part exists for sufficiently large η. The case $\eta \to -\infty$ can be treated in the same way. Now we show that for any $h > 0$ and $\eta \in \mathbf{R}$, there are no pure imaginary roots, which, combined with the continuity of the roots with respect to the coefficient of the characteristic equation, proves that the number of roots with a negative real part does not change.

We separate the characteristic equation into its real and imaginary parts. Setting $\lambda = a + \mathrm{i}b$ we have

$$\begin{aligned} a^3 - 3ab^2 - \varepsilon ha + \varepsilon\eta b - \frac{h}{\alpha} &= 0, \\ -b^3 + 3a^2b - \varepsilon hb - \varepsilon\eta a - \frac{\eta}{\alpha} &= 0. \end{aligned}$$

Suppose that a pure imaginary root exists, which corresponds to a root with $a = 0$, we then have

$$\varepsilon\eta b = \frac{h}{\alpha},\ \ -b^3 - \varepsilon hb - \frac{\eta}{\alpha} = 0.$$

From the first equation we have $\eta b = \frac{h}{\alpha\varepsilon}$. Substituting this into the second equation yields

$$-b^4 - \varepsilon hb^2 - \frac{h}{\alpha^2\varepsilon} = 0. \tag{3.3.7}$$

Since we are considering $h > 0$ and $\varepsilon > 0$, (3.3.7) is a contradiction. Thus, no such root exists. □

From Lemma 3.3.1, we see that the Laplace transform of a square integrable solution to (3.3.2) can be expressed as

$$\tilde{\boldsymbol{u}}_2(x,\tau) = \frac{1}{\mu}\tilde{\boldsymbol{\Phi}}(\tau)\mathrm{e}^{\mu x},$$

where μ is the root of the characteristic equation mentioned in Lemma 3.3.1. We denote the dependence of μ on τ as $\mu(\tau)$. We estimate $\boldsymbol{u}_2$ in Sobolev–Slobodetskiĭ spaces. To estimate $\boldsymbol{u}_2$ in $H_h^{l,l/2}(Q_\infty)$, we use the following norm.

$$\sum_{j=0}^{l}\int_{-\infty}^{\infty}\left\|\frac{\partial^j\tilde{\boldsymbol{u}}_2}{\partial x^j}(\cdot,\tau)\right\|^2|\tau|^{l-j}\mathrm{d}\eta.$$

Since

$$\frac{\partial^j \tilde{\boldsymbol{u}}_2}{\partial x^j} = \mu(\tau)^{j-1}\tilde{\boldsymbol{\Phi}}(\tau)\mathrm{e}^{\mu x},$$

we have

$$\begin{aligned}
\left\| \frac{\partial^j \tilde{\boldsymbol{u}}_2}{\partial x^j}(\cdot,\tau) \right\|^2 &= \int_0^\infty |\mu|^{2(j-1)}|\tilde{\boldsymbol{\Phi}}|^2|\mathrm{e}^{\mu x}|^2 \mathrm{d}x \\
&= |\mu|^{2(j-1)}|\tilde{\boldsymbol{\Phi}}|^2 \left(-\frac{1}{2\Re\mu}\right).
\end{aligned}$$

Thus we have

$$\int_{-\infty}^{\infty} \left\| \frac{\partial^j \tilde{\boldsymbol{u}}_2}{\partial x^j}(\cdot,\tau) \right\|^2 |\tau|^{l-j}\mathrm{d}\eta = \int_{-\infty}^{\infty} |\tilde{\boldsymbol{\Phi}}(\tau)|^2|\mu(\tau)|^{2(j-1)} \left(\frac{1}{2|\Re\mu(\tau)|}\right)|\tau|^{l-j}\mathrm{d}\eta.$$

We divide the above integral domain into two parts, namely the part with $|\eta| \geq \eta_0$ and $|\eta| \leq \eta_0$, where η_0 is a constant appearing in Lemma 3.3.1. From Lemma 3.3.1, in the domain $|\eta| \geq \eta_0$, we have

$$\left| \mu + \sqrt{\frac{\varepsilon}{2}}\,(1+\mathrm{i})\,|\eta|^{1/2} \right| \leq C,$$

which implies, by taking η_0 larger if necessary, $|\frac{\tau}{|\mu|^2}| \leq C$. We then obtain

$$\int_{|\eta|\geq\eta_0} |\tilde{\boldsymbol{\Phi}}(\tau)|^2|\mu(\tau)|^{2(j-1)} \left(\frac{1}{2|\Re\mu(\tau)|}\right)|\tau|^{l-j}\mathrm{d}\eta \leq C\int_{|\eta|\geq\eta_0} |\tilde{\boldsymbol{\Phi}}(\tau)|^2|\tau|^{l-3/2}\mathrm{d}\eta,$$

$$\int_{|\eta|\leq\eta_0} |\tilde{\boldsymbol{\Phi}}(\tau)|^2|\mu(\tau)|^{2(j-1)} \left(\frac{1}{2|\Re\mu(\tau)|}\right)|\tau|^{l-j}\mathrm{d}\eta \leq C\int_{-\infty}^{\infty} |\tilde{\boldsymbol{\Phi}}(\tau)|^2\mathrm{d}\eta.$$

Combining these estimates, we have

$$\begin{aligned}
|||\boldsymbol{u}_2|||^2_{H_h^{l,l/2}(Q_\infty)} &\leq C\|\boldsymbol{u}_{1x}(0,\cdot)\|^2_{H^{\frac{l}{2}-\frac{3}{4}}(0,\infty)} \\
&\leq C|||\boldsymbol{u}_{1x}|||^2_{H_h^{l-1,l/2-1/2}(Q_\infty)} \\
&\leq C|||\boldsymbol{u}_1|||^2_{H_h^{l,l/2}(Q_\infty)}.
\end{aligned}$$

Here, we have used a trace theorem proved in [38] for functions belonging to the Sobolev–Slobodetskiĭ space. Choosing $l = 2k$ for an integer k and from Sobolev's embedding theorem, we see that

$$\begin{aligned}
&\boldsymbol{u}_2 \in H_h^{2k,k}(Q_T) \hookrightarrow C\big([0,T];H^{2k-2}(\mathbf{R}_+)\big), \\
&\frac{\partial^m \boldsymbol{u}_2}{\partial t^m} \in H_h^{2(k-m),k-m}(Q_T) \hookrightarrow C\big([0,T];H^{2(k-m)-2}(\mathbf{R}_+)\big),
\end{aligned} \tag{3.3.8}$$

for $m \le k$. We mentioned in Subsection 3.2.1 that the given data can be taken as being smooth while satisfying the necessary compatibility conditions, so from the above arguments, for an arbitrary integer l, we have constructed a solution of (3.1.8) such that

$$\boldsymbol{u} \in \bigcap_{j=0}^{l} C^j\big([0,T]; H^{2(l-j)}(\mathbf{R}_+)\big).$$

To prove the uniqueness, we derive an energy estimate for the solution of (3.1.8). Direct calculations yield

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\{\|\boldsymbol{u}\|^2 + \alpha^2\varepsilon^2\|\boldsymbol{u}_x\|^2\big\} &= (\boldsymbol{u}, \boldsymbol{u}_t) + \alpha^2\varepsilon^2(\boldsymbol{u}_x, \boldsymbol{u}_{tx}) \\ &\le -\alpha\boldsymbol{u}(0,t)\cdot\boldsymbol{u}_{xx}(0,t) - \frac{|\alpha|\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}(0,t)|^2 - \alpha^2\varepsilon\|\boldsymbol{u}_{xx}\|^2 \\ &\qquad + \frac{1}{2}\big(\|\boldsymbol{u}\|^2 + \alpha^2\varepsilon^2\|\boldsymbol{u}_x\|^2\big) + \|\boldsymbol{g}\|^2,\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\{\|\boldsymbol{u}_x\|^2 + \alpha^2\varepsilon^2\|\boldsymbol{u}_{xx}\|^2\big\} &= (\boldsymbol{u}_x, \boldsymbol{u}_{xt}) + \alpha^2\varepsilon^2(\boldsymbol{u}_{xx}, \boldsymbol{u}_{xxt}) \\ &= -\frac{|\alpha|}{2}|\boldsymbol{u}_{xx}(0,t)|^2 - \alpha^2\varepsilon\|\boldsymbol{u}_{xxx}\|^2 - (\boldsymbol{u}_{xx}, \boldsymbol{g}) - \alpha\varepsilon(\boldsymbol{u}_{xxx}, \boldsymbol{g}) \\ &\qquad - \alpha^2\varepsilon\boldsymbol{u}_{xx}(0,t)\cdot\boldsymbol{u}_{xxx}(0,t) - \alpha\varepsilon\boldsymbol{u}_{xx}(0,t)\cdot\boldsymbol{g}(0,t) \\ &= -\frac{|\alpha|}{2}|\boldsymbol{u}_{xx}(0,t)|^2 - \alpha^2\varepsilon\|\boldsymbol{u}_{xxx}\|^2 - (\boldsymbol{u}_{xx}, \boldsymbol{g}) - \alpha\varepsilon(\boldsymbol{u}_{xxx}, \boldsymbol{g}) \\ &\qquad - \alpha\varepsilon\boldsymbol{u}_{xx}(0,t)\cdot\boldsymbol{u}_t(0,t).\end{aligned}$$

On the other hand, we have from the equation,

$$\|\boldsymbol{u}_t + \alpha\varepsilon\boldsymbol{u}_{tx}\|^2 = \|\alpha\boldsymbol{u}_{xxx} + \boldsymbol{g}\|^2.$$

Expanding the left-hand side gives

$$\begin{aligned}\|\boldsymbol{u}_t + \alpha\varepsilon\boldsymbol{u}_{tx}\|^2 &= \|\boldsymbol{u}_t\|^2 + 2\alpha\varepsilon(\boldsymbol{u}_t, \boldsymbol{u}_{tx}) + \alpha^2\varepsilon^2\|\boldsymbol{u}_{tx}\|^2 \\ &= \|\boldsymbol{u}_t\|^2 + |\alpha|\varepsilon|\boldsymbol{u}_t(0,t)|^2 + \alpha^2\varepsilon^2\|\boldsymbol{u}_{tx}\|^2.\end{aligned}$$

Thus we have

$$|\alpha|\varepsilon|\boldsymbol{u}_t(0,t)|^2 \le \|\alpha\boldsymbol{u}_{xxx} + \boldsymbol{g}\|^2.$$

Utilizing the above estimate, we have for any positive γ

$$|\alpha|\varepsilon|\boldsymbol{u}_{xx}(0,t)\cdot\boldsymbol{u}_t(0,t)| \le |\alpha|\gamma|\boldsymbol{u}_{xx}(0,t)|^2 + \frac{5\alpha^2\varepsilon}{18\gamma}\|\boldsymbol{u}_{xxx}\|^2 + C\|\boldsymbol{g}\|^2.$$

By choosing $\frac{5}{18} < \gamma < \frac{1}{2}$, both $|\boldsymbol{u}_{xx}(0,t)|^2$ and $\|\boldsymbol{u}_{xxx}\|^2$ can be dealt with in the estimates. Combining all the above estimates, we arrive at

$$\|\boldsymbol{u}(t)\|_2^2 + \int_0^t \|\boldsymbol{u}_{xx}(r)\|_1^2 \mathrm{d}r \le C\|\boldsymbol{u}_0\|_2^2 + C\int_0^t \|\boldsymbol{g}(r)\|^2 \mathrm{d}r. \tag{3.3.9}$$

By taking the t derivative of the equation, applying the same estimate as above, and estimating $\|\partial_t^k \boldsymbol{u}(\cdot,0)\|$ as we did in the estimate of $\boldsymbol{u}_1$, we have

$$\begin{aligned}\|\partial_t^k \boldsymbol{u}(t)\|_2^2 &+ \int_0^t \|\partial_t^k \boldsymbol{u}_{xx}(r)\|_1^2 \mathrm{d}r \\ &\le C\left(\|\boldsymbol{u}_0\|_{2+2k}^2 + \sum_{j=0}^{k-1} \|\partial_t^j \boldsymbol{g}(t)\|_{2+2(k-1-j)}^2 + \int_0^t \|\partial_t^k \boldsymbol{g}(r)\|^2 \mathrm{d}r\right)\end{aligned}$$

for $k \ge 1$. By using the equation to convert the time regularity into regularity in x, we have for any k satisfying $0 \le k \le l$

$$\begin{aligned}\sup_{0\le t\le T} \|\partial_t^k \boldsymbol{u}(t)\|_{2+2(l-k)}^2 &+ \int_0^T \|\partial_t^k \boldsymbol{u}_{xx}(t)\|_{1+2(l-k)}^2 \mathrm{d}t \\ &\le C\left(\|\boldsymbol{u}_0\|_{2+2l}^2 + \sum_{j=0}^{l-1} \sup_{0\le t\le T} \|\partial_t^j \boldsymbol{g}(t)\|_{2+2(l-1-j)}^2 + \int_0^T \|\partial_t^l \boldsymbol{g}(t)\|^2 \mathrm{d}t\right).\end{aligned}$$

Up to this point, we have assumed that the given data are smooth. Through an approximation argument, we can relax the assumption on the data and prove the following.

Lemma 3.3.2 *For an arbitrary natural number l, if $\boldsymbol{u}_0 \in H^{2+2l}(\mathbf{R}_+)$, $\boldsymbol{g} \in \bigcap_{j=0}^{l-1} C^j\big([0,T]; H^{2+2(l-1-j)}(\mathbf{R}_+)\big)$, and $\partial_t^l \boldsymbol{g} \in L^2(Q_T)$ satisfy the compatibility conditions up to order l, there exists a unique solution $\boldsymbol{u}$ to* (3.1.8) *satisfying*

$$\begin{aligned}\sum_{k=0}^{l} \left(\sup_{0\le t\le T} \|\partial_t^k \boldsymbol{u}(t)\|_{2+2(l-k)}^2 + \int_0^T \|\partial_t^k \boldsymbol{u}_{xx}(t)\|_{1+2(l-k)}^2 \mathrm{d}t\right) \\ \le C\left(\|\boldsymbol{u}_0\|_{2+2l}^2 + \sum_{j=0}^{l-1} \sup_{0\le t\le T} \|\partial_t^j \boldsymbol{g}(t)\|_{2+2(l-1-j)}^2 + \int_0^T \|\partial_t^l \boldsymbol{g}(t)\|^2 \mathrm{d}t\right).\end{aligned}$$

3.4 Solving the Parabolic-Dispersive System

In this section, we construct the solution $\boldsymbol{u}$ of (3.2.1) such that for a natural number l

$$\boldsymbol{u} \in \bigcap_{j=0}^{l} \left\{C^j\big([0,T]; H^{2+2(l-j)}(\mathbf{R}_+)\big) \cap H^j\big(0,T; H^{3+2(l-j)}(\mathbf{R}_+)\big)\right\}, \tag{3.4.1}$$

by iteration. For $n \geq 1$, we define $\boldsymbol{u}^{(n)}$ as the solution of the following problem.

$$\begin{cases} \boldsymbol{u}_t^{(n)} = \alpha \boldsymbol{u}_{xxx}^{(n)} + \alpha\varepsilon \boldsymbol{u}_{tx}^{(n)} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u}^{(n-1)} + \boldsymbol{g}, & x > 0, t > 0, \\ \boldsymbol{u}^{(n)}(x, 0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}_x^{(n)}(0, t) = \boldsymbol{0}, & t > 0. \end{cases}$$

$\boldsymbol{u}^{(0)}$ must be defined in a specific way so that the compatibility conditions for each successive iteration are satisfied. First, again by the approximation argument, we will assume that $\boldsymbol{u}_0$ and $\boldsymbol{g}$ satisfy the compatibility conditions for (3.2.1) up to some fixed order N and are smooth. We introduce the following notation.

$$Q(\boldsymbol{v}) := \alpha \boldsymbol{v}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{v} + \boldsymbol{g}.$$

We define $\boldsymbol{u}^{(0)}$ as

$$\boldsymbol{u}^{(0)}(x, t) := \boldsymbol{u}_0(x) + \sum_{j=1}^{N} \frac{t^j}{j!} \left(\frac{\partial^j}{\partial t^j} Q(\boldsymbol{v}) \right)(x, 0),$$

where $\boldsymbol{v}(x, 0) := \boldsymbol{u}_0(x)$ and $\boldsymbol{\psi}_k(x) := \partial_t^k \boldsymbol{v}(x, 0)$ for $k \geq 1$ are defined as the solution of the following linear ordinary differential equation, under the constraint that $\boldsymbol{\psi}_k$ is integrable over $\mathbf{R}_+$.

$$\boldsymbol{\psi}_k' = -\frac{1}{\alpha\varepsilon}\boldsymbol{\psi}_k + \frac{1}{\alpha\varepsilon}\left(\alpha \boldsymbol{\psi}_{k-1}''' + \sum_{j=0}^{k-1} \begin{pmatrix} k-1 \\ j \end{pmatrix} \left(\partial_t^j \mathrm{A}(\boldsymbol{w}(\cdot, 0), \partial_x) \right) \boldsymbol{\psi}_{k-1-j} + \partial_t^{k-1} \boldsymbol{g}(\cdot, 0) \right).$$

N is chosen to accommodate the necessary order of compatibility conditions and regularity. By defining $\boldsymbol{u}^{(0)}$ as such, the compatibility conditions for each successive iteration are automatically satisfied. Then, Lemma 3.3.2 guarantees that $\{\boldsymbol{u}^{(n)}\}_{n=0}^{\infty}$ is well-defined. Now, we prove the convergence of $\{\boldsymbol{u}^{(n)}\}_{n=0}^{\infty}$ in the desired function space. From the way that we constructed $\boldsymbol{u}^{(0)}$ we have

$$\sum_{k=0}^{l} \sup_{0 \leq t \leq T} \|\partial_t^k \boldsymbol{u}^{(0)}(t)\|_{2+2(l-k)}^2 \leq C_0 \left(\|\boldsymbol{u}_0\|_{2+2l+3N}^2 + \sum_{j=0}^{l-1} \sup_{0 \leq t \leq T} \|\partial_t^j \boldsymbol{g}(t)\|_{2+2(l-1-j)+3N}^2 \right).$$

Setting $\boldsymbol{z}^{(n)} := \boldsymbol{u}^{(n)} - \boldsymbol{u}^{(n-1)}$ for $n = 1, 2, 3, \ldots$, we have

$$\begin{cases} \boldsymbol{z}_t^{(n)} = \alpha \boldsymbol{z}_{xxx}^{(n)} - \alpha\varepsilon \boldsymbol{z}_{tx}^{(n)} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{z}^{(n-1)}, & x > 0, t > 0, \\ \boldsymbol{z}^{(n)}(x, 0) = \boldsymbol{0}, & x > 0, \\ \boldsymbol{z}_x^{(n)}(0, t) = \boldsymbol{0}, & t > 0. \end{cases}$$

In the same way that we derived (3.3.9), we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\boldsymbol{z}^{(n)}(t)\|_2^2 + \int_0^T \|\boldsymbol{z}_{xx}^{(n)}(t)\|_1^2 \mathrm{d}t &\leq C \int_0^T \|\boldsymbol{z}_{xx}^{(n-1)}(t)\|^2 \mathrm{d}t \\ &\leq \frac{(CT)^{n-1}}{(n-1)!}. \end{aligned}$$

The above estimate proves that $\boldsymbol{u}^{(n)}$ converges in $C\big([0,T];H^2(\mathbf{R}_+)\big)\cap L^2\big(0,T;H^3(\mathbf{R}_+)\big)$. Since $\partial_t^k \boldsymbol{z}^{(n)}(x,0)=\mathbf{0}$, we can prove in the same way as above that $\partial_t^k\boldsymbol{u}^{(n)}$ converges in $C\big([0,T];H^2(\mathbf{R}_+)\big)\cap L^2\big(0,T;H^3(\mathbf{R}_+)\big)$. Using the equation we can prove that for $0\le k\le l$, $\partial_t^k\boldsymbol{u}^{(n)}$ converges in $C\left([0,T];H^{2+2(l-k)}(\mathbf{R}_+)\right)\cap L^2\big(0,T;H^{3+2(l-k)}(\mathbf{R}_+)\big)$. Thus, for an arbitrary l, we have constructed a solution of (3.2.1) satisfying (3.4.1).

Now we consider the limit $\varepsilon\to+0$ for problem (3.2.1). For this, we derive an estimate of the solution that is uniform in ε. The energy form we use is the same as the estimate we obtained before, but we use the elliptic term to make the estimate uniform in ε. We are still assuming that the given data are smooth as necessary. We estimate as follows.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\{\|\boldsymbol{u}\|^2+\alpha^2\varepsilon^2\|\boldsymbol{u}_x\|^2\big\}\le-\alpha\boldsymbol{u}(0,t)\cdot\boldsymbol{u}_{xx}(0,t)-\frac{|\alpha|\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}(0,t)|^2-\frac{\delta}{2}\|\boldsymbol{u}_x\|^2-\alpha^2\varepsilon\|\boldsymbol{u}_{xx}\|^2$$
$$+C\|\boldsymbol{u}\|_1^2+\varepsilon^2\|\mathrm{A}(\boldsymbol{w},\partial_x)\boldsymbol{u}\|^2+\|\boldsymbol{g}\|^2.$$

Here, we have used the estimate $(\mathrm{A}(\boldsymbol{w},\partial_x)\boldsymbol{u},\boldsymbol{u})\le-\frac{\delta}{2}\|\boldsymbol{u}_x\|^2+C\|\boldsymbol{u}\|^2$, which follows from the strong ellipticity of $\mathrm{A}(\boldsymbol{w},\partial_x)$. We choose $\varepsilon_1>0$ such that $\varepsilon_1\|\mathrm{A}_0(\boldsymbol{w})\|^2_{L^\infty(0,T;L^\infty(\mathbf{R}_+))}\le\frac{\alpha^2}{2}$. Then, for $0<\varepsilon\le\varepsilon_1$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\{\|\boldsymbol{u}\|^2+\alpha^2\varepsilon^2\|\boldsymbol{u}_x\|^2\big\}\le-\alpha\boldsymbol{u}(0,t)\cdot\boldsymbol{u}_{xx}(0,t)-\frac{|\alpha|\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}(0,t)|^2-\frac{\delta}{2}\|\boldsymbol{u}_x\|^2-\frac{\alpha^2\varepsilon}{2}\|\boldsymbol{u}_{xx}\|^2$$
$$+C\|\boldsymbol{u}\|_1^2+\|\boldsymbol{g}\|^2.$$

Next, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{u}_x\|^2+\alpha^2\varepsilon^2\|\boldsymbol{u}_{xx}\|^2\right\}\le&-\frac{|\alpha|}{6}|\boldsymbol{u}_{xx}(0,t)|^2-\frac{\delta}{2}\|\boldsymbol{u}_{xx}\|^2-\frac{\alpha^2\varepsilon}{12}\|\boldsymbol{u}_{xxx}\|^2\\&+\varepsilon C_0\|\mathrm{A}_0(\boldsymbol{w})\|^2_{L^\infty(0,T;L^\infty(\mathbf{R}_+))}\|\boldsymbol{u}_x\|\|\boldsymbol{u}_{xxx}\|\\&-\alpha\varepsilon(\boldsymbol{u}_{xxx},\mathrm{A}(\boldsymbol{w},\partial_x)\boldsymbol{u})+C\big(\|\boldsymbol{u}\|_1^2+\|\boldsymbol{g}\|^2\big),\end{aligned}$$

where we have used the interpolation inequality $\|\boldsymbol{u}_{xx}\|^2\le C\|\boldsymbol{u}_x\|\|\boldsymbol{u}_{xxx}\|$. Now we choose $\varepsilon_2>0$ so that $\varepsilon_2C_0\|\mathrm{A}_0(\boldsymbol{w})\|^2_{L^\infty(0,T;L^\infty(\mathbf{R}_+))}\le\frac{|\alpha|}{48}$ and $48\varepsilon_2\|\mathrm{A}_0(\boldsymbol{w})\|^2_{L^\infty(0,T;L^\infty(\mathbf{R}_+))}\le\frac{\delta}{4}$. Then, for $0\le\varepsilon\le\varepsilon_2$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{u}_x\|^2+\alpha^2\varepsilon^2\|\boldsymbol{u}_{xx}\|^2\right\}\le-\frac{|\alpha|}{6}|\boldsymbol{u}_{xx}(0,t)|^2-\frac{\delta}{4}\|\boldsymbol{u}_{xx}\|^2-\frac{\alpha^2\varepsilon}{24}\|\boldsymbol{u}_{xxx}\|^2+C\big(\|\boldsymbol{u}\|_1^2+\|\boldsymbol{g}\|^2\big).$$

Finally we estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_{xx}\|^2\le-\frac{|\alpha|}{2}|\boldsymbol{u}_{xxx}(0,t)|^2-\frac{\varepsilon}{2}\|\boldsymbol{u}_{tx}\|^2-\frac{\delta}{4}\|\boldsymbol{u}_{xxx}\|^2+C\|\boldsymbol{u}\|_2^2+\|\boldsymbol{g}_x\|^2.$$

In each estimate, the constant C is independent of $\varepsilon\in(0,\varepsilon_0]$, where $\varepsilon_0:=\min\{\varepsilon_1,\varepsilon_2\}$. Combining all the estimates yields for $0\le\varepsilon\le\varepsilon_0$,

$$\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|_2^2+\int_0^T\big(\delta\|\boldsymbol{u}_{xxx}(t)\|^2+\varepsilon\|\boldsymbol{u}_{tx}(t)\|^2+|\boldsymbol{u}_{xx}(0,t)|^2+|\boldsymbol{u}_{xxx}(0,t)|^2\big)\mathrm{d}t$$
$$\le C\left(\|\boldsymbol{u}_0\|_2^2+\int_0^T\|\boldsymbol{g}(t)\|_1^2\mathrm{d}t\right).$$

