Model updating for uncertain structures with interval parameters

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ABSTRACT:

This paper presents a model updating strategy for uncertain structures and its associated uncertainties by utilizing interval experimental modal data (e.g., natural frequencies and mode shapes). According to the eigenequation with respect to the interval parameters, the proposed methodology transforms model updating problem into two equivalent deterministic constrained optimization problems regarding the midpoint and uncertainties of interval parameters. Both the midpoint and associated uncertainties of the interval structural parameters could be obtained in iterative processes. The numerical results show the effectiveness of the proposed method. And the updated finite element model can generate reasonable interval modal data even when the experimental modal data are incomplete.

INTRODUCTION

Worldwide authorities have recognized the importance of structural management, damage identification, and structural health monitoring (SHM) in securing proper operation during the structure's lifetime. But even with the great advances in the field of structural modeling, an initial finite element model is often a poor reflection of actual structure. A significant discrepancy can be found when predicted dynamic properties from the FE model analysis are compared with the experimental results. The discrepancy could be attributed to several reasons[1]: (1) inaccuracy in the FE model discretization; (2) uncertainties in geometry and boundary conditions; (3) variations in the material properties; (4) environmental variability (such as temperature and wind) and (5) errors associated with measured signals and post processing techniques. During traditional model updating process, the experimental modal data is usually chosen as reference value which means that the experimental modal data is considered to be accurate. However, in most practical situation, the modal properties and mechanical properties are uncertain due to manufacturing errors, measurement errors and other factors. Therefore, the concept of uncertainty plays an important role in the investigation of various engineering problems. The most common approach to problems of uncertainty is to model the structural

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parameters as random variables. Under the circumstance, all information of the structural parameters is provided by joint probability density function (or distribution function) of structural parameters. But the probabilistic modeling is not the only way to describe the uncertainty, and also uncertainty is not equal to randomness. Moreover, the probabilistic approaches are not able to deliver the reliable results at the required precision without sufficient experimental data regarding the joint probability densities of the random variables or functions involved.

Since Moor in his monograph [2] established the basic theory for the interval analysis in 1966, the interval analysis has become a tool in many fields in recent years [3-5]. This paper assumes that the experimental modal data, the structural parameters are uncertain and treated as interval parameters. Based on the assumption, this paper presents a model updating strategy for uncertain structures by using experimental interval modal data. The proposed methodology transforms model updating problem into two deterministic constrained optimization problems regarding the midpoint and radius of interval parameters. The numerical results show the effectiveness of the method.

MATHEMATICAL FORMULATION

Interval characteristic matrices for structures with interval parameters

Assume that the interval parameters of structures are denoted by

$$\mathbf{p} = (p_1, p_2, \cdots, p_m)^T \in \mathbf{p^I} = \left(p_1^I, p_2^I, \cdots, p_m^I\right)^T \tag{1}$$

The midpoint of the vector $\mathbf{p^I}$ is

$$\mathbf{p^c} = \left(p_1^c, p_2^c, \cdots, p_m^c\right)^T \tag{2}$$

For each component of the vector, $p_j \in p_j^I = [\underline{p}_j, \overline{p}_j] = p_j^c + \Delta p_j e_j$, where the midpoint is $p_j^c = (\overline{p}_j + \underline{p}_j)/2$, the interval uncertainty (or the maximum width) is $\Delta p_j = (\overline{p}_j - \underline{p}_j)/2$, and $e_j = [-1,1]$. It can be seen from what has been discussed above that an interval parameter can be determined by the midpoint p_j^c and the interval uncertainty Δp_j uniquely. The following discussion will be limited to the cases where the interval uncertainties of the interval parameters are small compared to the midpoints, and the changes of parameters do not lead to the changes of shape of element. For any $p_j \in p_j^I$, using the first-order Taylor expansion, the characteristic matrices of a structure can be expressed as [6,7]

$$\begin{aligned}\mathbf{M(p)} &= \mathbf{M(p}^c) + \sum_{j=1}^{m} \left(\frac{\partial \mathbf{M(p)}}{\partial p_j} \right)_{\mathbf{p}=\mathbf{p}^c} (p_j - p_j^c) = \mathbf{M(p}^c) + \sum_{j=1}^{m} \left(\frac{\partial \mathbf{M(p)}}{\partial p_j} \right)_{\mathbf{p}=\mathbf{p}^c} \Delta p_j e_j \\ &= \mathbf{M(p}^c) + \Delta\mathbf{M(p)}\end{aligned} \tag{3}$$