Now we take the derivative of the equation m times $(1 \leq m \leq l)$ with respect to t and set $\boldsymbol{v}_m := \partial_t^m \boldsymbol{u}$. Then, $\boldsymbol{v}_m$ satisfies

$$\begin{cases} \boldsymbol{v}_{mt} = \alpha \boldsymbol{v}_{mxxx} - \alpha\varepsilon \boldsymbol{v}_{mxt} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{v}_m + \partial_t^m \boldsymbol{g} + \boldsymbol{F}_m, & x > 0, t > 0, \\ \boldsymbol{v}_m(x, 0) = \boldsymbol{\phi}_m(x), & x > 0, \\ \boldsymbol{v}_{mx}(0, t) = \boldsymbol{0}, & t > 0, \end{cases}$$

where $\boldsymbol{F}_m = \sum_{j=0}^{m-1} \begin{pmatrix} m-1 \\ j \end{pmatrix} \big(\partial_t^{m-1-j} \mathrm{A}(\boldsymbol{w}, \partial_x)\big) \boldsymbol{v}_j$. We derive the uniform estimate by induction on m. The case $m = 0$ was just derived. Suppose that for $0 \leq j \leq m-1$

$$\begin{aligned} &\sup_{0 \leq t \leq T} \|\boldsymbol{v}_j(t)\|_2^2 + \delta \int_0^T \|\boldsymbol{v}_{jxxx}(t)\|^2 \mathrm{d}t \\ &\leq C \left\{ \|\boldsymbol{u}_0\|_{2+3j}^2 + \sup_{0 \leq t \leq T} \left(\|\partial_t^{j-1} \boldsymbol{g}(\cdot, t)\|_2^2 + \sum_{k=0}^{j-2} \|\partial_t^k \boldsymbol{g}(\cdot, t)\|_{2+3(m-2-k)}^2 \right) + \int_0^T \|\partial_t^j \boldsymbol{g}(t)\|_1^2 \mathrm{d}t \right\} \end{aligned}$$

holds with C independent of ε. Estimating in the same way as before, we have

$$\sup_{0 \leq t \leq T} \|\boldsymbol{v}_m(t)\|_2^2 + \delta \int_0^T \|\boldsymbol{v}_{mxxx}(t)\|^2 \mathrm{d}t \leq C \left(\|\boldsymbol{\phi}_m\|_2^2 + \int_0^T \big(\|\partial_t^m \boldsymbol{g}(t)\|_1^2 + \|\boldsymbol{F}_m(t)\|_1^2 \big) \mathrm{d}t \right).$$

Now we estimate each term on the right-hand side.

$$\|\boldsymbol{F}_m(t)\|_1^2 \leq C \sum_{j=0}^{m-1} \|\boldsymbol{v}_j(t)\|_3^2,$$

where C depends on the norm of $\boldsymbol{w}$ in $\bigcap_{j=0}^{m-1} W^{j,\infty}\big(0, T; H^1(\mathbf{R}_+)\big)$. The expression for $\boldsymbol{\phi}_m$ and its derivatives are

$$\partial_x^k \boldsymbol{\phi}_m(x) = -\frac{1}{\alpha\varepsilon} \int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)} \partial_y^k \big\{ \alpha \boldsymbol{\phi}_{m-1}'''(y) + \boldsymbol{F}_{m-1}(y, 0) + \partial_t^{m-1} \boldsymbol{g}(y, 0) \big\} \mathrm{d}y.$$

Through direct calculation, we see that

$$\left\| -\frac{1}{\alpha\varepsilon} \int_\cdot^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(\cdot - y)} \boldsymbol{\Phi}(y) \mathrm{d}y \right\| \leq \|\boldsymbol{\Phi}\|.$$

Thus, we can prove that

$$\|\boldsymbol{\phi}_m\|_2 \leq C \left(\|\boldsymbol{u}_0\|_{2+3m} + \|\partial_t^{m-1} \boldsymbol{g}(\cdot, 0)\|_2 + \sum_{j=0}^{m-2} \|\partial_t^j \boldsymbol{g}(\cdot, 0)\|_{2+3(m-2-j)} \right).$$

Here, C depends on the norm of $\boldsymbol{w}$ in $\bigcap_{j=0}^{m-1} C^j\big([0, T]; H^{2+3(m-1-j)}(\mathbf{R}_+)\big)$ and the norm of $\partial_t^m \boldsymbol{w}$ in $L^\infty\big(0, T; H^1(\mathbf{R}_+)\big)$, but is independent of ε. Combining these estimates and using the equation yield

$$
\begin{aligned}
(3.4.2)\qquad &\sum_{j=0}^{l}\Big\{\sup_{0\le t\le T}\|\partial_t^j\boldsymbol{u}(t)\|_{2+3(l-j)}^2+\delta\int_0^T\|\partial_t^j\boldsymbol{u}(t)\|_{3+3(l-j)}^2\mathrm{d}t\Big\}\\
&\le C\Big\{\|\boldsymbol{u}_0\|_{2+3l}^2+\sum_{j=0}^{l-1}\Big(\sup_{0\le t\le T}\|\partial_t^j\boldsymbol{g}(t)\|_{2+3(l-1-j)}^2+\int_0^T\|\partial_t^j\boldsymbol{g}(t)\|_{3+3(l-1-j)}^2\mathrm{d}t\Big)\\
&\qquad\qquad+\int_0^T\|\partial_t^l\boldsymbol{g}(t)\|_1^2\mathrm{d}t\Big\}.
\end{aligned}
$$

Again, we emphasize that C is independent of ε. Now we denote the solution of (3.2.1) as $\boldsymbol{u}^\varepsilon$ to emphasize that the solution depends on ε. We also recall that $\boldsymbol{g}$ was a correction of $\boldsymbol{f}$ which depends on ε, so we denote it as $\boldsymbol{g}^\varepsilon$. We assume that $\boldsymbol{u}_0\in H^{2+3N}(\mathbf{R}_+)$, $\boldsymbol{g}^\varepsilon\in Y_T^N$, $\boldsymbol{w}\in Z_T^N$ for $N>l+1$, and $\boldsymbol{g}^\varepsilon\to\boldsymbol{f}$ in Y_T^{l+1}. Thus, we know the existence of a unique solution $\boldsymbol{u}^\varepsilon\in X_T^{l+1}$ of (3.2.1) with a uniform bound in X_T^{l+1}. For $0<\varepsilon<\varepsilon'\le\varepsilon_0$, we set $\boldsymbol{z}:=\boldsymbol{u}^{\varepsilon'}-\boldsymbol{u}^\varepsilon$. Then, $\boldsymbol{z}$ satisfies

$$
\begin{cases}
\boldsymbol{z}_t=\alpha\boldsymbol{z}_{xxx}-\alpha\varepsilon'\boldsymbol{z}_{tx}+\mathrm{A}(\boldsymbol{w},\partial_x)\boldsymbol{z}-\alpha(\varepsilon'-\varepsilon)\boldsymbol{u}_{tx}^\varepsilon+\boldsymbol{g}^{\varepsilon'}-\boldsymbol{g}^\varepsilon, & x>0,t>0,\\
\boldsymbol{z}(x,0)=\mathbf{0}, & x>0,\\
\boldsymbol{z}_x(0,t)=\mathbf{0}, & t>0.
\end{cases}
$$

From (3.4.2), we have

$$
\begin{aligned}
\|\boldsymbol{z}\|_{X_T^l}^2&\le C(\varepsilon'+\varepsilon)^2\Big\{\sum_{j=0}^{l-1}\Big(\sup_{0\le t\le T}\|\partial_t^{j+1}\boldsymbol{u}_x^\varepsilon(t)\|_{2+3(l-1-j)}^2+\int_0^T\|\partial_t^{j+1}\boldsymbol{u}_x^\varepsilon(t)\|_{3+3(l-1-j)}^2\mathrm{d}t\Big)\\
&\qquad+\int_0^T\|\partial_t^{l+1}\boldsymbol{u}_x^\varepsilon(t)\|_1^2\mathrm{d}t\Big\}+\|(\boldsymbol{g}^{\varepsilon'}-\boldsymbol{g}^\varepsilon)(t)\|_{Y_T^{l+1}}^2\\
&\le C(\varepsilon'+\varepsilon)^2+\|(\boldsymbol{g}^{\varepsilon'}-\boldsymbol{g}^\varepsilon)(t)\|_{Y_T^{l+1}}^2.
\end{aligned}
$$

Thus we see that there exists a $\boldsymbol{u}$ such that $\boldsymbol{u}^\varepsilon\to\boldsymbol{u}$ in X_T^l, and $\boldsymbol{u}$ is a solution of (3.1.1). We derive an energy estimate for $\boldsymbol{u}$ to prove the uniqueness of the solution. Through a standard energy estimate, we obtain the following.

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2&\le-\alpha\boldsymbol{u}(0,t)\cdot\boldsymbol{u}_{xx}(0,t)-\frac{\delta}{2}\|\boldsymbol{u}_{xx}\|^2+C\big(\|\boldsymbol{u}\|^2+\|\boldsymbol{f}\|^2\big),\\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_x\|^2&\le-\frac{|\alpha|}{2}|\boldsymbol{u}_{xx}(0,t)|^2-\frac{\delta}{2}\|\boldsymbol{u}_{xx}\|^2+C\big(\|\boldsymbol{u}\|_1^2+\|\boldsymbol{f}\|^2\big),\\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_{xx}\|^2&\le-\frac{|\alpha|}{2}|\boldsymbol{u}_{xxx}(0,t)|^2-\frac{\delta}{2}\|\boldsymbol{u}_{xxx}\|^2+C\big(\|\boldsymbol{u}\|_2^2+\|\boldsymbol{f}_x\|^2\big).
\end{aligned}
$$

Combining these estimates, we have

$$
\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|_2^2+\int_0^T\|\boldsymbol{u}_x(t)\|_2^2\mathrm{d}t\le C\Big(\|\boldsymbol{u}_0\|_2^2+\int_0^T\|\boldsymbol{f}(t)\|_1^2\mathrm{d}t\Big).
$$

As before, taking the derivative with respect to t of the equation, applying the above estimate, and converting the regularity in t into x via the equation, we have

$$\|\boldsymbol{u}\|_{X_T^l} \leq C\big(\|\boldsymbol{u}_0\|_{2+3l} + \|\boldsymbol{f}\|_{Y_T^l}\big). \tag{3.4.3}$$

Here, C depends on $\|\boldsymbol{w}\|_{Z_T^l}$, T, and δ.

As in Lemma 3.3.2, we can relax the condition on the given data by taking approximating sequences $\{\boldsymbol{u}_{0n}\}_{n=0}^{\infty}$ in H^{2+3N}, $\{\boldsymbol{f}_n\}_{n=0}^{\infty}$ in Y_T^N, and $\{\boldsymbol{w}_n\}_{n=0}^{\infty}$ in Z_T^N with $\boldsymbol{u}_{0n} \to \boldsymbol{u}_0$ in $H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f}_n \to \boldsymbol{f}$ in Y_T^l, and $\boldsymbol{w}_n \to \boldsymbol{w}$ in Z_T^l. Applying (3.4.3), and passing to the limit, we arrive at the first main theorem. The proof of Theorem 3.1.1 is complete.

3.5 Remark on the Case $\alpha > 0$

The case $\alpha > 0$ can be treated by a standard argument. We start by considering the following regularized problem.

$$\begin{cases} \boldsymbol{u}_t = -\varepsilon \boldsymbol{u}_{xxxx} + \boldsymbol{g}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}(0,t) = \boldsymbol{e}, & t > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0. \end{cases} \tag{3.5.1}$$

As before, we explicitly construct the solution of (3.5.1) in the form $\boldsymbol{u} = \boldsymbol{u}^1 + \boldsymbol{u}^2$, where $\boldsymbol{u}^1$ is defined as the solution of

$$\begin{cases} \boldsymbol{u}^1_t = -\varepsilon \boldsymbol{u}^1_{xxxx} + \boldsymbol{G}, & x \in \mathbf{R}, t > 0, \\ \boldsymbol{u}^1(x,0) = \boldsymbol{U}_0(x), & x \in \mathbf{R}, \end{cases}$$

and $\boldsymbol{u}^2$ is defined as the solution of

$$\begin{cases} \boldsymbol{u}^2_t = -\varepsilon \boldsymbol{u}^2_{xxxx}, & x > 0, t > 0, \\ \boldsymbol{u}^2(x,0) = \boldsymbol{0}, & x > 0, \\ \boldsymbol{u}^2(0,t) = \boldsymbol{e} - \boldsymbol{u}^1(0,t), & t > 0, \\ \boldsymbol{u}^2_x(0,t) = -\boldsymbol{u}^1_x(0,t), & t > 0. \end{cases}$$

The solutions can be constructed by using Fourier transform and Laplace transform as in the case $\alpha < 0$. We note that in estimating $\boldsymbol{u}^2$, we slightly modify the Sobolev–Slobodetskiĭ space for the fourth order parabolic system. For an integer m, we define the space $H_h^{m,m/4}(Q_T)$ analogous to $H_h^{m,m/2}(Q_T)$, and we use the case $m = 4l$ and the norm

$$\|\boldsymbol{u}\|^2_{H_h^{4l,l}(Q_T)} = \sum_{j=0}^{l} \int_{-\infty}^{\infty} \left\| \frac{\partial^{4j} \tilde{\boldsymbol{u}}}{\partial x^{4j}}(\cdot, \tau) \right\|^2 |\tau|^{l-j} \mathrm{d}\eta.$$

Then we construct the solution $\boldsymbol{u} \in \bigcap_{j=0}^{l} C^j\big([0,T]; H^{2+4(l-j)}(\mathbf{R}_+)\big) \cap H^j\big(0,T; H^{3+4(l-j)}(\mathbf{R}_+)\big)$ of

$$\begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} - \varepsilon \boldsymbol{u}_{xxxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}(0,t) = \boldsymbol{e}, & t > 0, \\ \boldsymbol{u}_x(0,t) = \mathbf{0}, & t > 0 \end{cases}$$

through iteration. Now we need an estimate uniform in ε. Via energy method, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2 \le -\alpha \boldsymbol{u}(0,t)\cdot \boldsymbol{u}_{xx}(0,t) + C\|\boldsymbol{u}\|_2^2 + \varepsilon \boldsymbol{u}(0,t)\cdot \boldsymbol{u}_{xxx}(0,t) - \delta\|\boldsymbol{u}_{xx}\|^2 + \|\boldsymbol{f}\|^2,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_x\|^2 \le \frac{\alpha}{2}|\boldsymbol{u}_{xx}(0,t)|^2 - \varepsilon\|\boldsymbol{u}_{xxx}\|^2 - \delta\|\boldsymbol{u}_{xx}\|^2 + \varepsilon \boldsymbol{u}_{xx}(0,t)\cdot \boldsymbol{u}_{xxx}(0,t) + C\|\boldsymbol{u}_{xx}\|^2 + \|\boldsymbol{f}\|^2,$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_{xx}\|^2 \le{}& \frac{\alpha}{2}|\boldsymbol{u}_{xxx}(0,t)|^2 - \varepsilon\|\partial_x^4\boldsymbol{u}\|^2 - \delta\|\boldsymbol{u}_{xxx}\|^2 - \varepsilon \boldsymbol{u}_{xxx}(0,t)\cdot \partial_x^4\boldsymbol{u}(0,t) \\ & - \big(\boldsymbol{u}_{xxx}, (\partial_x \mathrm{A}_0(\boldsymbol{w}))\boldsymbol{u}_{xx}\big) - \big(\boldsymbol{u}_{xxx}, \boldsymbol{f}_x\big) + C\|\boldsymbol{u}\|_2^2.\end{aligned}$$

Using the equation, we can also obtain

$$-\varepsilon \boldsymbol{u}_{xxx}(0,t)\cdot \partial_x^4\boldsymbol{u}(0,t) = -\alpha|\boldsymbol{u}_{xxx}(0,t)|^2 - \boldsymbol{u}_{xxx}(0,t)\cdot \big(\mathrm{A}(\boldsymbol{w},\partial_x)\boldsymbol{u}\big)(0,t) - \boldsymbol{u}_{xxx}(0,t)\cdot \boldsymbol{f}(0,t).$$

From the above estimate, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_{xx}\|^2 \le -\frac{\alpha}{4}|\boldsymbol{u}_{xxx}(0,t)|^2 - \varepsilon\|\partial_x^4\boldsymbol{u}\|^2 - \frac{\delta}{4}\|\boldsymbol{u}_{xxx}\|^2 + C\big(\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{f}\|_1^2\big),$$

which combined with the other two estimates yields

$$\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|_2^2 + \int_0^T \big(\varepsilon\|\boldsymbol{u}_{xx}(t)\|_2^2 + \delta\|\boldsymbol{u}_x(t)\|_2^2\big)\mathrm{d}t \le C\left\{\|\boldsymbol{u}_0\|_2^2 + \int_0^T \|\boldsymbol{f}(t)\|_1^2\mathrm{d}t\right\},$$

where C is independent of ε. Taking the t derivatives of the equation and estimating in the same way as above, we have for $0 \le m \le l$,

$$\begin{aligned}&\sup_{0\le t\le T}\|\partial_t^m\boldsymbol{u}(t)\|_{2+3(l-m)}^2 + \int_0^T \|\partial_t^m\boldsymbol{u}_x(t)\|_{2+3(l-m)}^2\mathrm{d}t \\ &\qquad \le C\left\{\|\boldsymbol{u}_0\|_{2+4l}^2 + \sum_{j=0}^{l-1}\|\partial_t^j\boldsymbol{f}(\cdot,0)\|_{2+4(l-1-j)}^2 + \sum_{j=0}^{l}\int_0^T \|\partial_t^j\boldsymbol{f}(t)\|_1^2\mathrm{d}t\right\}.\end{aligned}$$

After passing to the limit $\varepsilon \to +0$, we obtain the solution of the limit problem. Similarly to the above, we see that the solution satisfies for $0 \le m \le l$,

$$\begin{aligned}&\sup_{0\le t\le T}\|\partial_t^m\boldsymbol{u}(t)\|_{2+3(l-m)}^2 + \int_0^T \|\partial_t^m\boldsymbol{u}_x(t)\|_{2+3(l-m)}^2\mathrm{d}t \\ &\qquad \le C\left\{\|\boldsymbol{u}_0\|_{2+3l}^2 + \sum_{j=0}^{l-1}\|\partial_t^j\boldsymbol{f}(\cdot,0)\|_{2+3(l-1-j)}^2 + \sum_{j=0}^{l}\int_0^T \|\partial_t^j\boldsymbol{f}(t)\|_1^2\mathrm{d}t\right\}.\end{aligned}$$

Thus, we have proven Theorem 3.1.2.

Chapter 4

Motion of a Vortex Filament with Axial Flow in the Half Space

4.1 Problem Setting

In this chapter, we prove the unique solvability locally in time of the following initial-boundary value problems. For $\alpha < 0$,

$$
(4.1.1) \qquad \begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + \alpha \Big\{ \boldsymbol{x}_{sss} + \dfrac{3}{2} \boldsymbol{x}_{ss} \times \big(\boldsymbol{x}_s \times \boldsymbol{x}_{ss} \big) \Big\}, & s > 0, t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_{ss}(0,t) = \boldsymbol{0}, & t > 0. \end{cases}
$$

For $\alpha > 0$,

$$
(4.1.2) \qquad \begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + \alpha \Big\{ \boldsymbol{x}_{sss} + \dfrac{3}{2} \boldsymbol{x}_{ss} \times \big(\boldsymbol{x}_s \times \boldsymbol{x}_{ss} \big) \Big\}, & s > 0, t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_s(0,t) = \boldsymbol{e}_3, & t > 0, \\ \boldsymbol{x}_{ss}(0,t) = \boldsymbol{0}, & t > 0. \end{cases}
$$

Here, $\boldsymbol{x}(s,t) = (x^1(s,t), x^2(s,t), x^3(s,t))$ is the position vector of the vortex filament parameterized by its arc length s at time t, the symbol $\times$ is the exterior product in the three dimensional Euclidean space, α is a non-zero constant that describes the magnitude of the effect of axial flow, $\boldsymbol{e}_3 = (0,0,1)$, and subscripts denote derivatives with their respective variables. Later in this chapter, we will also use ∂_s and ∂_t to denote partial derivatives as well. We will refer to the equation in (4.1.1) and (4.1.2) as the vortex filament equation. We recall that the number of boundary conditions depends on the sign of α as is the case for the KdV and KdV–Burgers equation. We prove the unique solvability of the problems locally in time of (4.1.1) and (4.1.2) based on the existence theorems we obtained in the previous chapter.

For convenience, we introduce a new variable $\boldsymbol{v}(s,t) := \boldsymbol{x}_s(s,t)$ and rewrite the problems in terms of $\boldsymbol{v}$. Setting $\boldsymbol{v}_0(s) := \boldsymbol{x}_{0s}(s)$, we have for $\alpha < 0$,

$$
(4.1.3)\begin{cases}
\boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\Big\{\boldsymbol{v}_{sss} + \dfrac{3}{2}\boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s\big) & \\
\qquad\qquad\qquad\qquad + \dfrac{3}{2}\boldsymbol{v}_s \times \big(\boldsymbol{v} \times \boldsymbol{v}_{ss}\big)\Big\}, & s>0, t>0,\\
\boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s>0,\\
\boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t>0.
\end{cases}
$$

For $\alpha > 0$,

$$
(4.1.4)\begin{cases}
\boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\Big\{\boldsymbol{v}_{sss} + \dfrac{3}{2}\boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s\big) & \\
\qquad\qquad\qquad\qquad + \dfrac{3}{2}\boldsymbol{v}_s \times \big(\boldsymbol{v} \times \boldsymbol{v}_{ss}\big)\Big\}, & s>0, t>0,\\
\boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s>0,\\
\boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t>0,\\
\boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t>0.
\end{cases}
$$

Once we obtain a solution for (4.1.3) and (4.1.4), we can construct $\boldsymbol{x}(s,t)$ from the formula

$$
\boldsymbol{x}(s,t) = \boldsymbol{x}_0(s) + \int_0^t \Big\{\boldsymbol{v} \times \boldsymbol{v}_s + \alpha\boldsymbol{v}_{ss} + \frac{3}{2}\alpha\boldsymbol{v}_s \times \big(\boldsymbol{v} \times \boldsymbol{v}_s\big)\Big\}(s,\tau)\mathrm{d}\tau,
$$

and $\boldsymbol{x}(s,t)$ will satisfy (4.1.1) and (4.1.2) respectively, in other words, (4.1.1) is equivalent to (4.1.3) and (4.1.2) is equivalent to (4.1.4). Hence, we will concentrate on the solvability of (4.1.3) and (4.1.4). Our approach for solving (4.1.3) and (4.1.4) is to rewrite the nonlinear problem utilizing the property $|\boldsymbol{v}| = 1$ and linearizing the equation. We rewrite the nonlinear problem first because if we directly linearize the equation around a function $\boldsymbol{w}$ we have

$$
\boldsymbol{v}_t = \boldsymbol{w} \times \boldsymbol{v}_{ss} + \alpha\Big\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss} \times \big(\boldsymbol{w} \times \boldsymbol{w}_s\big) + \frac{3}{2}\boldsymbol{w}_s \times \big(\boldsymbol{w} \times \boldsymbol{v}_{ss}\big)\Big\}.
$$

Directly considering the initial-boundary value problem for the above equation seems hard. When we try to estimate the solution in Sobolev spaces, the term $\boldsymbol{w}_s \times \big(\boldsymbol{w} \times \boldsymbol{v}_{ss}\big)$ causes a loss of regularity because of the form of the coefficient. We were able to overcome this by using the fact that if the initial datum is parameterized by its arc length, i.e. $|\boldsymbol{v}_0| \equiv 1$, a sufficiently smooth solution of (4.1.3) or (4.1.4) satisfies $|\boldsymbol{v}| \equiv 1$. The same property was proved in Nishiyama and Tani [33] for the initial value problem. This allows us to use the identity

$$
\boldsymbol{v}_s \times \big(\boldsymbol{v} \times \boldsymbol{v}_{ss}\big) = \boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s\big) - |\boldsymbol{v}_s|^2\boldsymbol{v}_s.
$$

Linearizing the equation in (4.1.3) and (4.1.4) after using the above identity yields

$$\boldsymbol{v}_t = \boldsymbol{w} \times \boldsymbol{v}_{ss} + \alpha\big\{\boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss} \times \big(\boldsymbol{w} \times \boldsymbol{w}_s\big)\big\} + \boldsymbol{f}. \tag{4.1.5}$$

The term causing the loss of regularity is gone, but still, the existence of a solution to the initial-boundary value problem of the above third order dispersive equation is not trivial. We can overcome this point by utilizing the results of the previous chapter.

Finally, we state the main existence theorems here.

Theorem 4.1.1 (M. Aiki and T. Iguchi [2]) *For $\alpha < 0$ and a natural number k, if $\boldsymbol{x}_{0ss} \in H^{1+3k}(\mathbf{R}_+)$, $|\boldsymbol{x}_{0s}| \equiv 1$, and $\boldsymbol{x}_{0s}$ satisfies the compatibility conditions for* (4.2.3) *up to order k, then there exists $T > 0$ such that* (4.1.1) *has a unique solution $\boldsymbol{x}$ satisfying*

$$\boldsymbol{x}_{ss} \in \bigcap_{j=0}^{k} W^{j,\infty}\big(0,T; H^{1+3j}(\mathbf{R}_+)\big)$$

and $|\boldsymbol{x}_s| \equiv 1$. Here, T depends on $\|\boldsymbol{x}_{0ss}\|_3$.