$$\mathbf{K}(\mathbf{p}) = \mathbf{K}(\mathbf{p}^c) + \sum_{j=1}^{m}\left(\frac{\partial \mathbf{K}(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{p}^c}(p_j - p_j^c) = \mathbf{K}(\mathbf{p}^c) + \sum_{j=1}^{m}\left(\frac{\partial \mathbf{K}(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{p}^c}\Delta p_j e_j \quad (4)$$
$$= \mathbf{K}(\mathbf{p}^c) + \Delta\mathbf{K}(\mathbf{p})$$

Generally speaking, it is difficult to express the stiffness and mass matrices coefficients as explicit functions of design variables. The calculations of $(\partial \mathbf{M}(\mathbf{p})/\partial p_j)_{\mathbf{p}=\mathbf{p}^c}$ and $(\partial \mathbf{K}(\mathbf{p})/\partial p_j)_{\mathbf{p}=\mathbf{p}^c}$ can be approximated as follows

$$\partial \mathbf{K}(\mathbf{p})/\partial p_j = \frac{\Delta \mathbf{K}_j^c}{\Delta P_j} \quad \partial \mathbf{M}(\mathbf{p})/\partial p_j = \frac{\Delta \mathbf{M}_j^c}{\Delta P_j} \quad (5)$$

Where $\Delta \mathbf{K}_j^c = \mathbf{K}(p_1^c,\cdots,p_j^c+\Delta P_j,\cdots,p_m^c) - \mathbf{K}(p_1^c,\cdots,p_j^c,\cdots,p_m^c)$

$$\Delta \mathbf{M}_j^c = \mathbf{M}\left(p_1^c,\cdots,p_j^c+\Delta P_j,\cdots,p_m^c\right) - \mathbf{M}\left(p_1^c,\cdots,p_j^c,\cdots,p_m^c\right) \quad (6)$$

Method of FE model updating for uncertain structures based on interval eigenvalue analysis

For a discretized continuous uncertain structure, the i^{th} eigenvalue $\lambda_i(\mathbf{p})$ and the corresponding eigenvector (mode shape) $\boldsymbol{\varphi}_i(\mathbf{p})$ can be obtained from the *n* degrees of freedom finite element model by solving the eigenvalue equation

$$[\mathbf{K}(\mathbf{p}) - \lambda_i(\mathbf{p})\mathbf{M}(\mathbf{p})]\boldsymbol{\varphi}_i(\mathbf{p}) = 0 \quad (7)$$

where $\mathbf{K}(\mathbf{p})$ and $\mathbf{M}(\mathbf{p})$ are the interval characteristic stiffness and mass matrices, respectively. $\mathbf{p}$ is a set of interval structural parameters including geometric and material properties as well as boundary conditions. Substituting Eqs.(3)(4) (the interval characteristic stiffness and mass matrices $\mathbf{K}(\mathbf{p})$ and $\mathbf{M}(\mathbf{p})$ into Eq.(7),one can obtain[9,10]

$$\{[\mathbf{K}(\mathbf{p}^c) + \Delta\mathbf{K}(\mathbf{p})] - [\lambda_i(\mathbf{p}^c) + \lambda_i^I(\mathbf{p})][\mathbf{M}(\mathbf{p}^c) + \Delta\mathbf{M}(\mathbf{p})]\}[\boldsymbol{\varphi}_i(\mathbf{p}^c) + \Delta\boldsymbol{\varphi}_i(\mathbf{p})] = 0 \quad (8)$$

Expanding and neglecting the higher order parts

$$[\mathbf{K}(\mathbf{p}^c) - \lambda_i(\mathbf{p}^c)\mathbf{M}(\mathbf{p}^c)]\boldsymbol{\varphi}_i(\mathbf{p}) + \{[\Delta\mathbf{K}(\mathbf{p}) - \lambda_i(\mathbf{p}^c)\Delta\mathbf{M}(\mathbf{p})]\boldsymbol{\varphi}_i(\mathbf{p}) - \lambda_i^I(\mathbf{p})\mathbf{M}(\mathbf{p}^c)\boldsymbol{\varphi}_i(\mathbf{p})\} = 0 \quad (9)$$

Multiplying $\boldsymbol{\varphi}_i(\mathbf{p}^c)^T$ at the left of the equation, one can obtain eigenvalue with respect to the interval parameters as Eq.10. The following study based on the assumption that $\boldsymbol{\varphi}_i(\mathbf{p}) \approx \boldsymbol{\varphi}_i(\mathbf{p}^c)$ and $\boldsymbol{\varphi}_i(\mathbf{p}^c)^T\mathbf{M}\boldsymbol{\varphi}_i(\mathbf{p}) \approx 1$, because the eigenvectors are not sensitive to the small perturbation of structural parameters.

$$\lambda_i(\mathbf{p}) = \lambda_i(\mathbf{p}^c) + \lambda_i^I(\mathbf{p}) = \lambda_i(\mathbf{p}^c) + \frac{\boldsymbol{\varphi}_i(\mathbf{p}^c)^T[\Delta\mathbf{K}(\mathbf{p}) - \lambda_i(\mathbf{p}^c)\Delta\mathbf{M}(\mathbf{p})]\boldsymbol{\varphi}_i(\mathbf{p})}{\boldsymbol{\varphi}_i(\mathbf{p}^c)^T\mathbf{M}\boldsymbol{\varphi}_i(\mathbf{p})}$$
$$= \lambda_i(\mathbf{p}^c) + \boldsymbol{\varphi}_i(\mathbf{p}^c)^T[\sum_{j=1}^{m}\left(\frac{\partial \mathbf{K}(\mathbf{p})}{\partial P_j}\right)_{\mathbf{p}=\mathbf{p}^c}\Delta p_j e_j - \lambda_i(\mathbf{p}^c)\sum_{j=1}^{m}\left(\frac{\partial \mathbf{M}(\mathbf{p})}{\partial P_j}\right)_{\mathbf{p}=\mathbf{p}^c}\Delta p_j e_j]\boldsymbol{\varphi}_i(\mathbf{p}^c)$$

$$= \lambda_i(\mathbf{p}^c) + \sum_{j=1}^{m} \left| \boldsymbol{\varphi}_i(\mathbf{p}^c)^T \left[\left(\frac{\partial \mathbf{K}(\mathbf{p})}{\partial P_j} \right) - \lambda_i(\mathbf{p}^c) \left(\frac{\partial \mathbf{M}(\mathbf{p})}{\partial P_j} \right) \right] \boldsymbol{\varphi}_i(\mathbf{p}^c) \right|_{\mathbf{p}=\mathbf{p}^c} \Delta p_j [-1,1]$$

$$= \lambda_i(\mathbf{p}^c) + \left(\sum_{j=1}^{m} S_{ij} \Delta p_j \right) [-1,1] = \left[\lambda_i(\mathbf{p}^c) - \sum_{j=1}^{m} S_{ij} \Delta p_j , \lambda_i(\mathbf{p}^c) + \sum_{j=1}^{m} S_{ij} \Delta p_j \right] \tag{10}$$

So the eigenvalue vector $\Delta\boldsymbol{\Lambda}(\mathbf{p})$ can be expressed as $\Delta\boldsymbol{\Lambda}(\mathbf{p})_{N\times 1} = S_{N\times m}\Delta\mathbf{p}_{m\times 1}$

where $\Delta\boldsymbol{\Lambda}(\mathbf{p})_{N\times 1} = (\lambda_1^I(\mathbf{p}), \lambda_2^I(\mathbf{p}), \cdots, \lambda_N^I(\mathbf{p}))^T$, $\Delta\mathbf{p}_{m\times 1} = (\Delta p_1, \Delta p_2, \cdots, \Delta p_m)^T$ and

$$S_{ij} = \left| \boldsymbol{\varphi}_i(\mathbf{p}^c)^T \left[\left(\frac{\partial \mathbf{K}(\mathbf{p})}{\partial p_j} \right) - \lambda_i(\mathbf{p}^c) \left(\frac{\partial \mathbf{M}(\mathbf{p})}{\partial p_j} \right) \right] \boldsymbol{\varphi}_i(\mathbf{p}^c) \right|_{\mathbf{p}=\mathbf{p}^c}$$