Theorem 4.1.2 (M. Aiki and T. Iguchi [2]) *For $\alpha > 0$ and a natural number k, if $\boldsymbol{x}_{0ss} \in H^{2+3k}(\mathbf{R}_+)$, $|\boldsymbol{x}_{0s}| \equiv 1$, and $\boldsymbol{x}_{0s}$ satisfies the compatibility conditions for* (4.2.2) *up to order k, then there exists $T > 0$ such that* (4.1.2) *has a unique solution $\boldsymbol{x}$ satisfying*

$$\boldsymbol{x}_{ss} \in \bigcap_{j=0}^{k} W^{j,\infty}\big(0,T; H^{2+3j}(\mathbf{R}_+)\big)$$

and $|\boldsymbol{x}_s| \equiv 1$. Here, T depends on $\|\boldsymbol{x}_{0ss}\|_2$.

The contents of this chapter are as follows. In Section 4.2, we consider the compatibility conditions for regularized nonlinear problems and the necessary corrections of the initial datum. In Section 4.3, we review the existence theorems obtained in Chapter 3, which will be applied to the nonlinear problems. In Section 4.4, we prove an existence theorem for the case $\alpha < 0$, and in Section 4.5, we prove an existence theorem for the case $\alpha > 0$.

4.2 Regularized Nonlinear Problem and its Compatibility Conditions

We consider the following problems: for $\alpha < 0$,

$$\begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\Big\{\boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s\big) - \dfrac{3}{2}|\boldsymbol{v}_s|^2\boldsymbol{v}_s\Big\}, & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t > 0; \end{cases} \tag{4.2.1}$$

for $\alpha > 0$,

$$
(4.2.2) \begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha \Big\{ \boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s \big) - \dfrac{3}{2} |\boldsymbol{v}_s|^2 \boldsymbol{v}_s \Big\}, & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t > 0, \\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t > 0, \end{cases}
$$

which are equivalent to (4.1.1) and (4.1.2) with $|\boldsymbol{v}| \equiv 1$ respectively, and construct the solutions by passing to the limit $\delta \to +0$ in the following regularized problems: for $\alpha < 0$,

$$
(4.2.3) \begin{cases} \boldsymbol{v}^\delta_t = \boldsymbol{v}^\delta \times \boldsymbol{v}^\delta_{ss} + \alpha \Big\{ \boldsymbol{v}^\delta_{sss} + 3\boldsymbol{v}^\delta_{ss} \times \big(\boldsymbol{v}^\delta \times \boldsymbol{v}^\delta_s \big) \\ \qquad\qquad - \dfrac{3}{2} |\boldsymbol{v}^\delta_s|^2 \boldsymbol{v}^\delta_s \Big\} + \delta \big(\boldsymbol{v}^\delta_{ss} + |\boldsymbol{v}^\delta_s|^2 \boldsymbol{v}^\delta \big), & s > 0, t > 0, \\ \boldsymbol{v}^\delta(s,0) = \boldsymbol{v}^\delta_0(s), & s > 0, \\ \boldsymbol{v}^\delta_s(0,t) = \boldsymbol{0}, & t > 0, \end{cases}
$$

for $\alpha > 0$,

$$
(4.2.4) \begin{cases} \boldsymbol{v}^\delta_t = \boldsymbol{v}^\delta \times \boldsymbol{v}^\delta_{ss} + \alpha \Big\{ \boldsymbol{v}^\delta_{sss} + 3\boldsymbol{v}^\delta_{ss} \times \big(\boldsymbol{v}^\delta \times \boldsymbol{v}^\delta_s \big) \\ \qquad\qquad - \dfrac{3}{2} |\boldsymbol{v}^\delta_s|^2 \boldsymbol{v}^\delta_s \Big\} + \delta \big(\boldsymbol{v}^\delta_{ss} + |\boldsymbol{v}^\delta_s|^2 \boldsymbol{v}^\delta \big), & s > 0, t > 0, \\ \boldsymbol{v}^\delta(s,0) = \boldsymbol{v}^\delta_0(s), & s > 0, \\ \boldsymbol{v}^\delta(0,t) = \boldsymbol{e}_3, & t > 0, \\ \boldsymbol{v}^\delta_s(0,t) = \boldsymbol{0}, & t > 0. \end{cases}
$$

For the proceeding analysis, we assume that $|\boldsymbol{v}^\delta_0| \equiv 1$ holds, i.e. the initial datum is parameterized by its arc length. Since we modified the problem, we must make corrections to the initial datum to ensure that the compatibility conditions hold for each problem.

4.2.1 Compatibility Conditions for (4.2.1) and (4.2.2)

First, we derive the compatibility conditions for (4.2.1) and (4.2.2). We set $\boldsymbol{Q}_{(0)}(\boldsymbol{v}) = \boldsymbol{v}$ and

$$
\boldsymbol{Q}_{(1)}(\boldsymbol{v}) = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha \Big\{ \boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s \big) - \frac{3}{2} |\boldsymbol{v}_s|^2 \boldsymbol{v}_s \Big\}.
$$

We will also use the notations $\boldsymbol{Q}_{(1)}(s,t)$ and $\boldsymbol{Q}_{(1)}$ instead of $\boldsymbol{Q}_{(1)}(\boldsymbol{v})$ for convenience. For $n \geq 2$, we successively define $\boldsymbol{Q}_{(n)}$ as

$$\begin{aligned}
\boldsymbol{Q}_{(n)} &= \sum_{j=0}^{n-1} \begin{pmatrix} n-1 \\ j \end{pmatrix} \boldsymbol{Q}_{(j)} \times \boldsymbol{Q}_{(n-1-j)ss} + \alpha \boldsymbol{Q}_{(n-1)sss} \\
&\quad + 3\alpha \Biggl\{ \sum_{j=0}^{n-1} \sum_{k=0}^{n-1-j} \begin{pmatrix} n-1 \\ j \end{pmatrix} \begin{pmatrix} n-1-j \\ k \end{pmatrix} \boldsymbol{Q}_{(j)ss} \times \Bigl(\boldsymbol{Q}_{(k)} \times \boldsymbol{Q}_{(n-1-j-k)s} \Bigr) \Biggr\} \\
&\quad - \frac{3}{2}\alpha \left\{ \sum_{j=0}^{n-1} \sum_{k=0}^{n-1-j} \begin{pmatrix} n-1 \\ j \end{pmatrix} \begin{pmatrix} n-1-j \\ k \end{pmatrix} \left(\boldsymbol{Q}_{(j)s} \cdot \boldsymbol{Q}_{(k)s} \right) \boldsymbol{Q}_{(n-1-j-k)s} \right\}.
\end{aligned}$$

The above definition of $\boldsymbol{Q}_{(n)}(\boldsymbol{v})$ gives an expression for $\partial_t^n \boldsymbol{v}$ in terms of $\boldsymbol{v}$ and its s derivatives only. It is obvious that the term with the highest order derivative in $\boldsymbol{Q}_{(n)}$ is $\alpha^n \partial_s^{3n} \boldsymbol{v}$. From the boundary conditions of (4.2.1) and (4.2.2), we arrive at the following compatibility conditions.

Definition 4.2.1 (Compatibility conditions for (4.2.1)) *For $n \in \mathbf{N} \cup \{0\}$, we say that $\boldsymbol{v}_0$ satisfies the n-th order compatibility condition for* (4.2.1) *if $\boldsymbol{v}_{0s} \in H^{1+3n}(\mathbf{R}_+)$ and*

$$\bigl(\partial_s \boldsymbol{Q}_{(n)}(\boldsymbol{v}_0)\bigr)(0) = \mathbf{0}.$$

We also say that $\boldsymbol{v}_0$ satisfies the compatibility conditions for (4.2.1) *up to order n if it satisfies the k-th order compatibility condition for all k with $0 \leq k \leq n$.*

Definition 4.2.2 (Compatibility conditions for (4.2.2)) *For $n \in \mathbf{N} \cup \{0\}$, we say that $\boldsymbol{v}_0$ satisfies the n-th order compatibility condition for* (4.2.2) *if $\boldsymbol{v}_{0s} \in H^{2+3n}(\mathbf{R}_+)$ and*

$$\boldsymbol{v}_0(0) = \boldsymbol{e}_3, \ \ \boldsymbol{v}_{0s}(0) = \mathbf{0},$$

when $n = 0$, and

$$\bigl(\boldsymbol{Q}_{(n)}(\boldsymbol{v}_0)\bigr)(0) = \mathbf{0}, \ \ \bigl(\partial_s \boldsymbol{Q}_{(n)}(\boldsymbol{v}_0)\bigr)(0) = \mathbf{0},$$

when $n \geq 1$. We also say that $\boldsymbol{v}_0$ satisfies the compatibility conditions for (4.2.2) *up to order n if it satisfies the k-th order compatibility condition for all k with $0 \leq k \leq n$.*

Note that the regularity imposed on $\boldsymbol{v}_{0s}$ in Definition 4.2.2 is not the minimal regularity required for the trace at $s = 0$ to have meaning, but we defined it as above so that it coincides with the regularity assumption in the existence theorem that we obtain later. Also note that the regularity assumption is made on $\boldsymbol{v}_{0s}$ instead of $\boldsymbol{v}_0$ because $|\boldsymbol{v}_0| = 1$ and thus, $\boldsymbol{v}_0$ is not square integrable.

4.2.2 Compatibility Conditions for (4.2.3) and (4.2.4)

We derive the compatibility conditions for (4.2.3) and (4.2.4) in the same way as those for (4.2.1) and (4.2.2). Set $\boldsymbol{P}_{(0)}(\boldsymbol{v}) = \boldsymbol{v}$ and define $\boldsymbol{P}_{(1)}(\boldsymbol{v})$ by

$$\boldsymbol{P}_{(1)}(\boldsymbol{v}) = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\Big\{\boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s\big) - \frac{3}{2}|\boldsymbol{v}_s|^2\boldsymbol{v}_s\Big\} + \delta\big(\boldsymbol{v}_{ss} + |\boldsymbol{v}_s|^2\boldsymbol{v}\big).$$

We successively define $\boldsymbol{P}_{(n)}$ for $n \geq 2$ by

$$\begin{aligned}
\boldsymbol{P}_{(n)} &= \sum_{j=0}^{n-1} \binom{n-1}{j} \boldsymbol{P}_{(j)} \times \boldsymbol{P}_{(n-1-j)ss} + \alpha \boldsymbol{P}_{(n-1)sss} \\
&\quad + 3\alpha\Big\{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j} \binom{n-1}{j}\binom{n-1-j}{k} \boldsymbol{P}_{(j)ss} \times \Big(\boldsymbol{P}_{(k)} \times \boldsymbol{P}_{(n-1-j-k)s}\Big)\Big\} \\
&\quad - \frac{3}{2}\alpha\left\{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j} \binom{n-1}{j}\binom{n-1-j}{k} \big(\boldsymbol{P}_{(j)s} \cdot \boldsymbol{P}_{(k)s}\big)\, \boldsymbol{P}_{(n-1-j-k)s}\right\} \\
&\quad + \delta\left\{\boldsymbol{P}_{(n-1)ss} + \sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j} \binom{n-1}{j}\binom{n-1-j}{k} \big(\boldsymbol{P}_{(j)s} \cdot \boldsymbol{P}_{(k)s}\big)\, \boldsymbol{P}_{(n-1-j-k)}\right\}.
\end{aligned}$$

We arrive at the following compatibility conditions.

Definition 4.2.3 (Compatibility conditions for (4.2.3)) *For $n \in \mathbf{N} \cup \{0\}$, we say that $\boldsymbol{v}_0^\delta$ satisfies the n-th order compatibility condition for* (4.2.3) *if $\boldsymbol{v}_{0s}^\delta \in H^{1+3n}(\mathbf{R}_+)$ and*

$$\big(\partial_s \boldsymbol{P}_{(n)}(\boldsymbol{v}_0^\delta)\big)(0) = \mathbf{0}.$$

We also say that $\boldsymbol{v}_0^\delta$ satisfies the compatibility conditions for (4.2.3) *up to order n if it satisfies the k-th order compatibility condition for all k with $0 \leq k \leq n$.*

Definition 4.2.4 (Compatibility conditions for (4.2.4)) *For $n \in \mathbf{N} \cup \{0\}$, we say that $\boldsymbol{v}_0^\delta$ satisfies the n-th order compatibility condition for* (4.2.4) *if $\boldsymbol{v}_{0s}^\delta \in H^{2+3n}(\mathbf{R}_+)$ and*

$$\boldsymbol{v}_0^\delta(0) = \boldsymbol{e}_3,\ \boldsymbol{v}_{0s}^\delta(0) = \mathbf{0},$$

when $n = 0$, and

$$\big(\boldsymbol{P}_{(n)}(\boldsymbol{v}_0^\delta)\big)(0) = \mathbf{0},\ \big(\partial_s \boldsymbol{P}_{(n)}(\boldsymbol{v}_0^\delta)\big)(0) = \mathbf{0},$$

when $n \geq 1$. We also say that $\boldsymbol{v}_0^\delta$ satisfies the compatibility conditions for (4.2.4) *up to order n if it satisfies the k-th order compatibility condition for all k with $0 \leq k \leq n$.*

4.2.3 Corrections to the Initial Data

We construct a corrected initial datum $\boldsymbol{v}_0^\delta$ such that given an initial datum $\boldsymbol{v}_0$ satisfying the compatibility conditions for (4.2.1) or (4.2.2), $\boldsymbol{v}_0^\delta$ satisfies the compatibility conditions for (4.2.3) or (4.2.4), and $\boldsymbol{v}_0^\delta \to \boldsymbol{v}_0$ $(\delta \to +0)$ in the appropriate function space. As it will be shown later, a sufficiently smooth solution of (4.2.3) or (4.2.4) with $\delta \geq 0$ satisfies $|\boldsymbol{v}^\delta| = 1$ if $|\boldsymbol{v}_0^\delta| = 1$. Thus, the correction of the initial datum must be done in a way that this property is preserved. Since the same method for the construction of $\boldsymbol{v}_0^\delta$ holds for the cases $\alpha > 0$ and $\alpha < 0$, we show the details for the case $\alpha < 0$ only.

Suppose that an initial datum $\boldsymbol{v}_0$ such that $\boldsymbol{v}_{0s} \in H^{1+3m}(\mathbf{R}_+)$ satisfies the compatibility conditions for (4.2.1) up to order m. We will construct $\boldsymbol{v}_0^\delta$ in the form

$$\boldsymbol{v}_0^\delta = \frac{\boldsymbol{v}_0 + \boldsymbol{h}_\delta}{|\boldsymbol{v}_0 + \boldsymbol{h}_\delta|} \tag{4.2.5}$$

with $\boldsymbol{h}_\delta \to \mathbf{0}$ as $\delta \to +0$. The method to construct $\boldsymbol{h}_\delta$ is standard, i.e. we substitute (4.2.5) into the compatibility conditions for (4.2.3) to determine the differential coefficients of $\boldsymbol{h}_\delta$ at $s = 0$ and then extend them to $s > 0$ so that $\boldsymbol{h}_\delta$ belongs to the appropriate Sobolev space and its differential coefficients have the desired values.

We introduce some notations. We set

$$\begin{aligned}
&\boldsymbol{g}_0^\delta(\boldsymbol{V}) := \boldsymbol{V}, \\
&\boldsymbol{g}_1^\delta(\boldsymbol{V}) := \boldsymbol{V} \times \boldsymbol{V}_{ss} + \alpha\Big\{\boldsymbol{V}_{sss} + 3\boldsymbol{V}_{ss} \times (\boldsymbol{V} \times \boldsymbol{V}_s) - \frac{3}{2}|\boldsymbol{V}_s|^2\boldsymbol{V}_s\Big\} + \delta(\boldsymbol{V}_{ss} + |\boldsymbol{V}_s|^2\boldsymbol{V}), \\
&\boldsymbol{g}_{m+1}^\delta(\boldsymbol{V}) := \mathrm{D}\boldsymbol{g}_m^\delta(\boldsymbol{V})[\boldsymbol{g}_1^\delta(\boldsymbol{V})],
\end{aligned}$$

where $m \geq 1$ and D is the derivative with respect to $\boldsymbol{V}$, i.e. $\mathrm{D}\boldsymbol{g}_m^\delta(\boldsymbol{V})[\boldsymbol{W}] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\boldsymbol{g}_m^\delta(\boldsymbol{V} + \varepsilon\boldsymbol{W})\big|_{\varepsilon=0}$. Note that under these notations, the m-th order compatibility condition for (4.2.3) can be expressed as $\partial_s\boldsymbol{g}_m^\delta(\boldsymbol{v}_0^\delta)(0) = \mathbf{0}$, because $\boldsymbol{P}_{(m)}(\boldsymbol{V}) = \boldsymbol{g}_m^\delta(\boldsymbol{V})$. We gave another notation because it is more convenient for the following calculations.

First we prove that if $|\boldsymbol{V}| \equiv 1$, then for any $m \geq 1$

$$\sum_{k=0}^{m} \binom{m}{k} \boldsymbol{g}_k^\delta(\boldsymbol{V}) \cdot \boldsymbol{g}_{m-k}^\delta(\boldsymbol{V}) \equiv 0 \tag{4.2.6}$$

by induction. From direct calculation, we can prove that

$$\boldsymbol{g}_1^\delta(\boldsymbol{V}) \cdot \boldsymbol{V} = \frac{\alpha}{2}(|\boldsymbol{V}|^2)_{sss} - 3\alpha(\boldsymbol{V} \cdot \boldsymbol{V}_{ss})(|\boldsymbol{V}|^2)_s - \frac{3}{2}|\boldsymbol{V}_s|^2(|\boldsymbol{V}|^2)_s + \frac{\delta}{2}(|\boldsymbol{V}|^2)_{ss} = 0,$$

which proves (4.2.6) with $m = 1$. Suppose that (4.2.6) holds up to some m with $m \geq 1$, i.e.,

$$\sum_{k=0}^{m} \binom{m}{k} \boldsymbol{g}_k^\delta\Big(\frac{\boldsymbol{V} + t\boldsymbol{W}}{|\boldsymbol{V} + t\boldsymbol{W}|}\Big) \cdot \boldsymbol{g}_{m-k}^\delta\Big(\frac{\boldsymbol{V} + t\boldsymbol{W}}{|\boldsymbol{V} + t\boldsymbol{W}|}\Big) \equiv 0 \quad \text{for any vector } \boldsymbol{W} \text{ and } t \in \mathbf{R}.$$

Differentiating this with respect to t and setting $t=0$ yield

$$\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix}\big\{\mathrm{D}\boldsymbol{g}_k^\delta(\boldsymbol{V})[\boldsymbol{W}-(\boldsymbol{V}\cdot\boldsymbol{W})\boldsymbol{V}]\cdot\boldsymbol{g}_{m-k}^\delta(\boldsymbol{V}) \\ +\boldsymbol{g}_k^\delta(\boldsymbol{V})\cdot\mathrm{D}\boldsymbol{g}_{m-k}^\delta(\boldsymbol{V})[\boldsymbol{W}-(\boldsymbol{V}\cdot\boldsymbol{W})\boldsymbol{V}]\big\}\equiv 0.$$

By choosing $\boldsymbol{W}=\boldsymbol{g}_1^\delta(\boldsymbol{V})$ we have

$$\begin{aligned} 0 &\equiv \sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix}\big\{\boldsymbol{g}_{k+1}^\delta(\boldsymbol{V})\cdot\boldsymbol{g}_{m-k}^\delta(\boldsymbol{V})+\boldsymbol{g}_k^\delta(\boldsymbol{V})\cdot\boldsymbol{g}_{m-k+1}^\delta(\boldsymbol{V})\big\} \\ &= \sum_{k=0}^{m+1}\begin{pmatrix} m+1 \\ k \end{pmatrix}\boldsymbol{g}_k^\delta(\boldsymbol{V})\cdot\boldsymbol{g}_{m+1-k}^\delta(\boldsymbol{V}), \end{aligned}$$

which proves (4.2.6) for the case $m+1$. Therefore (4.2.6) holds for any $m=1,2,3,\ldots$.

Next we introduce the following notations:

$$\begin{aligned} \boldsymbol{f}_0(\boldsymbol{V}) &:= \boldsymbol{V}, \\ \boldsymbol{f}_1(\boldsymbol{V}) &:= \boldsymbol{V}\times\boldsymbol{V}_{ss}+\alpha\Big\{\boldsymbol{V}_{sss}+3\boldsymbol{V}_{ss}\times(\boldsymbol{V}\times\boldsymbol{V}_s)-\frac{3}{2}|\boldsymbol{V}_s|^2\boldsymbol{V}_s\Big\}, \\ \boldsymbol{f}_{m+1}(\boldsymbol{V}) &:= \mathrm{D}\boldsymbol{f}_m(\boldsymbol{V})[\boldsymbol{f}_1(\boldsymbol{V})]. \end{aligned}$$

These correspond to $\boldsymbol{g}_m^\delta(\boldsymbol{V})$ with $\delta=0$, so that $\sum_{k=0}^m\begin{pmatrix} m \\ k \end{pmatrix}\boldsymbol{f}_k(\boldsymbol{V})\cdot\boldsymbol{f}_{m-k}(\boldsymbol{V})\equiv 0$ if $|\boldsymbol{V}|\equiv 1$, and the m-th order compatibility condition for (4.2.1) can be expressed as $\partial_s\boldsymbol{f}_m(\boldsymbol{v}_0)(0)=\boldsymbol{0}$ because $\boldsymbol{Q}_{(m)}(\boldsymbol{v}_0)=\boldsymbol{f}_m(\boldsymbol{v}_0)$.

Next, we show that

$$\boldsymbol{g}_m^\delta(\boldsymbol{V})=\boldsymbol{f}_m(\boldsymbol{V})+\delta\boldsymbol{r}_m^\delta(\boldsymbol{V}), \tag{4.2.7}$$

where $\boldsymbol{r}_1^\delta(\boldsymbol{V}):=\boldsymbol{V}_{ss}+|\boldsymbol{V}_s|^2\boldsymbol{V}$ and $\boldsymbol{r}_m^\delta(\boldsymbol{V}):=\mathrm{D}\boldsymbol{r}_{m-1}^\delta(\boldsymbol{V})[\boldsymbol{g}_1^\delta(\boldsymbol{V})]+\mathrm{D}\boldsymbol{f}_{m-1}(\boldsymbol{V})[\boldsymbol{r}_1^\delta(\boldsymbol{V})]$ for $m\ge 2$. Clearly, $\boldsymbol{r}_m^\delta(\boldsymbol{V})$ contains derivatives up to order $3m-1$.

It is obvious that (4.2.7) holds for $m=1$ from the definition of $\boldsymbol{g}_1^\delta$ and $\boldsymbol{f}_1$. Suppose that it holds up to $m-1$ for some $m\ge 2$. Then for any vector $\boldsymbol{W}$ and $t\in\mathbf{R}$,

$$\boldsymbol{g}_{m-1}^\delta(\boldsymbol{V}+t\boldsymbol{W})=\boldsymbol{f}_{m-1}(\boldsymbol{V}+t\boldsymbol{W})+\delta\boldsymbol{r}_{m-1}^\delta(\boldsymbol{V}+t\boldsymbol{W}).$$

Differentiating the above equation with respect to t and setting $t=0$ yield

$$\mathrm{D}\boldsymbol{g}_{m-1}^\delta(\boldsymbol{V})[\boldsymbol{W}]=\mathrm{D}\boldsymbol{f}_{m-1}(\boldsymbol{V})[\boldsymbol{W}]+\delta\mathrm{D}\boldsymbol{r}_{m-1}^\delta(\boldsymbol{V})[\boldsymbol{W}].$$

Finally, choosing $\boldsymbol{W}=\boldsymbol{g}_1^\delta(\boldsymbol{V})$ leads to

$$\begin{aligned} \boldsymbol{g}_m^\delta(\boldsymbol{V}) &= \mathrm{D}\boldsymbol{f}_{m-1}(\boldsymbol{V})[\boldsymbol{g}_1^\delta(\boldsymbol{V})]+\delta\mathrm{D}\boldsymbol{r}_{m-1}^\delta(\boldsymbol{V})[\boldsymbol{g}_1^\delta(\boldsymbol{V})] \\ &= \mathrm{D}\boldsymbol{f}_{m-1}(\boldsymbol{V})[\boldsymbol{f}_1(\boldsymbol{V})]+\delta\mathrm{D}\boldsymbol{f}_{m-1}(\boldsymbol{V})[\boldsymbol{r}_1^\delta(\boldsymbol{V})]+\delta\mathrm{D}\boldsymbol{r}_{m-1}^\delta(\boldsymbol{V})[\boldsymbol{g}_1^\delta(\boldsymbol{V})] \\ &= \boldsymbol{f}_m(\boldsymbol{V})+\delta\boldsymbol{r}_m^\delta(\boldsymbol{V}), \end{aligned}$$

which shows that (4.2.7) holds.