The interval experimental modal data are chosen as reference value, the model updating for uncertain structures based on interval analysis can be transformed into deterministic constrained optimization problems as follows

$$\text{Min} \qquad J(\mathbf{p}^c) = \sum_{i=1}^{N} W_{\varepsilon i} (\lambda_{ei}^c - \lambda_{ci}(\mathbf{p}^c))^2$$

$$\text{Subject to} \begin{cases} [\mathbf{K}(\mathbf{p}^c) - \lambda_{ci}(\mathbf{p}^c)\mathbf{M}(\mathbf{p}^c)]\boldsymbol{\varphi}_i(\mathbf{p}^c) = 0 \\ 1 - \dfrac{(\boldsymbol{\varphi}_{ci}{}^T \boldsymbol{\varphi}_i(\mathbf{p}^c))^2}{(\boldsymbol{\varphi}_{ci}{}^T \boldsymbol{\varphi}_{ci})(\boldsymbol{\varphi}_i(\mathbf{p}^c)^T \boldsymbol{\varphi}_i(\mathbf{p}^c))} \le \delta \\ \mathbf{p}_L \le \mathbf{p}^c \le \mathbf{p}_U \end{cases} \tag{11}$$

where λ_{ei}^c and $\lambda_{ci}(\mathbf{p}^c)$ are the i^{th} midpoint of interval experimental and computed eigenvalue with respect to the mean structural parameters $\mathbf{p}^c$. $\mathbf{p}_L$ and $\mathbf{p}_U$ are the lower and upper bound of the structural parameter vector **p**, which are large perturbation of the estimated value.

$$\text{Min} \quad J(\Delta\mathbf{p}) = (\Delta\boldsymbol{\Lambda}_e - \Delta\boldsymbol{\Lambda}(\mathbf{p}))^T \mathbf{W}_{\xi} (\Delta\boldsymbol{\Lambda}_e - \Delta\boldsymbol{\Lambda}(\mathbf{p})) = \sum_{i=1}^{N} W_{\xi i} (\lambda_{ei}^I - \lambda_i^I(\mathbf{p}))^2$$

$$\text{Subject to} \;\; 0 \le \Delta p_j \le \min(p^c - p_{jL}, p_{jU} - p^c), j = 1,2,\cdots,m \tag{12}$$

where λ_{ei}^I and $\lambda_i^I(\mathbf{p})$ are the interval uncertainty of i^{th} experimental and computed eigenvalue, $\mathbf{p}^c$ represents the best solution of optimization problem given by Eq. (11). It can be seen that the predictable part and associated uncertainties of structural parameter vector **p** can be achieved by solving the former two deterministic optimization problems.

NUMERICAL RESULTS

The correctness of proposed model updating procedure for uncertain structures

In order to demonstrate the correctness of the proposed method, two numerical examples are considered. The results are given in the Tables 1~6, where $\underline{f}_{\exp}$ and $\overline{f}_{\exp}$ are the lower and upper bounds of experimental frequencies; $\underline{f}_{opt}$ and $\overline{f}_{opt}$ are the lower and upper frequency bounds of updated finite element model. Because of the inevitable errors during the model updating procedure, the δ in the Eqs.(11)(12) is given a very small value (0.5% in this study) to ensure the quick searching capacity and the high precision of the undated results.

Example 1 Consider a frame of multistory structure shown in Fig.1. Suppose that the element stiffness and mass of the structure are all uncertain parameters. And all total five interval experimental modal parameters (frequencies and mode shapes) are chosen as reference value for model updating process. The bounds and updated value of the 10 interval structural parameters are given in Table 1. The true element stiffness are as follows: k_1= [2000, 2020]N/m, k_2=[1800, 1850]N/m, k_3=[1600, 1630]N/m, k_4=[1400, 1420]N/m, k_5=[1200, 1210]N/m and the true element mass were m_1=[29, 31]kg, m_2=[26, 28]kg, m_3=[26, 28]kg, m_4=[24, 26]kg, m_5=[17, 19]kg [8].