Next we prove that if $\boldsymbol{h}_\delta(0) = \boldsymbol{0}$ and $|\boldsymbol{v}_0| = 1$,

$$\text{(4.2.8)}\quad \boldsymbol{f}_m(\boldsymbol{v}_0^\delta)\big|_{s=0} = \left\{\boldsymbol{f}_m(\boldsymbol{v}_0) + \alpha^m\partial_s^{3m}\boldsymbol{h}_\delta - \alpha^m(\boldsymbol{v}_0\cdot\partial_s^{3m}\boldsymbol{h}_\delta)\boldsymbol{v}_0 + \boldsymbol{F}_m(\boldsymbol{v}_0,\boldsymbol{h}_\delta)\right\}\big|_{s=0},$$

where $\boldsymbol{F}_m(\boldsymbol{v}_0,\boldsymbol{h}_\delta)$ satisfies

$$|\boldsymbol{F}_m(\boldsymbol{v}_0,\boldsymbol{h}_\delta)| \le C(|\boldsymbol{h}_{\delta s}| + |\boldsymbol{h}_{\delta ss}| + \cdots + |\partial_s^{3m-1}\boldsymbol{h}_\delta|)$$

with a constant C depending on M and $\boldsymbol{v}_0$ if $|\boldsymbol{h}_{\delta s}| + |\boldsymbol{h}_{\delta ss}| + \cdots + |\partial_s^{3m-1}\boldsymbol{h}_\delta| \le M$. We see from the explicit form (4.2.5) of $\boldsymbol{v}_0^\delta$ that for a natural number n, $\partial_s^n\boldsymbol{v}_0^\delta$ has the form

$$\text{(4.2.9)}\qquad \partial_s^n\boldsymbol{v}_0^\delta\big|_{s=0} = \left\{\partial_s^n\boldsymbol{v}_0 + \partial_s^n\boldsymbol{h}_\delta - (\boldsymbol{v}_0\cdot\partial_s^n\boldsymbol{h}_\delta)\boldsymbol{v}_0 + \boldsymbol{q}_n(\boldsymbol{v}_0,\boldsymbol{h}_\delta)\right\}\big|_{s=0},$$

where $\boldsymbol{q}_n(\boldsymbol{v}_0,\boldsymbol{h}_\delta)$ contains the derivatives of $\boldsymbol{v}_0$ and $\boldsymbol{h}_\delta$ up to order $n-1$, and satisfies

$$|\boldsymbol{q}_n(\boldsymbol{v}_0,\boldsymbol{h}_\delta)| \le C(|\boldsymbol{h}_{\delta s}| + |\boldsymbol{h}_{\delta ss}| + \cdots + |\partial_s^{n-1}\boldsymbol{h}_\delta|),$$

if $\boldsymbol{h}_\delta(0) = \boldsymbol{0}$ and $|\boldsymbol{h}_{\delta s}| + |\boldsymbol{h}_{\delta ss}| + \cdots + |\partial_s^{n-1}\boldsymbol{h}_\delta| \le M$, for a constant C depending on M and $\boldsymbol{v}_0$. From the definition of $\boldsymbol{f}_m(\boldsymbol{v}_0^\delta)$, we see that the term with the highest order of derivative is $\alpha^m\partial_s^{3m}\boldsymbol{v}_0^\delta$, so combining this with (4.2.9) yields (4.2.8).

Finally, we prove by induction that the differential coefficients of $\boldsymbol{h}_\delta$ can be chosen so that $\boldsymbol{v}_0^\delta$ satisfies the compatibility conditions for (4.2.3), and all the coefficients are $O(\delta)$. First, let $\boldsymbol{h}_\delta(0) = \partial_s\boldsymbol{h}_\delta(0) = \boldsymbol{0}$. This insures that $\boldsymbol{v}_0^\delta$ satisfies the 0-th order compatibility condition. Suppose that the differential coefficients of $\boldsymbol{h}_\delta$ up to order $1 + 3(m-1)$ are chosen so that they are $O(\delta)$ and the compatibility conditions for (4.2.3) up to order $m-1$ are satisfied, i.e. $\boldsymbol{g}_k^\delta(\boldsymbol{v}_0^\delta)_s(0) = \boldsymbol{0}$ for all $0 \le k \le m-1$. By choosing $\boldsymbol{V} = \boldsymbol{v}_0^\delta$, (4.2.6) becomes

$$\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix}\boldsymbol{g}_k^\delta(\boldsymbol{v}_0^\delta)\cdot\boldsymbol{g}_{m-k}^\delta(\boldsymbol{v}_0^\delta) \equiv 0.$$

Taking the s derivative of the above and using the assumption of induction yield

$$\text{(4.2.10)}\qquad \boldsymbol{v}_0^\delta(0)\cdot\partial_s\boldsymbol{g}_m^\delta(\boldsymbol{v}_0^\delta)(0) = \boldsymbol{v}_0(0)\cdot\partial_s\boldsymbol{g}_m^\delta(\boldsymbol{v}_0^\delta)(0) = 0.$$

Now, from (4.2.7) at $s = 0$ and (4.2.8) lead to

$$\begin{aligned}
\partial_s\boldsymbol{g}_m^\delta(\boldsymbol{v}_0^\delta) &= \partial_s\boldsymbol{f}_m(\boldsymbol{v}_0^\delta) + \delta\partial_s\boldsymbol{r}_m^\delta(\boldsymbol{v}_0^\delta)\\
&= \partial_s\boldsymbol{f}_m(\boldsymbol{v}_0) + \alpha^m\partial_s^{3m+1}\boldsymbol{h}_\delta - \alpha^m(\boldsymbol{v}_0\cdot\partial_s^{3m+1}\boldsymbol{h}_\delta)\boldsymbol{v}_0 + \partial_s\boldsymbol{F}_m(\boldsymbol{v}_0,\boldsymbol{h}_\delta) + \delta\partial_s\boldsymbol{r}_m^\delta(\boldsymbol{v}_0^\delta)\\
&= \alpha^m\partial_s^{3m+1}\boldsymbol{h}_\delta - \alpha^m(\boldsymbol{v}_0\cdot\partial_s^{3m+1}\boldsymbol{h}_\delta)\boldsymbol{v}_0 + \partial_s\boldsymbol{F}_m(\boldsymbol{v}_0,\boldsymbol{h}_\delta) + \delta\boldsymbol{r}_m^\delta(\boldsymbol{v}_0^\delta)_s.
\end{aligned}$$

From this and (4.2.10), it follows that

$$\left(\partial_s \boldsymbol{F}_m(\boldsymbol{v}_0, \boldsymbol{h}_\delta) + \delta \partial_s \boldsymbol{r}_m^\delta(\boldsymbol{v}_0^\delta)\right) \cdot \boldsymbol{v}_0\big|_{s=0} = 0.$$

The assumption of induction implies

$$\partial_s \boldsymbol{F}_m(\boldsymbol{v}_0, \boldsymbol{h}_\delta) + \delta \partial_s \boldsymbol{r}_m^\delta(\boldsymbol{v}_0^\delta)\big|_{s=0} = O(\delta).$$

Thus by defining $\partial_s^{3m-1}\boldsymbol{h}_\delta(0) = \partial_s^{3m}\boldsymbol{h}_\delta(0) = \boldsymbol{0}$ and $\partial_s^{3m+1}\boldsymbol{h}_\delta(0) = -\frac{1}{\alpha^m}\big(\boldsymbol{F}_m(\boldsymbol{v}_0, \boldsymbol{h}_\delta)_s + \delta \boldsymbol{r}_m^\delta(\boldsymbol{v}_0^\delta)_s\big)\big|_{s=0}$, they are all $O(\delta)$ and $\partial_s \boldsymbol{g}^\delta(\boldsymbol{v}_0^\delta)(0) = \boldsymbol{0}$, i.e. the m-th order compatibility condition is satisfied. The differential coefficients are then used to define $\boldsymbol{h}_\delta(s)$ as

$$\boldsymbol{h}_\delta(s) = \phi(s)\left(\sum_{j=0}^{m} \frac{\partial_s^{3j+1}\boldsymbol{h}_\delta(0)}{(3j+1)!} s^{3j+1}\right),$$

where $\phi(s)$ is a smooth cut-off function that is 1 near $s = 0$. We summarize the arguments so far in the following statement.

Lemma 4.2.5 *For an initial datum $\boldsymbol{v}_0$ with $|\boldsymbol{v}_0| = 1$, $\boldsymbol{v}_{0s} \in H^{1+3m}(\mathbf{R}_+)$, and satisfying the compatibility conditions for* (4.2.1) *up to order m, we can construct a corrected initial datum $\boldsymbol{v}_0^\delta$ such that $|\boldsymbol{v}_0^\delta| = 1$, $\boldsymbol{v}_{0s}^\delta \in H^{1+3m}(\mathbf{R}_+)$, and it satisfies the compatibility conditions of* (4.2.3) *up to order m, and*

$$\boldsymbol{v}_0^\delta \to \boldsymbol{v}_0 \text{ in } L^\infty(\mathbf{R}_+),\ \boldsymbol{v}_{0s}^\delta \to \boldsymbol{v}_{0s} \text{ in } H^{1+3m}(\mathbf{R}_+)$$

as $\delta \to +0$.

Similar arguments can be used to construct an approximating sequence of $\boldsymbol{v}_0$ by a smoother function while satisfying the necessary compatibility conditions by following the method due to Rauch and Massey [36].

4.3 Existence Theorems for Associated Linear Problems

We consider the following linear problems associated to the regularized nonlinear problem. For $\alpha < 0$,

$$(4.3.1)\quad \begin{cases} \boldsymbol{v}_t = \alpha \boldsymbol{v}_{sss} + \delta \boldsymbol{v}_{ss} + \boldsymbol{w} \times \boldsymbol{v}_{ss} + 3\alpha \boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s) + \boldsymbol{f}, & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t > 0, \end{cases}$$

and for $\alpha > 0$,

$$
(4.3.2) \quad \begin{cases} \boldsymbol{v}_t = \alpha \boldsymbol{v}_{sss} + \delta \boldsymbol{v}_{ss} + \boldsymbol{w} \times \boldsymbol{v}_{ss} + 3\alpha \boldsymbol{v}_{ss} \times (\boldsymbol{w} \times \boldsymbol{w}_s) + \boldsymbol{f}, & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t > 0, \\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t > 0. \end{cases}
$$

The existence and uniqueness of solutions to (4.3.1) and (4.3.2) can be shown as an application of existence theorems for more general equations obtained in the previous chapter. There, we obtained existence theorems for a linear second order parabolic system with a third order dispersive term. The problems considered there are as follows. For $\alpha < 0$,

$$
(4.3.3) \qquad \begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0. \end{cases}
$$

For $\alpha > 0$,

$$
(4.3.4) \qquad \begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}(0,t) = \boldsymbol{e}, & t > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0. \end{cases}
$$

Here, $\boldsymbol{u}(x,t) = (u^1(x,t), u^2(x,t), \ldots, u^m(x,t))$ is the unknown vector valued function, $\boldsymbol{u}_0(x)$, $\boldsymbol{w}(x,t) = (w^1(x,t), w^2(x,t), \ldots, w^k(x,t))$, and $\boldsymbol{f}(x,t) = (f^1(x,t), f^2(x,t), \ldots, f^m(x,t))$ are known vector valued functions, $\boldsymbol{e}$ is an arbitrary constant vector, subscripts denote derivatives with the respective variables, and $\mathrm{A}(\boldsymbol{w}, \partial_x)$ is a second order differential operator of the form $\mathrm{A}(\boldsymbol{w}, \partial_x) = \mathrm{A}_0(\boldsymbol{w})\partial_x^2 + \mathrm{A}_1(\boldsymbol{w})\partial_x + \mathrm{A}_2(\boldsymbol{w})$. We assume that $\mathrm{A}_0,\ \mathrm{A}_1,\ \mathrm{A}_2$ are smooth matrices and $\mathrm{A}(\boldsymbol{w}, \partial_x)$ is strongly elliptic in the sense that for any bounded domain E in $\mathbf{R}^k$, there is a positive constant δ such that for any $\boldsymbol{w} \in E$

$$
\mathrm{A}_0(\boldsymbol{w}) + \mathrm{A}_0(\boldsymbol{w})^* \geq \delta \mathrm{I},
$$

where I is the unit matrix and $*$ denotes the adjoint of a matrix. For the above problems we obtained the following theorems in the previous chapter.

Theorem 4.3.1 (Aiki and Iguchi [2]) *Let $\alpha < 0$. For any $T > 0$ and an arbitrary non-negative integer l, if $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y_T^l$, and $\boldsymbol{w} \in Z_T^l$ satisfy the compatibility conditions up to order l, a unique solution $\boldsymbol{u}$ of* (4.3.3) *exists such that $\boldsymbol{u} \in X_T^l$. Furthermore, $\boldsymbol{u}$ satisfies*

$$
\|\boldsymbol{u}\|_{X_T^l} \leq C\big(\|\boldsymbol{u}_0\|_{2+3l} + \|\boldsymbol{f}\|_{Y_T^l}\big),
$$

where C depends on α, T, and $\|\boldsymbol{w}\|_{Z_T^l}$.

Theorem 4.3.2 (Aiki and Iguchi [2]) *Let $\alpha > 0$. For any $T > 0$ and an arbitrary non-negative integer l, if $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y^l_T$, and $\boldsymbol{w} \in Z^l_T$ satisfy the compatibility conditions up to order l, a unique solution $\boldsymbol{u}$ of* (4.3.4) *exists such that $\boldsymbol{u} \in X^l_T$. Furthermore, $\boldsymbol{u}$ satisfies*

$$\|\boldsymbol{u}\|_{X^l_T} \le C\big(\|\boldsymbol{u}_0\|_{2+3l} + \|\boldsymbol{f}\|_{Y^l_T}\big),$$

where C depends on α, T, and $\|\boldsymbol{w}\|_{Z^l_T}$.

We apply these theorems with

$$\mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{v} = \delta \boldsymbol{v}_{xx} + \boldsymbol{w} \times \boldsymbol{v}_{xx} + 3\alpha \boldsymbol{v}_{xx} \times (\boldsymbol{w} \times \boldsymbol{w}_x), \tag{4.3.5}$$

which obviously satisfies the assumptions on the elliptic operator. Thus we have existence and uniqueness of the solutions to (4.3.1) and (4.3.2). Based on these linear existence theorems, we construct the solution to (4.2.3) and (4.2.4).

4.4 Construction of Solution in the Case $\alpha < 0$

4.4.1 Existence of Solution

We construct the solution by the following iteration scheme. For $n \ge 2$ and $R \ge 1$, we define $\boldsymbol{v}^{(n),R}$ as the solution of

$$\begin{cases} \boldsymbol{v}^{(n),R}_t = \alpha \boldsymbol{v}^{(n),R}_{sss} + \mathrm{A}(\boldsymbol{v}^{(n-1),R}, \partial_s)\boldsymbol{v}^{(n),R} - \dfrac{3}{2}\alpha |\boldsymbol{v}^{(n-1),R}_s|^2 \boldsymbol{v}^{(n-1),R}_s & \\ \qquad\qquad + \delta |\boldsymbol{v}^{(n-1),R}_s|^2 \boldsymbol{v}^{(n-1),R}, & s > 0, t > 0, \\ \boldsymbol{v}^{(n),R}(s,0) = \boldsymbol{v}^{\delta,R}_0(s), & s > 0, \\ \boldsymbol{v}^{(n),R}_s(0,t) = \mathbf{0}, & t > 0, \end{cases}$$

where $\mathrm{A}(\boldsymbol{v}^{(n-1),R}, \partial_s)$ is the operator (4.3.5) in the previous section and $\boldsymbol{v}^{\delta,R}_0(s) = \phi(\frac{s}{R})\boldsymbol{v}^\delta_0(s)$. Here, $\boldsymbol{v}^\delta_0$ is the modified initial datum constructed in Subsection 4.2.3 and $\phi(s)$ is a smooth cut-off function satisfying $0 \le \phi \le 1$, $\phi(s) = 1$ for $0 \le s \le 1$, and $\phi(s) = 0$ for $s > 2$. Now, $\boldsymbol{v}^{(1),R}$ is appropriately chosen so as to satisfy the necessary compatibility conditions at each iteration step. For this we choose

$$\boldsymbol{v}^{(1),R}(s,t) = \boldsymbol{v}^{\delta,R}_0(s) + \sum_{j=1}^m \frac{t^j}{j!}\boldsymbol{P}_{(j)}(\boldsymbol{v}^{\delta,R}_0(s)), \tag{4.4.1}$$

where m is a fixed natural number and $\boldsymbol{P}_{(j)}$ is defined in Section 4.2. Note that multiplying the initial datum by ϕ has no influence on the compatibility conditions for (4.2.3). Recall that $\boldsymbol{v}^\delta_0$ is assumed to be smooth, to satisfy the compatibility conditions up to an arbitrary fixed order, and $\boldsymbol{v}_0^\delta \to \boldsymbol{v}_0$ in an appropriate function space. More specifically, we assume that $\boldsymbol{v}_0^\delta$ is smooth enough so that $\boldsymbol{v}^{(1),R} \in X_T^N$ for a large $N(> m)$ to be determined later. For each $R \geq 1$ and natural number n, $\boldsymbol{v}^{(n),R}$ is well-defined by Theorem 4.3.1 and $\boldsymbol{v}^{(n),R} \in X_T^m$. We introduce the function space $\tilde{X}_T^m$ as

$$\begin{aligned}\tilde{X}_T^m := \big\{\boldsymbol{v}; \boldsymbol{v}_s \in C\big([0,T]; H^{1+3m}(\mathbf{R}_+)\big)\big\} \cap \left\{\bigcap_{j=1}^{m} C^j\big([0,T]; H^{2+3j}(\mathbf{R}_+)\big)\right\} \\ \cap\, C\big([0,T]; L^\infty(\mathbf{R}_+)\big).\end{aligned}$$

We seek a solution to the nonlinear problem in this function space. It is easy to see that from (4.4.1), we have

$$\|\boldsymbol{v}^{(1),R}\|_{\tilde{X}_T^m} \leq 1 + C\|\boldsymbol{v}_{0s}^{\delta,R}\|_{2+6m}(1 + \|\boldsymbol{v}_{0s}^{\delta,R}\|_{2+6m})^{1+2m} =: M_0$$

with a positive constant C depending on α and T, but not on $\boldsymbol{v}_0^{\delta,R}$, $\boldsymbol{v}_0^{\delta,R} \to \boldsymbol{v}_0^\delta$ in $\tilde{X}_T^m$ as $R \to +\infty$, and there is a positive constant C independent of $R \geq 1$ such that

$$\|\boldsymbol{v}_{0s}^{\delta,R}\|_{1+3m} \leq C\|\boldsymbol{v}_{0s}^\delta\|_{1+3m}. \tag{4.4.2}$$

Note that the uniform estimate (4.4.2) does not hold for $\|\boldsymbol{v}_0^{\delta,R}\|$ because $\boldsymbol{v}_0^\delta$ does not belong to $L^2(\mathbf{R}_+)$. We show the uniform boundedness of $\{\boldsymbol{v}^{(n),R}\}_{n=0}^\infty$ with respect to n and R on some time interval $[0, T_0]$ by induction. Suppose that $\|\boldsymbol{v}^{(j),R}\|_{\tilde{X}_T^m} \leq M$ for any j with $1 \leq j \leq n-1$. Then, by a standard energy estimate, we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_s^{(n),R}\|^2 &\leq -\frac{|\alpha|}{2}|\boldsymbol{v}_{ss}^{(n),R}(0,t)|^2 + C\|\boldsymbol{v}_{ss}^{(n),R}\|^2 - \delta\|\boldsymbol{v}_{ss}^{(n),R}\|^2 + CM^3, \\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_{ss}^{(n),R}\|^2 &\leq -\frac{|\alpha|}{2}|\boldsymbol{v}_{sss}^{(n),R}(0,t)|^2 - \frac{\delta}{4}\|\boldsymbol{v}_{sss}^{(n),R}\|^2 + CM^2\|\boldsymbol{v}_{ss}^{(n),R}\|^2 + CM^3,
\end{aligned}$$

where C is independent of M and n. Combining these estimates yields for any $0 \leq t \leq T$,

$$\|\boldsymbol{v}_s^{(n),R}(t)\|_1^2 + \int_0^t \|\boldsymbol{v}_s^{(n),R}(\tau)\|_2^2 \mathrm{d}\tau \leq C\mathrm{e}^{M^2T}\big(\|\boldsymbol{v}_{0s}^{\delta,R}\|_1^2 + M^3T\big),$$

where C is independent of T, M, and n. For a natural number k with $1 \le k \le m$, we set $\boldsymbol{v}^{(n),k} := \partial_t^k \boldsymbol{v}^{(n),R}$. Then, $\boldsymbol{v}^{(n),k}$ satisfies

$$
\begin{aligned}
\boldsymbol{v}^{(n),k}_t &= \alpha \boldsymbol{v}^{(n),k}_{sss} + \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (\boldsymbol{v}^{(n-1),j} \times \boldsymbol{v}^{(n),k-j}_{ss}) \\
&\qquad + 3\alpha \left\{ \sum_{j=0}^{k} \sum_{i=0}^{j} \begin{pmatrix} k \\ j \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} \boldsymbol{v}^{(n),k-j}_{ss} \times (\boldsymbol{v}^{(n-1),i} \times \boldsymbol{v}^{(n-1),j-i}_{s}) \right\} \\
&\qquad - \frac{3}{2}\alpha \left\{ \sum_{j=0}^{k} \sum_{i=0}^{j} \begin{pmatrix} k \\ j \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} (\boldsymbol{v}^{(n-1),i}_{s} \cdot \boldsymbol{v}^{(n-1),j-i}_{s}) \boldsymbol{v}^{(n-1),k-j}_{s} \right\} \\
&\qquad + \delta \boldsymbol{v}^{(n),k}_{ss} + \delta \left\{ \sum_{j=0}^{k} \sum_{i=0}^{j} \begin{pmatrix} k \\ j \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} (\boldsymbol{v}^{(n-1),i}_{s} \cdot \boldsymbol{v}^{(n-1),j-i}_{s}) \boldsymbol{v}^{(n-1),k-j} \right\} \\
&=: \alpha \boldsymbol{v}^{(n),k}_{sss} + \boldsymbol{v}^{(n-1),R} \times \boldsymbol{v}^{(n),k}_{ss} + 3\alpha \boldsymbol{v}^{(n),k}_{ss} \times (\boldsymbol{v}^{(n-1),R} \times \boldsymbol{v}^{(n-1)}_{s}) + \delta \boldsymbol{v}^{(n),k}_{ss} + \boldsymbol{F}^k.
\end{aligned}
$$

By a similar energy estimate, we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{v}^{(n),k}\|_2^2 &\le CM^2(1+M^2)\bigl\{ \|\boldsymbol{v}^{(n),k}\|_2^2 + (1+M^2)^5 + \|\boldsymbol{F}^k\|_1^2 \bigr\} \\
&\le CM^2(1+M^2)\bigl\{ \|\boldsymbol{v}^{(n),k}\|_2^2 + (1+M^2)^5 \bigr\},
\end{aligned}
$$

where we have used $\|\boldsymbol{v}^{(j)}\|_{\tilde{X}^m_T} \le M$ for $1 \le j \le n-1$ to estimate $\boldsymbol{F}^k$. Thus we have

$$
\|\boldsymbol{v}^{(n),k}\|_2^2 \le C\mathrm{e}^{CM^2(1+M^2)T} \bigl\{ \|\boldsymbol{v}^{(n),k}(\cdot,0)\|_2^2 + (1+M^2)^5 T \bigr\}.
$$

From the equation, we obtain

$$
\|\boldsymbol{v}^{(n),k}(\cdot,0)\|_2^2 \le C \|\boldsymbol{v}^{\delta,R}_{0s}\|_{1+3k}^2 (1 + \|\boldsymbol{v}^{\delta,R}_{0s}\|_{1+3k})^{2+4m},
$$

and hence

$$
\|\boldsymbol{v}^{(n),k}\|_2^2 \le C\mathrm{e}^{CM^2(1+M^2)T} \bigl\{ \|\boldsymbol{v}^{\delta,R}_{0s}\|_{1+3k}^2 (1 + \|\boldsymbol{v}^{\delta,R}_{0s}\|_{1+3k})^{2+4m} + (1+M^2)^5 T \bigr\}.
$$

Finally, from the equation and the above estimates, we can convert the regularity in t into the regularity in s and obtain for $1 \le j \le m$,

$$
\|\boldsymbol{v}^{(n),j}\|_{1+3(m-j)}^2 \le C\mathrm{e}^{CM^2(1+M^2)T} \bigl\{ \|\boldsymbol{v}^{\delta,R}_{0s}\|_{1+3m}^2 (1 + \|\boldsymbol{v}^{\delta,R}_{0s}\|_{1+3m})^{2+4m} + (1+M^2)^5 T \bigr\}.
$$

Thus, by choosing $M := C_0 M_0$, with a sufficiently large $C_0 > 0$ independent of n and R, there is a $T_0 > 0$ such that

$$
\|\boldsymbol{v}^{(n),R}_s(t)\|_{1+3m}^2 + \sum_{j=1}^{m} \|\partial_t^j \boldsymbol{v}^{(n),R}(t)\|_{2+3(m-j)}^2 \le \frac{C_0 M_0}{2}.
$$

Next, we estimate the solution in $C\big([0,T];L^\infty(\mathbf{R}_+)\big)$. To do this, we introduce a new variable $\boldsymbol{W}^{(n),R} := \boldsymbol{v}^{(n),R} - \boldsymbol{v}_0^{\delta,R}$. Then, $\boldsymbol{W}^{(n),R}$ satisfies

$$\begin{cases} \boldsymbol{W}_t^{(n),R} = \alpha \boldsymbol{W}_{sss}^{(n),R} + \boldsymbol{v}^{(n-1),R} \times \boldsymbol{W}_{ss}^{(n),R} + 3\alpha \boldsymbol{W}_{ss}^{(n),R} \times (\boldsymbol{v}^{(n-1),R} \times \boldsymbol{v}_s^{(n-1),R}) + \delta \boldsymbol{W}_{ss}^{(n),R} \\ \qquad -\dfrac{3}{2}\alpha |\boldsymbol{v}_s^{(n-1),R}|^2 \boldsymbol{v}_s^{(n-1),R} + \delta |\boldsymbol{v}_s^{(n-1),R}|^2 \boldsymbol{v}^{(n-1),R} + \alpha \boldsymbol{v}_{0sss}^{\delta,R} + \boldsymbol{v}^{(n-1),R} \times \boldsymbol{v}_{0ss}^{\delta,R} \\ \qquad +3\alpha \boldsymbol{v}_{0ss}^{\delta,R} \times (\boldsymbol{v}^{(n-1),R} \times \boldsymbol{v}_s^{(n-1),R}) + \delta \boldsymbol{v}_{0ss}^{\delta,R}, & s>0, t>0, \\ \boldsymbol{W}^{(n),R}(s,0) = \boldsymbol{0}, & s>0, \\ \boldsymbol{W}_s^{(n),R}(0,t) = \boldsymbol{0}, & t>0. \end{cases}$$

We have by a direct calculation,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{W}^{(n),R}\|^2 \le -\frac{\delta}{2}\|\boldsymbol{W}_s^{(n),R}\|^2 + C\big(\|\boldsymbol{W}^{(n),R}\|^2 + (1+M)^3\big).$$

Thus we have

$$\begin{aligned} \|\boldsymbol{v}^{(n),R}\|_{L^\infty(\mathbf{R}_+)} &\le \|\boldsymbol{W}^{(n),R}\|_{L^\infty(\mathbf{R}_+)} + \|\boldsymbol{v}_0^{\delta,R}\|_{L^\infty(\mathbf{R}_+)} \\ &\le C\big\{\|\boldsymbol{W}^{(n),R}\|_1 + 1\big\} \\ &\le C\big\{(1+M^2)^5 T + 1\big\}. \end{aligned}$$

Thus, by choosing T_0 smaller if necessary, we have a uniform estimate of the form $\|\boldsymbol{v}^{(n),R}\|^2_{\tilde{X}^m_{T_0}} \le C_0 M_0^2$.

Now, we show that $\{\boldsymbol{v}^{(n),R}\}_{n=1}^\infty$ converges. Set $\boldsymbol{V}^{(n),R} := \boldsymbol{v}^{(n),R} - \boldsymbol{v}^{(n-1),R}$ for $n \ge 2$. Then, $\boldsymbol{V}^{(n),R}$ satisfies

$$\begin{cases} \boldsymbol{V}_t^{(n),R} = \alpha \boldsymbol{V}_{sss}^{(n),R} + \boldsymbol{v}^{(n-1),R} \times \boldsymbol{V}_{ss}^{(n),R} + 3\alpha \boldsymbol{V}_{ss}^{(n),R} \times (\boldsymbol{v}^{(n-1),R} \times \boldsymbol{v}_s^{(n-1),R}) \\ \qquad\qquad +\delta \boldsymbol{V}_{ss}^{(n),R} + \boldsymbol{G}_n^R, & s>0, t>0, \\ \boldsymbol{V}^{(n),R}(s,0) = \boldsymbol{0}, & s>0, \\ \boldsymbol{V}_s^{(n),R}(0,t) = \boldsymbol{0}, & t>0, \end{cases}$$

where $\boldsymbol{G}_n^R$ are terms depending linearly on $\boldsymbol{V}^{(n-1),R}$. In the same way as for $\boldsymbol{v}^{(n),R}$, we have

$$\begin{aligned} \|\boldsymbol{V}^{(n),R}(t)\|_2^2 + \int_0^t \|\boldsymbol{V}_s^{(n),R}(\tau)\|_2^2 \mathrm{d}\tau &\le C\int_0^t \|\boldsymbol{V}^{(n-1),R}(\tau)\|_2^2 \mathrm{d}\tau \\ &\le \frac{(CT_0)^{n-1}}{(n-1)!} M_0, \end{aligned}$$

which implies that $\boldsymbol{v}^{(n),R}$ converges to some $\boldsymbol{v}^R$ in $\tilde{X}^0_{T_0}$. Combining this convergence, uniform estimate, and the interpolation inequality, we see that $\boldsymbol{v}^{(n),R}$ converges to $\boldsymbol{v}^R$ in $\tilde{X}^{m-1}_{T_0}$. Since the initial datum has been approximated as smooth as we desire, the above argument implies that for any natural number m, we can construct a solution $\boldsymbol{v}^R$ to (4.2.3) in $\tilde{X}^m_{T_0}$ with the initial datum $\boldsymbol{v}_0^{\delta,R}$.

Finally, we pass to the limit $R \to +\infty$. First, from the estimate uniform in R, we have $\sum_{j=0}^{2}\sum_{k=0}^{2(2-j)} \sup_{0\le t\le T_0} \|\partial_t^j \partial_s^k \boldsymbol{v}^R(t)\|_{L^\infty(\mathbf{R}_+)} \le C$ with $C > 0$ independent of R. Therefore, by a standard compactness argument, we see that there is a subsequence $\{\boldsymbol{v}^{R_j}\}_{j=0}^{\infty}$ and $\boldsymbol{v}$ such that for $l = 0, 1$ and $0 \le k \le 2(1-l)$, $\partial_t^l \partial_s^k \boldsymbol{v}^{R_j} \to \partial_t^l \partial_s^k \boldsymbol{v}$ uniformly in any compact subset of $[0, T_0] \times \mathbf{R}_+$.

On the other hand, the uniform estimate implies that there is a subsequence of $\{\boldsymbol{v}^{R_j}\}_{j=0}^{\infty}$, which we also denote by $\{\boldsymbol{v}^{R_j}\}_{j=0}^{\infty}$, such that $\boldsymbol{v}^{R_j}$ converges to $\boldsymbol{v}$ weakly$*$ in $\tilde{\tilde{X}}_{T_0}^m := \big\{\boldsymbol{v}; \boldsymbol{v}_s \in L^\infty\big([0,T_0]; H^{1+3m}(\mathbf{R}_+)\big)\big\} \cap \big\{\bigcap_{j=1}^m W^{j,\infty}\big([0,T_0]; H^{2+3j}(\mathbf{R}_+)\big)\big\} \cap L^\infty\big([0,T_0] \times \mathbf{R}_+\big)$. From the above two convergence, we have a solution $\boldsymbol{v}$ of (4.2.3) with $\boldsymbol{v} \in \tilde{\tilde{X}}_{T_0}^m$. Let N be chosen large enough for a solution $\boldsymbol{v}$ to belong to $\tilde{\tilde{X}}_{T_0}^l$ with any fixed l. By taking $l > m+1$, $\boldsymbol{v} \in \tilde{X}_{T_0}^m$, according to Sobolev's embedding with respect to t.

We summarize the conclusion of this subsection.

Proposition 4.4.1 *For a natural number m and $\delta > 0$, there exists a $T_0 > 0$ such that a unique solution $\boldsymbol{v}^\delta \in \tilde{X}_{T_0}^m$ to (4.2.3) exists with a smooth initial datum $\boldsymbol{v}_0^\delta$.*

4.4.2 Uniform Estimate of Solution with respect to δ

In this subsection, we derive uniform estimates of the solution to pass to the limit $\delta \to +0$. We first show a property of the solution to (4.2.3) that is very important in the proceeding analysis. In the following, we omit the superscript δ on the solution for brevity.

Lemma 4.4.2 *If $\boldsymbol{v}$ is a solution of (4.2.3) with $\boldsymbol{v}_s \in C\big([0,T], H^2(\mathbf{R}_+)\big)$, $\boldsymbol{v} \in C\big([0,T]; L^\infty(\mathbf{R}_+)\big)$, and $|\boldsymbol{v}_0^\delta| = 1$, then $|\boldsymbol{v}| = 1$ in $\mathbf{R}_+ \times [0,T]$.*

Proof. Following Nishiyama and Tani [33], we set $h(s,t) = |\boldsymbol{v}(s,t)|^2 - 1$. From direct calculation and from the fact that $\boldsymbol{v}$ is a solution of (4.2.3), we have

$$
\begin{aligned}
h_t &= 2\boldsymbol{v} \cdot \boldsymbol{v}_t \\
&= 2\Big\{\alpha \boldsymbol{v} \cdot \boldsymbol{v}_{sss} + 3\alpha \boldsymbol{v} \cdot (\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_s)) - \frac{3}{2}\alpha |\boldsymbol{v}_s|^2 (\boldsymbol{v} \cdot \boldsymbol{v}_s) + \delta(\boldsymbol{v} \cdot \boldsymbol{v}_{ss}) + \delta |\boldsymbol{v}_s|^2 |\boldsymbol{v}|^2\Big\} \\
&= \alpha h_{sss} + \delta h_{ss} + \big(2\delta |\boldsymbol{v}_s|^2 + 3\alpha(\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss})\big) h.
\end{aligned}
$$

Thus, h satisfies

$$
\begin{cases}
h_t = \alpha h_{sss} + \delta h_{ss} + \big(2\delta |\boldsymbol{v}_s|^2 + 3\alpha(\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss})\big) h, & s > 0, t > 0, \\
h(s,0) = 0, & s > 0, \\
h_s(0,t) = 0, & t > 0.
\end{cases}
$$

By a standard energy method, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 \le -\alpha h(0)\cdot h_{ss}(0) - \delta\|h_s\|^2 + C\|h\|_1^2,$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_s\|^2 &= -(h_{ss}, h_t)\\ &\le -\frac{|\alpha|}{2}|h_{ss}(0)|^2 - \frac{\delta}{2}\|h_{ss}\|^2 + C\|h\|_1^2,\end{aligned}$$

where C depends on $\sup_{0\le t\le T}\|\boldsymbol{v}_s(t)\|_2$. Combining the two estimates and applying Gronwall's inequality, we obtain $h \equiv 0$. $\square$

Now that we have established that $|\boldsymbol{v}| = 1$, we rewrite the nonlinear terms in (4.2.3) into its original form.

$$(4.4.3)\begin{cases} \boldsymbol{v}_t = \boldsymbol{v}\times\boldsymbol{v}_{ss} + \alpha\Big\{\boldsymbol{v}_{sss} + \dfrac{3}{2}\boldsymbol{v}_{ss}\times\big(\boldsymbol{v}\times\boldsymbol{v}_s\big) & \\ \qquad\qquad + \dfrac{3}{2}\boldsymbol{v}_s\times(\boldsymbol{v}\times\boldsymbol{v}_{ss})\Big\} + \delta\big(\boldsymbol{v}_{ss} + |\boldsymbol{v}_s|^2\boldsymbol{v}\big), & s>0, t>0,\\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0^\delta(s), & s>0,\\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t>0.\end{cases}$$

We will refer to this form of the problem when estimating the solution.

The following two equalities were derived from the property $|\boldsymbol{v}| = 1$ in Nishiyama and Tani [33] which will be used to derive the uniform estimate.

$$(4.4.4)\qquad \boldsymbol{v}\cdot\partial_s^n\boldsymbol{v} = -\frac{1}{2}\sum_{j=1}^{n-1}\begin{pmatrix} n\\ j\end{pmatrix}\partial_s^j\boldsymbol{v}\cdot\partial_s^{n-j}\boldsymbol{v}.$$

$$(4.4.5)\qquad \boldsymbol{v}_s\times\partial_s^n\boldsymbol{v} = -[\boldsymbol{v}\cdot\partial_s^n\boldsymbol{v}](\boldsymbol{v}\times\boldsymbol{v}_s) + [(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\partial_s^n\boldsymbol{v}]\boldsymbol{v} \qquad \text{for } n\ge 2.$$

(4.4.4) is derived by differentiating the equality $|\boldsymbol{v}|^2 = 1$ with respect to s. We show (4.4.5) in a little more detail for the convenience of the reader. Suppose $\boldsymbol{v}_s \ne \boldsymbol{0}$. Then, since $|\boldsymbol{v}| = 1$ and $\boldsymbol{v}\cdot\boldsymbol{v}_s = 0$, the triplet $\big\{\boldsymbol{v}, \frac{\boldsymbol{v}_s}{|\boldsymbol{v}_s|}, \frac{\boldsymbol{v}\times\boldsymbol{v}_s}{|\boldsymbol{v}_s|}\big\}$ forms an orthonormal frame of $\mathbf{R}^3$. Thus for $n\ge 2$, we have

$$\partial_s^n\boldsymbol{v} = [\boldsymbol{v}\cdot\partial_s^n\boldsymbol{v}]\boldsymbol{v} + \left[\frac{\boldsymbol{v}_s}{|\boldsymbol{v}_s|}\cdot\partial_s^n\boldsymbol{v}\right]\frac{\boldsymbol{v}_s}{|\boldsymbol{v}_s|} + \left[\frac{(\boldsymbol{v}\times\boldsymbol{v}_s)}{|\boldsymbol{v}_s|}\cdot\partial_s^n\boldsymbol{v}\right]\frac{\boldsymbol{v}\times\boldsymbol{v}_s}{|\boldsymbol{v}_s|}.$$

Taking the exterior product with $\boldsymbol{v}_s$ from the left yields

$$\begin{aligned}\boldsymbol{v}_s\times\partial_s^n\boldsymbol{v} &= -[\boldsymbol{v}\cdot\partial_s^n\boldsymbol{v}](\boldsymbol{v}\times\boldsymbol{v}_s) + \left[\frac{(\boldsymbol{v}\times\boldsymbol{v}_s)}{|\boldsymbol{v}_s|}\cdot\partial_s^n\boldsymbol{v}\right]\frac{[\boldsymbol{v}_s\times(\boldsymbol{v}\times\boldsymbol{v}_s)]}{|\boldsymbol{v}_s|}\\ &= -[\boldsymbol{v}\cdot\partial_s^n\boldsymbol{v}](\boldsymbol{v}\times\boldsymbol{v}_s) + [(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\partial_s^n\boldsymbol{v}]\boldsymbol{v}.\end{aligned}$$

When $\boldsymbol{v}_s = \boldsymbol{0}$, each term in (4.4.5) is zero, so that (4.4.5) holds in either case.

Now we estimate the solution. We first derive the basic estimate.

Proposition 4.4.3 *Let $M, T > 0$. Suppose that $\boldsymbol{v}$ is a solution of (4.4.3) with $\boldsymbol{v}_{0s}^\delta \in H^4(\mathbf{R}_+)$, $|\boldsymbol{v}_0^\delta| = 1$, and $\|\boldsymbol{v}_{0s}^\delta\|_1 \le M$ satisfying $\boldsymbol{v}_s \in C\big([0,T]; H^4(\mathbf{R}_+)\big)$ and $\boldsymbol{v} \in C\big([0,T]; L^\infty(\mathbf{R}_+)\big)$. Then, there exist $\delta_* > 0$ and $C_* > 0$ such that for $\delta \in (0, \delta_*]$, the following estimate holds.*

$$\sup_{0 \le t \le T} \|\boldsymbol{v}_s(t)\|_1 \le C_*.$$

We emphasize that C_* depends on M and T, but not on $\delta \in (0, \delta_*]$.

Proof. From Lemma 4.4.2, we have $|\boldsymbol{v}| = 1$. We make use of quantities which are conserved for the initial value problem in $\mathbf{R}$ with $\delta = 0$. First we estimate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_s\|^2 &= -(\boldsymbol{v}_t, \boldsymbol{v}_{ss}) \\
&= -\alpha\left\{(\boldsymbol{v}_{sss}, \boldsymbol{v}_{ss}) + \frac{3}{2}(\boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{ss}), \boldsymbol{v}_{ss})\right\} - \delta\big\{\|\boldsymbol{v}_{ss}\|^2 + (|\boldsymbol{v}_s|^2\boldsymbol{v}, \boldsymbol{v}_{ss})\big\} \\
&= -\frac{|\alpha|}{2}|\boldsymbol{v}_{ss}(0)|^2 - \delta\|\boldsymbol{v}_{ss}\|^2 + \delta\|\boldsymbol{v}_s\|_{L^4(\mathbf{R}_+)}^4 \\
&\le -\frac{|\alpha|}{2}|\boldsymbol{v}_{ss}(0)|^2 - \frac{\delta}{2}\|\boldsymbol{v}_{ss}\|^2 + C\delta\|\boldsymbol{v}_s\|^6.
\end{aligned}$$

Here, C is independent of δ and is determined from the interpolation inequality $\|\boldsymbol{v}_s\|_{L^4(\mathbf{R}_+)} \le C\|\boldsymbol{v}_s\|^{3/4}\|\boldsymbol{v}_{ss}\|^{1/4}$. Thus, we have $\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_s\|^2 \le C\delta\|\boldsymbol{v}_s\|^6$. On the other hand, the ordinary differential equation

$$\begin{cases} r_t = C\delta r^3, & t > 0, \\ r(0) = \|\boldsymbol{v}_{0s}^\delta\|^2 \end{cases}$$

has the explicit solution $r(t) = \big(\|\boldsymbol{v}_{0s}^\delta\|^{-4} - 2C\delta t\big)^{-1/2}$ as long as $\|\boldsymbol{v}_{0s}^\delta\|^{-4} > C\delta t$. Thus, if we choose $\delta_* > 0$ such that $M^{-4} > C\delta_* T$ holds, $r(t)$ is well-defined on $[0,T]$ and from the comparison principle,

$$\begin{aligned}
\|\boldsymbol{v}_s(t)\| \le r(t)^{1/2} &= \big(\|\boldsymbol{v}_{0s}^\delta\|^{-4} - C\delta t\big)^{-1/4} \\
&\le \big(M^{-4} - C\delta_* T\big)^{-1/4} =: C_1,
\end{aligned}$$

which is a uniform estimate for $\|\boldsymbol{v}_s\|$. Next we derive a uniform estimate for $\|\boldsymbol{v}_{ss}\|$. For the initial value problem with $\delta = 0$, this was achieved by fully utilizing the conserved quantity $\|\boldsymbol{v}_{ss}\|^2 - \frac{5}{4}\||\boldsymbol{v}_s|\|^2$. We also use this quantity for the initial-boundary value problem, while taking care of boundary terms.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{v}_{ss}\|^2 - \frac{5}{4}\||\boldsymbol{v}_s|^2\|^2\right\} &= 2(\boldsymbol{v}_{ss}, \boldsymbol{v}_{sst}) - 5(|\boldsymbol{v}_s|^2\boldsymbol{v}_s, \boldsymbol{v}_{st}) \\
&= -2(\boldsymbol{v}_{sss}, \boldsymbol{v}_{st}) - 5(|\boldsymbol{v}_s|^2\boldsymbol{v}_s, \boldsymbol{v}_{st}) \\
&=: I_1 + \alpha I_2 + \delta I_3.
\end{aligned}$$

We estimate each term separately.

$$
\begin{aligned}
I_1 &= -2(\boldsymbol{v}_{sss}, \boldsymbol{v}_s \times \boldsymbol{v}_{ss}) - 5(|\boldsymbol{v}_s|^2\boldsymbol{v}_s, \boldsymbol{v} \times \boldsymbol{v}_{sss}) \\
&= -2\int_{\mathbf{R}_+} (\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss})\big[\boldsymbol{v}_s \cdot (\boldsymbol{v} \times \boldsymbol{v}_{ss})\big]\mathrm{d}s + 4\int_{\mathbf{R}_+} (\boldsymbol{v} \cdot \boldsymbol{v}_{ss})\big[\boldsymbol{v}_{sss} \cdot (\boldsymbol{v} \times \boldsymbol{v}_s)\big]\mathrm{d}s \\
&\qquad\qquad - 5\int_{\mathbf{R}_+} |\boldsymbol{v}_s|^2\boldsymbol{v}_s \cdot (\boldsymbol{v} \times \boldsymbol{v}_{sss})\mathrm{d}s \\
&= -2\int_{\mathbf{R}_+} (\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss})\big[\boldsymbol{v}_s \cdot (\boldsymbol{v} \times \boldsymbol{v}_{ss})\big]\mathrm{d}s - \int_{\mathbf{R}_+} |\boldsymbol{v}_s|^2\big[\boldsymbol{v}_s \cdot (\boldsymbol{v} \times \boldsymbol{v}_{sss})\big]\mathrm{d}s \\
&= -\int_{\mathbf{R}_+} \big\{|\boldsymbol{v}_s|^2\boldsymbol{v}_s \cdot (\boldsymbol{v} \times \boldsymbol{v}_{ss})\big\}_s \mathrm{d}s = 0.
\end{aligned}
$$

Here, we have used (4.4.4), (4.4.5), and integration by parts. From here on, integration with respect to s is assumed to be taken over $\mathbf{R}_+$. Next we have

$$
\begin{aligned}
I_2 &= -2\int \boldsymbol{v}_{sss} \cdot \boldsymbol{v}_{ssss}\mathrm{d}s - 6\int \boldsymbol{v}_{sss} \cdot \big[\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_{ss})\big]\mathrm{d}s - 3\int \boldsymbol{v}_{sss} \cdot \big[\boldsymbol{v}_s \times (\boldsymbol{v}_s \times \boldsymbol{v}_{ss})\big]\mathrm{d}s \\
&\quad - 3\int \boldsymbol{v}_{sss} \cdot \big[\boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{sss})\big]\mathrm{d}s - 5\int |\boldsymbol{v}_s|^2\boldsymbol{v}_s \cdot \boldsymbol{v}_{ssss}\mathrm{d}s \\
&\quad - \frac{15}{2}\int |\boldsymbol{v}_s|^2\boldsymbol{v}_s \cdot \big[\boldsymbol{v}_{sss} \times (\boldsymbol{v} \times \boldsymbol{v}_s)\big]\mathrm{d}s - 15\int |\boldsymbol{v}_s|^2\boldsymbol{v}_s \cdot \big[\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_{ss})\big]\mathrm{d}s \\
&= |\boldsymbol{v}_{sss}(0)|^2 + 9\int (|\boldsymbol{v}_s|^2)_s|\boldsymbol{v}_{ss}|^2\mathrm{d}s - 3\int |\boldsymbol{v}_s|^2(|\boldsymbol{v}_{ss}|^2)_s\mathrm{d}s - \frac{3}{2}\int (\boldsymbol{v}_s \cdot \boldsymbol{v}_{sss})(|\boldsymbol{v}_s|^2)_s\mathrm{d}s \\
&\quad + \frac{3}{2}\int |\boldsymbol{v}_s|^2(|\boldsymbol{v}_{ss}|^2)_s\mathrm{d}s - \frac{9}{2}\int (|\boldsymbol{v}_s|^2)_s|\boldsymbol{v}_{ss}|^2\mathrm{d}s - 5\int |\boldsymbol{v}_s|^2\boldsymbol{v}_s \cdot \boldsymbol{v}_{ssss}\mathrm{d}s \\
&\quad - \frac{45}{4}\int |\boldsymbol{v}_s|^4(|\boldsymbol{v}_s|^2)_s\mathrm{d}s - \frac{15}{2}\int |\boldsymbol{v}_s|^4(|\boldsymbol{v}_s|^2)_s\mathrm{d}s \\
&= |\boldsymbol{v}_{sss}(0)|^2 + \frac{9}{2}\int (|\boldsymbol{v}_s|^2)_s|\boldsymbol{v}_{ss}|^2\mathrm{d}s - \frac{3}{2}\int |\boldsymbol{v}_s|^2(|\boldsymbol{v}_{ss}|^2)_s\mathrm{d}s - \frac{25}{4}\int \big(|\boldsymbol{v}_s|^6\big)_s\mathrm{d}s \\
&\quad - \frac{3}{2}\int (\boldsymbol{v}_s \cdot \boldsymbol{v}_{sss})(|\boldsymbol{v}_s|^2)_s\mathrm{d}s - 5\int |\boldsymbol{v}_s|^2\boldsymbol{v}_s \cdot \boldsymbol{v}_{ssss}\mathrm{d}s \\
&= |\boldsymbol{v}_{sss}(0)|^2 + \int \big\{|\boldsymbol{v}_s|^2|\boldsymbol{v}_{ss}|^2\big\}_s\mathrm{d}s + \frac{7}{2}\int (|\boldsymbol{v}_s|^2)_s|\boldsymbol{v}_{ss}|^2\mathrm{d}s + \frac{7}{2}\int (\boldsymbol{v}_s \cdot \boldsymbol{v}_{sss})(|\boldsymbol{v}_s|^2)_s\mathrm{d}s \\
&= |\boldsymbol{v}_{sss}(0)|^2.
\end{aligned}
$$

Again, we have used (4.4.4), (4.4.5), and integration by parts. Finally, we calculate

$$
\begin{aligned}
I_3 &= -\|\boldsymbol{v}_{sss}\|^2 - 2(\boldsymbol{v}_{sss}, |\boldsymbol{v}_s|^2\boldsymbol{v}_s) - 4(\boldsymbol{v}_{sss}, (\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss})\boldsymbol{v}) - 5(|\boldsymbol{v}_s|^2\boldsymbol{v}_s, \boldsymbol{v}_{sss}) \\
&\quad - 5(|\boldsymbol{v}_s|^2\boldsymbol{v}_s, |\boldsymbol{v}_s|^2\boldsymbol{v}_s) - 10(|\boldsymbol{v}_s|^2\boldsymbol{v}_s, (\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss})\boldsymbol{v}) \\
&\le -\frac{1}{2}\|\boldsymbol{v}_{sss}\|^2 + C\big(\|\boldsymbol{v}_s\|_{L^6(\mathbf{R}_+)}^6 + \|\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss}\|^2\big) \\
&\le -\frac{1}{4}\|\boldsymbol{v}_{sss}\|^2 + C_2.
\end{aligned}
$$

Here, C_2 is a constant depending on C_1. We also used the interpolation inequalities $\|\boldsymbol{v}_s\|_{L^6(\mathbf{R}_+)} \le C\|\boldsymbol{v}_s\|^{2/3}\|\boldsymbol{v}_{ss}\|^{1/3}$, $\|\boldsymbol{v}_s\|_{L^\infty(\mathbf{R}_+)} \le C\|\boldsymbol{v}_s\|^{1/2}\|\boldsymbol{v}_{ss}\|^{1/2}$, and $\|\boldsymbol{v}_{ss}\| \le C\|\boldsymbol{v}_s\|^{1/2}\| \boldsymbol{v}_{sss}\|^{1/2}$. Combining these three estimates, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{v}_{ss}\|^2 - \frac{5}{4}\||\boldsymbol{v}_s|^2\|^2\right\} \le -|\alpha||\boldsymbol{v}_{sss}(0,t)|^2 - \frac{\delta}{4}\|\boldsymbol{v}_{sss}\|^2 + C_2.$$

Integrating over $[0,t]$ yields

$$\begin{aligned}\|\boldsymbol{v}_{ss}(t)\|^2 &+ \int_0^t \left(|\alpha||\boldsymbol{v}_{sss}(0,\tau)|^2 + \frac{\delta}{4}\|\boldsymbol{v}_{sss}(\tau)\|^2\right)\mathrm{d}\tau \\ &\le \|\boldsymbol{v}_{0ss}^\delta\|^2 + \frac{5}{4}\||\boldsymbol{v}_s(t)|^2\|^2 + C_2 t \\ &\le C\|\boldsymbol{v}_{0s}^\delta\|_1^2 + \frac{1}{2}\|\boldsymbol{v}_{ss}(t)\|^2 + C\|\boldsymbol{v}_s(t)\|^6 + C_2 t,\end{aligned}$$

where we have used $\|\boldsymbol{v}_s\|_{L^4(\mathbf{R}_+)} \le C\|\boldsymbol{v}_s\|^{3/4}\|\boldsymbol{v}_{ss}\|^{1/4}$ again. Thus we have

$$\sup_{0\le t\le T}\|\boldsymbol{v}_{ss}(t)\|^2 + \int_0^T \big(|\alpha||\boldsymbol{v}_{sss}(0,t)|^2 + \delta\|\boldsymbol{v}_{sss}(t)\|^2\big)\mathrm{d}t \le C\|\boldsymbol{v}_{0s}^\delta\|_1^2 + C_3 + C_2 T,$$

where C_3 is a constant depending on C_1. Thus if we choose $C_*^2 := CM^2 + C_1^2 + C_3 + C_2 T$, we see that the proposition holds. □

Based on the estimate derived in Proposition 4.4.3, we derive the higher order estimate.