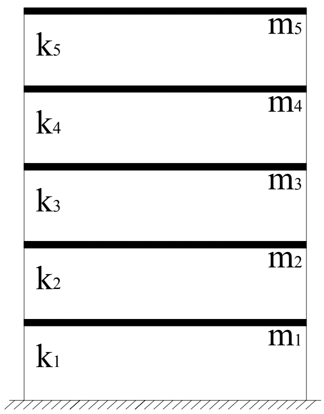

Figure 1: Frame of a multistory structure

Table 1: Midpoint of structural parameters for updated uncertain structures

Parameters	Initial Value	Lower Bounds	Upper Bounds	Updated Value	Midpoint error
m_1^c	Rand(27.0,33.0)	27.0	33.0	29.95	−0.183%
m_2^c	Rand(24.3,29.7)	24.3	29.7	26.96	−0.141%
m_3^c	Rand(24.3,29.7)	24.3	29.7	26.92	−0.289%
m_4^c	Rand(22.5,27.5)	22.5	27.5	24.96	−0.168%
m_5^c	Rand(16.2,19.8)	16.2	19.8	17.95	−0.306%
k_1^c	Rand(1809.0,2211.0)	1809.0	2211.0	2009.8	−0.010%
k_2^c	Rand(1642.5,2007.5)	1642.5	2007.5	1825.3	0.016%
k_3^c	Rand(1453.5,1776.5)	1453.5	1776.5	1613.8	−0.074%
k_4^c	Rand(1269.0,1551.0)	1269.0	1551.0	1410.2	0.014%
k_5^c	Rand(1084.5,1325.5)	1084.5	1325.5	1204.4	−0.050%

Table 2: Uncertainties of parameters

Uncertainty	Initial Value	Lower Bounds	Upper Bounds	Updated $\Delta\boldsymbol{p}$	$\underline{P}$ error	$\overline{P}$ error
Δm_1	Rand(0,2.945)	0	2.945	1.07	−0.427%	0.045%
Δm_2	Rand(0,2.662)	0	2.662	1.04	−0.307%	0.014%
Δm_3	Rand(0,2.622)	0	2.622	0.84	0.326%	−0.859%
Δm_4	Rand(0,2.458)	0	2.458	1.05	−0.378%	0.025%
Δm_5	Rand(0,1.745)	0	1.745	1.04	−0.578%	−0.062%
Δk_1	Rand(0,200.8)	0	200.8	7.42	0.119%	−0.138%
Δk_2	Rand(0,182.2)	0	182.2	24.76	0.030%	0.003%
Δk_3	Rand(0,160.3)	0	160.3	18.25	−0.278%	0.126%
Δk_4	Rand(0,140.8)	0	140.8	6.92	0.234%	−0.203%
Δk_5	Rand(0,119.9)	0	119.9	3.85	0.046%	−0.145%

Table 3: Frequency comparisons between the true values and computation

Modes	$\underline{f}_{\exp}$ (Hz)	$\overline{f}_{\exp}$ (Hz)	$\underline{f}_{opt}$ (Hz)	$\overline{f}_{opt}$ (Hz)	Midpoint error	$\underline{f}$ error	$\overline{f}$ error
1	0.385	0.406	0.386	0.406	0.111%	0.112%	0.111%
2	1.032	1.083	1.033	1.084	0.104%	0.107%	0.102%
3	1.582	1.659	1.584	1.661	0.096%	0.101%	0.090%
4	2.001	2.098	2.003	2.100	0.100%	0.100%	0.100%
5	2.304	2.414	2.306	2.416	0.089%	0.087%	0.091%

It can be seen from the results that the midpoint differences between the true and updated uncertain structural parameters are no more than 1% as shown in Table 1 and the frequency response differences are less than 0.2% as shown in Table 3. The proposed model updating procedure can generate frequency response close to the true values.

The influence of incomplete modal properties on the model updating results

The experimental modal parameters are limited in real application while the number of structural parameters is relatively much large. The situation that the modal parameters are incomplete during the finite element model updating process is very common. So it is important to understand the influence of incomplete modal properties on the updated results.