Proposition 4.4.4 *For a natural number k and $M > 0$, let $\boldsymbol{v}$ be a solution of* (4.4.3) *with $|\boldsymbol{v}_0^\delta| = 1$, $\boldsymbol{v}_{0s}^\delta \in H^{1+3k}(\mathbf{R}_+)$, and $\|\boldsymbol{v}_{0s}^\delta\|_{H^{1+3k}(\mathbf{R}_+)} \le M$ satisfying $\boldsymbol{v}_s \in C\big([0,T]; H^{1+3k}(\mathbf{R}_+)\big)$ and $\boldsymbol{v} \in C\big([0,T]; L^\infty(\mathbf{R}_+)\big)$. Then, there is a positive constant C_{**} and $T_1 \in (0,T]$ such that for $\delta \in (0,\delta_*)$, $\boldsymbol{v}$ satisfies*

$$\sup_{0\le t\le T_1}\|\boldsymbol{v}_s(t)\|_{1+3k} \le C_{**}.$$

Here, T_1 depends on $\|\boldsymbol{v}_{0s}\|_3$ and C_{**} depends on C_* and δ_*, but not on $\delta \in (0,\delta_*]$. C_* and δ_* are constants appearing in Proposition 4.4.3.

Proof. From Proposition 4.4.3, we have a $C_* > 0$ and $\delta_* > 0$ such that

$$\sup_{0\le t\le T}\|\boldsymbol{v}_s(t)\|_1 \le C_*$$

holds for $\delta \in (0,\delta_*]$. We also know from Lemma 4.4.2 that $|\boldsymbol{v}| = 1$.

Now, for an integer m with $4 \le m \le 1+3k$, differentiating equation (4.4.3) m times with respect to s yields

$$\begin{aligned}\partial_s^m \boldsymbol{v}_t &= \boldsymbol{v} \times \partial_s^{m+2}\boldsymbol{v} + m\boldsymbol{v}_s \times \partial_s^{m+1}\boldsymbol{v} + \alpha\Big\{\partial_s^{m+3}\boldsymbol{v} + \frac{3}{2}\big(\partial_s^{m+2}\boldsymbol{v}\big) \times (\boldsymbol{v} \times \boldsymbol{v}_s) \\ &\quad + \frac{3}{2}(m+1)\big(\partial_s^{m+1}\boldsymbol{v}\big) \times (\boldsymbol{v} \times \boldsymbol{v}_{ss}) + \frac{3}{2}(m+1)\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \partial_s^{m+1}\boldsymbol{v}) \\ &\quad + \frac{3}{2}\boldsymbol{v}_s \times (\boldsymbol{v} \times \partial_s^{m+2}\boldsymbol{v}) + \frac{3m}{2}\boldsymbol{v}_s \times (\boldsymbol{v}_s \times \partial_s^{m+1}\boldsymbol{v})\Big\} \\ &\quad + \delta\Big\{\partial_s^{m+2}\boldsymbol{v} + 2(\boldsymbol{v}_s \cdot \partial_s^{m+1}\boldsymbol{v})\boldsymbol{v} + \boldsymbol{z}_m\Big\} + \boldsymbol{w}_m,\end{aligned}$$

where $\boldsymbol{z}_m$ and $\boldsymbol{w}_m$ are terms that contain derivatives of $\boldsymbol{v}$ up to order m and are independent of δ. We estimate the solution in the following way.

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_s^{m+1}\boldsymbol{v}\|^2 &= -\big(\partial_s^m\boldsymbol{v}_t \cdot \partial_s^{m+1}\boldsymbol{v}\big)(0,t) - (\partial_s^m\boldsymbol{v}_t, \partial_s^{m+2}\boldsymbol{v}) \\ &= -\big(\partial_s^m\boldsymbol{v}_t \cdot \partial_s^{m+1}\boldsymbol{v}\big)(0,t) - m(\boldsymbol{v}_s \times \partial_s^{m+1}\boldsymbol{v}, \partial_s^{m+2}\boldsymbol{v}) - \alpha\Big\{(\partial_s^{m+3}\boldsymbol{v}, \partial_s^{m+2}\boldsymbol{v}) \\ &\quad + \frac{3}{2}(m+1)(\partial_s^{m+1}\boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{v}_{ss}), \partial_s^{m+2}\boldsymbol{v}) + \frac{3}{2}(m+1)(\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \partial_s^{m+1}\boldsymbol{v}), \partial_s^{m+2}\boldsymbol{v}) \\ &\quad + \frac{3}{2}(\boldsymbol{v}_s \times (\boldsymbol{v} \times \partial_s^{m+2}\boldsymbol{v}), \partial_s^{m+2}\boldsymbol{v}) + \frac{3m}{2}(\boldsymbol{v}_s \times (\boldsymbol{v}_s \times \partial_s^{m+1}\boldsymbol{v}), \partial_s^{m+2}\boldsymbol{v})\Big\} \\ &\quad - \delta\Big\{(\partial_s^{m+2}\boldsymbol{v}, \partial_s^{m+2}\boldsymbol{v}) + 2((\boldsymbol{v}_s \cdot \partial_s^{m+1}\boldsymbol{v})\boldsymbol{v}, \partial_s^{m+2}\boldsymbol{v}) + (\boldsymbol{z}_m, \partial_s^{m+2}\boldsymbol{v})\Big\} - (\boldsymbol{w}_m, \partial_s^{m+2}\boldsymbol{v}).\end{aligned}$$

Each term is estimated by using the fact that $|\boldsymbol{v}| = 1$, (4.4.4), and (4.4.5). The usage of these properties is sometimes hard to notice and somewhat complicated, hence we give a detailed calculation for such term even though the calculus itself is elementary. Set $m_* := \max\{3, m-3\}$. First we have

$$\begin{aligned}-m(\boldsymbol{v}_s \times \partial_s^{m+1}\boldsymbol{v}, \partial_s^{m+2}\boldsymbol{v}) &= m(\boldsymbol{v}_s \times \partial_s^{m+2}\boldsymbol{v}, \partial_s^{m+1}\boldsymbol{v}) \\ &= -m((\boldsymbol{v} \cdot \partial_s^{m+2}\boldsymbol{v})\boldsymbol{v} \times \boldsymbol{v}_s, \partial_s^{m+1}\boldsymbol{v}) + m([(\boldsymbol{v} \times \boldsymbol{v}_s) \cdot \partial_s^{m+2}\boldsymbol{v}]\boldsymbol{v}, \partial_s^{m+1}\boldsymbol{v}) \\ &= \frac{1}{2}m\sum_{j=1}^{m+1}\begin{pmatrix} m+2 \\ j \end{pmatrix}((\partial_s^j\boldsymbol{v} \cdot \partial_s^{m+2-j}\boldsymbol{v})\boldsymbol{v} \times \boldsymbol{v}_s, \partial_s^{m+1}\boldsymbol{v}) \\ &\quad - m([(\boldsymbol{v} \times \boldsymbol{v}_{ss}) \cdot \partial_s^{m+1}\boldsymbol{v}]\boldsymbol{v}, \partial_s^{m+1}\boldsymbol{v}) - m([(\boldsymbol{v} \times \boldsymbol{v}_s) \cdot \partial_s^{m+1}\boldsymbol{v}]\boldsymbol{v}_s, \partial_s^{m+1}\boldsymbol{v}) \\ &\quad + \frac{m}{2}\sum_{j=1}^{m+1}\begin{pmatrix} m+2 \\ j \end{pmatrix}([(\boldsymbol{v} \times \boldsymbol{v}_s) \cdot \partial_s^{m+1}\boldsymbol{v}]\partial_s^j\boldsymbol{v}, \partial_s^{m+2-j}\boldsymbol{v}) \\ &\le C\|\boldsymbol{v}_s\|_m^2,\end{aligned}$$

where C depends on $\|\boldsymbol{v}_s\|_{m_*}$. Next we have

$$
\begin{aligned}
&\frac{3}{2}(m+1)(\partial_s^{m+1}\boldsymbol{v}\times(\boldsymbol{v}\times\boldsymbol{v}_{ss}),\partial_s^{m+2}\boldsymbol{v})\\
&\qquad=\frac{3}{2}(m+1)\Big\{((\boldsymbol{v}_{ss}\cdot\partial_s^{m+1}\boldsymbol{v})\boldsymbol{v},\partial_s^{m+2}\boldsymbol{v})-((\boldsymbol{v}\cdot\partial_s^{m+1}\boldsymbol{v})\boldsymbol{v}_{ss},\partial_s^{m+2}\boldsymbol{v})\Big\}\\
&\qquad=\frac{3}{2}(m+1)\Big\{-\frac{1}{2}\sum_{j=1}^{m+1}\begin{pmatrix} m+2\\ j\end{pmatrix}((\boldsymbol{v}_{ss}\cdot\partial_s^{m+1}\boldsymbol{v})\partial_s^{j}\boldsymbol{v},\partial_s^{m+2-j}\boldsymbol{v})\\
&\qquad\quad+((\boldsymbol{v}_s\cdot\partial_s^{m+1}\boldsymbol{v})\boldsymbol{v}_{ss},\partial_s^{m+1}\boldsymbol{v})+((\boldsymbol{v}\cdot\partial_s^{m+2}\boldsymbol{v})\boldsymbol{v}_{ss},\partial_s^{m+1}\boldsymbol{v})\\
&\qquad\quad+((\boldsymbol{v}\cdot\partial_s^{m+1}\boldsymbol{v})\boldsymbol{v}_{sss},\partial_s^{m+1}\boldsymbol{v})+\big((\boldsymbol{v}\cdot\partial_s^{m+1}\boldsymbol{v})\boldsymbol{v}_{ss}\cdot\partial_s^{m+1}\boldsymbol{v}\big)(0,t)\Big\}\\
&\qquad\le C\big(\|\boldsymbol{v}_s\|_m^2+|\partial_s^{m+1}\boldsymbol{v}(0,t)|^2\big),
\end{aligned}
$$

where C depends on $\|\boldsymbol{v}_s\|_{m_*}$. We continue with

$$
\begin{aligned}
&\frac{3}{2}(m+1)(\boldsymbol{v}_{ss}\times(\boldsymbol{v}\times\partial_s^{m+1}),\partial_s^{m+2}\boldsymbol{v})\\
&\qquad=\frac{3}{2}(m+1)\Big\{((\boldsymbol{v}_{ss}\cdot\partial_s^{m+1}\boldsymbol{v})\boldsymbol{v},\partial_s^{m+2}\boldsymbol{v})-((\boldsymbol{v}\cdot\boldsymbol{v}_{ss})\partial_s^{m+1}\boldsymbol{v},\partial_s^{m+2}\boldsymbol{v})\Big\}\\
&\qquad=\frac{3}{2}(m+1)\Big\{-\frac{1}{2}\sum_{j=1}^{m+1}\begin{pmatrix} m+2\\ j\end{pmatrix}((\boldsymbol{v}_{ss}\cdot\partial_s^{m+1}\boldsymbol{v})\partial_s^{j}\boldsymbol{v},\partial_s^{m+2-j}\boldsymbol{v})\\
&\qquad\quad+((\boldsymbol{v}\cdot\boldsymbol{v}_{ss})_s\partial_s^{m+1}\boldsymbol{v},\partial_s^{m+1}\boldsymbol{v})-\big((\boldsymbol{v}\cdot\boldsymbol{v}_{ss})|\partial_s^{m+1}\boldsymbol{v}|^2\big)(0,t)\Big\}\\
&\qquad\le C\|\boldsymbol{v}_s\|_m^2,
\end{aligned}
$$

where, again, C depends on $\|\boldsymbol{v}_s\|_{m_*}$. From here on, it is assumed that generic constants C depend on $\|\boldsymbol{v}_s\|_{m_*}$ unless explicitly mentioned otherwise. We calculate furthermore

$$
\begin{aligned}
&\frac{3}{2}(\boldsymbol{v}_s\times(\boldsymbol{v}\times\partial_s^{m+2}\boldsymbol{v}),\partial_s^{m+2}\boldsymbol{v})=\frac{3}{2}((\boldsymbol{v}_s\cdot\partial_s^{m+2}\boldsymbol{v})\boldsymbol{v},\partial_s^{m+2}\boldsymbol{v})\\
&\quad=\frac{3}{2}\Big\{-\frac{1}{2}((\boldsymbol{v}_s\cdot\partial_s^{m+2}\boldsymbol{v})\boldsymbol{v}_s,\partial_s^{m+1}\boldsymbol{v})-\frac{1}{2}\sum_{j=2}^{m}\begin{pmatrix} m+2\\ j\end{pmatrix}((\boldsymbol{v}_s\cdot\partial_s^{m+2}\boldsymbol{v})\partial_s^{j}\boldsymbol{v},\partial_s^{m+2-j}\boldsymbol{v})\Big\}\\
&\quad=\frac{3}{4}\Big\{((\boldsymbol{v}_{ss}\cdot\partial_s^{m+1}\boldsymbol{v})\boldsymbol{v}_s,\partial_s^{m+1}\boldsymbol{v})+\sum_{j=2}^{m}\begin{pmatrix} m+2\\ j\end{pmatrix}\big[((\boldsymbol{v}_{ss}\cdot\partial_s^{m+1}\boldsymbol{v})\partial_s^{j}\boldsymbol{v},\partial_s^{m+2-j}\boldsymbol{v})\\
&\qquad+((\boldsymbol{v}_s\cdot\partial_s^{m+1}\boldsymbol{v})\partial_s^{j+1}\boldsymbol{v},\partial_s^{m+2-j}\boldsymbol{v})+((\boldsymbol{v}_s\cdot\partial_s^{m+1}\boldsymbol{v})\partial_s^{j}\boldsymbol{v},\partial_s^{m+3-j}\boldsymbol{v})\big]\Big\}\\
&\quad\le C\|\boldsymbol{v}_s\|_m^2,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{3m}{2}(\boldsymbol{v}_s \times (\boldsymbol{v}_s \times \partial_s^{m+1}\boldsymbol{v}), \partial_s^{m+2}\boldsymbol{v}) \\
&\qquad = \frac{3m}{2}\Big\{((\boldsymbol{v}_s \cdot \partial_s^{m+1}\boldsymbol{v})\boldsymbol{v}_s, \partial_s^{m+2}\boldsymbol{v}) - ((\boldsymbol{v}_s \cdot \boldsymbol{v}_s)\partial_s^{m+1}\boldsymbol{v}, \partial_s^{m+2}\boldsymbol{v})\Big\} \\
&\qquad = \frac{3m}{2}\Big\{ - ((\boldsymbol{v}_{ss} \cdot \partial_s^{m+1}\boldsymbol{v})\boldsymbol{v}_s, \partial_s^{m+1}\boldsymbol{v}) + \frac{1}{2}((\boldsymbol{v}_s \cdot \boldsymbol{v}_s)_s\partial_s^{m+1}\boldsymbol{v}, \partial_s^{m+1}\boldsymbol{v})\Big\} \\
&\qquad \le C\|\boldsymbol{v}_s\|_m^2.
\end{aligned}
$$

Next, we estimate the boundary terms.

$$
\begin{aligned}
\partial_s^m\boldsymbol{v}_t \cdot \partial_s^{m+1}\boldsymbol{v} &= (\boldsymbol{v} \times \partial_s^{m+2}\boldsymbol{v}) \cdot \partial_s^{m+1}\boldsymbol{v} + \alpha\Big\{\partial_s^{m+3}\boldsymbol{v} \cdot \partial_s^{m+1}\boldsymbol{v} + \frac{3}{2}[\partial_s^{m+2}\boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{v}_s)] \cdot \partial_s^{m+1}\boldsymbol{v} \\
&\quad + \frac{3}{2}(m+1)[\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \partial_s^{m+1}\boldsymbol{v})] \cdot \partial_s^{m+1}\boldsymbol{v} + \frac{3}{2}[\boldsymbol{v}_s \times (\boldsymbol{v} \times \partial_s^{m+2}\boldsymbol{v})] \cdot \partial_s^{m+1}\boldsymbol{v} \\
&\quad + \frac{3m}{2}[\boldsymbol{v}_s \times (\boldsymbol{v}_s \times \partial_s^{m+1}\boldsymbol{v})] \cdot \partial_s^{m+1}\boldsymbol{v}\Big\} + \delta\Big\{\partial_s^{m+2}\boldsymbol{v} \cdot \partial_s^{m+1}\boldsymbol{v} \\
&\quad + 2(\boldsymbol{v}_s \cdot \partial_s^{m+1}\boldsymbol{v})(\boldsymbol{v} \cdot \partial_s^{m+1}\boldsymbol{v}) + \boldsymbol{z}_m \cdot \partial_s^{m+1}\boldsymbol{v}\Big\} + \boldsymbol{w}_m \cdot \partial_s^{m+1}\boldsymbol{v},
\end{aligned}
$$

thus we have

$$
\begin{aligned}
\big(\partial_s^m\boldsymbol{v}_t \cdot \partial_s^{m+1}\boldsymbol{v}\big)(0,t) &= [(\boldsymbol{v} \times \partial_s^{m+2}\boldsymbol{v}) \cdot \partial_s^{m+1}\boldsymbol{v}](0,t) \\
&\quad + \alpha\Big\{\partial_s^{m+3}\boldsymbol{v} \cdot \partial_s^{m+1}\boldsymbol{v} + \frac{3}{2}(m+1)[\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \partial_s^{m+1}\boldsymbol{v})] \cdot \partial_s^{m+1}\boldsymbol{v}\Big\}(0,t) \\
&\quad + \delta\Big\{\partial_s^{m+2}\boldsymbol{v} \cdot \partial_s^{m+1}\boldsymbol{v} + \boldsymbol{z}_m \cdot \partial_s^{m+1}\boldsymbol{v}\Big\}(0,t) + (\boldsymbol{w}_m \cdot \partial_s^{m+1}\boldsymbol{v})(0,t).
\end{aligned}
$$

Again, we estimate each term separately. For example,

$$
\begin{aligned}
\big\{[\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \partial_s^{m+1}\boldsymbol{v})] \cdot \partial_s^{m+1}\boldsymbol{v}\big\}(0,t) &= \Big\{(\boldsymbol{v}_{ss} \cdot \partial_s^{m+1}\boldsymbol{v})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{v}_{ss})\partial_s^{m+1}\boldsymbol{v}\Big\} \cdot \partial_s^{m+1}\boldsymbol{v}\Big|_{s=0} \\
&= -\frac{1}{2}\sum_{j=1}^{m}\begin{pmatrix} m+1 \\ j \end{pmatrix}(\boldsymbol{v}_{ss} \cdot \partial_s^{m+1}\boldsymbol{v})(\partial_s^j\boldsymbol{v} \cdot \partial_s^{m+1-j}\boldsymbol{v})\Big|_{s=0} \\
&\le C\big(\|\boldsymbol{v}_s\|_m^2 + |\partial_s^{m+1}\boldsymbol{v}(0,t)|^2\big)
\end{aligned}
$$

holds. Combining the estimates yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_s^{m+1}\boldsymbol{v}\|^2 + \frac{1}{2}|\alpha||\partial_s^{m+2}\boldsymbol{v}(0,t)|^2 \\
&\qquad \le C\big(\|\boldsymbol{v}_s\|_m^2 + |\partial_s^{m+1}\boldsymbol{v}(0,t)|^2\big) + (\boldsymbol{v} \times \partial_s^{m+2}\boldsymbol{v}) \cdot \partial_s^{m+1}\boldsymbol{v}\Big|_{s=0} + \alpha\partial_s^{m+3}\boldsymbol{v} \cdot \partial_s^{m+1}\boldsymbol{v}\Big|_{s=0} \\
&\qquad\qquad + \delta(\partial_s^{m+2}\boldsymbol{v} \cdot \partial_s^{m+1}\boldsymbol{v})\Big|_{s=0}.
\end{aligned}
$$

On the other hand, from the boundary condition we see that the solution satisfies $\partial_t^j\boldsymbol{v}_s(0,t)$ $= \boldsymbol{0}$ for any j with $0 \le j \le k$. Rewriting this by virtue of (4.4.3) yields $\alpha^j(\partial_s^{3j+1}\boldsymbol{v})(0,t) =$

$\boldsymbol{F}(\boldsymbol{v}, \boldsymbol{v}_s, \dots, \partial_s^{3j}\boldsymbol{v})(0,t)$, i.e. boundary terms with $(3j+1)$-th order derivative can be expressed in terms of boundary terms with derivatives up to order $3j$. For $m = 3j+1$, we have

$$
\begin{aligned}
(4.4.6) \quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_s^{3j+2}\boldsymbol{v}\|^2 + \frac{|\alpha|}{2}|\partial_s^{3j+3}\boldsymbol{v}(0,t)|^2 \\
&\le C\big(\|\boldsymbol{v}_s\|_{1+3j}^2 + |\partial_s^{3j+2}\boldsymbol{v}(0,t)|^2\big) + (\boldsymbol{v}\times\partial_s^{3j+3}\boldsymbol{v})\cdot\partial_s^{3j+2}\boldsymbol{v}\Big|_{s=0} \\
&\quad + \alpha\partial_s^{3(j+1)+1}\boldsymbol{v}\cdot\partial_s^{3j+2}\boldsymbol{v}\Big|_{s=0} + \delta(\partial_s^{3j+3}\boldsymbol{v}\cdot\partial_s^{3j+2}\boldsymbol{v})\Big|_{s=0} \\
&\le C\big(\|\boldsymbol{v}_s\|_{1+3j}^2 + |\partial_s^{3j+2}\boldsymbol{v}(0,t)|^2\big) + (\boldsymbol{v}\times\partial_s^{3j+3}\boldsymbol{v})\cdot\partial_s^{3j+2}\boldsymbol{v}\Big|_{s=0} \\
&\quad + C\big|\partial_s^{3j+3}\boldsymbol{v}\big|\big|\partial_s^{3j+2}\boldsymbol{v}\big|\Big|_{s=0} + \delta\big|\partial_s^{3j+3}\boldsymbol{v}\big|\big|\partial_s^{3j+2}\boldsymbol{v}\big|\Big|_{s=0}.
\end{aligned}
$$

By a similar estimate, we can show that

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_s^{j+1}\boldsymbol{v}\|^2 + \frac{|\alpha|}{2}|\partial_s^{j+2}\boldsymbol{v}(0,t)|^2 \\
&\le C\big(\|\boldsymbol{v}_s\|_3^3 + |\partial_s^{j+1}\boldsymbol{v}(0,t)|^2\big) + (\boldsymbol{v}\times\partial_s^{j+2}\boldsymbol{v})\cdot\partial_s^{j+1}\boldsymbol{v}\Big|_{s=0} \\
&\quad + |\alpha|\big|\partial_s^{j+3}\boldsymbol{v}\big|\big|\partial_s^{j+1}\boldsymbol{v}\big|\Big|_{s=0} + \delta\big|\partial_s^{j+1}\boldsymbol{v}\big|\big|\partial_s^{j+2}\boldsymbol{v}\big|\Big|_{s=0}
\end{aligned}
$$

holds for $j = 0, 1, 2, 3$. Thus, for $\eta > 0$ we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_s^{j+1}\boldsymbol{v}\|^2 + \frac{|\alpha|}{4}\big|\partial_s^{j+2}\boldsymbol{v}(0,t)\big|^2 \le C\big(\|\boldsymbol{v}_s\|_3^3 + \big|\partial_s^{j+1}\boldsymbol{v}(0,t)\big|^2\big) + \eta\big|\partial_s^{j+3}\boldsymbol{v}(0,t)\big|^2
$$

for $j = 1, 2$, and

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_s^4\boldsymbol{v}\|^2 + \frac{|\alpha|}{4}\big|\partial_s^5\boldsymbol{v}(0,t)\big|^2 &\le C\|\boldsymbol{v}_s\|_3^3 + C\big|\partial_s^4\boldsymbol{v}(0,t)\big|^2, \\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_s\|^2 + \frac{|\alpha|}{4}\big|\partial_s^2\boldsymbol{v}(0,t)\big|^2 &\le C\|\boldsymbol{v}_s\|_3^3.
\end{aligned}
$$

Here, C depends on η and C_*. Combining these estimates, we arrive at

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_s\|_3^2 \le C\|\boldsymbol{v}_s\|_3^3,
$$

where C depends on C_* and δ_* but not on $\delta \in (0, \delta_*]$. As before, the above estimate gives a time-local uniform estimate in $C\big([0,T_1]; H^3(\mathbf{R}_+)\big)$ for some $T_1 \in (0,T]$. From the H^3-estimate and (4.4.6), we can derive the uniform estimate in $C\big([0,T_1]; H^{1+3k}(\mathbf{R}_+)\big)$ in the same manner. Here, T_1 is determined from the H^3-estimate and only depends on $\|\boldsymbol{v}_{0s}\|_3$. $\square$

4.4.3 Passing to the limit $\delta \to +0$

Now, in virtue of passing to the limit $\delta \to +0$, we prove the existence theorem for the case $\alpha < 0$.

Proof of Theorem 4.1.1. Since $\boldsymbol{v}_{0s}^{\delta} \to \boldsymbol{v}_{0s}$ in $H^{1+3k}(\mathbf{R}_+)$ and $\boldsymbol{v}_0^{\delta} \to \boldsymbol{v}_0$ in $L^\infty(\mathbf{R}_+)$ as $\delta \to +0$, by taking $\delta_* > 0$ smaller if necessary, we have $\|\boldsymbol{v}_{0s}^{\delta}\|_{1+3k} \leq 2\|\boldsymbol{v}_{0s}\|_{1+3k}$ for any $\delta \in (0, \delta_*]$. For such δ, the solution $\boldsymbol{v}^{\delta}$ constructed in Subsection 4.4.1 with initial datum $\boldsymbol{v}_0^{\delta}$ satisfies the assumptions of Lemma 4.4.2 and Proposition 4.4.3 with $M = 2\|\boldsymbol{v}_{0s}\|_{1+3k}$, i.e. $|\boldsymbol{v}^{\delta}| = 1$ and a uniform estimate in $C\big([0,T]; H^{1+3k}(\mathbf{R}_+)\big)$ for some $T > 0$ hold. For any $\delta, \delta' \in (0, \delta_*]$, $\boldsymbol{V} := \boldsymbol{v}^{\delta'} - \boldsymbol{v}^{\delta} - (\boldsymbol{v}_0^{\delta'} - \boldsymbol{v}_0^{\delta})$ satisfies

$$
\begin{cases}
\boldsymbol{V}_t = \alpha \boldsymbol{V}_{sss} + \boldsymbol{v}^{\delta'} \times \boldsymbol{V}_{ss} + 3\alpha \boldsymbol{V}_{ss} \times (\boldsymbol{v}^{\delta'} \times \boldsymbol{v}_s^{\delta'}) + \delta'(\boldsymbol{v}_{ss}^{\delta'} + |\boldsymbol{v}_s^{\delta'}|^2 \boldsymbol{v}^{\delta'}), & \\
\qquad -\delta(\boldsymbol{v}_{ss}^{\delta} + |\boldsymbol{v}_s^{\delta}|^2 \boldsymbol{v}^{\delta}) + \boldsymbol{F}, & s > 0, t > 0, \\
\boldsymbol{V}(s, 0) = \mathbf{0}, & s > 0, \\
\boldsymbol{V}_s(0, t) = \mathbf{0}, & t > 0,
\end{cases}
$$

where $\boldsymbol{F}$ is the sum of lower order terms of $\boldsymbol{V}$ and depends linearly on $\boldsymbol{V}_0 := \boldsymbol{v}_0^{\delta'} - \boldsymbol{v}_0^{\delta}$. By a standard energy method, we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{V}\|^2 &\leq \alpha \boldsymbol{V}(0,t) \cdot \boldsymbol{V}_{ss}(0,t) + C\|\boldsymbol{V}\|_1^2 + C\big[(\delta + \delta') + \|\boldsymbol{V}_0\|_{L^\infty(\mathbf{R}_+)}^2 + \|\boldsymbol{V}_{0s}\|_2^2\big], \\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{V}_s\|^2 &\leq -\frac{|\alpha|}{2}|\boldsymbol{V}_{ss}(0,t)|^2 + C\|\boldsymbol{V}\|_1^2 + C\big[(\delta + \delta') + \|\boldsymbol{V}_0\|_{L^\infty(\mathbf{R}_+)}^2 + \|\boldsymbol{V}_{0s}\|_3^2\big],
\end{aligned}
$$

where C is independent of δ and δ'. Here, we have used $\boldsymbol{v}^{\delta'} \cdot (\boldsymbol{v}^{\delta'} - \boldsymbol{v}^{\delta})_{ss} = -\boldsymbol{v}_s^{\delta'} \cdot (\boldsymbol{v}^{\delta'} - \boldsymbol{v}^{\delta})_s - (\boldsymbol{v}^{\delta'} - \boldsymbol{v}^{\delta})_s \cdot \boldsymbol{v}_s^{\delta} - (\boldsymbol{v}^{\delta'} - \boldsymbol{v}^{\delta}) \cdot \boldsymbol{v}_{ss}^{\delta}$, which follows from the fact that $|\boldsymbol{v}^{\delta}| = |\boldsymbol{v}^{\delta'}| = 1$. The above estimate implies

$$
\|\boldsymbol{V}\|_1^2 \leq CT\big[(\delta + \delta') + \|\boldsymbol{V}_0\|_{L^\infty(\mathbf{R}_+)}^2 + \|\boldsymbol{V}_{0s}\|_3^2\big],
$$

where C is independent of δ and δ'. Thus, there is a $\boldsymbol{v}$ such that $\boldsymbol{v}^{\delta} \to \boldsymbol{v}$ in $C\big([0,T]; L^\infty(\mathbf{R}_+)\big)$ and $\boldsymbol{v}_s^{\delta} \to \boldsymbol{v}_s$ in $C\big([0,T]; L^2(\mathbf{R}_+)\big)$ as $\delta \to +0$. Combining these convergence with the uniform estimate, we have a solution $\boldsymbol{v}$ to (4.2.1) such that $\boldsymbol{v}_s \in \bigcap_{j=0}^{k} W^{j,\infty}\big(0,T; H^{1+3j}(\mathbf{R}_+)\big)$ and $|\boldsymbol{v}| = 1$. Again, since the initial datum can be approximated by a smooth function, we have a solution $\boldsymbol{v} \in \tilde{X}_T^k$, i.e. the continuity with respect to t can be recovered. The uniform estimate obtained in the previous subsection is essentially the energy estimate for $\boldsymbol{v}$, from which the uniqueness of the solution follows. Based on this estimate of the solution and a sequence of smooth initial datum $\{\boldsymbol{v}_0^n\}_{n=1}^{\infty}$ such that $\boldsymbol{v}_0^n \to \boldsymbol{v}_0$ in $L^\infty(\mathbf{R}_+)$ and $\boldsymbol{v}_{0s}^n \to \boldsymbol{v}_{0s}$ in $H^{1+3k}(\mathbf{R}_+)$ as $n \to +\infty$, we have a solution $\boldsymbol{v}$ satisfying $\boldsymbol{v}_s \in \bigcap_{j=0}^{k} W^{j,\infty}\big(0,T; H^{1+3j}(\mathbf{R}_+)\big)$ and $|\boldsymbol{v}| = 1$ with initial datum $\boldsymbol{v}_0$ satisfying $\boldsymbol{v}_{0s} \in H^{1+3k}(\mathbf{R}_+)$ and $|\boldsymbol{v}_0| = 1$ in the same manner as we did with $\delta \to +0$.

Since a compactness argument is used, the continuity in t is lost and we are unable to recover the continuity in t. One of the standard method to recover it is to prove the strong continuity of $\boldsymbol{v}$ at $t = 0$ via solving the problem reverse in time. Unfortunately, our problem is not reversible in time and we do not have any new ideas to recover the continuity.

Finally, as we mentioned in the introduction, we can construct $\boldsymbol{x}$ from $\boldsymbol{v}$. $\square$

4.5 Construction of Solution in the Case $\alpha > 0$

4.5.1 Existence of Solution

We construct the solution in a similar manner as in the case $\alpha < 0$. For $n \geq 2$, we define $\boldsymbol{v}^{(n)}$ by

$$\begin{cases} \boldsymbol{v}_t^{(n)} = \alpha \boldsymbol{v}_{sss}^{(n)} + \mathrm{A}(\boldsymbol{v}^{(n-1)}, \partial_s)\boldsymbol{v}^{(n)} - \dfrac{3}{2}\alpha|\boldsymbol{v}_s^{(n-1)}|^2\boldsymbol{v}_s^{(n-1)} + \delta|\boldsymbol{v}_s^{(n-1)}|^2\boldsymbol{v}^{(n-1)}, & s > 0, t > 0, \\ \boldsymbol{v}^{(n)}(s,0) = \boldsymbol{v}_0^{\delta,R}(s), & s > 0, \\ \boldsymbol{v}^{(n)}(0,t) = \boldsymbol{e}_3, & t > 0, \\ \boldsymbol{v}_s^{(n)}(0,t) = \boldsymbol{0}, & t > 0, \end{cases}$$

where $\boldsymbol{e}_3 = (0,0,1)$, $\boldsymbol{v}_0^{\delta,R}$ is the same initial datum that is defined in Subsection 4.4.1, and the operator $\mathrm{A}(\boldsymbol{v}^{(n-1)}, \partial_s)$ is the same as in the case $\alpha < 0$. The dependence of $\boldsymbol{v}$ on R is suppressed for brevity. Again, we define $\boldsymbol{v}^{(1)}$ by

$$\boldsymbol{v}^{(1)}(s,t) = \boldsymbol{v}_0^{\delta,R}(s) + \sum_{j=1}^{m} \frac{t^j}{j!}\boldsymbol{Q}_{(j)}(\boldsymbol{v}_0^{\delta,R}(s))$$

so that the compatibility conditions are satisfied at each iteration step. By Theorem 4.3.2, each $\boldsymbol{v}^{(n)}$ is well-defined.

Since the arguments for the uniform estimate and the convergence with respect to n and R are the same as in the case $\alpha < 0$, we omit most of the details and just show the basic energy estimates used to derive the uniform estimates. For any $\eta > 0$ we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_s^{(n)}\|^2 &= -(\boldsymbol{v}_{ss}^{(n)}, \boldsymbol{v}_t^{(n)}) \\ &\leq \frac{\alpha}{2}|\boldsymbol{v}_{ss}^{(n)}(0,t)|^2 - \frac{\delta}{2}\|\boldsymbol{v}_{ss}^{(n)}\|^2 + C\|\boldsymbol{v}_s^{(n-1)}\|_1^2 \\ &\leq \eta\|\boldsymbol{v}_{sss}^{(n)}\|^2 + C_\eta\|\boldsymbol{v}_{ss}^{(n)}\|^2 - \frac{\delta}{2}\|\boldsymbol{v}_{ss}^{(n)}\|^2 + C\|\boldsymbol{v}_s^{(n-1)}\|_1^2, \\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_{ss}^{(n)}\|^2 &= -(\boldsymbol{v}_{sss}^{(n)}, \boldsymbol{v}_{st}^{(n)}) \\ &\leq \frac{\alpha}{2}|\boldsymbol{v}_{sss}^{(n)}(0,t)|^2 - \delta\|\boldsymbol{v}_{sss}^{(n)}\|^2 + \eta\|\boldsymbol{v}_{sss}^{(n)}\|^2 + C_\eta(\|\boldsymbol{v}_{ss}^{(n)}\|^2 + \|\boldsymbol{v}_s^{(n-1)}\|_1^2). \end{aligned}$$

The equation and Sobolev's embedding theorem imply

$$|\boldsymbol{v}_{sss}^{(n)}(0,t)|^2 \le \eta\|\boldsymbol{v}_{sss}^{(n)}\|^2 + C_\eta\|\boldsymbol{v}_{ss}^{(n)}\|^2 + C(1+\|\boldsymbol{v}_s^{(n-1)}\|_1)^2.$$

Combining all the estimates yields

$$\sup_{0\le t\le T}\|\boldsymbol{v}_s^{(n)}(t)\|_1^2 + \int_0^T \|\boldsymbol{v}_s^{(n)}(t)\|_2^2 \mathrm{d}t \le C\int_0^T \|\boldsymbol{v}_s^{(n-1)}(t)\|_1^2 \mathrm{d}t + CT,$$

where C depends on $\|\boldsymbol{v}^{(n-1)}(t)\|_{L^\infty([0,T];L^\infty(\mathbf{R}_+))}$. From this, estimates uniform in n and R can be obtained by induction with respect to n.

4.5.2 Uniform Estimate of Solution with respect to δ

As before, we derive a uniform estimate. First we prove the following.

Lemma 4.5.1 *If $\boldsymbol{v}$ is a solution of* (4.2.4) *with $\boldsymbol{v}_s \in C\big([0,T],H^2(\mathbf{R}_+)\big)$, $\boldsymbol{v}\in C\big([0,T];L^\infty(\mathbf{R}_+)\big)$, and $|\boldsymbol{v}_0^\delta|=1$, then $|\boldsymbol{v}|=1$ in $\mathbf{R}_+\times[0,T]$.*

Proof. As in the proof of Lemma 4.4.2, if we set $h(s,t) := |\boldsymbol{v}(s,t)|^2-1$, h satisfies

$$\begin{cases} h_t = \alpha h_{sss} + \delta h_{ss} + (2\delta|\boldsymbol{v}_s|^2 + 3\alpha(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss}))h, & s>0, t>0,\\ h(s,0)=0, & s>0,\\ h(0,t)=0, & t>0,\\ h_s(0,t)=0, & t>0. \end{cases}$$

It is easy to see that for any $\eta>0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 \le C_\eta\|h\|^2 - \delta\|h_s\|^2 + \eta\|h_s\|^2$$

holds. Thus, after choosing $\eta>0$ sufficiently small, $h\equiv 0$ follows. □

As before, we rewrite the nonlinear terms in (4.2.4) into its original form.

$$(4.5.1)\quad \begin{cases} \boldsymbol{v}_t = \boldsymbol{v}\times\boldsymbol{v}_{ss} + \alpha\Big\{\boldsymbol{v}_{sss} + \dfrac{3}{2}\boldsymbol{v}_{ss}\times(\boldsymbol{v}\times\boldsymbol{v}_s) \\ \qquad\qquad + \dfrac{3}{2}\boldsymbol{v}_s\times(\boldsymbol{v}\times\boldsymbol{v}_{ss})\Big\} + \delta\big(\boldsymbol{v}_{ss}+|\boldsymbol{v}_s|^2\boldsymbol{v}\big), & s>0, t>0,\\ \boldsymbol{v}(s,0)=\boldsymbol{v}_0^\delta(s), & s>0,\\ \boldsymbol{v}(0,t)=\boldsymbol{e}_3, & t>0,\\ \boldsymbol{v}_s(0,t)=\mathbf{0}, & t>0. \end{cases}$$

Now, we derive a basic uniform estimate with respect to δ. The main method and properties used for it are the same as in the case $\alpha<0$, namely, utilizing $|\boldsymbol{v}|=1$, (4.4.4), and

(4.4.5), but the energy is slightly modified and is different from the higher order conserved quantity. First we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_s\|^2 &= \frac{\alpha}{2}|\boldsymbol{v}_{ss}(0,t)|^2 - \delta\|\boldsymbol{v}_{ss}\|^2,\\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_{ss}\|^2 &\leq C\|\boldsymbol{v}_s\|_2^2(1+\|\boldsymbol{v}_s\|_2) + \frac{\alpha}{2}|\boldsymbol{v}_{sss}(0,t)|^2 - \delta\|\boldsymbol{v}_{sss}\|^2\\
&\leq C\|\boldsymbol{v}_s\|_2^2(1+\|\boldsymbol{v}_s\|_2) - \delta\|\boldsymbol{v}_{sss}\|^2,
\end{aligned}$$

where we have used $|\boldsymbol{v}_{sss}(0,t)|^2 \leq C\|\boldsymbol{v}_s\|_2^2(1+\|\boldsymbol{v}_s\|_2)$, which follows from the boundary condition and the equation. To close the estimate, we will derive estimates for $\boldsymbol{v}_{sss}$. However, like the estimates above, the boundary terms have a bad sign unlike in the case $\alpha < 0$. Thus, we must modify the energy to obtain the desired estimate. Specifically, to obtain an estimate for $\boldsymbol{v}_{sss}$, we use the following.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{v}_{sss}\|^2 + \frac{2}{\alpha}(\boldsymbol{v}\times\boldsymbol{v}_{ss},\boldsymbol{v}_{sss}) + \frac{2\delta}{\alpha}(\boldsymbol{v}_{ss},\boldsymbol{v}_{sss})\right\} \leq C\|\boldsymbol{v}_s\|_2^2(1+\|\boldsymbol{v}_s\|_2^2).$$

In each estimate, C is independent of δ. Combining the three estimates, we obtain a uniform estimate for $\|\boldsymbol{v}_s\|_2$ for sufficiently small δ. We denote this threshold as δ_*.

The reason we modified the energy from the Sobolev norm is to take care of the boundary term. If we directly estimate $\|\boldsymbol{v}_{sss}\|^2$, boundary term of the form $\boldsymbol{v}_{sss}(0)\cdot\partial_s^5\boldsymbol{v}(0)$ comes out and the order of derivative is too high to estimate. We can cancel out this term by adding a lower order modification term in the energy. This kind of modification is needed to close the estimate for $\|\boldsymbol{v}_s\|_{2+3k}$ with $k\in\mathbf{N}$. We use the above energy that we just derived an estimate for as an example to demonstrate the idea behind finding the correct modifying term. Taking the trace $s=0$ in the equation yields

$$\alpha\boldsymbol{v}_{sss}(0,t) + (\boldsymbol{v}\times\boldsymbol{v}_{ss})(0,t) + \delta\boldsymbol{v}_{ss}(0,t) = \mathbf{0}$$

for any $t>0$. Thus, replacing $\|\boldsymbol{v}_{sss}\|^2$ with $\|\boldsymbol{v}_{sss}\|^2 + \frac{2}{\alpha}(\boldsymbol{v}\times\boldsymbol{v}_{ss},\boldsymbol{v}_{sss}) + \frac{2\delta}{\alpha}(\boldsymbol{v}_{ss},\boldsymbol{v}_{sss})$ changes the boundary term from $\boldsymbol{v}_{sss}(0,t)\cdot\partial_s^5\boldsymbol{v}(0,t)$ to $\big(\boldsymbol{v}_{sss}(0,t) + \frac{1}{\alpha}\boldsymbol{v}\times\boldsymbol{v}_{ss}(0,t) + \delta\boldsymbol{v}_{ss}(0,t)\big)\cdot\partial_s^5\boldsymbol{v}(0,t)$, which is zero.

We continue the estimate in this pattern. Suppose that we have a uniform estimate $\sup_{0\leq t\leq T}\|\boldsymbol{v}_s(t)\|_{2+3(i-1)} \leq M$ for some $i\geq 1$. For $j=1,2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_s^{3i+j}\boldsymbol{v}\|^2 \leq C(1+\|\boldsymbol{v}_s\|_{2+3i}^2),$$

where we have used $|\partial_s^{3(i+1)}\boldsymbol{v}(0)|^2 \leq C\|\boldsymbol{v}_s\|_{2+3i}^2$. Here, C depends on M and δ_*, but not on δ. Set $\boldsymbol{W}_{(m)}(\boldsymbol{v}) := \boldsymbol{P}_{(m)}(\boldsymbol{v}) - \alpha^m\partial_s^{3m}\boldsymbol{v}$, which is $\boldsymbol{P}_{(m)}(\boldsymbol{v})$ without the highest order derivative term. Then, the final estimate is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\partial_s^{3(i+1)}\boldsymbol{v}\|^2 + \frac{2}{\alpha^{i+1}}(\boldsymbol{W}_{(i+1)}(\boldsymbol{v}),\partial_s^{3(i+1)}\boldsymbol{v})\right\} \leq C\|\boldsymbol{v}_s\|_{2+3i}^2 + C,$$

where, again, C depends on M, but not on δ. Thus, we have proven the following time-local uniform estimate by induction.

Proposition 4.5.2 *For a natural number k and $M > 0$, let $\boldsymbol{v}$ be a solution of* (4.5.1) *with $|\boldsymbol{v}_0^\delta| = 1$, $\boldsymbol{v}_{0s}^\delta \in H^{2+3k}(\mathbf{R}_+)$, and $\|\boldsymbol{v}_{0s}^\delta\|_{H^{2+3k}(\mathbf{R}_+)} \le M$ satisfying $\boldsymbol{v}_s \in C\big([0,T]; H^{2+3k}(\mathbf{R}_+)\big)$ and $\boldsymbol{v} \in C\big([0,T]; L^\infty(\mathbf{R}_+)\big)$. Then, there is a $C_{**} > 0$ and $T_1 \in (0,T]$ such that for $0 \le \delta \le \delta_*$, $\boldsymbol{v}$ satisfies*

$$\sup_{0 \le t \le T_1} \|\boldsymbol{v}_s(t)\|_{2+3k} \le C_{**}.$$

Here, T_1 depends on $\|\boldsymbol{v}_{0s}\|_2$ and C_{**} is independent of $\delta \in (0, \delta_*]$.

4.5.3 Passing to the limit $\delta \to +0$

Now we pass to the limit $\delta \to +0$. For $\delta', \delta \in (0, \delta_*]$, we set the difference of the corresponding solutions as $\boldsymbol{V} := \boldsymbol{v}^{\delta'} - \boldsymbol{v}^\delta - (\boldsymbol{v}_0^{\delta'} - \boldsymbol{v}_0^\delta)$. Then, $\boldsymbol{V}$ satisfies

$$\begin{cases} \boldsymbol{V}_t = \boldsymbol{v}^{\delta'} \times \boldsymbol{V}_{ss} + \alpha\big\{\boldsymbol{V}_{sss} + 3\boldsymbol{V}_{ss} \times (\boldsymbol{v}^{\delta'} \times \boldsymbol{v}^{\delta'})\big\} + \delta'\boldsymbol{V}_{ss} + \boldsymbol{G}, & s > 0, t > 0, \\ \boldsymbol{V}(s,0) = \mathbf{0}, & s > 0, \\ \boldsymbol{V}(0,t) = \mathbf{0}, & t > 0, \\ \boldsymbol{V}_s(0,t) = \mathbf{0}, & t > 0, \end{cases}$$

where $\boldsymbol{G}$ is the collection of terms that are lower order in $\boldsymbol{V}$ and depends linearly on $\boldsymbol{V}_0 := \boldsymbol{v}_0^{\delta'} - \boldsymbol{v}_0^\delta$. By a standard energy method, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{V}\|_3^2 \le C\|\boldsymbol{V}\|_3^2 + C\big[(\delta' + \delta) + \|\boldsymbol{V}_0\|_{L^\infty(\mathbf{R}_+)}^2 + \|\boldsymbol{V}_{0s}\|_3^2\big],$$

where C depends on C_{**} defined in Proposition 4.5.2. Here, we have used identities such as

$$\boldsymbol{v}_s^{\delta'} \times \partial_s^4(\boldsymbol{v}^{\delta'} - \boldsymbol{v}^\delta) = \boldsymbol{v}_s^{\delta'} \times \partial_s^4\boldsymbol{v}^{\delta'} - \boldsymbol{v}_s^\delta \times \partial_s^4\boldsymbol{v}^\delta - (\boldsymbol{v}^{\delta'} - \boldsymbol{v}^\delta)_s \times \partial_s^4\boldsymbol{v}^\delta,$$

to obtain the estimate. From this estimate, we see that $\boldsymbol{v}^\delta \to \boldsymbol{v}$ in $C\big([0,T]; L^\infty(\mathbf{R}_+)\big)$ and $\boldsymbol{v}_s^\delta \to \boldsymbol{v}_s$ in $C\big([0,T]; H^2(\mathbf{R}_+)\big)$ as $\delta \to +0$, and $\boldsymbol{v}$ is the solution to (4.2.2). Combining this with the uniform estimate, we see that $\boldsymbol{v}_s \in \bigcap_{j=0}^k W^{j,\infty}\big(0,T; H^{2+3(k-j)}(\mathbf{R}_+)\big)$. As before, the uniform estimate is essentially the energy estimate of the solution to the limit problem, and after an approximation argument on the initial datum, the regularity assumption on the initial datum can be relaxed. Thus we have proven Theorem 4.1.2.

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Appendix A

Generalized Hasimoto Transformation

A.1 Remark on the generalized Hasimoto Transformation

In this appendix, we use our initial-boundary value problem for the LIE to demonstrate the generalized Hasimoto transformation. As mentioned in Chapter 1, this transformation was first constructed by Koiso [24] in a more geometrically generalized setting than our problem. More specifically, the unknown variable $\boldsymbol{v}$ takes values in a general manifold. Recall that in our analysis, $|\boldsymbol{v}| = 1$, i.e. $\boldsymbol{v}$ takes values in the unit sphere $\boldsymbol{S}^2$.