In order to check the influence of incomplete modal properties on the updated results, the experimental modal parameters are supposed to be noise free. Tables 3-6 illustrate model updating results for Example 1 using only five interval frequencies as reference values. Only two of the midpoint errors are bigger than 1%, all the other midpoint error fall within 1% as shown in Table 4. The midpoint frequency differences between the experimental and that computed by the updated model are no more than 0.15% as shown in Table 6.

Table 4: Midpoint of structural parameters for updated uncertain structures

Parameters	Initial Value	Lower Bounds	Upper Bounds	Updated Value	Midpoint error
m_1^c	Rand(27.0,33.0)	27.0	33.0	29.85	−0.497%
m_2^c	Rand(24.3,29.7)	24.3	29.7	27.22	0.800%
m_3^c	Rand(24.3,29.7)	24.3	29.7	26.84	−0.611%

m_4^c	Rand(22.5,27.5)	22.5	27.5	25.28	1.124%
m_5^c	Rand(16.2,19.8)	16.2	19.8	17.97	−0.161%
k_1^c	Rand(1809.0,2211.0)	1809.0	2211.0	2021.6	0.577%
k_2^c	Rand(1642.5,2007.5)	1642.5	2007.5	1850.9	1.419%
k_3^c	Rand(1453.5,1776.5)	1453.5	1776.5	1602.7	−0.762%
k_4^c	Rand(1269.0,1551.0)	1269.0	1551.0	1415.6	0.397%
k_5^c	Rand(1084.5,1325.5)	1084.5	1325.5	1205.4	0.033%

Table 5: Uncertainties of parameters

Uncertainty	Initial Value	Lower Bounds	Upper Bounds	Updated $\Delta \boldsymbol{p}$	$\underline{P}$ error	$\overline{P}$ error
Δm_1	Rand(0,2.945)	0	2.945	0.96	−0.361%	−0.624%
Δm_2	Rand(0,2.662)	0	2.852	1.04	0.693%	0.899%
Δm_3	Rand(0,2.622)	0	2.484	0.92	−0.324%	−0.878%
Δm_4	Rand(0,2.458)	0	2.535	1.07	0.886%	1.343%
Δm_5	Rand(0,1.745)	0	2.219	0.91	0.386%	−0.650%
Δk_1	Rand(0,200.8)	0	1.771	18.01	0.179%	0.971%
Δk_2	Rand(0,182.2)	0	189.4	19.29	1.756%	1.092%
Δk_3	Rand(0,160.3)	0	156.7	20.11	−1.088%	−0.441%
Δk_4	Rand(0,140.8)	0	149.2	9.81	0.414%	0.381%
Δk_5	Rand(0,119.9)	0	135.4	6.71	−0.109%	0.174%

Table 6: Frequency comparisons between the true values and compututation

Modes	$\underline{f}_{\exp}$ (Hz)	$\overline{f}_{\exp}$ (Hz)	$\underline{f}_{opt}$ (Hz)	$\overline{f}_{opt}$ (Hz)	Midpoint error	$\underline{f}$ error	$\overline{f}$ error
1	0.385	0.406	0.386	0.406	0.110%	0.117%	0.103%
2	1.032	1.083	1.033	1.084	0.095%	0.097%	0.092%
3	1.582	1.659	1.584	1.661	0.093%	0.082%	0.102%
4	2.001	2.098	2.003	2.100	0.098%	0.110%	0.086%
5	2.304	2.414	2.306	2.416	0.087%	0.091%	0.083%

CONCLUSIONS

In this paper, a model updating method for uncertain structures with interval parameters is proposed. The proposed method can generate the interval structural parameters using experimental modal data. Numerical results reveal that the proposed method can obtain precise structural interval parameters when the reference modal properties are over determined. Even if the reference modal properties are incomplete, the updated uncertain structures can generate interval modal frequencies close to the true value and reflect the real modal properties. The proposed method has strong numerical stability and the capacity of noise resistance.

The model updating method discussed above is general enough that it can be applied to the model updating for real uncertain structures with interval parameters. Because the present approach is based on the first order Taylor expansion, the application of the approach is limited to the cases where the interval uncertainties of the interval parameters are small compared to the midpoints. If the interval uncertainties of the interval parameters are fairly large, in order to obtain higher computing accuracy, the second order Taylor expansion should be considered.

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