We restate the problems for convenience.

$$\begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss}, & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t > 0, \end{cases} \tag{A.1.1}$$

$$\begin{cases} \mathrm{i}q_t = q_{ss} + \dfrac{1}{2}|q|^2 q, & s > 0, t > 0, \\ q(s,0) = q_0(s), & s > 0, \\ q_s(0,t) = 0, & t > 0. \end{cases} \tag{A.1.2}$$

Here, $\mathrm{i} = \sqrt{-1}$ and we assume that $|\boldsymbol{v}_0| \equiv 1$ and the compatibility conditions mentioned in Chapter 2 are satisfied. We first derive compatibility conditions for (A.1.2).

Lemma A.1.1 *The compatibility conditions for* (A.1.2) *are that for* $n \in \mathbf{N}$,

$$\partial_s^{2n+1} q_0(0) = 0.$$

Proof. We prove that a smooth solution q of (A.1.2) satisfies $\partial_s^{2n+1}q(0,t)=0$ for $n\in\mathbf{N}$ and $t>0$ by induction. It is obvious for $n=1$. Assume that it holds up to $n-1$ for some $n\geq 2$. Differentiating the equation $(2n-1)$ times with respect to s, we have

$$\mathrm{i}\partial_s^{2n-1}q_t=\partial_s^{2n+1}q+\frac{1}{2}\partial_s^{2n-1}\left\{|q|^2q\right\}.$$

Since the last term always contains derivatives with order less then or equal to $2n-1$, we get for any $t>0$

$$\partial_s^{2n+1}q(0,t)=0,$$

and the trace at $t=0$ yields the desired assertion. $\square$

A.2 LIE to the nonlinear Schrödinger equation

Given a q_0 satisfying the compatibility conditions, we first transform (A.1.1) to (A.1.2). Assume that we have a smooth solution of (A.1.1) with an appropriate initial datum which will be specified later. The solution will necessarily satisfy $|\boldsymbol{v}(s,t)|\equiv 1$. The idea is to construct a basis of the tangent space of the unit sphere $\boldsymbol{S}^2$ that is parallel to the curve $\boldsymbol{v}$ on $\boldsymbol{S}^2$. First we construct a vector $\boldsymbol{e}(s,t)$ orthogonal to $\boldsymbol{v}$ with unit length satisfying

$$\nabla_s\boldsymbol{e}=\mathbf{0},$$

where ∇_s is the covariant derivative along $\boldsymbol{v}$. Suppose that such a vector $\boldsymbol{e}$ exists. Since we know that $\boldsymbol{v}$ is the unit normal of $\boldsymbol{S}^2$, we have

$$\nabla_s\boldsymbol{e}=\boldsymbol{e}_s-(\boldsymbol{e}_s\cdot\boldsymbol{v})\boldsymbol{v}=\boldsymbol{e}_s+(\boldsymbol{e}\cdot\boldsymbol{v}_s)\boldsymbol{v}=0,$$

where we have used $\boldsymbol{e}\cdot\boldsymbol{v}\equiv 0$. The above relation is a necessary condition that $\boldsymbol{e}$ should satisfy. Conversely, for any $t\geq 0$, let $\boldsymbol{e}(s,t)$ be the solution of the following linear ordinary differential equation in s

$$\text{(A.2.1)}\qquad \begin{cases} \boldsymbol{e}_s+(\boldsymbol{e}\cdot\boldsymbol{v}_s)\boldsymbol{v}=\mathbf{0}, & s>0,\\ \boldsymbol{e}(0,t)=\boldsymbol{e}_1, \end{cases}$$

where $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ denotes the standard orthonormal basis of $\mathbf{R}^3$. We see that

$$(\boldsymbol{e}\cdot\boldsymbol{v})_s=\boldsymbol{e}_s\cdot\boldsymbol{v}+\boldsymbol{e}\cdot\boldsymbol{v}_s=0,$$

and $\boldsymbol{e}\cdot\boldsymbol{v}\equiv 0$. This and

$$(\boldsymbol{e}\cdot\boldsymbol{e})_s=2(\boldsymbol{e}_s\cdot\boldsymbol{e})=-2(\boldsymbol{e}\cdot\boldsymbol{v}_s)\boldsymbol{v}\cdot\boldsymbol{e}=0,$$

yield $|\boldsymbol{e}| \equiv 1$, and $\nabla_s \boldsymbol{e} = \mathbf{0}$. Thus, the solution to (A.2.1) is the desired vector. From this, we see that $\{\boldsymbol{v}, \boldsymbol{e}, \boldsymbol{v} \times \boldsymbol{e}\}$ is an orthonormal basis in $\mathbf{R}^3$ for every $s \geq 0$ and $t \geq 0$. Since $\boldsymbol{v} \cdot \boldsymbol{v}_s \equiv 0$ and $\boldsymbol{v} \cdot \boldsymbol{v}_t \equiv 0$ from $|\boldsymbol{v}| \equiv 1$, we can decompose $\boldsymbol{v}_s$ and $\boldsymbol{v}_t$ as

$$\boldsymbol{v}_s = q_1 \boldsymbol{e} + q_2(\boldsymbol{v} \times \boldsymbol{e}), \quad \boldsymbol{v}_t = p_1 \boldsymbol{e} + p_2(\boldsymbol{v} \times \boldsymbol{e}). \tag{A.2.2}$$

The q_i and p_i $(i = 1, 2)$ are functions of s and t. From (A.2.1) and (A.2.2) it follows that

$$\boldsymbol{e}_s = -(\boldsymbol{e} \cdot \boldsymbol{v}_s)\boldsymbol{v} = -q_1 \boldsymbol{v}.$$

From $\boldsymbol{e} \cdot \boldsymbol{v} \equiv 0$ and (A.2.2) we deduce $\boldsymbol{e}_t \cdot \boldsymbol{v} = -\boldsymbol{e} \cdot \boldsymbol{v}_t = -p_1$, so that with the help of $|\boldsymbol{e}| \equiv 1$, we get

$$\boldsymbol{e}_t = -p_1 \boldsymbol{v} + \alpha(\boldsymbol{v} \times \boldsymbol{e}),$$

where α is an unknown function. From the equality $\boldsymbol{v}_{st} = \boldsymbol{v}_{ts}$ and comparing the components, we see that

$$q_{1t} = p_{1s} + \alpha q_2, \quad q_{2t} = p_{2s} - \alpha q_1.$$

On the other hand, from $\boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss}$ we get

$$p_1 = -q_{2s}, \ p_2 = q_{1s}.$$

Finally from $\boldsymbol{e}_{ts} = \boldsymbol{e}_{st}$ we have

$$\alpha_s = p_1 q_2 - q_1 p_2 = -\frac{1}{2}\left\{(q_1)^2_s + (q_2)^2_s\right\},$$

so that $\alpha = -\frac{1}{2}\left\{(q_1)^2 + (q_2)^2\right\} + \alpha(0, t)$. Since $\boldsymbol{e}(0, t) = \boldsymbol{e}_1$, we see that $\boldsymbol{e}_t(0, t) \equiv \mathbf{0}$, and hence $\alpha(0, t) = 0$. Then, $q := q_1 - \mathrm{i}q_2$ satisfies

$$\mathrm{i}q_t = q_{ss} + \frac{1}{2}|q|^2 q.$$

Since $\boldsymbol{v}(0, t) = \boldsymbol{e}_3$, differentiating this with respect to t yields $\boldsymbol{v}_t(0, t) = \mathbf{0}$, so that the boundary condition for q becomes $q_s(0, t) = 0$. We are left to determine $\boldsymbol{v}_0$ from a given initial datum $q_0 = q_{01} - \mathrm{i}q_{02}$ of (A.1.2). From the arguement above, we naturally arrive at defining $\boldsymbol{v}_0$ as the solution of

$$\begin{cases} \boldsymbol{v}_{0s} = q_{01}\boldsymbol{e}^1 + q_{02}\boldsymbol{e}^2, & s > 0, \\ \boldsymbol{e}^1_s = -q_{01}\boldsymbol{v}_0, & s > 0, \\ \boldsymbol{e}^2_s = -q_{02}\boldsymbol{v}_0, & s > 0, \\ (\boldsymbol{v}_0, \boldsymbol{e}^1, \boldsymbol{e}^2)(0) = (\boldsymbol{e}_3, \boldsymbol{e}_1, \boldsymbol{e}_2). \end{cases} \tag{A.2.3}$$

We have to check that $\boldsymbol{v}_0$ satisfies the compatibility conditions for (A.1.1), which is $\left(\boldsymbol{v}_0 \times \partial_s^{2n}\boldsymbol{v}_0\right)(0) = \mathbf{0}$ for $n \in \mathbf{N}$. Defining the matrix A as A$=$ $(\boldsymbol{v}_0, \boldsymbol{e}^1, \boldsymbol{e}^2)$, (A.2.3) can be written as

$$\mathrm{A}_s = \mathrm{A}\begin{pmatrix} 0 & q_{2s} & -q_{1s} \\ -q_{2s} & 0 & \dfrac{1}{2}|q|^2 \\ q_{1s} & -\dfrac{1}{2}|q|^2 & 0 \end{pmatrix} =: \mathrm{AP}.$$

Since P is anti-symmetric,

$$\left(\mathrm{AA}^T\right)_s = \mathrm{APA}^T + \mathrm{A}(\mathrm{AP})^T = \mathrm{A}(\mathrm{P}+\mathrm{P}^T)\mathrm{A} = \mathrm{O},$$

where A^T is the transpose matrix of A. Thus we have $\mathrm{AA}^T(s) = \mathrm{AA}^T(0) = \mathrm{I}_3$, where I_3 is the 3×3 unit matrix. This shows that $\{\boldsymbol{v}_0, \boldsymbol{e}^1, \boldsymbol{e}^2\}_{s\geq 0}$ is an orthonormal basis of $\mathbf{R}^3$. $\boldsymbol{e}^2$ is actually $\boldsymbol{v}_0 \times \boldsymbol{e}^1$, but we use $\boldsymbol{e}^2$ for simplicity. Differentiating the equation for $\boldsymbol{v}_0$ and using the other two equations, we get

$$\boldsymbol{v}_{0ss} = q_{01s}\boldsymbol{e}^1 - (q_{01})^2\boldsymbol{v}_0 + q_{02s}\boldsymbol{e}^2 - (q_{02})^2\boldsymbol{v}_0.$$

Taking the exterior product with $\boldsymbol{v}_0$ and setting $s=0$, we see that $(\boldsymbol{v}_0 \times \boldsymbol{v}_{0ss})(0) = \mathbf{0}$. Thus the condition is true for $n=1$. Suppose that it holds up to $n-1$. Differentiating the equation $2n-1$ times yields

$$\partial_s^{2n}\boldsymbol{v}_0 = \sum_{k_1=0}^{2n-1}\begin{pmatrix} 2n-1 \\ k_1 \end{pmatrix}(\partial_s^{k_1}q_{01})(\partial_s^{2n-1-k_1}\boldsymbol{e}^1) + \sum_{k_1=0}^{2n-1}\begin{pmatrix} 2n-1 \\ k_1 \end{pmatrix}(\partial_s^{k_1}q_{02})(\partial_s^{2n-1-k_1}\boldsymbol{e}^2).$$

At $s=0$, the terms where k_1 is odd are zero from the compatibility condition for q. When k_1 is even, $2n-1-k_1$ is an odd number greater than or equal to one. Setting $m_1 := 2n-1-k_1$, we have for $i=1,2$

$$\partial_s^{m_1}\boldsymbol{e}^i = -\sum_{k_2=0}^{m_1-1}\begin{pmatrix} m_1-1 \\ k_2 \end{pmatrix}(\partial_s^{k_2}q_{0i})(\partial_s^{m_1-1-k_2}\boldsymbol{v}_0).$$

Again only terms where k_2 is even remain. Then, m_1-1-k_2 is an even number less than or equal to $2(n-1)$ so that setting $k_1 = 2j_1$ and $k_2 = 2j_2$, we have

$$\begin{aligned}\partial_s^{2n}\boldsymbol{v}_0(0) = &\sum_{j_1=0}^{n-1}\begin{pmatrix} 2n-1 \\ 2j_1 \end{pmatrix}(\partial_s^{2j_1}q_{01})\left\{-\sum_{j_2=0}^{\frac{1}{2}(m_1-1)}\begin{pmatrix} m_1-1 \\ 2j_2 \end{pmatrix}(\partial_s^{2j_2}q_{01})(\partial_s^{m_1-1-2j_2}\boldsymbol{v}_0)\right\}\Bigg|_{s=0} \\ &+\sum_{j_1=0}^{n-1}\begin{pmatrix} 2n-1 \\ 2j_1 \end{pmatrix}(\partial_s^{2j_1}q_{02})\left\{-\sum_{j_2=0}^{\frac{1}{2}(m_1-1)}\begin{pmatrix} m_1-1 \\ 2j_2 \end{pmatrix}(\partial_s^{2j_2}q_{02})(\partial_s^{m_1-1-2j_2}\boldsymbol{v}_0)\right\}\Bigg|_{s=0}.\end{aligned}$$

Since the derivative of $\boldsymbol{v}_0$ is of even order less than or equal to $2(n-1)$ on the right-hand side, taking the exterior product with $\boldsymbol{v}_0$ yields $(\boldsymbol{v}_0 \times \partial_s^{2n}\boldsymbol{v}_0)(0) = \mathbf{0}$ according to the assumption of induction. Therefore, the $\boldsymbol{v}_0$ constructed here satisfies the compatibility conditions for (A.1.1).

A.3 Nonlinear Schrödinger equation to the LIE

In this section, given an initial datum $\boldsymbol{v}_0$, we construct the solution of (A.1.1) from the solution of (A.1.2) with an appropriate initial datum. First we define $\tilde{\boldsymbol{e}}^1(s)$ as the solution of

$$\begin{cases} \tilde{\boldsymbol{e}}^1_s + (\tilde{\boldsymbol{e}}^1 \cdot \boldsymbol{v}_{0s})\boldsymbol{v}_0 = \mathbf{0}, & s > 0, \\ \tilde{\boldsymbol{e}}^1(0) = \boldsymbol{e}_1. \end{cases}$$

In the same way as before, we see that $\tilde{\boldsymbol{e}}^1 \cdot \boldsymbol{v}_0 \equiv 0$ and $|\tilde{\boldsymbol{e}}^1| \equiv 1$. Thus, $\boldsymbol{v}_{0s}$ can be expressed as $\boldsymbol{v}_{0s} = q_{01}\tilde{\boldsymbol{e}}^1 + q_{02}(\boldsymbol{v}_0 \times \tilde{\boldsymbol{e}}^1)$. We use $q_0 := q_{01} - \mathrm{i}q_{02}$ as the initial datum. We first check that q_0 satisfies the compatibility conditions for (A.1.2). As before, set $\tilde{\boldsymbol{e}}^2 := \boldsymbol{v}_0 \times \tilde{\boldsymbol{e}}^1$. Then, $\boldsymbol{v}_0, \tilde{\boldsymbol{e}}^1, \tilde{\boldsymbol{e}}^2$ satisfies (A.2.3). Differentiating the equation with respect to s, we have

$$\boldsymbol{v}_{0ss} = q_{01s}\tilde{\boldsymbol{e}}^1 - (q_{01})^2\boldsymbol{v}_0 + q_{02s}\tilde{\boldsymbol{e}}^2 - (q_{02})^2\boldsymbol{v}_0.$$

From a compatibility condition for $\boldsymbol{v}_0$ we get

$$\begin{aligned} \mathbf{0} = (\boldsymbol{v}_0 \times \boldsymbol{v}_{0ss})(0) &= \left\{ q_{01s}(\boldsymbol{v}_0 \times \tilde{\boldsymbol{e}}^1) + q_{02s}(\boldsymbol{v}_0 \times \tilde{\boldsymbol{e}}^2) \right\}\Big|_{s=0} \\ &= \left\{ q_{01}\tilde{\boldsymbol{e}}^2 - q_{02s}\tilde{\boldsymbol{e}}^1 \right\}\Big|_{s=0}. \end{aligned}$$

Since $\tilde{\boldsymbol{e}}^1$ and $\tilde{\boldsymbol{e}}^2$ are orthogonal, $q_{01s}(0) = q_{02s}(0) = 0$. Suppose that the compatibility conditions up to order $n-1$ hold. As we did previously, taking into account the compatibility conditions that $\boldsymbol{v}_0$ satisfy and the assumption of induction, we arrive at

$$\begin{aligned} \partial_s^{2n}\boldsymbol{v}_0(0) &= (\partial_s^{2n-1}q_{01})\tilde{\boldsymbol{e}}^1 + (\partial_s^{2n-1}q_{02})\tilde{\boldsymbol{e}}^2\Big|_{s=0} \\ &\quad + \sum_{j_1=0}^{2n-2} \begin{pmatrix} 2n-1 \\ 2j_1 \end{pmatrix} (\partial_s^{2j_1}q_{01}) \left\{ - \sum_{j_2=0}^{\frac{1}{2}(m_1-1)} \begin{pmatrix} m_1-1 \\ 2j_2 \end{pmatrix} (\partial_s^{2j_2}q_{01})(\partial_s^{m_1-1-2j_2}\boldsymbol{v}_0) \right\}\Bigg|_{s=0} \\ &\quad + \sum_{j_1=0}^{2n-2} \begin{pmatrix} 2n-1 \\ 2j_1 \end{pmatrix} (\partial_s^{2j_1}q_{02}) \left\{ - \sum_{j_2=0}^{\frac{1}{2}(m_1-1)} \begin{pmatrix} m_1-1 \\ 2j_2 \end{pmatrix} (\partial_s^{2j_2}q_{02})(\partial_s^{m_1-1-2j_2}\boldsymbol{v}_0) \right\}\Bigg|_{s=0}, \end{aligned}$$

where $m_1 = 2n-1-2j_1$. Since $m_1 - 1 - 2j_2$ is even, $(\boldsymbol{v}_0 \times \partial_s^{m_1-1-2j_2}\boldsymbol{v}_0)(0) = \mathbf{0}$. Thus, we have

$$\mathbf{0} = (\boldsymbol{v}_0 \times \partial_s^{2n}\boldsymbol{v}_0)(0) = (\partial_s^{2n-1}q_{01})\tilde{\boldsymbol{e}}^2 - (\partial_s^{2n-1}q_{02})\tilde{\boldsymbol{e}}^1,$$

from which $\partial_s^{2n-1}q_0(0)=0$ follows. This implies that q_0 satisfies the compatibility conditions for (A.1.2).

Suppose that we have a smooth solution $q(s,t)$ of (A.1.2) with the initial datum just obtained. Set $q(s,t)=q_1(s,t)-\mathrm{i}q_2(s,t)$. For any $s\geq 0$, we extend the vectors $\boldsymbol{v}_0$, $\tilde{\boldsymbol{e}}$, and $\boldsymbol{v}_0\times\tilde{\boldsymbol{e}}$ in the t direction as the solution of

$$\begin{cases} \boldsymbol{v}_t=-q_{2s}\boldsymbol{e}^1+q_{1s}\boldsymbol{e}^2, & t>0,\\ \boldsymbol{e}^1_t=q_{2s}\boldsymbol{v}-\dfrac{1}{2}|q|^2\boldsymbol{e}^2, & t>0,\\ \boldsymbol{e}^2_t=-q_{1s}\boldsymbol{v}+\dfrac{1}{2}|q|^2\boldsymbol{e}^1, & t>0,\\ (\boldsymbol{v},\boldsymbol{e}^1,\boldsymbol{e}^2)(s,0)=(\boldsymbol{v}_0(s),\tilde{\boldsymbol{e}}(s),(\boldsymbol{v}_0\times\tilde{\boldsymbol{e}})(s)). \end{cases}$$

We express $\boldsymbol{v},\boldsymbol{e}^1,\boldsymbol{e}^2$ as column vectors. Then we have

$$\begin{aligned}(\boldsymbol{v},\boldsymbol{e}^1,\boldsymbol{e}^2)_t&=\Big(-q_{2s}\boldsymbol{e}^1+q_{1s}\boldsymbol{e}^2,q_{2s}\boldsymbol{v}-\frac{1}{2}|q|^2\boldsymbol{e}^2,-q_{1s}\boldsymbol{v}+\frac{1}{2}|q|^2\boldsymbol{e}^1\Big)\\ &=(\boldsymbol{v},\boldsymbol{e}^1,\boldsymbol{e}^2)\begin{pmatrix} 0 & q_{2s} & -q_{1s}\\ -q_{2s} & 0 & \frac{1}{2}|q|^2\\ q_{1s} & -\frac{1}{2}|q|^2 & 0\end{pmatrix}.\end{aligned}$$

As before, since the coefficient matrix is anti-symmetric, $\{\boldsymbol{v},\boldsymbol{e}^1,\boldsymbol{e}^2\}$ forms an orthonormal basis and $\boldsymbol{e}^2=\boldsymbol{v}\times\boldsymbol{e}^1$. From here we denote $\boldsymbol{e}^1$ as simply $\boldsymbol{e}$. Since $0=(\frac{1}{2}|\boldsymbol{v}|^2)_s=\boldsymbol{v}\cdot\boldsymbol{v}_s$, $\boldsymbol{v}_s$ can be expressed as

$$\boldsymbol{v}_s=\tilde{q}_1\boldsymbol{e}+\tilde{q}_2(\boldsymbol{v}\times\boldsymbol{e}).$$

From $|\boldsymbol{e}|\equiv 1$ and $\boldsymbol{e}\cdot\boldsymbol{v}\equiv 0$, we see that

$$\boldsymbol{e}_s=-\tilde{q}_1\boldsymbol{v}+\alpha(\boldsymbol{v}\times\boldsymbol{e}),\quad (\boldsymbol{v}\times\boldsymbol{e})_s=-\tilde{q}_2\boldsymbol{v}-\alpha\boldsymbol{e},$$

where $\tilde{q}_i$ and α are unknown functions. From the way we constructed $\tilde{\boldsymbol{e}}$, we see that at $t=0$

$$\tilde{q}_1=q_{01},\ \tilde{q}_2=q_{02},\ \alpha=0.$$

As before, from $\boldsymbol{v}_{st}=\boldsymbol{v}_{ts}$ and $\boldsymbol{e}_{st}=\boldsymbol{e}_{ts}$ we have

$$\begin{cases} \tilde{q}_{1t}=-q_{2ss}-\dfrac{1}{2}|q|^2\tilde{q}_2-\alpha q_{1s}, & t>0,\\ \tilde{q}_{2t}=q_{1ss}+\dfrac{1}{2}|q|^2\tilde{q}_1-\alpha q_{2s}, & t>0,\\ \alpha_t=\tilde{q}_1q_{1s}+\tilde{q}_2q_{2s}-\big(\dfrac{1}{2}|q|^2\big)_s, & t>0,\\ (\tilde{q}_1,\tilde{q}_2,\alpha)(s,0)=(q_{01}(s),q_{02}(s),0). \end{cases}$$

Setting $W_1 := \tilde{q}_1 - q_1$ and $W_2 := \tilde{q}_2 - q_2$, we have

$$\begin{cases} W_{1t} = -\dfrac{1}{2}|q|^2 W_2 - \alpha q_{1s}, & t > 0, \\ W_{2t} = \dfrac{1}{2}|q|^2 W_1 - \alpha q_{2s}, & t > 0, \\ \alpha_t = q_{1s} W_1 + q_{2s} W_2, & t > 0, \\ (W_1, W_2, \alpha)(s, 0) = (0, 0, 0). \end{cases}$$

This is represented in terms of $\boldsymbol{W} := (W_1, W_2, \alpha)^T$ as

$$\boldsymbol{W}_t = \begin{pmatrix} 0 & -\dfrac{1}{2}|q|^2 & -q_{1s} \\ \dfrac{1}{2}|q|^2 & 0 & -q_{2s} \\ q_{1s} & q_{2s} & 0 \end{pmatrix} \boldsymbol{W}.$$

Since the coefficient matrix is anti-symmetric, we have $|\boldsymbol{W}(s,t)| \equiv |\boldsymbol{W}(s,0)| = 0$, which is equivalent to $\tilde{q}_i = q_i$ for $i = 1, 2$ and $\alpha \equiv 0$. From direct calculation we have

$$\boldsymbol{v} \times \boldsymbol{v}_{ss} = (\boldsymbol{v} \times \boldsymbol{v}_s)_s = \left\{ q_1 (\boldsymbol{v} \times \boldsymbol{e}) - q_2 \boldsymbol{e} \right\}_s = q_{1s} (\boldsymbol{v} \times \boldsymbol{e}) - q_{2s} \boldsymbol{e} = \boldsymbol{v}_t.$$

From the boundary condition imposed on q, we see that

$$\boldsymbol{v}_t(0,t) = -q_{2s}(0,t)\boldsymbol{e} + q_{1s}(0,t)(\boldsymbol{v} \times \boldsymbol{e}) = \boldsymbol{0}.$$

Integrating this in t yields

$$\boldsymbol{v}(0,t) = \boldsymbol{v}_0(0) = \boldsymbol{e}_3.$$

Hence this function $\boldsymbol{v}$ is a solution of (A.1.1). We summarize the above results.

Theorem A.3.1 *Given an initial datum $\boldsymbol{v}_0$, the solution to* (A.1.1) *can be constructed from the solution to* (A.1.2) *with an appropriate initial datum.*

Conversely, given an initial datum q_0, the solution to (A.1.2) *can be constructed from the solution to* (A.1.1) *with an appropriate initial datum.*